**Computational Hydraulics**
**ProfessorAnirban Dhar**
**Department of Civil Engineering**
**Indian Institute of Technology Kharagpur**
**Lecture 11**
**Partial Differential Equation: Numerical Stability of IBVP**

Welcome to this lecture number 11 of the course computational hydraulics.We are in model number 2, numerical methods. And in this lecture we will be covering unit 7, partial differential equation, numerical stability of initial boundary value problem.

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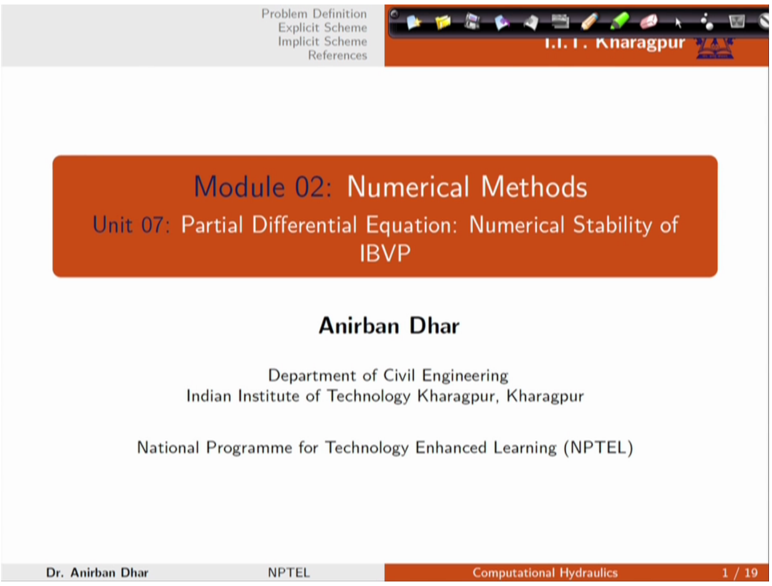


So what is the learning objective for this particular unit? At the end of this unit students will be able to analyze the numerical stability of the discretized partial differential equation.

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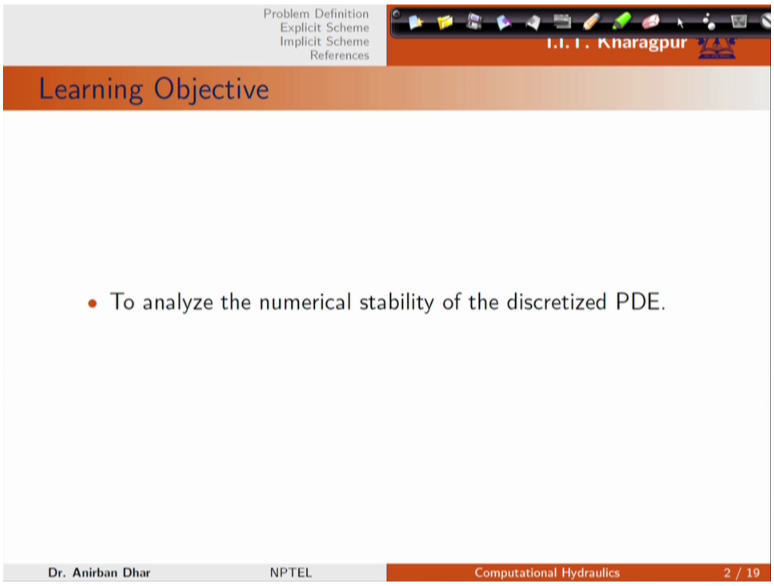


In lecture 10 we have discussed the initialboundary value problem, IBVP in terms of this temporal derivative and 2, second order spatial derivatives and one source sink term.

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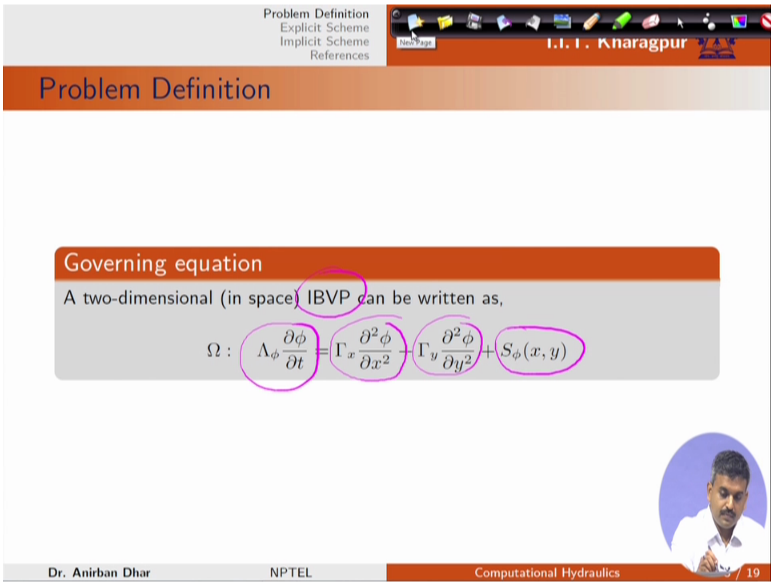


In this case we have discretized the equation and we have utilized 3 schemes, explicit, implicit and Crank Nicolson. All these themes, we need to definethis del x, del y and del t for each schemes. So depending on the value of del x, del y, del t whether there will be changed in thefinal result, we need to see that thing from numerical stability.

This is the initial problem definition. Initial condition and boundary condition for the problemwith rectangular domain.

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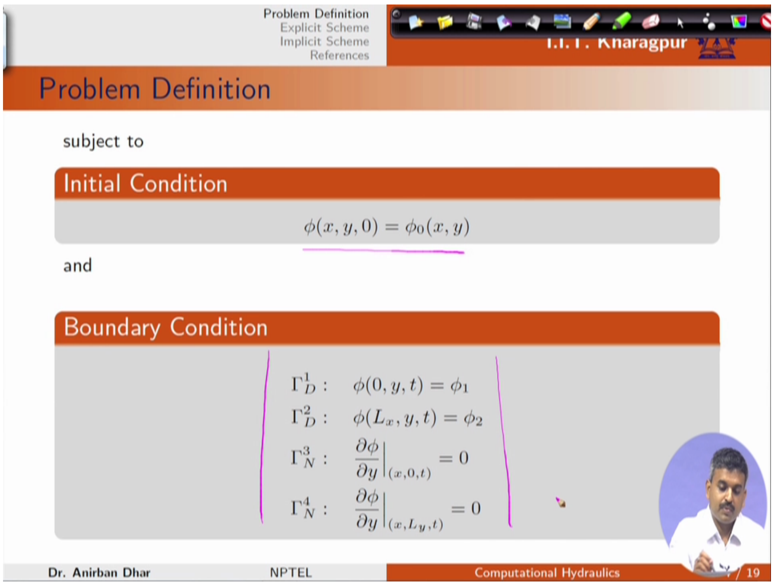


Now we have errors. We can define this discretization error as the difference between analytical solution of the PDE which is the closed form solution and the exact solution of the finite difference equation obtained on hypothetical infinite precision computer. This is involves truncation error, error due to treatment of boundary condition.

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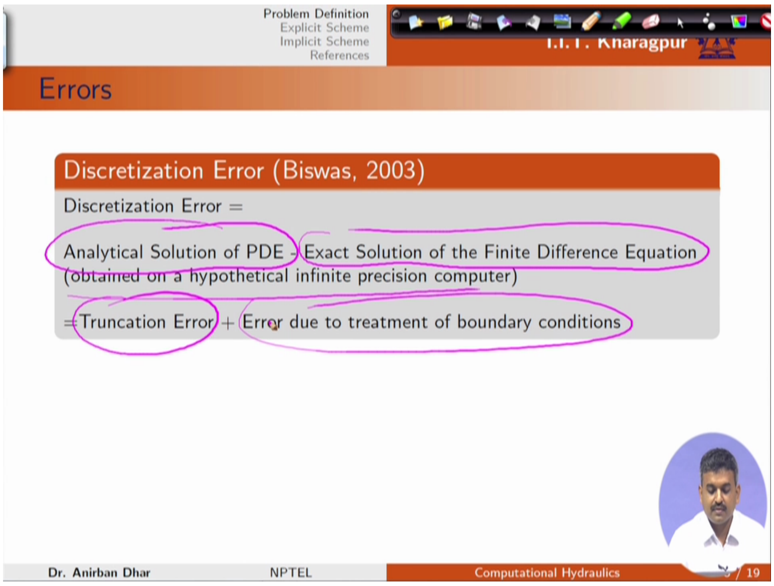


Now if you consider the round off error, round off error is numerical solution of the finite difference equation obtained from finite precision computer and exact solution of the finite differences equation obtained on a hypothetical infinite precision computer.

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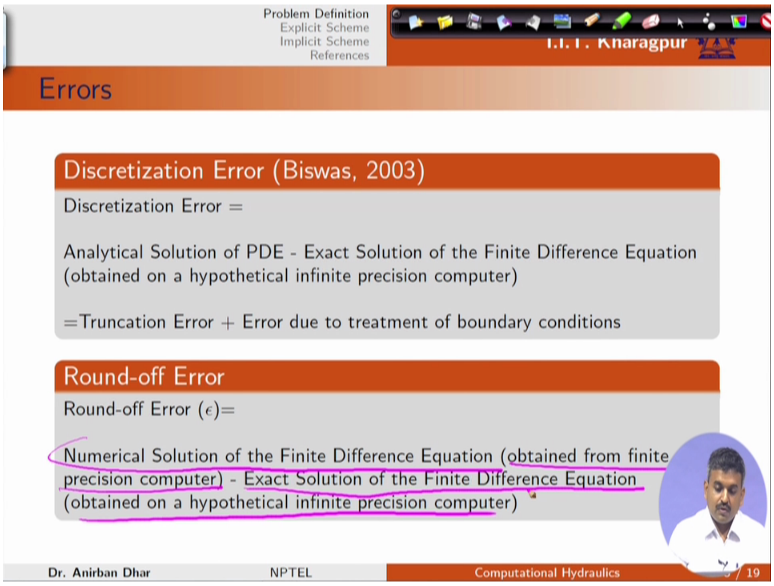


So for any problem if we discretize equation and we are solving the equation infinite precision computer there will be some amount of error involved due to round off error. We need to see few things. For this numerical errors, every algorithm requires repeated operations that is plus, minus, or addition, subtraction, multiplication, division. So there will be accumulation of round off error. And in time stepping algorithm whatever we have seen in our lecture number 10, accumulated round off error may magnify or reduce with every step.

Error may increase exponentially. It is known as numerical instability.Numerical stability or (in) instability is a property of the algorithm and discretization of partial differential equation plus boundary conditions. And it does not depends on the computer used. We need to check our discretization scheme to check the numerical instability for the problem.

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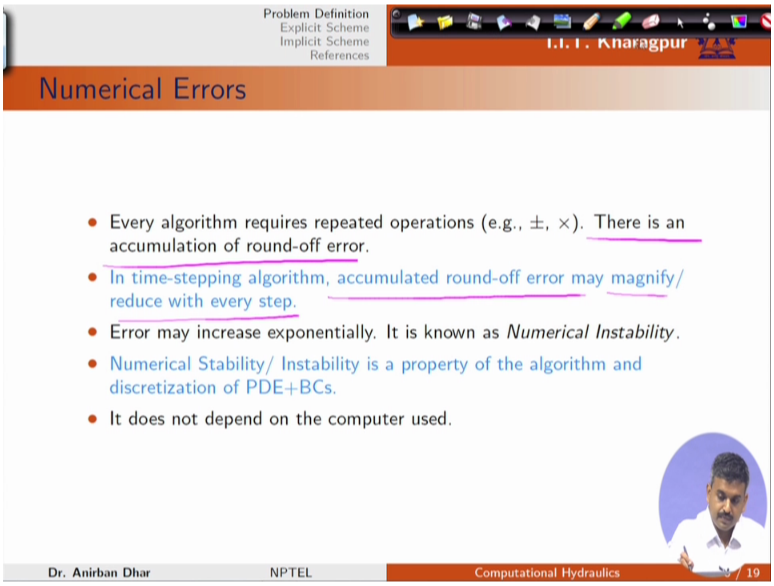


Stability analysis. In stability analysis of linear PDE, we analyze only one arbitrary Fourier mode.Let us consider that error can be represented in form of Fourier Series and single arbitrary term can be written as,epsilon ij n and An is the amplitude. in this case, the omega x and omega y are the wave numbers corresponding to X and Y directions respectively. And square root of minus 1 is the imaginary number. And i and j, these are corresponding toX and Y directions.

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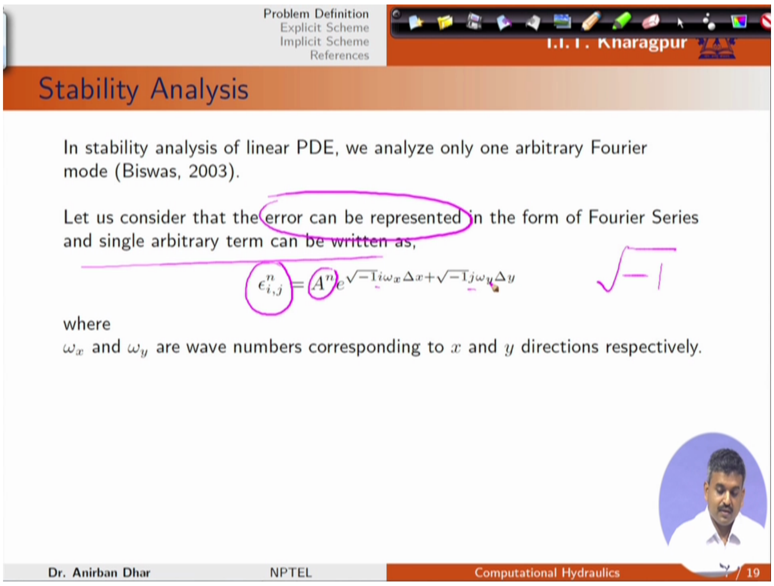


With this information we can say that if we take modulus of this error, obviously this depends on the amplitude term. Not on this term because modulus of this term will be obviously 1.

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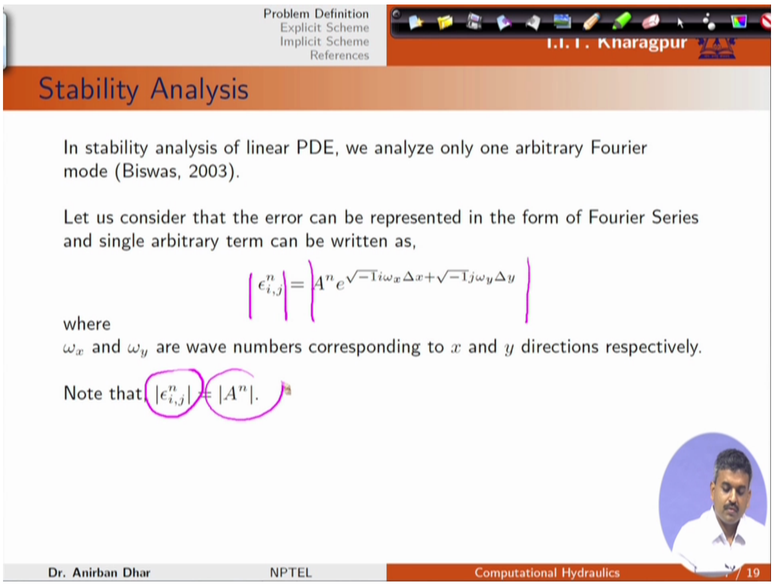


So with this we can simplify the error term. We can write it in the form of face values where phi x, where phi y for X and Y directions. And this is basically our omega x into del x and where phi y is omega y into del y.

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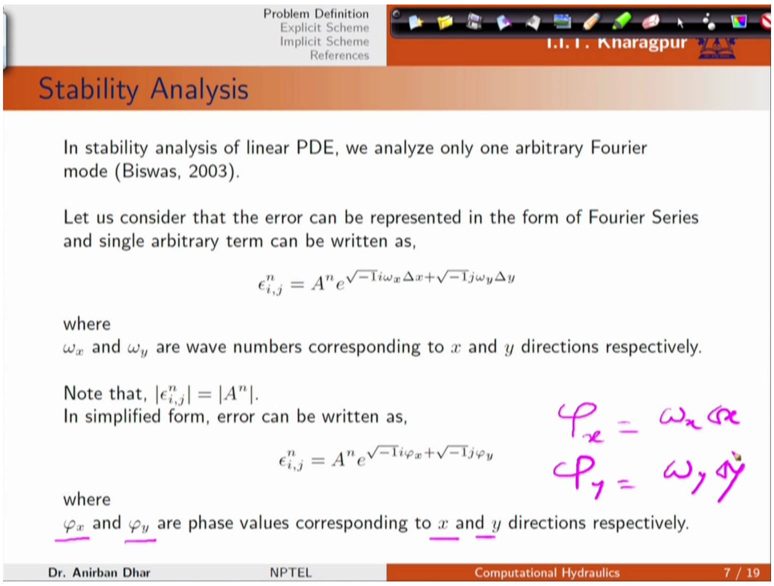


We can define this amplification term. It governs the growth of the Fourier components and in this casewe can define Von Neumann stability condition. This is modulus of G should be less than equal to 1. If modulus of G is greater than 1, error grows. This is unstable scheme. If we have modulus of G less than one, error reduces. This is stable scheme. If we have modulus of G equals to 1, error remains same. This is neutrally stable scheme.

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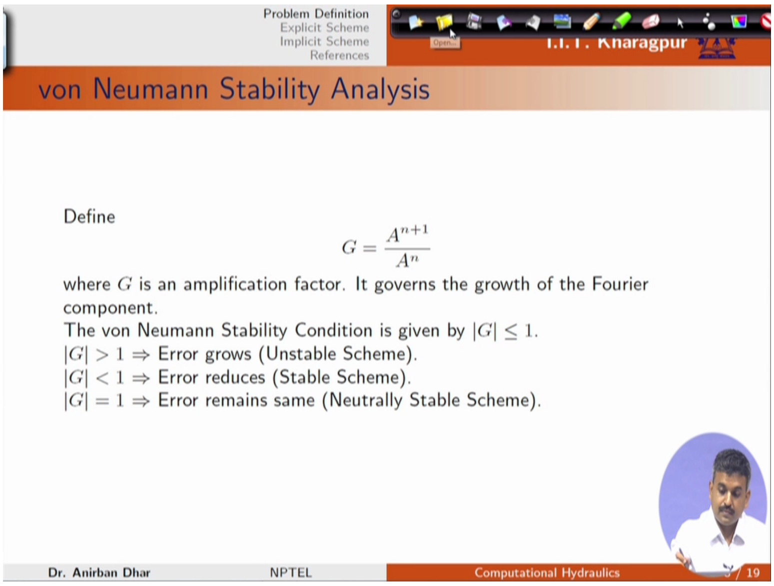


Now we have discretized our initial boundary value problem using explicit scheme.We can write the same thing here. Now phi ij n plus 1 or phi ij n, this is obtained from finite precision computer.

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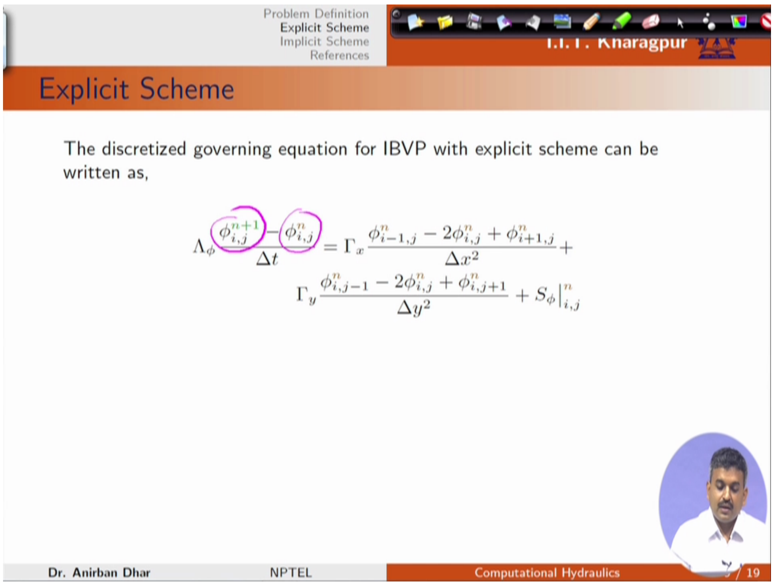


Sowe can write the general variable phi in terms of this phi hat ij n and epsilon ij n where phi ij n is a numerical solution obtained from finite precision computer and phi hat ijn is the exact discrete solution obtained on hypothetical infinite precision computer. And epsilon ij n is the accumulated round off error at level n.

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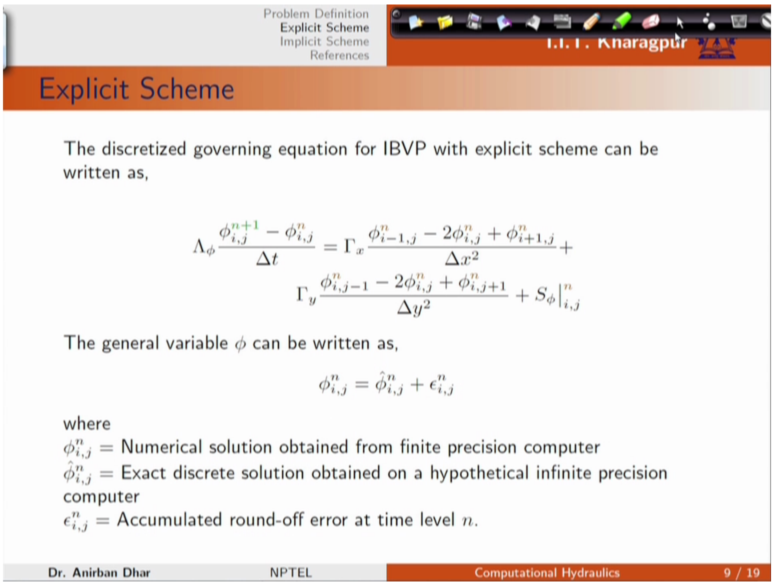


With this we can write our discretized governing equation with explicit scheme as, we can replace this phi ij n with phi ij hat n plus epsilon ij n. So this is actually our discretized governing equation.

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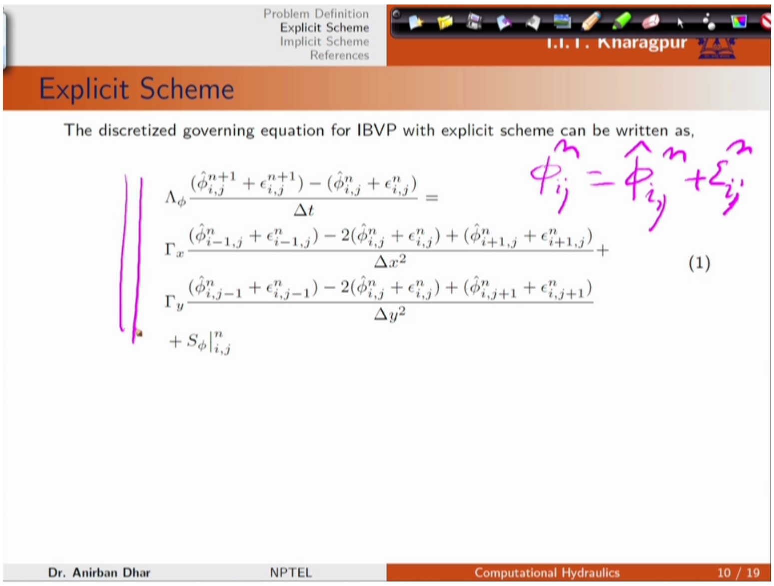


But we have discretized our governing equation with assumption that we will get the infinite solution from infinite precision computer. So we can writeour governing equation with this exact discrete solution phi hat ij n plus phi ij n. So ideally this should be satisfied. This equation number 2.

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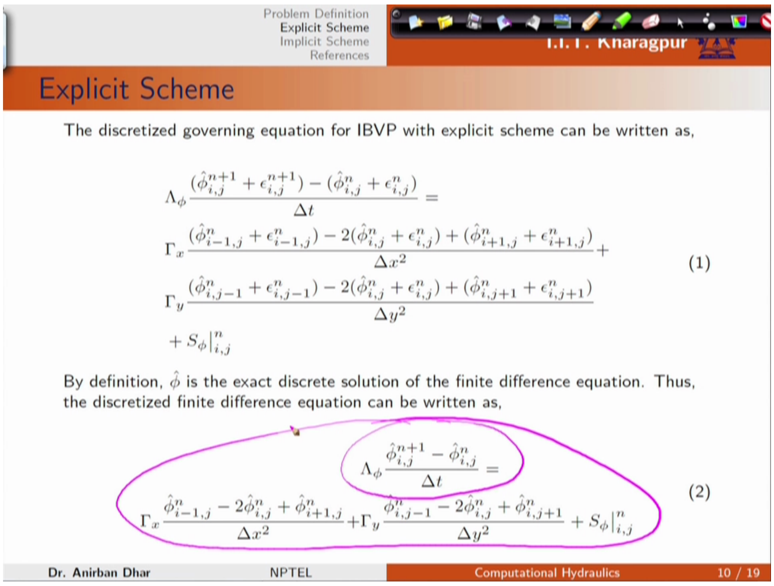


So if we subtract equation number 2 from 1, then we can getthis error equation. Error equation is similar to our originaldiscretized form but without source sink term, because there will be no error involved there due to discretization. So in simplified form we can write it as epsilon ij n plus 1 and other terms on the right hand side with alpha x, alpha y like this.

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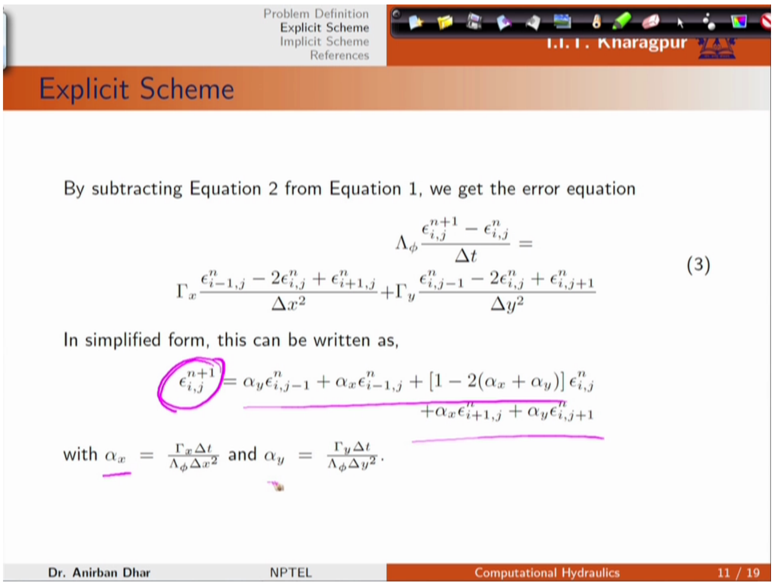


Now in explicit scheme, we can define this epsilon ij n plus 1as An plus 1. Only change in amplitude but there is no change in X or Y direction in this values. And for epsilon ij n, only change in amplitude. Epsilon I minus 1 jn, change in the index for X. in this case again change in the index for X. Change for the index for Y, change in the index for Y.

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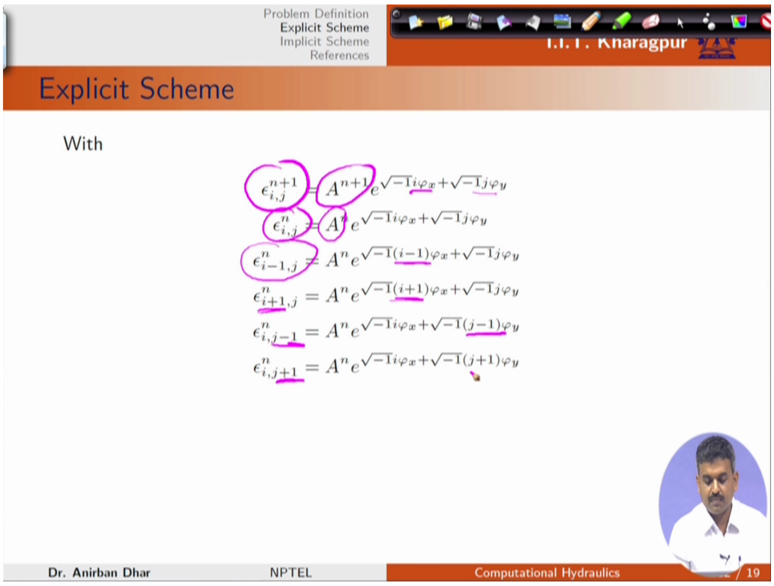


With this information we can write our error equation. With this simplification, like this.

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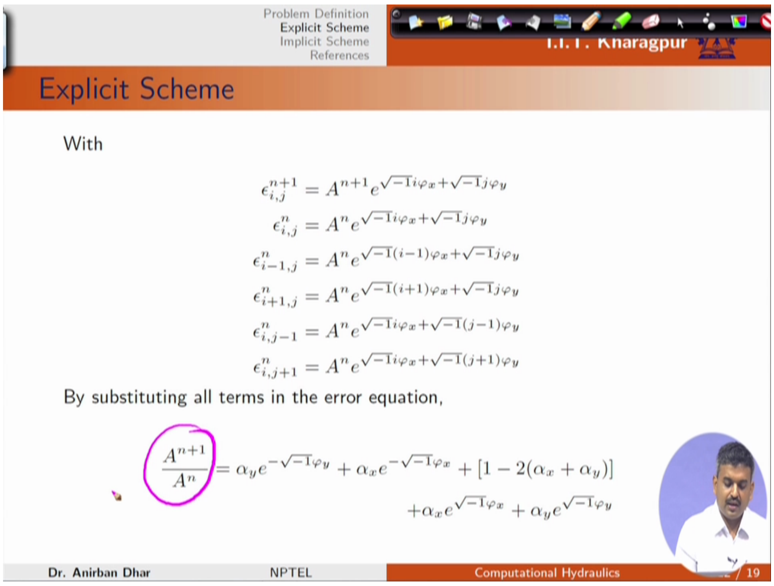


This is essentially our amplification term and these are the known things on the right hand side.

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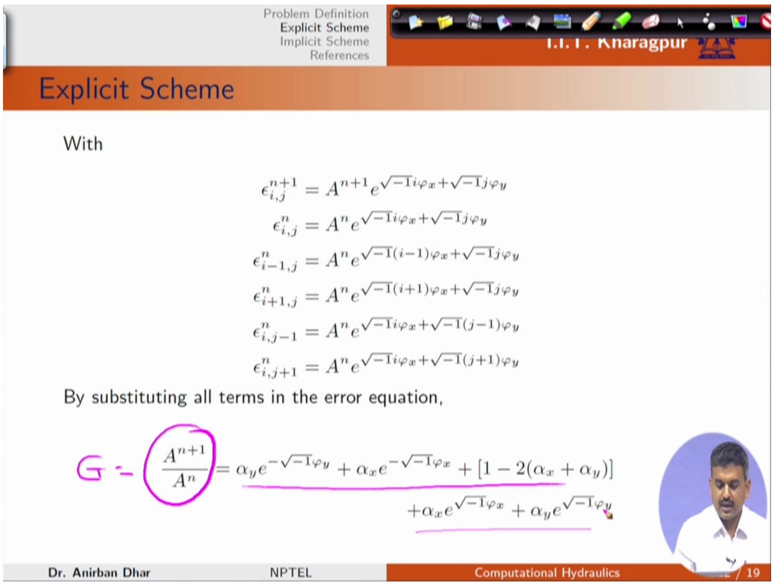


The growth factor or amplification term can be written like this. Essentially in our last equation. This is into the power minus imaginary number into phi y. If we combine these two term we will get 2 cos phi y into alpha y.

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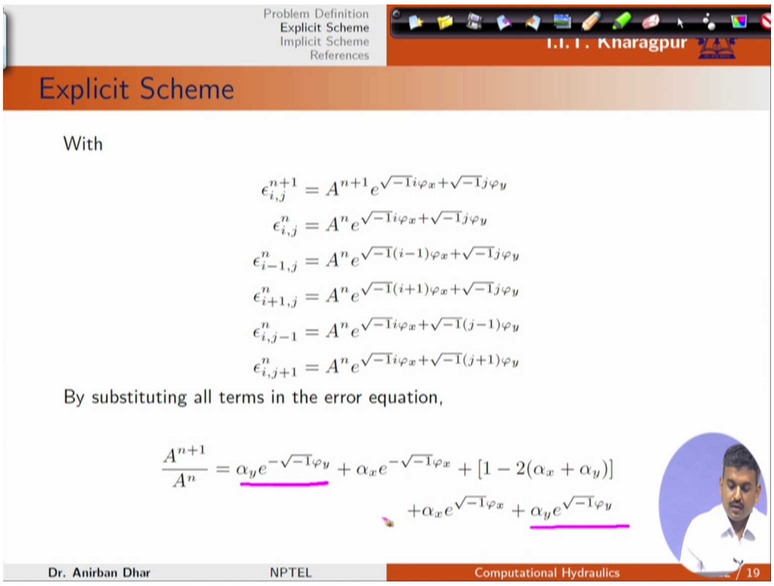


In this case if we combine these two related to X we will get, cosof var phi x. So I can just write it here this exponential of minus, minus 1 var phi x plus e to the power minus1 var phi x. This will be 2 cos var phi x.

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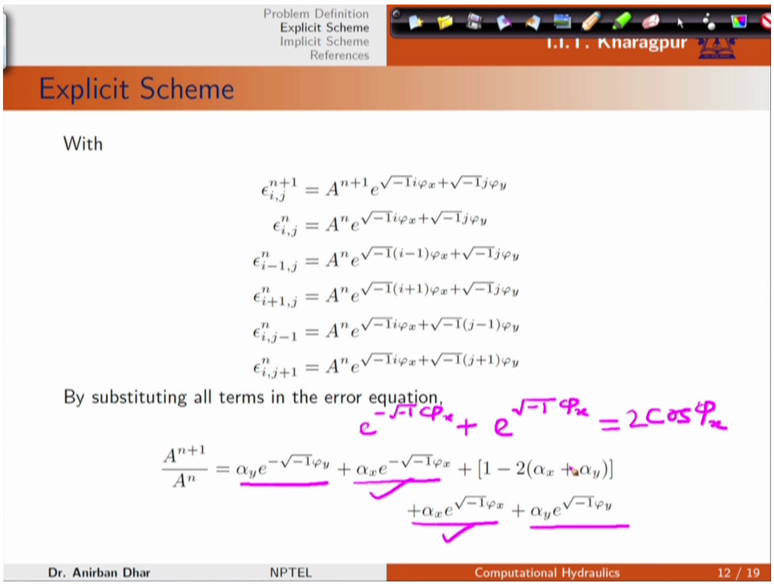


Similarly for Y direction. Now we can simplify the right hand side write it like this. And again for this cos phi var phi y minus 1, we can write it as minus 2 sin square var phi y by 2. So with this, this is our growth term or amplification factor, 1 minus 4 alpha y sin square var phi y by 2, minus 4 x sin square var phi x by 2.

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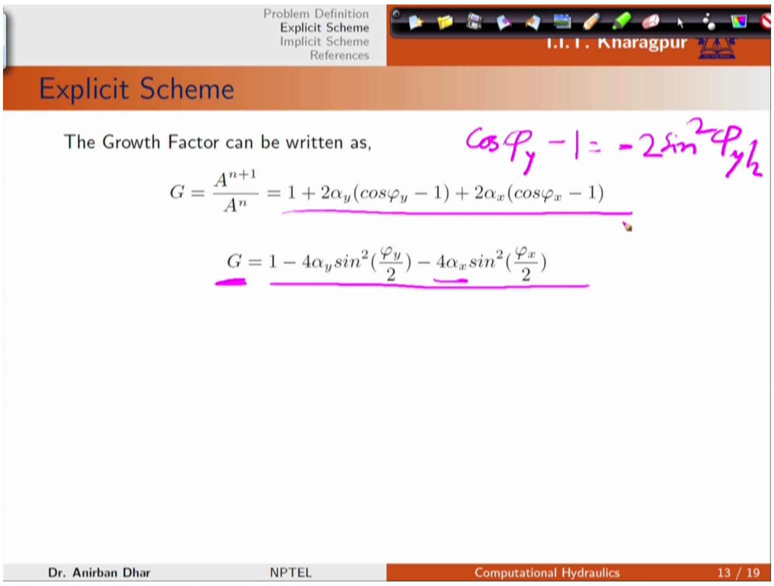


In this caseif you want to check the Von Neumann stability condition,we have mode of this term here. And this should be less than equal to 1 which is minus 1 less than equal to 1 and this is withinthis limit.

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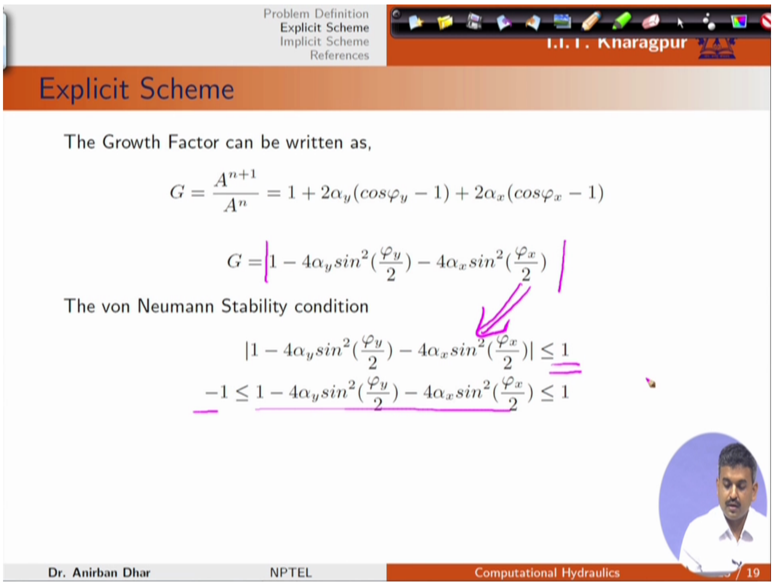


We have 2 cases uh or extreme ones, where sin phi x by 2, sin var phi y by 2,these value are zero. And this means that G is equal to 1.The scheme is neutrally stable.

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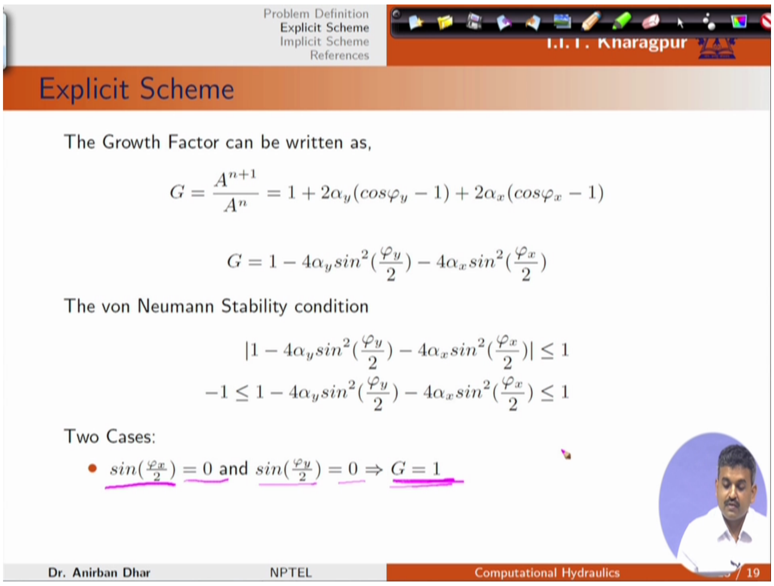


However if we apply this condition thatsin var phi x by 2 equals to 1 and var phi y by 2 equals to 1, then comes this G equals to 1 minus 4 alpha x plus alpha y. And this is less than equal to half. Because minus 1, this is 1 minus 4 alpha x minus 4 alpha y. If we change sides, this will be 4 alpha x plus 4 alpha y less than equals to 2. And from here it is coming alpha x plus alpha y is less than equals to half.

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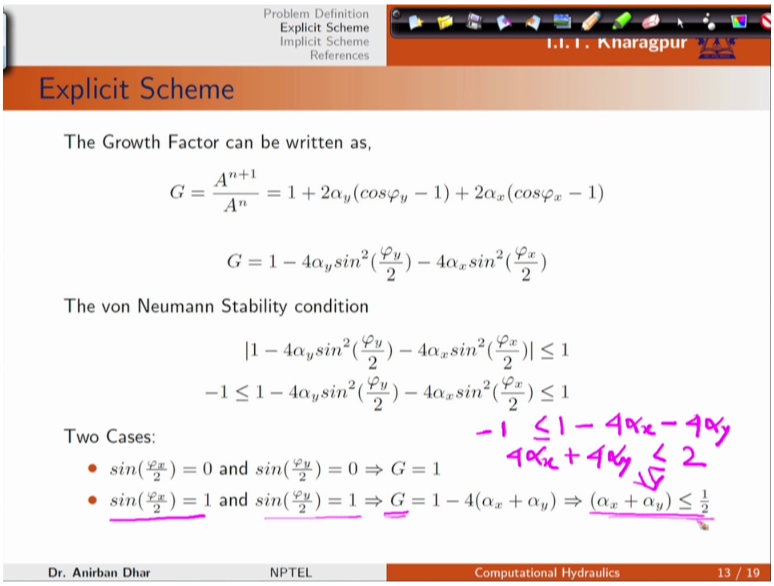


So we can comment on the stability of the scheme here that explicit scheme is conditionally stable. This alpha x, alpha y, this addition should be less than equals to half. Interestingly this alpha x is gamma x delta t divided by lambda phi into del x square. So in this case there is this delta t term and del x square term. Similarly for alphay, we have alpha y, we have gamma y, del t divided by this lambda phi del y square. So it relates our delt, delx, del y.

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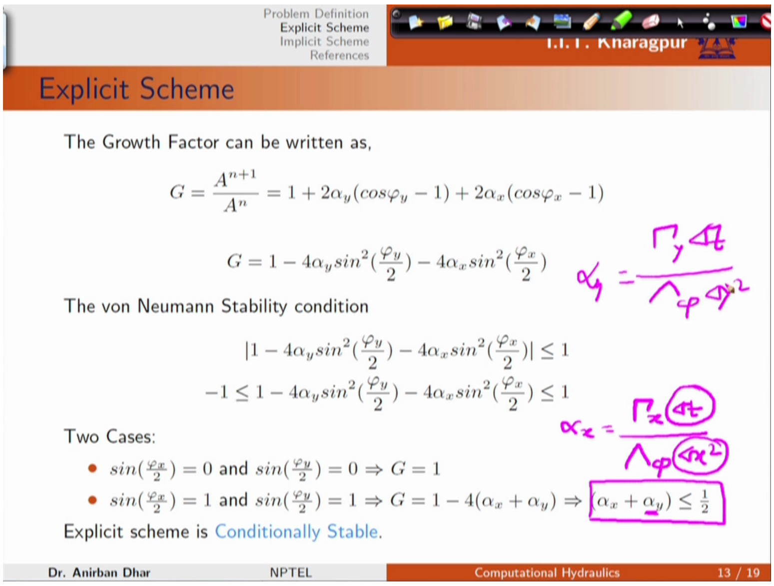


So we cannot specify arbitrary values for del t in case of explicit scheme. That should be related to del x. So that is a condition. Now if you consider implicit scheme again we can write our main discretized equation in terms of exact discrete value anderror term.

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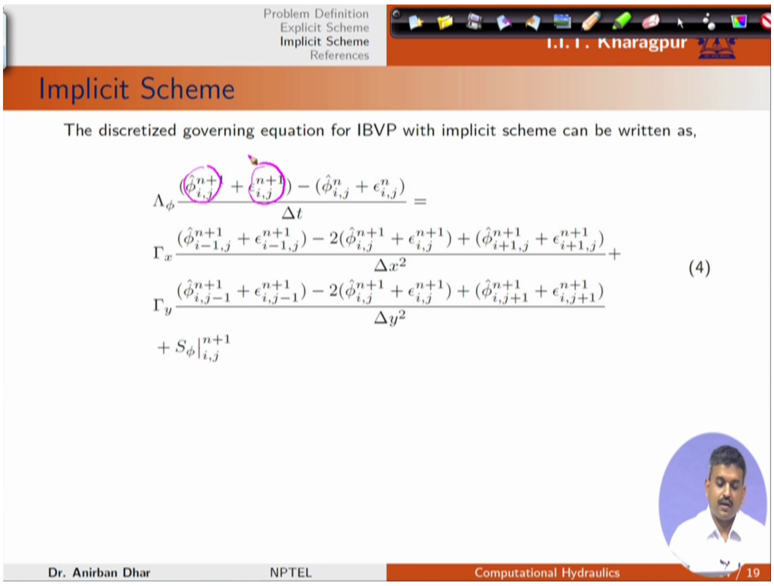


And similarly we can define our exactdiscrete solution of the finite difference equation in terms ofthese finite difference equation.

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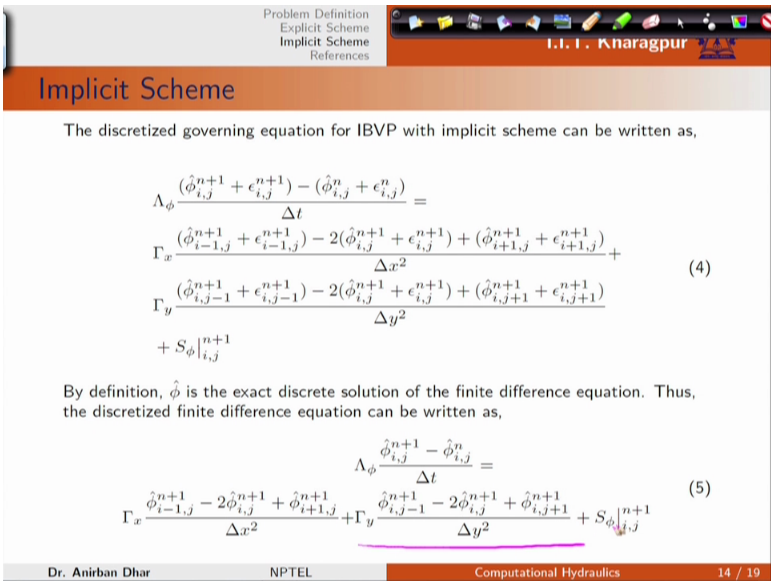


Now if we subtract 5 from 4, obviously we will be getting error equation here.

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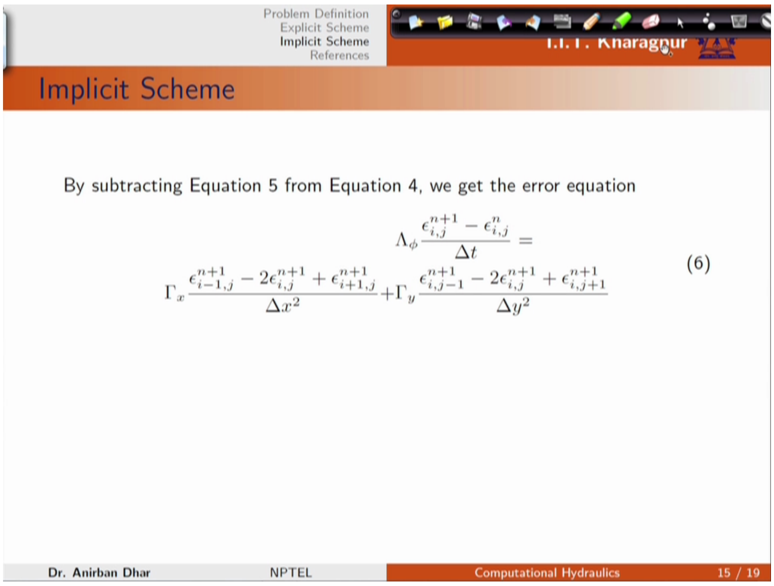


And in simplified form we can getthis thing.Interestingly in this case left hand side, these values are unknown values. This is known on the right hand side.

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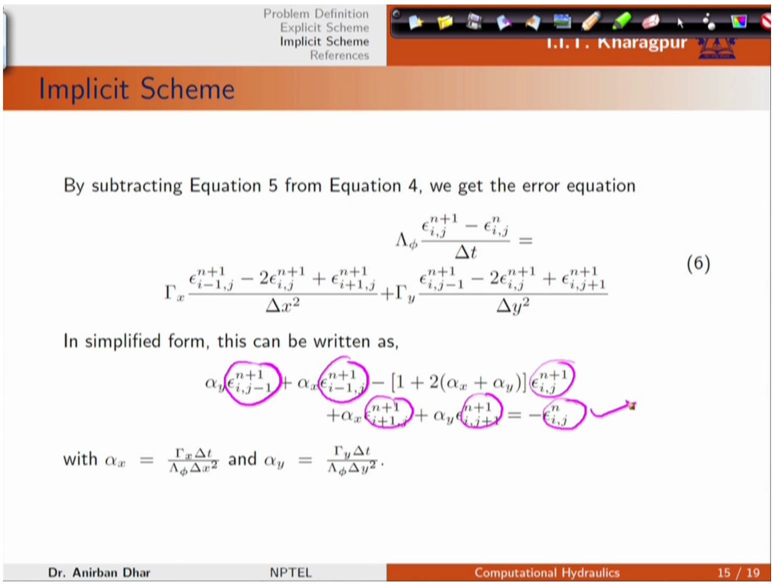


With this if we expand, we can again utilize the information. The only change is due to change in thetime index, n plus 1. But there is no change in these values compared to our explicit scheme.

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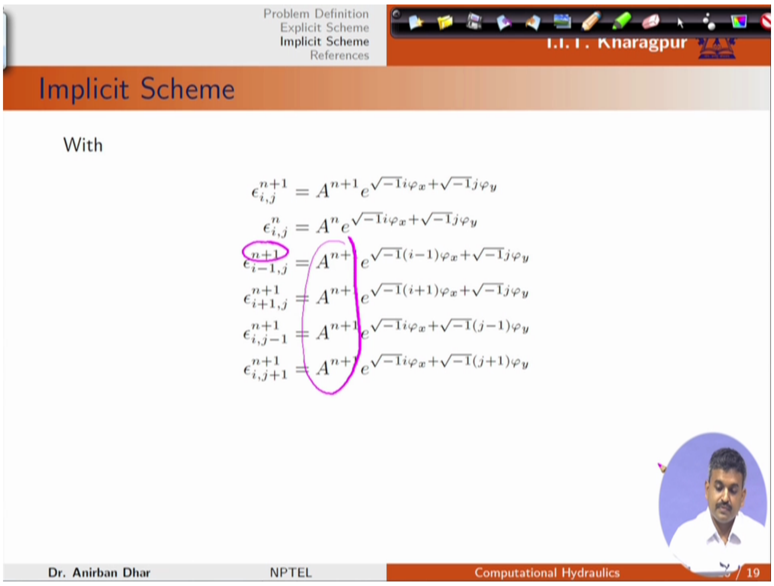


With this if we write our error equation. So error equation becomes, on the left hand side we have An plus 1 divided by An and minus 1 on the right hand side.

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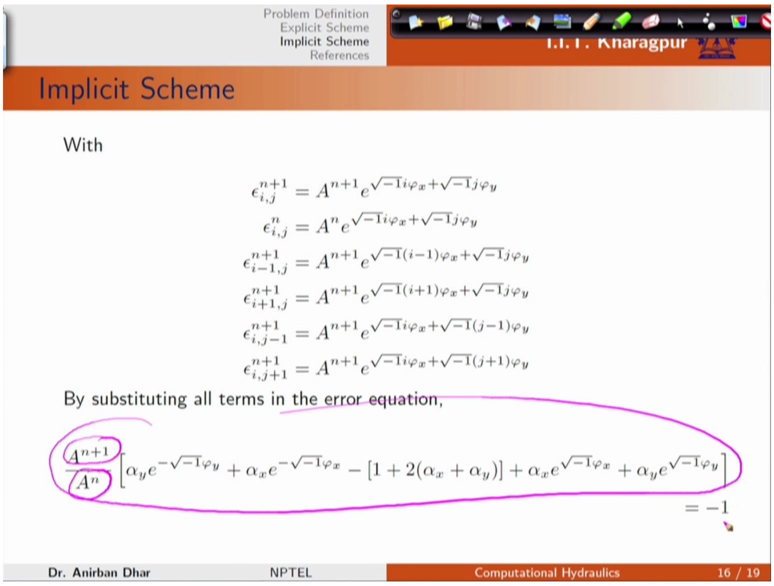


Againwe can combine this term, this term. That means terms related to Y. And if we add these two terms we will get 2 cosvar phi y. And if we combine this X terms, we will get 2 alpha x cos var phi x.

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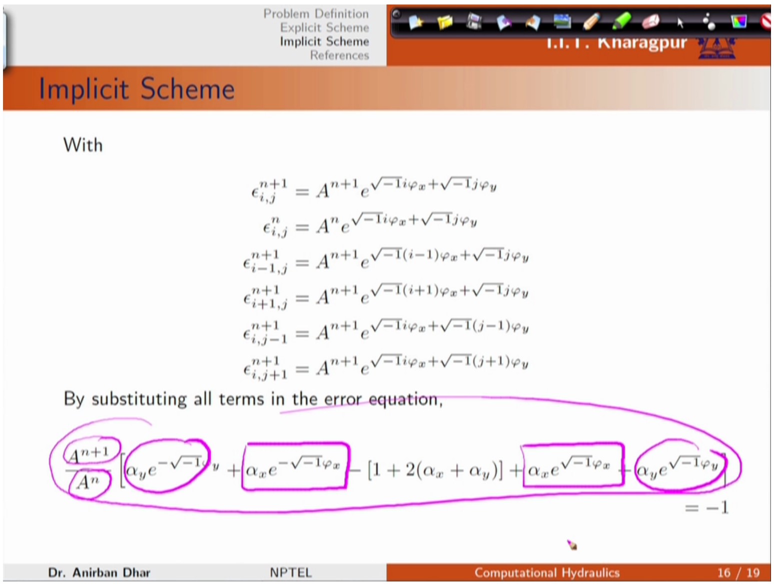


So with this if we simplify our error equation, then growth factor or amplification factor, we can write like this, minus 1 divided byminus 1 plus 2 alpha cos var phi y minus 1.

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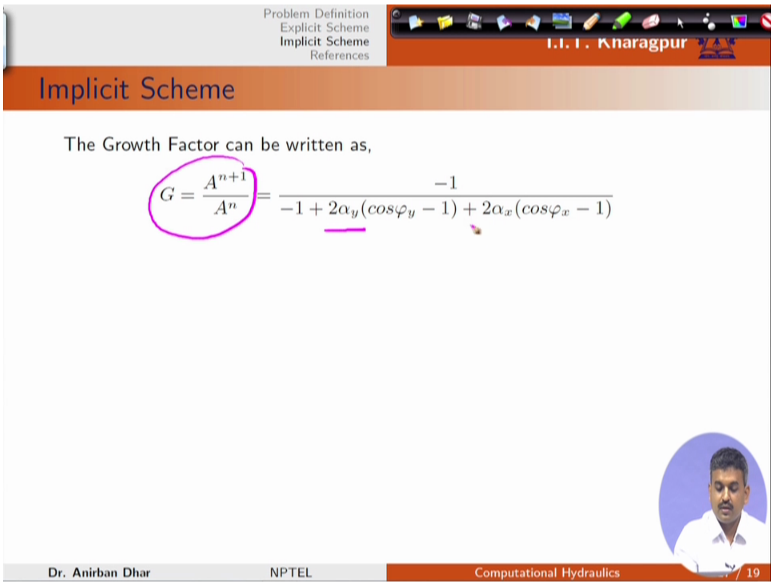


So if we again simplify this,coscar phi y minus 1 equals to minus 2 sin square var phi y by 2. And this one as minus 2 sin square var phi x divided by 2.

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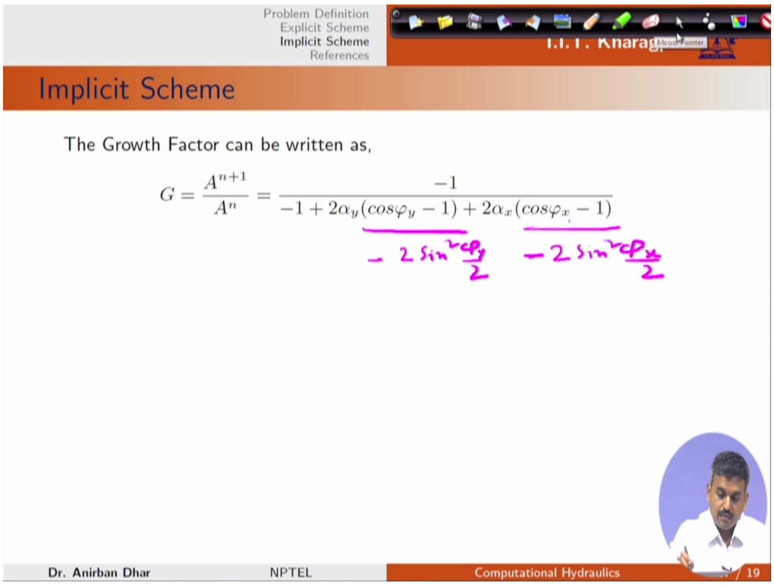


We can write this growth factor or amplification factor like this. Where alpha y and alpha x, these values are positive values obviously. So 1 by 1 plus some positive values and sin square.That means square of any term will be always positive. So 1 plus some positive value and on numerator we have only 1.

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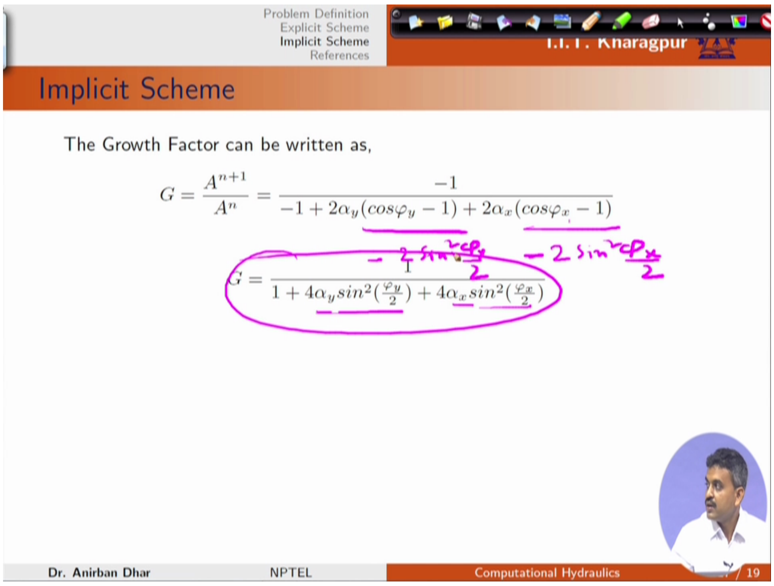


So obviously this value is always less than 1.From Von Neumann stability condition, 1 by 1 plus 4 alpha y sin square var phi by 2. So with this if we proceedfor the two cases.  if sin values are zero then we haveG equals to 1. Neutrally stable condition.

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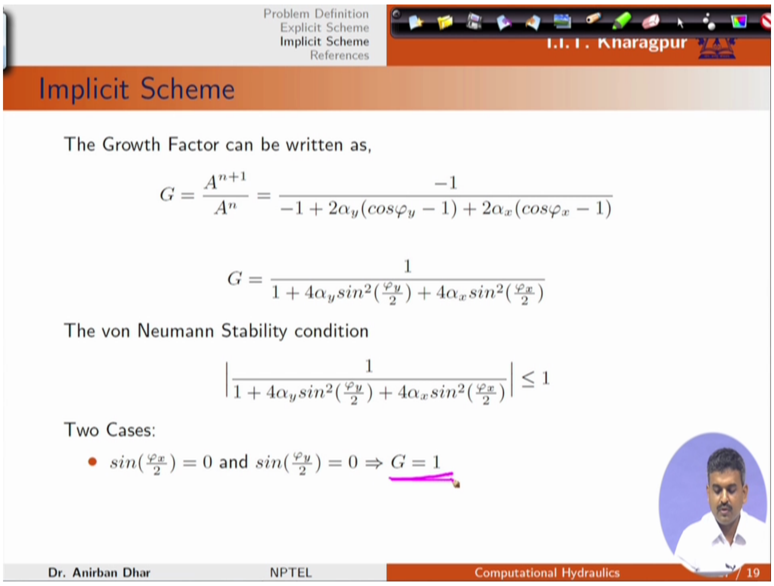


And if we have sin values equal to 1, then we have G equals to 1 by 1 plus 4 alpha y plus 4 alpha x. This is less than 1. This is obviously without imposing any condition, we are getting this. So we can say that implicit scheme is unconditionally stable. We don't need to put any restriction on del t compared to del x. Although we need to use small values of del t for this problems.

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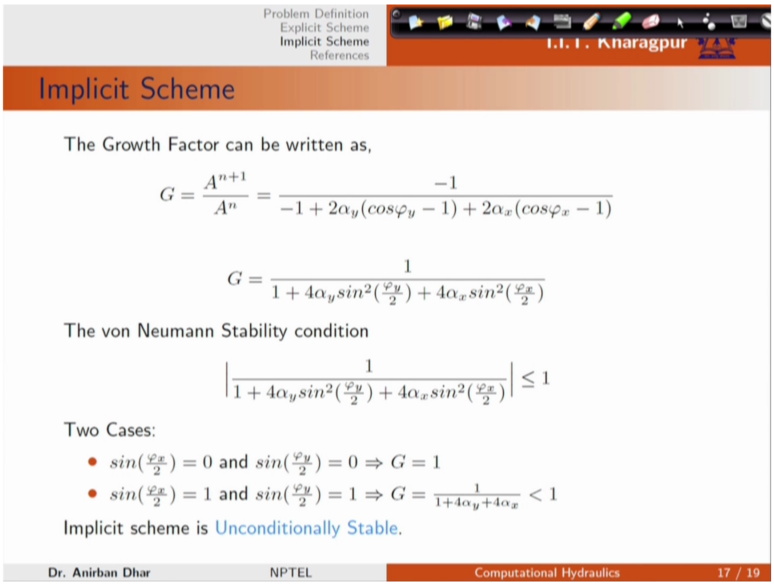


Now we can extend this approach for Crank Nicolson scheme and we can check the numerical stability of the Crank Nicolson scheme. Thank you.