

**Computational Hydraulics**  
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**Lecture 11**  
**Partial Differential Equation: Numerical Stability of IBVP**

Welcome to this lecture number 11 of the course computational hydraulics. We are in model number 2, numerical methods. And in this lecture we will be covering unit 7, partial differential equation, numerical stability of initial boundary value problem.

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Problem Definition  
Explicit Scheme  
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References

I.I.T. Kharagpur

**Module 02: Numerical Methods**  
**Unit 07: Partial Differential Equation: Numerical Stability of IBVP**

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So what is the learning objective for this particular unit? At the end of this unit students will be able to analyze the numerical stability of the discretized partial differential equation.

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### Learning Objective

- To analyze the numerical stability of the discretized PDE.

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In lecture 10 we have discussed the initial boundary value problem, IBVP in terms of this temporal derivative and 2, second order spatial derivatives and one source sink term.

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Problem Definition

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### Problem Definition

Governing equation

A two-dimensional (in space) IBVP can be written as,

$$\Omega : \Lambda_{\phi} \frac{\partial \phi}{\partial t} = \Gamma_x \frac{\partial^2 \phi}{\partial x^2} + \Gamma_y \frac{\partial^2 \phi}{\partial y^2} + S_{\phi}(x, y)$$

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In this case we have discretized the equation and we have utilized 3 schemes, explicit, implicit and Crank Nicolson. All these themes, we need to definethis del x, del y and del t for each schemes. So depending on the value of del x, del y, del t whether there will be changed in thefinal result, we need to see that thing from numerical stability.

This is the initial problem definition. Initial condition and boundary condition for the problemwith rectangular domain.

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### Problem Definition

subject to

**Initial Condition**

$$\phi(x, y, 0) = \phi_0(x, y)$$

and

**Boundary Condition**

$$\left( \begin{array}{l} \Gamma_D^1 : \phi(0, y, t) = \phi_1 \\ \Gamma_D^2 : \phi(L_x, y, t) = \phi_2 \\ \Gamma_N^3 : \frac{\partial \phi}{\partial y} \Big|_{(x, 0, t)} = 0 \\ \Gamma_N^4 : \frac{\partial \phi}{\partial y} \Big|_{(x, L_y, t)} = 0 \end{array} \right)$$

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Now we have errors. We can define this discretization error as the difference between analytical solution of the PDE which is the closed form solution and the exact solution of the finite difference equation obtained on hypothetical infinite precision computer. This involves truncation error, error due to treatment of boundary condition.

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### Errors

**Discretization Error (Biswas, 2003)**

Discretization Error =

Analytical Solution of PDE - Exact Solution of the Finite Difference Equation  
(obtained on a hypothetical infinite precision computer)

= Truncation Error + Error due to treatment of boundary conditions

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Now if you consider the round off error, round off error is numerical solution of the finite difference equation obtained from finite precision computer and exact solution of the finite differences equation obtained on a hypothetical infinite precision computer.

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## Errors

### Discretization Error (Biswas, 2003)

Discretization Error =

Analytical Solution of PDE - Exact Solution of the Finite Difference Equation  
(obtained on a hypothetical infinite precision computer)

= Truncation Error + Error due to treatment of boundary conditions

### Round-off Error

Round-off Error ( $\epsilon$ ) =

Numerical Solution of the Finite Difference Equation (obtained from finite precision computer) - Exact Solution of the Finite Difference Equation  
(obtained on a hypothetical infinite precision computer)

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So for any problem if we discretize equation and we are solving the equation infinite precision computer there will be some amount of error involved due to round off error. We need to see few things. For this numerical errors, every algorithm requires repeated operations that is plus, minus, or addition, subtraction, multiplication, division. So there will be accumulation of round off error. And in time stepping algorithm whatever we have seen in our lecture number 10, accumulated round off error may magnify or reduce with every step.

Error may increase exponentially. It is known as numerical instability. Numerical stability or (in) instability is a property of the algorithm and discretization of partial differential equation plus boundary conditions. And it does not depends on the computer used. We need to check our discretization scheme to check the numerical instability for the problem.



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## Numerical Errors

- Every algorithm requires repeated operations (e.g.,  $\pm$ ,  $\times$ ). There is an accumulation of round-off error.
- In time-stepping algorithm, accumulated round-off error may magnify/reduce with every step.
- Error may increase exponentially. It is known as *Numerical Instability*.
- Numerical Stability/ Instability is a property of the algorithm and discretization of PDE+BCs.
- It does not depend on the computer used.

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Stability analysis. In stability analysis of linear PDE, we analyze only one arbitrary Fourier mode. Let us consider that error can be represented in form of Fourier Series and single arbitrary term can be written as,  $\epsilon_{i,j}^n$  and  $A^n$  is the amplitude. In this case, the  $\omega_x$  and  $\omega_y$  are the wave numbers corresponding to  $X$  and  $Y$  directions respectively. And square root of minus 1 is the imaginary number. And  $i$  and  $j$ , these are corresponding to  $X$  and  $Y$  directions.

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## Stability Analysis

In stability analysis of linear PDE, we analyze only one arbitrary Fourier mode (Biswas, 2003).

Let us consider that the error can be represented in the form of Fourier Series and single arbitrary term can be written as,

$$\epsilon_{i,j}^n = A^n e^{\sqrt{-1}i\omega_x \Delta x + \sqrt{-1}j\omega_y \Delta y}$$

where  $\omega_x$  and  $\omega_y$  are wave numbers corresponding to  $x$  and  $y$  directions respectively.

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With this information we can say that if we take modulus of this error, obviously this depends on the amplitude term. Not on this term because modulus of this term will be obviously 1.

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References

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### Stability Analysis


In stability analysis of linear PDE, we analyze only one arbitrary Fourier mode (Biswas, 2003).

Let us consider that the error can be represented in the form of Fourier Series and single arbitrary term can be written as,

$$e_{i,j}^n = A^n e^{\sqrt{-1}i\omega_x \Delta x + \sqrt{-1}j\omega_y \Delta y}$$

where  $\omega_x$  and  $\omega_y$  are wave numbers corresponding to  $x$  and  $y$  directions respectively.

Note that  $|e_{i,j}^n| = |A^n|$ .



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So with this we can simplify the error term. We can write it in the form of face values where  $\phi_x$ , where  $\phi_y$  for  $X$  and  $Y$  directions. And this is basically our  $\omega_x \Delta x$  into  $\phi_x$  and where  $\phi_y$  is  $\omega_y \Delta y$  into  $\phi_y$ .

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### Stability Analysis

In stability analysis of linear PDE, we analyze only one arbitrary Fourier mode (Biswas, 2003).

Let us consider that the error can be represented in the form of Fourier Series and single arbitrary term can be written as,

$$e_{i,j}^n = A^n e^{\sqrt{-1}i\omega_x \Delta x + \sqrt{-1}j\omega_y \Delta y}$$

where  $\omega_x$  and  $\omega_y$  are wave numbers corresponding to  $x$  and  $y$  directions respectively.

Note that,  $|e_{i,j}^n| = |A^n|$ .  
In simplified form, error can be written as,

$$e_{i,j}^n = A^n e^{\sqrt{-1}i\phi_x + \sqrt{-1}j\phi_y}$$

where  $\phi_x$  and  $\phi_y$  are phase values corresponding to  $x$  and  $y$  directions respectively.

$\phi_x = \omega_x \Delta x$   
 $\phi_y = \omega_y \Delta y$

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We can define this amplification term. It governs the growth of the Fourier components and in this case we can define Von Neumann stability condition. This is modulus of  $G$  should be less than equal to 1. If modulus of  $G$  is greater than 1, error grows. This is unstable scheme. If we have modulus of  $G$  less than one, error reduces. This is stable scheme. If we have modulus of  $G$  equals to 1, error remains same. This is neutrally stable scheme.

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## von Neumann Stability Analysis

Define

$$G = \frac{A^{n+1}}{A^n}$$

where  $G$  is an amplification factor. It governs the growth of the Fourier component.


The von Neumann Stability Condition is given by  $|G| \leq 1$ .

$|G| > 1 \Rightarrow$  Error grows (Unstable Scheme).

$|G| < 1 \Rightarrow$  Error reduces (Stable Scheme).

$|G| = 1 \Rightarrow$  Error remains same (Neutrally Stable Scheme).

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Now we have discretized our initial boundary value problem using explicit scheme. We can write the same thing here. Now  $\phi_{ij}^{n+1}$  or  $\phi_{ij}^n$ , this is obtained from finite precision computer.

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## Explicit Scheme

The discretized governing equation for IBVP with explicit scheme can be written as,

$$\Lambda_\phi \frac{\phi_{i,j}^{n+1} - \phi_{i,j}^n}{\Delta t} = \Gamma_x \frac{\phi_{i-1,j}^n - 2\phi_{i,j}^n + \phi_{i+1,j}^n}{\Delta x^2} + \Gamma_y \frac{\phi_{i,j-1}^n - 2\phi_{i,j}^n + \phi_{i,j+1}^n}{\Delta y^2} + S_\phi|_{i,j}^n$$

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So we can write the general variable  $\phi$  in terms of this  $\hat{\phi}_{ij}^n$  and  $\epsilon_{ij}^n$  where  $\hat{\phi}_{ij}^n$  is a numerical solution obtained from finite precision computer and  $\phi_{ij}^n$  is the exact discrete solution obtained on hypothetical infinite precision computer. And  $\epsilon_{ij}^n$  is the accumulated round off error at level  $n$ .

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### Explicit Scheme

The discretized governing equation for IBVP with explicit scheme can be written as,

$$\Lambda_\phi \frac{\phi_{i,j}^{n+1} - \phi_{i,j}^n}{\Delta t} = \Gamma_x \frac{\phi_{i-1,j}^n - 2\phi_{i,j}^n + \phi_{i+1,j}^n}{\Delta x^2} + \Gamma_y \frac{\phi_{i,j-1}^n - 2\phi_{i,j}^n + \phi_{i,j+1}^n}{\Delta y^2} + S_\phi|_{i,j}^n$$

The general variable  $\phi$  can be written as,

$$\phi_{i,j}^n = \hat{\phi}_{i,j}^n + \epsilon_{i,j}^n$$

where  
 $\phi_{i,j}^n$  = Numerical solution obtained from finite precision computer  
 $\hat{\phi}_{i,j}^n$  = Exact discrete solution obtained on a hypothetical infinite precision computer  
 $\epsilon_{i,j}^n$  = Accumulated round-off error at time level  $n$ .

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With this we can write our discretized governing equation with explicit scheme as, we can replace this  $\phi_{i,j}^n$  with  $\hat{\phi}_{i,j}^n + \epsilon_{i,j}^n$ . So this is actually our discretized governing equation.

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### Explicit Scheme

The discretized governing equation for IBVP with explicit scheme can be written as,

$$\Lambda_\phi \frac{(\hat{\phi}_{i,j}^{n+1} + \epsilon_{i,j}^{n+1}) - (\hat{\phi}_{i,j}^n + \epsilon_{i,j}^n)}{\Delta t} = \Gamma_x \frac{(\hat{\phi}_{i-1,j}^n + \epsilon_{i-1,j}^n) - 2(\hat{\phi}_{i,j}^n + \epsilon_{i,j}^n) + (\hat{\phi}_{i+1,j}^n + \epsilon_{i+1,j}^n)}{\Delta x^2} + \Gamma_y \frac{(\hat{\phi}_{i,j-1}^n + \epsilon_{i,j-1}^n) - 2(\hat{\phi}_{i,j}^n + \epsilon_{i,j}^n) + (\hat{\phi}_{i,j+1}^n + \epsilon_{i,j+1}^n)}{\Delta y^2} + S_\phi|_{i,j}^n \quad (1)$$

$\phi_{ij}^n = \hat{\phi}_{ij}^n + \epsilon_{ij}^n$

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But we have discretized our governing equation with assumption that we will get the infinite solution from infinite precision computer. So we can write our governing equation with this exact discrete solution  $\hat{\phi}_{i,j}^n + \epsilon_{i,j}^n$ . So ideally this should be satisfied. This equation number 2.

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### Explicit Scheme

The discretized governing equation for IBVP with explicit scheme can be written as,

$$\Lambda_\phi \frac{(\hat{\phi}_{i,j}^{n+1} + \epsilon_{i,j}^{n+1}) - (\hat{\phi}_{i,j}^n + \epsilon_{i,j}^n)}{\Delta t} = \Gamma_x \frac{(\hat{\phi}_{i-1,j}^n + \epsilon_{i-1,j}^n) - 2(\hat{\phi}_{i,j}^n + \epsilon_{i,j}^n) + (\hat{\phi}_{i+1,j}^n + \epsilon_{i+1,j}^n)}{\Delta x^2} + \Gamma_y \frac{(\hat{\phi}_{i,j-1}^n + \epsilon_{i,j-1}^n) - 2(\hat{\phi}_{i,j}^n + \epsilon_{i,j}^n) + (\hat{\phi}_{i,j+1}^n + \epsilon_{i,j+1}^n)}{\Delta y^2} + S_\phi|_{i,j}^n \quad (1)$$

By definition,  $\hat{\phi}$  is the exact discrete solution of the finite difference equation. Thus, the discretized finite difference equation can be written as,

$$\Lambda_\phi \frac{\hat{\phi}_{i,j}^{n+1} - \hat{\phi}_{i,j}^n}{\Delta t} = \Gamma_x \frac{\hat{\phi}_{i-1,j}^n - 2\hat{\phi}_{i,j}^n + \hat{\phi}_{i+1,j}^n}{\Delta x^2} + \Gamma_y \frac{\hat{\phi}_{i,j-1}^n - 2\hat{\phi}_{i,j}^n + \hat{\phi}_{i,j+1}^n}{\Delta y^2} + S_\phi|_{i,j}^n \quad (2)$$

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So if we subtract equation number 2 from 1, then we can get this error equation. Error equation is similar to our original discretized form but without source sink term, because there will be no error involved there due to discretization. So in simplified form we can write it as epsilon ij n plus 1 and other terms on the right hand side with alpha x, alpha y like this.

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Problem Definition  
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### Explicit Scheme

By subtracting Equation 2 from Equation 1, we get the error equation

$$\Lambda_\phi \frac{\epsilon_{i,j}^{n+1} - \epsilon_{i,j}^n}{\Delta t} = \Gamma_x \frac{\epsilon_{i-1,j}^n - 2\epsilon_{i,j}^n + \epsilon_{i+1,j}^n}{\Delta x^2} + \Gamma_y \frac{\epsilon_{i,j-1}^n - 2\epsilon_{i,j}^n + \epsilon_{i,j+1}^n}{\Delta y^2} \quad (3)$$

In simplified form, this can be written as,

$$\epsilon_{i,j}^{n+1} = \alpha_y \epsilon_{i,j-1}^n + \alpha_x \epsilon_{i-1,j}^n + [1 - 2(\alpha_x + \alpha_y)] \epsilon_{i,j}^n + \alpha_x \epsilon_{i+1,j}^n + \alpha_y \epsilon_{i,j+1}^n$$

with  $\alpha_x = \frac{\Gamma_x \Delta t}{\Lambda_\phi \Delta x^2}$  and  $\alpha_y = \frac{\Gamma_y \Delta t}{\Lambda_\phi \Delta y^2}$ .

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Now in explicit scheme, we can define this epsilon ij n plus 1 as An plus 1. Only change in amplitude but there is no change in X or Y direction in this values. And for epsilon ij n, only change in amplitude. Epsilon I minus 1 jn, change in the index for X. in this case again change in the index for X. Change for the index for Y, change in the index for Y.


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### Explicit Scheme

With

$$\begin{aligned} \epsilon_{i,j}^{n+1} &= A^{n+1} e^{\sqrt{-1}i\varphi_x + \sqrt{-1}j\varphi_y} \\ \epsilon_{i,j}^n &= A^n e^{\sqrt{-1}i\varphi_x + \sqrt{-1}j\varphi_y} \\ \epsilon_{i-1,j}^n &= A^n e^{\sqrt{-1}(i-1)\varphi_x + \sqrt{-1}j\varphi_y} \\ \epsilon_{i+1,j}^n &= A^n e^{\sqrt{-1}(i+1)\varphi_x + \sqrt{-1}j\varphi_y} \\ \epsilon_{i,j-1}^n &= A^n e^{\sqrt{-1}i\varphi_x + \sqrt{-1}(j-1)\varphi_y} \\ \epsilon_{i,j+1}^n &= A^n e^{\sqrt{-1}i\varphi_x + \sqrt{-1}(j+1)\varphi_y} \end{aligned}$$


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With this information we can write our error equation. With this simplification, like this.

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### Explicit Scheme

With

$$\begin{aligned} \epsilon_{i,j}^{n+1} &= A^{n+1} e^{\sqrt{-1}i\varphi_x + \sqrt{-1}j\varphi_y} \\ \epsilon_{i,j}^n &= A^n e^{\sqrt{-1}i\varphi_x + \sqrt{-1}j\varphi_y} \\ \epsilon_{i-1,j}^n &= A^n e^{\sqrt{-1}(i-1)\varphi_x + \sqrt{-1}j\varphi_y} \\ \epsilon_{i+1,j}^n &= A^n e^{\sqrt{-1}(i+1)\varphi_x + \sqrt{-1}j\varphi_y} \\ \epsilon_{i,j-1}^n &= A^n e^{\sqrt{-1}i\varphi_x + \sqrt{-1}(j-1)\varphi_y} \\ \epsilon_{i,j+1}^n &= A^n e^{\sqrt{-1}i\varphi_x + \sqrt{-1}(j+1)\varphi_y} \end{aligned}$$

By substituting all terms in the error equation,

$$\frac{A^{n+1}}{A^n} = \alpha_y e^{-\sqrt{-1}\varphi_y} + \alpha_x e^{-\sqrt{-1}\varphi_x} + [1 - 2(\alpha_x + \alpha_y)] + \alpha_x e^{\sqrt{-1}\varphi_x} + \alpha_y e^{\sqrt{-1}\varphi_y}$$


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This is essentially our amplification term and these are the known things on the right hand side.

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### Explicit Scheme

With

$$\begin{aligned} \epsilon_{i,j}^{n+1} &= A^{n+1} e^{\sqrt{-1}i\varphi_x + \sqrt{-1}j\varphi_y} \\ \epsilon_{i,j}^n &= A^n e^{\sqrt{-1}i\varphi_x + \sqrt{-1}j\varphi_y} \\ \epsilon_{i-1,j}^n &= A^n e^{\sqrt{-1}(i-1)\varphi_x + \sqrt{-1}j\varphi_y} \\ \epsilon_{i+1,j}^n &= A^n e^{\sqrt{-1}(i+1)\varphi_x + \sqrt{-1}j\varphi_y} \\ \epsilon_{i,j-1}^n &= A^n e^{\sqrt{-1}i\varphi_x + \sqrt{-1}(j-1)\varphi_y} \\ \epsilon_{i,j+1}^n &= A^n e^{\sqrt{-1}i\varphi_x + \sqrt{-1}(j+1)\varphi_y} \end{aligned}$$

By substituting all terms in the error equation,

$$G = \frac{A^{n+1}}{A^n} = \alpha_y e^{-\sqrt{-1}\varphi_y} + \alpha_x e^{-\sqrt{-1}\varphi_x} + [1 - 2(\alpha_x + \alpha_y)] + \alpha_x e^{\sqrt{-1}\varphi_x} + \alpha_y e^{\sqrt{-1}\varphi_y}$$

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The growth factor or amplification term can be written like this. Essentially in our last equation. This is into the power minus imaginary number into phi y. If we combine these two term we will get 2 cos phi y into alpha y.

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### Explicit Scheme

With

$$\begin{aligned} \epsilon_{i,j}^{n+1} &= A^{n+1} e^{\sqrt{-1}i\varphi_x + \sqrt{-1}j\varphi_y} \\ \epsilon_{i,j}^n &= A^n e^{\sqrt{-1}i\varphi_x + \sqrt{-1}j\varphi_y} \\ \epsilon_{i-1,j}^n &= A^n e^{\sqrt{-1}(i-1)\varphi_x + \sqrt{-1}j\varphi_y} \\ \epsilon_{i+1,j}^n &= A^n e^{\sqrt{-1}(i+1)\varphi_x + \sqrt{-1}j\varphi_y} \\ \epsilon_{i,j-1}^n &= A^n e^{\sqrt{-1}i\varphi_x + \sqrt{-1}(j-1)\varphi_y} \\ \epsilon_{i,j+1}^n &= A^n e^{\sqrt{-1}i\varphi_x + \sqrt{-1}(j+1)\varphi_y} \end{aligned}$$

By substituting all terms in the error equation,

$$\frac{A^{n+1}}{A^n} = \alpha_y e^{-\sqrt{-1}\varphi_y} + \alpha_x e^{-\sqrt{-1}\varphi_x} + [1 - 2(\alpha_x + \alpha_y)] + \alpha_x e^{\sqrt{-1}\varphi_x} + \alpha_y e^{\sqrt{-1}\varphi_y}$$

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In this case if we combine these two related to X we will get, cosof var phi x. So I can just write it here this exponential of minus, minus 1 var phi x plus e to the power minus1 var phi x. This will be 2 cos var phi x.

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### Explicit Scheme

With

$$\begin{aligned} \epsilon_{i,j}^{n+1} &= A^{n+1} e^{\sqrt{-1}i\varphi_x + \sqrt{-1}j\varphi_y} \\ \epsilon_{i,j}^n &= A^n e^{\sqrt{-1}i\varphi_x + \sqrt{-1}j\varphi_y} \\ \epsilon_{i-1,j}^n &= A^n e^{\sqrt{-1}(i-1)\varphi_x + \sqrt{-1}j\varphi_y} \\ \epsilon_{i+1,j}^n &= A^n e^{\sqrt{-1}(i+1)\varphi_x + \sqrt{-1}j\varphi_y} \\ \epsilon_{i,j-1}^n &= A^n e^{\sqrt{-1}i\varphi_x + \sqrt{-1}(j-1)\varphi_y} \\ \epsilon_{i,j+1}^n &= A^n e^{\sqrt{-1}i\varphi_x + \sqrt{-1}(j+1)\varphi_y} \end{aligned}$$

By substituting all terms in the error equation,

$$\frac{A^{n+1}}{A^n} = \alpha_y e^{-\sqrt{-1}\varphi_y} + \alpha_x e^{-\sqrt{-1}\varphi_x} + [1 - 2(\alpha_x + \alpha_y)] + \alpha_x e^{\sqrt{-1}\varphi_x} + \alpha_y e^{\sqrt{-1}\varphi_y}$$

*Handwritten notes:*  $e^{-\sqrt{-1}\varphi_x} + e^{\sqrt{-1}\varphi_x} = 2\cos\varphi_x$

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Similarly for Y direction. Now we can simplify the right hand side write it like this. And again for this  $\cos\varphi_y \varphi_y - 1$ , we can write it as  $\cos\varphi_y - 1 = -2\sin^2(\varphi_y/2)$ . So with this, this is our growth term or amplification factor,  $1 - 4\alpha_y \sin^2(\varphi_y/2) - 4\alpha_x \sin^2(\varphi_x/2)$ .

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### Explicit Scheme

The Growth Factor can be written as,

$$G = \frac{A^{n+1}}{A^n} = 1 + 2\alpha_y(\cos\varphi_y - 1) + 2\alpha_x(\cos\varphi_x - 1)$$

$$G = 1 - 4\alpha_y \sin^2\left(\frac{\varphi_y}{2}\right) - 4\alpha_x \sin^2\left(\frac{\varphi_x}{2}\right)$$

*Handwritten notes:*  $\cos\varphi_y - 1 = -2\sin^2(\varphi_y/2)$

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In this case if you want to check the Von Neumann stability condition, we have made use of this term here. And this should be less than equal to 1 which is  $-1 \leq G \leq 1$  and this is within this limit.



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### Explicit Scheme

The Growth Factor can be written as,

$$G = \frac{A^{n+1}}{A^n} = 1 + 2\alpha_y(\cos\varphi_y - 1) + 2\alpha_x(\cos\varphi_x - 1)$$

$$G = \left| 1 - 4\alpha_y \sin^2\left(\frac{\varphi_y}{2}\right) - 4\alpha_x \sin^2\left(\frac{\varphi_x}{2}\right) \right|$$

The von Neumann Stability condition

$$\left| 1 - 4\alpha_y \sin^2\left(\frac{\varphi_y}{2}\right) - 4\alpha_x \sin^2\left(\frac{\varphi_x}{2}\right) \right| \leq 1$$

$$-1 \leq 1 - 4\alpha_y \sin^2\left(\frac{\varphi_y}{2}\right) - 4\alpha_x \sin^2\left(\frac{\varphi_x}{2}\right) \leq 1$$

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We have 2 cases uh or extreme ones, where  $\sin \phi_x$  by 2,  $\sin \varphi_y$  by 2, these value are zero. And this means that G is equal to 1. The scheme is neutrally stable.

(Refer Slide Time 16:59)

Problem Definition  
Explicit Scheme  
Implicit Scheme  
References

I.I.T. Kharagpur

### Explicit Scheme

The Growth Factor can be written as,

$$G = \frac{A^{n+1}}{A^n} = 1 + 2\alpha_y(\cos\varphi_y - 1) + 2\alpha_x(\cos\varphi_x - 1)$$

$$G = 1 - 4\alpha_y \sin^2\left(\frac{\varphi_y}{2}\right) - 4\alpha_x \sin^2\left(\frac{\varphi_x}{2}\right)$$

The von Neumann Stability condition

$$\left| 1 - 4\alpha_y \sin^2\left(\frac{\varphi_y}{2}\right) - 4\alpha_x \sin^2\left(\frac{\varphi_x}{2}\right) \right| \leq 1$$

$$-1 \leq 1 - 4\alpha_y \sin^2\left(\frac{\varphi_y}{2}\right) - 4\alpha_x \sin^2\left(\frac{\varphi_x}{2}\right) \leq 1$$

Two Cases:

- $\sin\left(\frac{\varphi_x}{2}\right) = 0$  and  $\sin\left(\frac{\varphi_y}{2}\right) = 0 \Rightarrow G = 1$

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However if we apply this condition that  $\sin \varphi_x$  by 2 equals to 1 and  $\varphi_y$  by 2 equals to 1, then comes this G equals to 1 minus 4 alpha x plus alpha y. And this is less than equal to half. Because minus 1, this is 1 minus 4 alpha x minus 4 alpha y. If we change sides, this will be 4 alpha x plus 4 alpha y less than equals to 2. And from here it is coming alpha x plus alpha y is less than equals to half.

(Refer Slide Time 18:06)

Problem Definition  
Explicit Scheme  
Implicit Scheme  
References

I.I.T. Kharagpur

### Explicit Scheme

The Growth Factor can be written as,

$$G = \frac{A^{n+1}}{A^n} = 1 + 2\alpha_y(\cos\varphi_y - 1) + 2\alpha_x(\cos\varphi_x - 1)$$

$$G = 1 - 4\alpha_y\sin^2\left(\frac{\varphi_y}{2}\right) - 4\alpha_x\sin^2\left(\frac{\varphi_x}{2}\right)$$

The von Neumann Stability condition

$$|1 - 4\alpha_y\sin^2\left(\frac{\varphi_y}{2}\right) - 4\alpha_x\sin^2\left(\frac{\varphi_x}{2}\right)| \leq 1$$

$$-1 \leq 1 - 4\alpha_y\sin^2\left(\frac{\varphi_y}{2}\right) - 4\alpha_x\sin^2\left(\frac{\varphi_x}{2}\right) \leq 1$$

Two Cases:

- $\sin\left(\frac{\varphi_x}{2}\right) = 0$  and  $\sin\left(\frac{\varphi_y}{2}\right) = 0 \Rightarrow G = 1$
- $\sin\left(\frac{\varphi_x}{2}\right) = 1$  and  $\sin\left(\frac{\varphi_y}{2}\right) = 1 \Rightarrow G = 1 - 4(\alpha_x + \alpha_y) \Rightarrow (\alpha_x + \alpha_y) \leq \frac{1}{2}$

Handwritten notes:  $-1 \leq 1 - 4\alpha_x - 4\alpha_y \Rightarrow 4\alpha_x + 4\alpha_y \leq 2$

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So we can comment on the stability of the scheme here that explicit scheme is conditionally stable. This alpha x, alpha y, this addition should be less than equals to half. Interestingly this alpha x is gamma x delta t divided by lambda phi into del x square. So in this case there is this delta t term and del x square term. Similarly for alphas, we have alpha y, we have gamma y, del t divided by this lambda phi del y square. So it relates our del t, del x, del y.

(Refer Slide Time 19:16)

Problem Definition  
Explicit Scheme  
Implicit Scheme  
References

I.I.T. Kharagpur

### Explicit Scheme

The Growth Factor can be written as,

$$G = \frac{A^{n+1}}{A^n} = 1 + 2\alpha_y(\cos\varphi_y - 1) + 2\alpha_x(\cos\varphi_x - 1)$$

$$G = 1 - 4\alpha_y\sin^2\left(\frac{\varphi_y}{2}\right) - 4\alpha_x\sin^2\left(\frac{\varphi_x}{2}\right)$$

The von Neumann Stability condition

$$|1 - 4\alpha_y\sin^2\left(\frac{\varphi_y}{2}\right) - 4\alpha_x\sin^2\left(\frac{\varphi_x}{2}\right)| \leq 1$$

$$-1 \leq 1 - 4\alpha_y\sin^2\left(\frac{\varphi_y}{2}\right) - 4\alpha_x\sin^2\left(\frac{\varphi_x}{2}\right) \leq 1$$

Two Cases:

- $\sin\left(\frac{\varphi_x}{2}\right) = 0$  and  $\sin\left(\frac{\varphi_y}{2}\right) = 0 \Rightarrow G = 1$
- $\sin\left(\frac{\varphi_x}{2}\right) = 1$  and  $\sin\left(\frac{\varphi_y}{2}\right) = 1 \Rightarrow G = 1 - 4(\alpha_x + \alpha_y) \Rightarrow \alpha_x + \alpha_y \leq \frac{1}{2}$

Handwritten notes:  $\alpha_y = \frac{\Gamma_y \Delta t}{\Lambda_\varphi \Delta x^2}$  and  $\alpha_x = \frac{\Gamma_x \Delta t}{\Lambda_\varphi \Delta x^2}$

Explicit scheme is **Conditionally Stable**.

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So we cannot specify arbitrary values for del t in case of explicit scheme. That should be related to del x. So that is a condition. Now if you consider implicit scheme again we can write our main discretized equation in terms of exact discrete value and error term.

(Refer Slide Time 19:46)

Problem Definition  
Explicit Scheme  
Implicit Scheme  
References

I.I.T. Kharagpur

### Implicit Scheme

The discretized governing equation for IBVP with implicit scheme can be written as,

$$\Lambda_\phi \frac{(\hat{\phi}_{i,j}^{n+1} + \epsilon_{i,j}^{n+1}) - (\hat{\phi}_{i,j}^n + \epsilon_{i,j}^n)}{\Delta t} = \Gamma_x \frac{(\hat{\phi}_{i-1,j}^{n+1} + \epsilon_{i-1,j}^{n+1}) - 2(\hat{\phi}_{i,j}^{n+1} + \epsilon_{i,j}^{n+1}) + (\hat{\phi}_{i+1,j}^{n+1} + \epsilon_{i+1,j}^{n+1})}{\Delta x^2} + \Gamma_y \frac{(\hat{\phi}_{i,j-1}^{n+1} + \epsilon_{i,j-1}^{n+1}) - 2(\hat{\phi}_{i,j}^{n+1} + \epsilon_{i,j}^{n+1}) + (\hat{\phi}_{i,j+1}^{n+1} + \epsilon_{i,j+1}^{n+1})}{\Delta y^2} + S_\phi|_{i,j}^{n+1} \quad (4)$$

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And similarly we can define our exact discrete solution of the finite difference equation in terms of these finite difference equation.

(Refer Slide Time 20:03)

Problem Definition  
Explicit Scheme  
Implicit Scheme  
References

I.I.T. Kharagpur

### Implicit Scheme

The discretized governing equation for IBVP with implicit scheme can be written as,

$$\Lambda_\phi \frac{(\hat{\phi}_{i,j}^{n+1} + \epsilon_{i,j}^{n+1}) - (\hat{\phi}_{i,j}^n + \epsilon_{i,j}^n)}{\Delta t} = \Gamma_x \frac{(\hat{\phi}_{i-1,j}^{n+1} + \epsilon_{i-1,j}^{n+1}) - 2(\hat{\phi}_{i,j}^{n+1} + \epsilon_{i,j}^{n+1}) + (\hat{\phi}_{i+1,j}^{n+1} + \epsilon_{i+1,j}^{n+1})}{\Delta x^2} + \Gamma_y \frac{(\hat{\phi}_{i,j-1}^{n+1} + \epsilon_{i,j-1}^{n+1}) - 2(\hat{\phi}_{i,j}^{n+1} + \epsilon_{i,j}^{n+1}) + (\hat{\phi}_{i,j+1}^{n+1} + \epsilon_{i,j+1}^{n+1})}{\Delta y^2} + S_\phi|_{i,j}^{n+1} \quad (4)$$

By definition,  $\hat{\phi}$  is the exact discrete solution of the finite difference equation. Thus, the discretized finite difference equation can be written as,

$$\Lambda_\phi \frac{\hat{\phi}_{i,j}^{n+1} - \hat{\phi}_{i,j}^n}{\Delta t} = \Gamma_x \frac{\hat{\phi}_{i-1,j}^{n+1} - 2\hat{\phi}_{i,j}^{n+1} + \hat{\phi}_{i+1,j}^{n+1}}{\Delta x^2} + \Gamma_y \frac{\hat{\phi}_{i,j-1}^{n+1} - 2\hat{\phi}_{i,j}^{n+1} + \hat{\phi}_{i,j+1}^{n+1}}{\Delta y^2} + S_\phi|_{i,j}^{n+1} \quad (5)$$

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Now if we subtract 5 from 4, obviously we will be getting error equation here.

(Refer Slide Time 20:18)

Problem Definition  
Explicit Scheme  
Implicit Scheme  
References

I.I.T. Kharagpur

### Implicit Scheme

By subtracting Equation 5 from Equation 4, we get the error equation

$$\Lambda_\phi \frac{\epsilon_{i,j}^{n+1} - \epsilon_{i,j}^n}{\Delta t} = \Gamma_x \frac{\epsilon_{i-1,j}^{n+1} - 2\epsilon_{i,j}^{n+1} + \epsilon_{i+1,j}^{n+1}}{\Delta x^2} + \Gamma_y \frac{\epsilon_{i,j-1}^{n+1} - 2\epsilon_{i,j}^{n+1} + \epsilon_{i,j+1}^{n+1}}{\Delta y^2} \quad (6)$$

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And in simplified form we can get this thing. Interestingly in this case left hand side, these values are unknown values. This is known on the right hand side.

(Refer Slide Time 20:37)

Problem Definition  
Explicit Scheme  
Implicit Scheme  
References

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### Implicit Scheme

By subtracting Equation 5 from Equation 4, we get the error equation

$$\Lambda_\phi \frac{\epsilon_{i,j}^{n+1} - \epsilon_{i,j}^n}{\Delta t} = \Gamma_x \frac{\epsilon_{i-1,j}^{n+1} - 2\epsilon_{i,j}^{n+1} + \epsilon_{i+1,j}^{n+1}}{\Delta x^2} + \Gamma_y \frac{\epsilon_{i,j-1}^{n+1} - 2\epsilon_{i,j}^{n+1} + \epsilon_{i,j+1}^{n+1}}{\Delta y^2} \quad (6)$$

In simplified form, this can be written as,

$$\alpha_x \epsilon_{i,j-1}^{n+1} + \alpha_x \epsilon_{i+1,j}^{n+1} - [1 + 2(\alpha_x + \alpha_y)] \epsilon_{i,j}^{n+1} + \alpha_x \epsilon_{i,j-1}^{n+1} + \alpha_y \epsilon_{i,j+1}^{n+1} = -\epsilon_{i,j}^n$$

with  $\alpha_x = \frac{\Gamma_x \Delta t}{\Lambda_\phi \Delta x^2}$  and  $\alpha_y = \frac{\Gamma_y \Delta t}{\Lambda_\phi \Delta y^2}$ .

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With this if we expand, we can again utilize the information. The only change is due to change in the time index, n plus 1. But there is no change in these values compared to our explicit scheme.


(Refer Slide Time 21:06)

Problem Definition  
Explicit Scheme  
Implicit Scheme  
References

I.I.T. Kharagpur

### Implicit Scheme

With

$$\begin{aligned}\epsilon_{i,j}^{n+1} &= A^{n+1} e^{\sqrt{-1}i\varphi_x + \sqrt{-1}j\varphi_y} \\ \epsilon_{i,j}^n &= A^n e^{\sqrt{-1}i\varphi_x + \sqrt{-1}j\varphi_y} \\ \epsilon_{i-1,j}^{n+1} &= A^{n+1} e^{\sqrt{-1}(i-1)\varphi_x + \sqrt{-1}j\varphi_y} \\ \epsilon_{i+1,j}^{n+1} &= A^{n+1} e^{\sqrt{-1}(i+1)\varphi_x + \sqrt{-1}j\varphi_y} \\ \epsilon_{i,j-1}^{n+1} &= A^{n+1} e^{\sqrt{-1}i\varphi_x + \sqrt{-1}(j-1)\varphi_y} \\ \epsilon_{i,j+1}^{n+1} &= A^{n+1} e^{\sqrt{-1}i\varphi_x + \sqrt{-1}(j+1)\varphi_y}\end{aligned}$$


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With this if we write our error equation. So error equation becomes, on the left hand side we have  $A^{n+1}$  plus 1 divided by  $A^n$  and minus 1 on the right hand side.

(Refer Slide Time 21:26)

Problem Definition  
Explicit Scheme  
Implicit Scheme  
References

I.I.T. Kharagpur

### Implicit Scheme

With

$$\begin{aligned}\epsilon_{i,j}^{n+1} &= A^{n+1} e^{\sqrt{-1}i\varphi_x + \sqrt{-1}j\varphi_y} \\ \epsilon_{i,j}^n &= A^n e^{\sqrt{-1}i\varphi_x + \sqrt{-1}j\varphi_y} \\ \epsilon_{i-1,j}^{n+1} &= A^{n+1} e^{\sqrt{-1}(i-1)\varphi_x + \sqrt{-1}j\varphi_y} \\ \epsilon_{i+1,j}^{n+1} &= A^{n+1} e^{\sqrt{-1}(i+1)\varphi_x + \sqrt{-1}j\varphi_y} \\ \epsilon_{i,j-1}^{n+1} &= A^{n+1} e^{\sqrt{-1}i\varphi_x + \sqrt{-1}(j-1)\varphi_y} \\ \epsilon_{i,j+1}^{n+1} &= A^{n+1} e^{\sqrt{-1}i\varphi_x + \sqrt{-1}(j+1)\varphi_y}\end{aligned}$$

By substituting all terms in the error equation,

$$\frac{A^{n+1}}{A^n} \left[ \alpha_y e^{-\sqrt{-1}\varphi_y} + \alpha_x e^{-\sqrt{-1}\varphi_x} - [1 + 2(\alpha_x + \alpha_y)] + \alpha_x e^{\sqrt{-1}\varphi_x} + \alpha_y e^{\sqrt{-1}\varphi_y} \right] = -1$$

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Again we can combine this term, this term. That means terms related to Y. And if we add these two terms we will get  $2 \cos \text{var } \phi_y$ . And if we combine this X terms, we will get  $2 \alpha_x \cos \text{var } \phi_x$ .

(Refer Slide Time 22:02)

Problem Definition  
Explicit Scheme  
Implicit Scheme  
References

I.I.T. Kharagpur

### Implicit Scheme

With

$$\begin{aligned}\epsilon_{i,j}^{n+1} &= A^{n+1} e^{\sqrt{-1}i\varphi_x + \sqrt{-1}j\varphi_y} \\ \epsilon_{i,j}^n &= A^n e^{\sqrt{-1}i\varphi_x + \sqrt{-1}j\varphi_y} \\ \epsilon_{i-1,j}^{n+1} &= A^{n+1} e^{\sqrt{-1}(i-1)\varphi_x + \sqrt{-1}j\varphi_y} \\ \epsilon_{i+1,j}^{n+1} &= A^{n+1} e^{\sqrt{-1}(i+1)\varphi_x + \sqrt{-1}j\varphi_y} \\ \epsilon_{i,j-1}^{n+1} &= A^{n+1} e^{\sqrt{-1}i\varphi_x + \sqrt{-1}(j-1)\varphi_y} \\ \epsilon_{i,j+1}^{n+1} &= A^{n+1} e^{\sqrt{-1}i\varphi_x + \sqrt{-1}(j+1)\varphi_y}\end{aligned}$$

By substituting all terms in the error equation,

$$\frac{A^{n+1}}{A^n} \left[ \alpha_y e^{-\sqrt{-1}j\varphi_y} + \alpha_x e^{-\sqrt{-1}i\varphi_x} - [1 + 2(\alpha_x + \alpha_y)] + \alpha_x e^{\sqrt{-1}i\varphi_x} + \alpha_y e^{\sqrt{-1}j\varphi_y} \right] = -1$$

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So with this if we simplify our error equation, then growth factor or amplification factor, we can write like this, minus 1 divided by minus 1 plus 2 alpha cos var phi y minus 1.


(Refer Slide Time 22:24)

Problem Definition  
Explicit Scheme  
Implicit Scheme  
References

I.I.T. Kharagpur

### Implicit Scheme

The Growth Factor can be written as,

$$G = \frac{A^{n+1}}{A^n} = \frac{-1}{-1 + 2\alpha_y(\cos\varphi_y - 1) + 2\alpha_x(\cos\varphi_x - 1)}$$


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So if we again simplify this, cos var phi y minus 1 equals to minus 2 sin square var phi y by 2. And this one as minus 2 sin square var phi x divided by 2.

(Refer Slide Time 22:58)

Problem Definition  
Explicit Scheme  
Implicit Scheme  
References

I.I.T. Kharagpur

### Implicit Scheme

The Growth Factor can be written as,

$$G = \frac{A^{n+1}}{A^n} = \frac{-1}{-1 + 2\alpha_y(\cos\phi_y - 1) + 2\alpha_x(\cos\phi_x - 1)}$$

$- 2 \sin^2 \frac{\phi_y}{2}$        $- 2 \sin^2 \frac{\phi_x}{2}$

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We can write this growth factor or amplification factor like this. Where alpha y and alpha x, these values are positive values obviously. So 1 by 1 plus some positive values and sin square. That means square of any term will be always positive. So 1 plus some positive value and on numerator we have only 1.

(Refer Slide Time 23:35)

Problem Definition  
Explicit Scheme  
Implicit Scheme  
References

I.I.T. Kharagpur

### Implicit Scheme

The Growth Factor can be written as,

$$G = \frac{A^{n+1}}{A^n} = \frac{-1}{-1 + 2\alpha_y(\cos\phi_y - 1) + 2\alpha_x(\cos\phi_x - 1)}$$

$- 2 \sin^2 \frac{\phi_y}{2}$        $- 2 \sin^2 \frac{\phi_x}{2}$

$$G = \frac{1}{1 + 4\alpha_y \sin^2(\frac{\phi_y}{2}) + 4\alpha_x \sin^2(\frac{\phi_x}{2})}$$

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So obviously this value is always less than 1. From Von Neumann stability condition, 1 by 1 plus 4 alpha y sin square var phi by 2. So with this if we proceed for the two cases. if sin values are zero then we have G equals to 1. Neutrally stable condition.

(Refer Slide Time 24:07)

Problem Definition  
Explicit Scheme  
Implicit Scheme  
References

I.I.T. Kharagpur

## Implicit Scheme

The Growth Factor can be written as,

$$G = \frac{A^{n+1}}{A^n} = \frac{-1}{-1 + 2\alpha_y(\cos\varphi_y - 1) + 2\alpha_x(\cos\varphi_x - 1)}$$

$$G = \frac{1}{1 + 4\alpha_y \sin^2\left(\frac{\varphi_y}{2}\right) + 4\alpha_x \sin^2\left(\frac{\varphi_x}{2}\right)}$$

The von Neumann Stability condition

$$\left| \frac{1}{1 + 4\alpha_y \sin^2\left(\frac{\varphi_y}{2}\right) + 4\alpha_x \sin^2\left(\frac{\varphi_x}{2}\right)} \right| \leq 1$$

Two Cases:

- $\sin\left(\frac{\varphi_x}{2}\right) = 0$  and  $\sin\left(\frac{\varphi_y}{2}\right) = 0 \Rightarrow G = 1$

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And if we have sin values equal to 1, then we have G equals to 1 by 1 plus 4 alpha y plus 4 alpha x. This is less than 1. This is obviously without imposing any condition, we are getting this. So we can say that implicit scheme is unconditionally stable. We don't need to put any restriction on del t compared to del x. Although we need to use small values of del t for this problems.

(Refer Slide Time 24:52)

Problem Definition  
Explicit Scheme  
Implicit Scheme  
References

I.I.T. Kharagpur

## Implicit Scheme

The Growth Factor can be written as,

$$G = \frac{A^{n+1}}{A^n} = \frac{-1}{-1 + 2\alpha_y(\cos\varphi_y - 1) + 2\alpha_x(\cos\varphi_x - 1)}$$

$$G = \frac{1}{1 + 4\alpha_y \sin^2\left(\frac{\varphi_y}{2}\right) + 4\alpha_x \sin^2\left(\frac{\varphi_x}{2}\right)}$$

The von Neumann Stability condition

$$\left| \frac{1}{1 + 4\alpha_y \sin^2\left(\frac{\varphi_y}{2}\right) + 4\alpha_x \sin^2\left(\frac{\varphi_x}{2}\right)} \right| \leq 1$$

Two Cases:

- $\sin\left(\frac{\varphi_x}{2}\right) = 0$  and  $\sin\left(\frac{\varphi_y}{2}\right) = 0 \Rightarrow G = 1$
- $\sin\left(\frac{\varphi_x}{2}\right) = 1$  and  $\sin\left(\frac{\varphi_y}{2}\right) = 1 \Rightarrow G = \frac{1}{1 + 4\alpha_y + 4\alpha_x} < 1$

Implicit scheme is **Unconditionally Stable**.

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Now we can extend this approach for Crank Nicolson scheme and we can check the numerical stability of the Crank Nicolson scheme. Thank you.