

Computational Hydraulics
Professor Anirban Dhar
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Lecture 10
Partial Differential Equation: IBVP

Welcome to this lecture number 10 of the course computational hydraulics. We are in the module number 2, numerical methods. And in this particular lecture we will be covering unit 6, partial differential equation IVBP that is initial boundary value problem.

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Problem Definition
Explicit Scheme
Implicit Scheme
θ Scheme

I.I.T. Kharagpur

Module 02: Numerical Methods
Unit 06: Partial Differential Equation: IBVP

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What is the learning objective for this particular unit? At the end of this unit lecture students will be able to discretize the spatial and temporal derivative of single valued multi-dimensional function using finite difference approximations. Also they will be able to derive the algebraic form using discretized partial differential equation, initial condition and boundary conditions.

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Learning Objectives

- To discretize the spatial and temporal derivatives of single-valued multi-dimensional functions using finite difference approximations.
- To derive the algebraic form using discretized PDE, IC and BCs.

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Let us consider our general equation. In our previous lecture we have considered only this term and source sink term for modelling the boundary value problem. Where phi is the general variable and in that case we have considered a two-dimensional system where this tensor is having four values or two directions.

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Problem Definition
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General Equation

A form of differential equation with a general variable ϕ :

$$\frac{\partial(\Lambda_{\phi}\phi)}{\partial t} + \nabla \cdot (\Upsilon_{\phi}\phi\mathbf{u}) = \nabla \cdot (\Gamma_{\phi}\nabla\phi) + F_{\phi o} + S_{\phi} \quad (1)$$

where

- ϕ = general variable
- $\Lambda_{\phi}, \Upsilon_{\phi}$ = problem dependent parameters
- Γ_{ϕ} = tensor
- $F_{\phi o}$ = other forces
- S_{ϕ} = source/sink term

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In this particular lecture we will introduce another term which is the temporal term. So we will have two spatial directions x, y and one temporal direction that is t.

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General Equation

A form of differential equation with a general variable ϕ :

$$\frac{\partial(\Lambda_\phi \phi)}{\partial t} + \nabla \cdot (\Upsilon_\phi \phi \mathbf{u}) = \nabla \cdot (\Gamma_\phi \nabla \phi) + F_{\phi_o} + S_\phi \quad (1)$$

where

- ϕ = general variable
- $\Lambda_\phi, \Upsilon_\phi$ = problem dependent parameters
- Γ_ϕ = tensor
- F_{ϕ_o} = other forces
- S_ϕ = source/sink term

01/2017

So with this information we can simplify this situation with temporal term and we can write this $\Lambda_\phi \phi$ let us consider that this is not varying with time. So we can write it as coefficient $\Delta \phi$ by Δt $\Gamma_x \Delta^2 \phi$ $\Gamma_y \Delta^2 \phi$ and this is again source sink term. And this is valid for Ω that is for our interior domain of the problem.

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Problem Definition

Governing equation

A two-dimensional (in space) IBVP can be written as,

$$\Omega : \Lambda_\phi \frac{\partial \phi}{\partial t} = \Gamma_x \frac{\partial^2 \phi}{\partial x^2} + \Gamma_y \frac{\partial^2 \phi}{\partial y^2} + S_\phi(x, y)$$

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We are considering two dimensional system where on top we have Neumann conditions, in bottom we have Neumann condition and left we have Dirichlet, right we have Dirichlet kind of condition. With this information we can define the boundary condition. However for initial

boundary value problem we need to specify the initial condition. This is two-dimensional in space.

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Problem Definition

Governing equation

A two-dimensional (in space) IBVP can be written as,

$$\Omega : \Lambda_\phi \frac{\partial \phi}{\partial t} = \Gamma_D \frac{\partial^2 \phi}{\partial x^2} + \Gamma_N \frac{\partial^2 \phi}{\partial y^2} + S_\phi(x, y)$$

Diagram showing a rectangular domain with boundary segments labeled Γ_D (left and right) and Γ_N (top and bottom).

So with this information we need to define this initial condition. Initial condition that is valid for x, y within the domain and zero is t equals to zero, that is initial time level. So this phi 0 is varying with x and y only. Boundary condition on left hand side that is x is equal to zero, we have this phi 1. Right hand side, lx y, t phi 2. On top and bottom this is at the bottom, for all x, y is equal to zero this value is zero. Again we have this x, y is equal to ly that is top boundary. We have specified value is equal to zero.

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Problem Definition

subject to

Initial Condition

$$\phi(x, y, 0) = \phi_0(x, y)$$

and

Boundary Condition

$$\Gamma_D^1 : \phi(0, y, t) = \phi_1$$

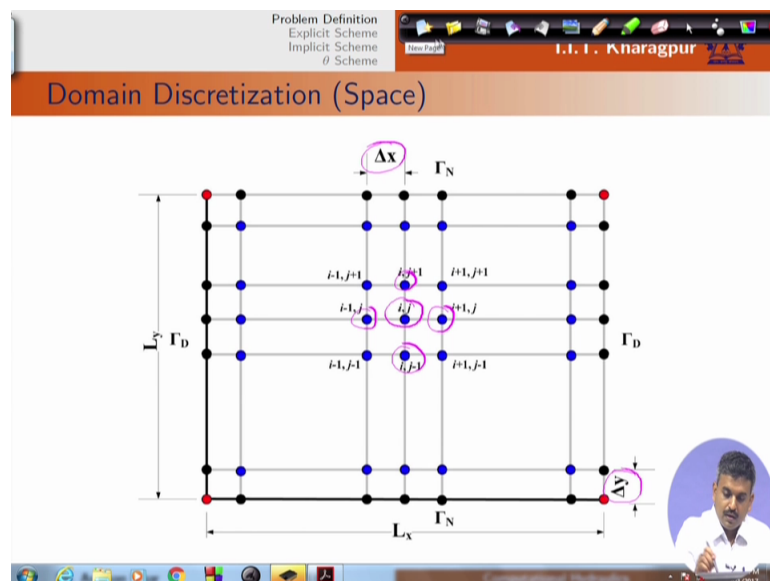
$$\Gamma_D^2 : \phi(L_x, y, t) = \phi_2$$

$$\Gamma_N^3 : \frac{\partial \phi}{\partial y} \Big|_{(x, 0, t)} = 0$$

$$\Gamma_N^4 : \frac{\partial \phi}{\partial y} \Big|_{(x, L_y, t)} = 0$$

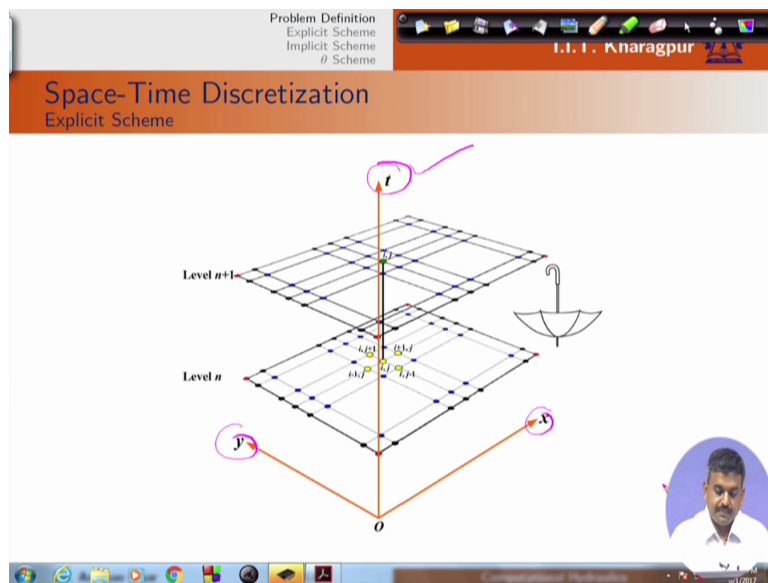
So we have completely defined the problem with our governing equation, initial condition and boundary condition. Now we need to discretize this system with finite difference approximation. So first of all if we consider domain discretization then we can define this L_x and L_y with Δy and Δx as space intervals, in y and x directions. With this for a general location ij these are the neighboring points. That means $i-1, j$, $i+1, j$, $i, j-1$, $i, j+1$.

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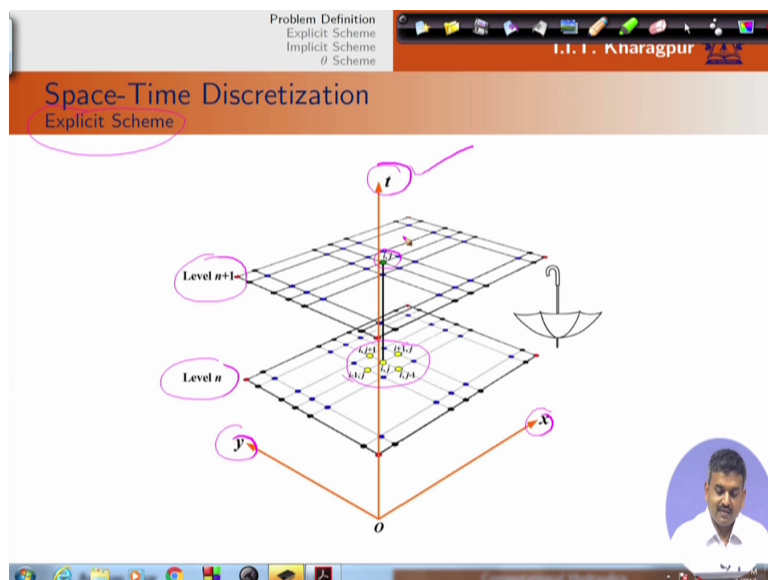
With this we need to introduce the temporal dimension. This is only spatial dimension discretization and this is valid for boundary value problem. And in this case also we can utilize it. However we need one extra dimension to discretize the IVBP problem. So with extra dimension we can define it, another axis t . This is x , y and this is our t direction, that is time direction. So at each time level we need to discretize our domain in spatial direction.

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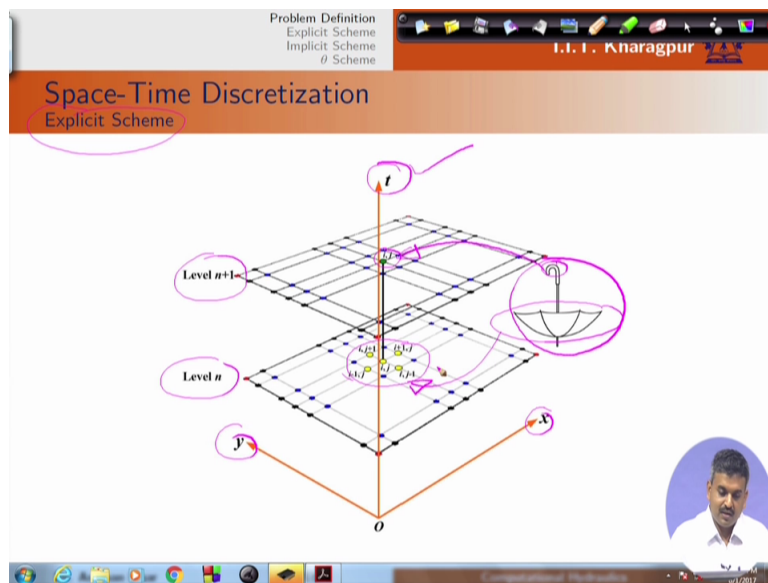
So this is level n this is level n plus 1 and we are in explicit scheme. What is this explicit? In explicit scheme we are considering the spatial values for neighboring cells and the neighboring points or nodes and the nodes for which we are discretizing at the present time level or n th level and we consider only the n plus 1 level value for the central or the point which is under consideration.

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This is like inverted umbrella kind of situation where we have this portion is considering the future time level value and in inverted umbrella this portion is considering the present time level spatial discretization.

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With this information if we start discretizing our governing equation then we are discretizing at the present time level. So that's why n, i, j . n is for time level i, j for x and y direction. So all are at n, i, j level. This source sink term, we need to consider at n, i, j level.

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$$\Lambda_\phi \frac{\partial \phi}{\partial t} \Big|_{i,j}^n = \Gamma_x \frac{\partial^2 \phi}{\partial x^2} \Big|_{i,j}^n + \Gamma_y \frac{\partial^2 \phi}{\partial y^2} \Big|_{i,j}^n + S_\phi(x, y) \Big|_{i,j}^n$$

With this if we discretize our temporal derivative then we can see that this is forward difference in time and we have discretization or truncation error of order Δt .

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Problem Definition
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Explicit Scheme

Governing Equation

$$\Lambda_\phi \frac{\partial \phi}{\partial t} \Big|_{i,j}^n = \Gamma_x \frac{\partial^2 \phi}{\partial x^2} \Big|_{i,j}^n + \Gamma_y \frac{\partial^2 \phi}{\partial y^2} \Big|_{i,j}^n + S_\phi(x, y) \Big|_{i,j}^n$$

Time Discretization

$$\frac{\partial \phi}{\partial t} \Big|_{i,j}^n = \frac{\phi_{i,j}^{n+1} - \phi_{i,j}^n}{\Delta t} + \mathcal{O}(\Delta t)$$

12:30 PM 6/1/2017

In explicit scheme the space derivatives are discretized at the present time level. So now we need to consider the space derivatives. One, two these two are space derivatives. So this should be discretized at the present time level or level n.

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Problem Definition
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Explicit Scheme

Governing Equation

$$\Lambda_\phi \frac{\partial \phi}{\partial t} \Big|_{i,j}^n = \Gamma_x \frac{\partial^2 \phi}{\partial x^2} \Big|_{i,j}^n + \Gamma_y \frac{\partial^2 \phi}{\partial y^2} \Big|_{i,j}^n + S_\phi(x, y) \Big|_{i,j}^n$$

Time Discretization

$$\frac{\partial \phi}{\partial t} \Big|_{i,j}^n = \frac{\phi_{i,j}^{n+1} - \phi_{i,j}^n}{\Delta t} + \mathcal{O}(\Delta t)$$

In *Explicit scheme*, space derivatives are discretized at the present time level (n).

12:30 PM 6/1/2017

So space discretization for first $\Delta^2 \phi / \Delta x^2$, we are considering $\phi_{i-1, j, n}$, $\phi_{i, j, n}$ and $\phi_{i+1, j, n}$. So we are only changing the index for i keeping other index as constant. So in this case we have second order accuracy for the truncation error in Δx .

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Problem Definition
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Explicit Scheme

Space Discretization

$$\frac{\partial^2 \phi}{\partial x^2} \Big|_{i,j}^n = \frac{\phi_{i-1,j}^n - 2\phi_{i,j}^n + \phi_{i+1,j}^n}{\Delta x^2} + \mathcal{O}(\Delta x^2)$$

If we consider the y direction, similarly we will have second order accuracy in y direction.

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Problem Definition
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Explicit Scheme

Space Discretization

$$\frac{\partial^2 \phi}{\partial x^2} \Big|_{i,j}^n = \frac{\phi_{i-1,j}^n - 2\phi_{i,j}^n + \phi_{i+1,j}^n}{\Delta x^2} + \mathcal{O}(\Delta x^2)$$
$$\frac{\partial^2 \phi}{\partial y^2} \Big|_{i,j}^n = \frac{\phi_{i,j-1}^n - 2\phi_{i,j}^n + \phi_{i,j+1}^n}{\Delta y^2} + \mathcal{O}(\Delta y^2)$$

Now we can integrate all these derivatives to construct the discretized form of the governing equation. So we have this lambda phi, gamma x, gamma y these values are as coefficients. Interestingly in this case n plus 1 level value is unknown and whatever value is available at present time step that are denoted with level n. So we have only one term which is unknown, I have marked it with this cross and for other terms values you are known.

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Explicit Scheme

Space Discretization

$$\frac{\partial^2 \phi}{\partial x^2} \Big|_{i,j}^n = \frac{\phi_{i-1,j}^n - 2\phi_{i,j}^n + \phi_{i+1,j}^n}{\Delta x^2} + \mathcal{O}(\Delta x^2)$$

$$\frac{\partial^2 \phi}{\partial y^2} \Big|_{i,j}^n = \frac{\phi_{i,j-1}^n - 2\phi_{i,j}^n + \phi_{i,j+1}^n}{\Delta y^2} + \mathcal{O}(\Delta y^2)$$

The corresponding difference equation can be written as,

$$\Lambda_\phi \frac{\phi_{i,j}^{n+1} - \phi_{i,j}^n}{\Delta t} = \Gamma_x \frac{\phi_{i-1,j}^n - 2\phi_{i,j}^n + \phi_{i+1,j}^n}{\Delta x^2} + \Gamma_y \frac{\phi_{i,j-1}^n - 2\phi_{i,j}^n + \phi_{i,j+1}^n}{\Delta y^2} + S_\phi \Big|_{i,j}^n + \mathcal{O}(\Delta x^2, \Delta y^2, \Delta t)$$

Handwritten annotations in pink circles highlight Λ_ϕ , $\phi_{i,j}^{n+1}$, $\phi_{i,j}^n$, Γ_x , Γ_y , S_ϕ , and the error term.

In this case we can see that overall truncation error for this scheme is Δx square, Δy square, because of two spatial derivatives. Second order spatial derivative for phi in x and y directions. And Δt , this is coming from the temporal discretization of the time derivative.

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Explicit Scheme

Space Discretization

$$\frac{\partial^2 \phi}{\partial x^2} \Big|_{i,j}^n = \frac{\phi_{i-1,j}^n - 2\phi_{i,j}^n + \phi_{i+1,j}^n}{\Delta x^2} + \mathcal{O}(\Delta x^2)$$

$$\frac{\partial^2 \phi}{\partial y^2} \Big|_{i,j}^n = \frac{\phi_{i,j-1}^n - 2\phi_{i,j}^n + \phi_{i,j+1}^n}{\Delta y^2} + \mathcal{O}(\Delta y^2)$$

The corresponding difference equation can be written as,

$$\Lambda_\phi \frac{\phi_{i,j}^{n+1} - \phi_{i,j}^n}{\Delta t} = \Gamma_x \frac{\phi_{i-1,j}^n - 2\phi_{i,j}^n + \phi_{i+1,j}^n}{\Delta x^2} + \Gamma_y \frac{\phi_{i,j-1}^n - 2\phi_{i,j}^n + \phi_{i,j+1}^n}{\Delta y^2} + S_\phi \Big|_{i,j}^n + \mathcal{O}(\Delta x^2, \Delta y^2, \Delta t)$$

Handwritten annotations in pink circles highlight Λ_ϕ , $\phi_{i,j}^{n+1}$, $\phi_{i,j}^n$, Γ_x , Γ_y , S_ϕ , and the error term.

With this if we start simplifying the equation we can write it like this only unknown term in the left inside others are known. So we can write it in simplified form with this notation. Alpha x, where alpha x is gamma x delta t, this lambda phi del x square, alpha y is gamma y del t lambda phi del y square. So with this if we write the right hand side so we have all values known. So now we can try to get the information about the future time level value with the available values of the present time level.

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Compact Form

In simplified form, this can be written as

$$\phi_{i,j}^{n+1} = \alpha_y \phi_{i,j-1}^n + \alpha_x \phi_{i-1,j}^n + [1 - 2(\alpha_x + \alpha_y)] \phi_{i,j}^n + \alpha_x \phi_{i+1,j}^n + \alpha_y \phi_{i,j+1}^n + \frac{\Delta t}{\Lambda_\phi} S_\phi |_{i,j}^n$$

with $\alpha_x = \frac{\Gamma_x \Delta t}{\Lambda_\phi \Delta x^2}$ and $\alpha_y = \frac{\Gamma_y \Delta t}{\Lambda_\phi \Delta y^2}$.

If we use our single index notation again we can transform this whole system into single index. With this we can write it as ϕ_l^{n+1} , where this j minus 1 will be l minus m plus 1, i minus 1 j that will be l minus 1. i plus 1 j is l plus 1, l plus m plus 1 is for ij plus 1. So with this information again this is ij is l . So with this information we can get the discretized form of our governing equation with single index notation.

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Single Index Notation

Single index l can be written as,

$$l = i + j \times (M + 1)$$

With single index notation, the equation can be written as,

$$\phi_l^{n+1} = \alpha_y \phi_{l-(M+1)}^n + \alpha_x \phi_{l-1}^n + [1 - 2(\alpha_x + \alpha_y)] \phi_l^n + \alpha_x \phi_{l+1}^n + \alpha_y \phi_{l+(M+1)}^n + \frac{\Delta t}{\Lambda_\phi} S_\phi |_{i,j}^n$$

Now we need to see how to get the solution with this explicit scheme. So what are the standard steps for this explicit schemes? Because we need time stepping algorithm. What is this time stepping algorithm? In time stepping algorithm we define the values with change in time in terms of our governing equation and rewrite it in algorithm format. In this case what

are the things that are available? We have λ_ϕ , γ_x , γ_y , S_ϕ , Δx , Δt , ϕ^n at time step n . That means we have n equals to zero.

That means initially we have initial condition available. And at the starting of anytime level, our ϕ^n value is specified. So with this what is the result required? In this case we need updated value of n plus 1 at time n plus 1.

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Standard Steps

Explicit Scheme: Time-stepping Algorithm

Data: $\lambda_\phi, \Gamma_x, \Gamma_y, S_\phi, \Delta x, \Delta y, \Delta t, \phi^n$ at time-step $n=0$
Result: Updated ϕ^{n+1} at time-step $n+1$

Now we can write this in algorithm format. This is for ij or l th point. And for any time level to predict the future time level we need to run this algorithm. That means t less than n time do. So first we need to solve the governing equation for interior points. For all interior points we need to get the $\phi^{L, n+1}$ level value with initial condition.

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Standard Steps

Explicit Scheme: Time-stepping Algorithm

Data: $\Lambda_\phi, \Gamma_x, \Gamma_y, S_\phi, \Delta x, \Delta y, \Delta t, \phi^n$ at time-step n
Result: Updated ϕ^{n+1} at time-step $n + 1$
while $t < \text{end time}$ **do**
 For interior points: $\phi_{i,j}^{n+1} = \alpha_y \phi_{l-(M+1)}^n + \alpha_x \phi_{l-1}^n + [1 - 2(\alpha_x + \alpha_y)] \phi_l^n + \alpha_x \phi_{l+1}^n + \alpha_y \phi_{l+(M+1)}^n + \frac{\Delta t}{\Lambda_\phi} S_\phi|_{i,j}^n$

Then we need to solve the boundary conditions at boundary points. And then we need to update the time level to $n + 1$ for calculation of the next time step or next time level value. So this one loop or two loop that will run until we reach our end time level. So in this explicit scheme important thing is that very first we need to solve the interior then we need to get the information about the boundary points. That is sequential. And then we need to update this time level from n to $n + 1$.

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Problem Definition
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Standard Steps

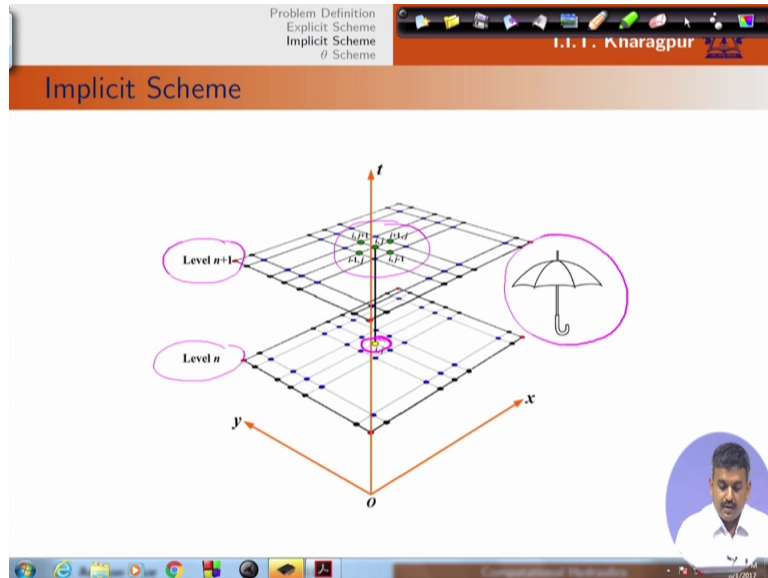
Explicit Scheme: Time-stepping Algorithm

Data: $\Lambda_\phi, \Gamma_x, \Gamma_y, S_\phi, \Delta x, \Delta y, \Delta t, \phi^n$ at time-step n
Result: Updated ϕ^{n+1} at time-step $n + 1$
while $t < \text{end time}$ **do**
 For interior points: $\phi_{i,j}^{n+1} = \alpha_y \phi_{l-(M+1)}^n + \alpha_x \phi_{l-1}^n + [1 - 2(\alpha_x + \alpha_y)] \phi_l^n + \alpha_x \phi_{l+1}^n + \alpha_y \phi_{l+(M+1)}^n + \frac{\Delta t}{\Lambda_\phi} S_\phi|_{i,j}^n$
 For boundary points: Use Boundary Conditions
 $n \leftarrow n + 1$
end

Now let us consider implicit scheme. What is this implicit? In implicit scheme we have $n + 1$ level value, n th level value. So n th level value, we are utilizing only one point that is the

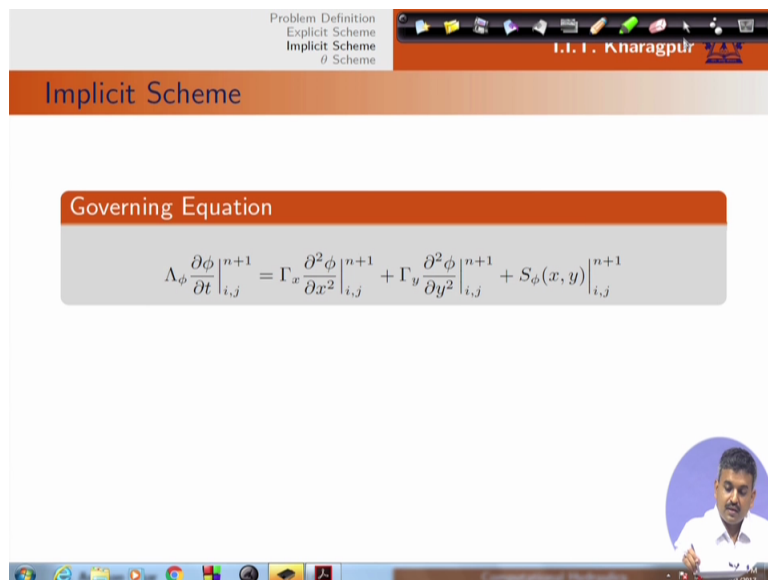
central point or central node. For others we will be utilizing the future time level value which is unknown. This is like straight umbrella.

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And if we discretize our equation, we need to discretize it at future time level value that is n plus 1, n plus 1.

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With this time discretization, this is backward in time because we are considering this thing. Again we have first order truncation error for time.

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The slide is titled "Implicit Scheme" and is part of a presentation by I.I.T. Kharagpur. It contains two main sections:

- Governing Equation:**
$$\Lambda_\phi \frac{\partial \phi}{\partial t} \Big|_{i,j}^{n+1} = \Gamma_x \frac{\partial^2 \phi}{\partial x^2} \Big|_{i,j}^{n+1} + \Gamma_y \frac{\partial^2 \phi}{\partial y^2} \Big|_{i,j}^{n+1} + S_\phi(x, y) \Big|_{i,j}^{n+1}$$
- Time Discretization:**
$$\frac{\partial \phi}{\partial t} \Big|_{i,j}^{n+1} = \frac{\phi_{i,j}^{n+1} - \phi_{i,j}^n}{\Delta t} + \mathcal{O}(\Delta t)$$

The time discretization equation has pink circles around the derivative term and the $\mathcal{O}(\Delta t)$ term. A small circular inset photo of a man is visible in the bottom right corner of the slide.

Implicit scheme, space derivatives are discretized at the future time level $n + 1$. So now if we discretize the space derivatives $\frac{\partial^2 \phi}{\partial x^2}$, $\frac{\partial^2 \phi}{\partial y^2}$, we should discretize it at future time level.

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The slide is titled "Implicit Scheme" and is part of a presentation by I.I.T. Kharagpur. It contains two main sections:

- Governing Equation:**
$$\Lambda_\phi \frac{\partial \phi}{\partial t} \Big|_{i,j}^{n+1} = \Gamma_x \frac{\partial^2 \phi}{\partial x^2} \Big|_{i,j}^{n+1} + \Gamma_y \frac{\partial^2 \phi}{\partial y^2} \Big|_{i,j}^{n+1} + S_\phi(x, y) \Big|_{i,j}^{n+1}$$
- Time Discretization:**
$$\frac{\partial \phi}{\partial t} \Big|_{i,j}^{n+1} = \frac{\phi_{i,j}^{n+1} - \phi_{i,j}^n}{\Delta t} + \mathcal{O}(\Delta t)$$

Below the time discretization equation, there is a note: "In *Implicit scheme*, space derivatives are discretized at the future time level ($n + 1$)." The space derivative terms in the governing equation are underlined in pink.

So space derivatives, these are discretized as $n + 1$ level, similarly for y direction.

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Implicit Scheme

Space Discretization

$$\frac{\partial^2 \phi}{\partial x^2} \Big|_{i,j}^{n+1} = \frac{\phi_{i-1,j}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i+1,j}^{n+1}}{\Delta x^2} + \mathcal{O}(\Delta x^2)$$

$$\frac{\partial^2 \phi}{\partial y^2} \Big|_{i,j}^{n+1} = \frac{\phi_{i,j-1}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i,j+1}^{n+1}}{\Delta y^2} + \mathcal{O}(\Delta y^2)$$

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Now we can write our governing equation in discretized form. In this case only unknown thing is or known thing is n level values. Also this s phi source sink term that may be defined for a particular system. So one, two, three, four, five, six, seven these terms are unknown.

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Implicit Scheme

Space Discretization

$$\frac{\partial^2 \phi}{\partial x^2} \Big|_{i,j}^{n+1} = \frac{\phi_{i-1,j}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i+1,j}^{n+1}}{\Delta x^2} + \mathcal{O}(\Delta x^2)$$

$$\frac{\partial^2 \phi}{\partial y^2} \Big|_{i,j}^{n+1} = \frac{\phi_{i,j-1}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i,j+1}^{n+1}}{\Delta y^2} + \mathcal{O}(\Delta y^2)$$

The corresponding difference equation can be written as,

$$\Lambda_\phi \frac{\phi_{i,j}^{n+1} - \phi_{i,j}^n}{\Delta t} = \Gamma_x \frac{\phi_{i-1,j}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i+1,j}^{n+1}}{\Delta x^2} + \Gamma_y \frac{\phi_{i,j-1}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i,j+1}^{n+1}}{\Delta y^2} + S_\phi \Big|_{i,j}^{n+1} + \mathcal{O}(\Delta x^2, \Delta y^2, \Delta t)$$

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Although this phi ij n plus 1, if we combine with this term and this term that will we be counted as single term.

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Implicit Scheme

Space Discretization

$$\frac{\partial^2 \phi}{\partial x^2} \Big|_{i,j}^{n+1} = \frac{\phi_{i-1,j}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i+1,j}^{n+1}}{\Delta x^2} + \mathcal{O}(\Delta x^2)$$

$$\frac{\partial^2 \phi}{\partial y^2} \Big|_{i,j}^{n+1} = \frac{\phi_{i,j-1}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i,j+1}^{n+1}}{\Delta y^2} + \mathcal{O}(\Delta y^2)$$

The corresponding difference equation can be written as,

$$\Lambda_\phi \frac{\phi_{i,j}^{n+1} - \phi_{i,j}^n}{\Delta t} = \Gamma_x \frac{\phi_{i-1,j}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i+1,j}^{n+1}}{\Delta x^2} + \Gamma_y \frac{\phi_{i,j-1}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i,j+1}^{n+1}}{\Delta y^2} + S_\phi \Big|_{i,j}^{n+1} + \mathcal{O}(\Delta x^2, \Delta y^2, \Delta t)$$

So we have one, two, three, four and five unknown values for this particular governing equation.

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Problem Definition
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θ Scheme

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Implicit Scheme

Space Discretization

$$\frac{\partial^2 \phi}{\partial x^2} \Big|_{i,j}^{n+1} = \frac{\phi_{i-1,j}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i+1,j}^{n+1}}{\Delta x^2} + \mathcal{O}(\Delta x^2)$$

$$\frac{\partial^2 \phi}{\partial y^2} \Big|_{i,j}^{n+1} = \frac{\phi_{i,j-1}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i,j+1}^{n+1}}{\Delta y^2} + \mathcal{O}(\Delta y^2)$$

The corresponding difference equation can be written as,

$$\Lambda_\phi \frac{\phi_{i,j}^{n+1} - \phi_{i,j}^n}{\Delta t} = \Gamma_x \frac{\phi_{i-1,j}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i+1,j}^{n+1}}{\Delta x^2} + \Gamma_y \frac{\phi_{i,j-1}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i,j+1}^{n+1}}{\Delta y^2} + S_\phi \Big|_{i,j}^{n+1} + \mathcal{O}(\Delta x^2, \Delta y^2, \Delta t)$$

And overall accuracy of the scheme is Δx^2 , Δy^2 in time, this is Δt .

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Problem Definition
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θ Scheme

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Implicit Scheme

Space Discretization

$$\frac{\partial^2 \phi}{\partial x^2} \Big|_{i,j}^{n+1} = \frac{\phi_{i-1,j}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i+1,j}^{n+1}}{\Delta x^2} + \mathcal{O}(\Delta x^2)$$

$$\frac{\partial^2 \phi}{\partial y^2} \Big|_{i,j}^{n+1} = \frac{\phi_{i,j-1}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i,j+1}^{n+1}}{\Delta y^2} + \mathcal{O}(\Delta y^2)$$

The corresponding difference equation can be written as,

$$\Lambda_\phi \frac{\phi_{i,j}^{n+1} - \phi_{i,j}^n}{\Delta t} = \Gamma_x \frac{\phi_{i-1,j}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i+1,j}^{n+1}}{\Delta x^2} + \Gamma_y \frac{\phi_{i,j-1}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i,j+1}^{n+1}}{\Delta y^2} + S_\phi \Big|_{i,j}^{n+1} + \mathcal{O}(\Delta x^2, \Delta y^2, \Delta t)$$

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In compact form again we can write it like this with alpha x and alpha y. In this case these are unknown values. One, two, three, four and five unknown values. And right hand side we have known value available. With this information we can construct the matrix form using a single index notation.

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Problem Definition
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Compact Form

In simplified form, this can be written as

$$\alpha_x \phi_{i,j-1}^{n+1} + \alpha_x \phi_{i-1,j}^{n+1} - [1 + 2(\alpha_x + \alpha_y)] \phi_{i,j}^{n+1} + \alpha_x \phi_{i+1,j}^{n+1} + \alpha_y \phi_{i,j+1}^{n+1} = -\phi_{i,j}^n - \frac{\Delta t}{\Lambda_\phi} S_\phi \Big|_{i,j}^{n+1}$$

with $\alpha_x = \frac{\Gamma_x \Delta t}{\Lambda_\phi \Delta x^2}$ and $\alpha_y = \frac{\Gamma_y \Delta t}{\Lambda_\phi \Delta y^2}$.

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For single index notation again we need to convert it with L minus n plus 1, L minus 1, L, L plus 1, L plus n plus 1 format.

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Single Index Notation

Single index l can be written as,

$$l = i + j \times (M + 1)$$

With single index notation, the equation can be written as,

$$\alpha_y \phi_{l-(M+1)}^{n+1} + \alpha_x \phi_{l-1}^{n+1} - [1 + 2(\alpha_x + \alpha_y)] \phi_l^{n+1} + \alpha_x \phi_{l+1}^{n+1} + \alpha_y \phi_{l+(M+1)}^{n+1} = -\phi_l^n - \frac{\Delta t}{\Lambda_\phi} S_\phi|_{i,j}^{n+1}$$

In this case these values are unknown, so matrix form is essential for this problem.

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Single Index Notation

Single index l can be written as,

$$l = i + j \times (M + 1)$$

With single index notation, the equation can be written as,

$$\alpha_y \phi_{l-(M+1)}^{n+1} + \alpha_x \phi_{l-1}^{n+1} - [1 + 2(\alpha_x + \alpha_y)] \phi_l^{n+1} + \alpha_x \phi_{l+1}^{n+1} + \alpha_y \phi_{l+(M+1)}^{n+1} = -\phi_l^n - \frac{\Delta t}{\Lambda_\phi} S_\phi|_{i,j}^{n+1}$$

So if you want to solve this implicit scheme, again we need to define this time stepping algorithm for this one. So data for this one is lambda phi, gamma x, gamma y, s phi, del x, del y and del t, phi n at time step n. So with this if we proceed what result is required. Result is updated value of phi n plus 1 at time step n plus 1. In this case we need to run the while loop, t less than n time.

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Problem Definition
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Standard Steps

Implicit Scheme: Time-stepping Algorithm

Data: $\Lambda_\phi, \Gamma_x, \Gamma_y, S_\phi, \Delta x, \Delta y, \Delta t, \phi^n$ at time-step n
Result: Updated ϕ^{n+1} at time-step $n + 1$
while $t < \text{end time}$ **do**
 For interior and boundary points: Solve governing equation and boundary conditions in discretized form.
 $n \leftarrow n + 1$
end

01/2017

In explicit scheme we have observed that first we need to solve the interior points using governing equation, then boundary points using boundary condition. In this case we need to solve interior points and (bo) boundary points simultaneously. So solve governing equation and boundary condition in discretized form. So simultaneously we need to solve the governing equation and boundary condition. Then we need to update the time level to get the future time level value or present time level value for the next time step.

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Problem Definition
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Standard Steps

Implicit Scheme: Time-stepping Algorithm

Data: $\Lambda_\phi, \Gamma_x, \Gamma_y, S_\phi, \Delta x, \Delta y, \Delta t, \phi^n$ at time-step n
Result: Updated ϕ^{n+1} at time-step $n + 1$
while $t < \text{end time}$ **do**
 For interior and boundary points: Solve governing equation and boundary conditions in discretized form.
 $n \leftarrow n + 1$
end

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Now we need to define the theta scheme. Theta scheme we consider some intermediate time step. In that intermediate time step first step is explicit in nature. Explicit means we are

defining at nth level. Then we define our implicit state. Implicit state is defined at n plus one level.

(Refer Slide Time 25:39)

The slide displays the governing equation for the θ -scheme. It is divided into two sections: 'Explicit Step' and 'Implicit Step'. The explicit step equation is $\Lambda_\phi \frac{\partial \phi}{\partial t} \Big|_{i,j}^n = \Gamma_x \frac{\partial^2 \phi}{\partial x^2} \Big|_{i,j}^n + \Gamma_y \frac{\partial^2 \phi}{\partial y^2} \Big|_{i,j}^n + S_\phi(x,y) \Big|_{i,j}^n$. The implicit step equation is $\Lambda_\phi \frac{\partial \phi}{\partial t} \Big|_{i,j}^{n+1} = \Gamma_x \frac{\partial^2 \phi}{\partial x^2} \Big|_{i,j}^{n+1} + \Gamma_y \frac{\partial^2 \phi}{\partial y^2} \Big|_{i,j}^{n+1} + S_\phi(x,y) \Big|_{i,j}^{n+1}$. The slide also includes a navigation menu at the top with options: Problem Definition, Explicit Scheme, Implicit Scheme, and θ Scheme. The I.I.T. Kharagpur logo is visible in the top right corner.

So with this we can write the explicit step for n plus theta level. Right hand side all values are known at nth line level. And implicit step we have all unknown at n plus 1 level and n plus theta is known from the first step.

(Refer Slide Time 26:14)

The slide displays the finite difference scheme for the θ -scheme. It is divided into two sections: 'Explicit Step' and 'Implicit Step'. The explicit step equation is $\Lambda_\phi \frac{\phi_{i,j}^{n+\theta} - \phi_{i,j}^n}{\theta \Delta t} = \Gamma_x \frac{\phi_{i-1,j}^n - 2\phi_{i,j}^n + \phi_{i+1,j}^n}{\Delta x^2} + \Gamma_y \frac{\phi_{i,j-1}^n - 2\phi_{i,j}^n + \phi_{i,j+1}^n}{\Delta y^2} + S_\phi \Big|_{i,j}^n + \mathcal{O}(\Delta x^2, \Delta y^2, \theta \Delta t)$. The implicit step equation is $\Lambda_\phi \frac{\phi_{i,j}^{n+1} - \phi_{i,j}^{n+\theta}}{(1-\theta)\Delta t} = \Gamma_x \frac{\phi_{i-1,j}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i+1,j}^{n+1}}{\Delta x^2} + \Gamma_y \frac{\phi_{i,j-1}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i,j+1}^{n+1}}{\Delta y^2} + S_\phi \Big|_{i,j}^{n+1} + \mathcal{O}(\Delta x^2, \Delta y^2, (1-\theta)\Delta t)$. The slide also includes a navigation menu at the top with options: Problem Definition, Explicit Scheme, Implicit Scheme, and θ Scheme. The I.I.T. Kharagpur logo is visible in the top right corner.

If we combine these two steps then we can write it as $i,j,n+1 - \theta \phi_{i,j,n}$ and this is some kind of weighted average $\theta, 1 - \theta$ for x direction. Again $\theta, 1 - \theta$ for y direction.

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θ -scheme

Finite Difference Scheme

By combining explicit and implicit discretizations,

$$\Lambda_\phi \frac{\phi_{i,j}^{n+1} - \phi_{i,j}^n}{\Delta t} = \Gamma_x \left[\theta \frac{\phi_{i-1,j}^n - 2\phi_{i,j}^n + \phi_{i+1,j}^n}{\Delta x^2} + (1-\theta) \frac{\phi_{i-1,j}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i+1,j}^{n+1}}{\Delta x^2} \right] + \Gamma_y \left[\theta \frac{\phi_{i,j-1}^n - 2\phi_{i,j}^n + \phi_{i,j+1}^n}{\Delta y^2} + (1-\theta) \frac{\phi_{i,j-1}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i,j+1}^{n+1}}{\Delta y^2} \right] + [\theta S_\phi]_{i,j}^n + (1-\theta) S_\phi|_{i,j}^{n+1} + \mathcal{O}(\Delta x^2, \Delta y^2, ?)$$


In this case important point is what will be the truncation error order for Δt or time? If we write it for explicit step it is in terms of n plus θ . We can write it. Again for implicit step we can write it for n plus θ and if we combine these two we can get the time derivative here.

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Truncation Error of θ -scheme

Time Discretization

Explicit Step

$$\phi_{i,j}^n = \phi_{i,j}^{n+\theta} - \theta \Delta t \frac{\partial \phi}{\partial t} \Big|_{i,j}^{n+\theta} + \frac{(\theta \Delta t)^2}{2!} \frac{\partial^2 \phi}{\partial t^2} \Big|_{i,j}^{n+\theta} - \frac{(\theta \Delta t)^3}{3!} \frac{\partial^3 \phi}{\partial t^3} \Big|_{i,j}^{n+\theta} + \dots$$

Implicit Step

$$\phi_{i,j}^{n+1} = \phi_{i,j}^{n+\theta} + (1-\theta) \Delta t \frac{\partial \phi}{\partial t} \Big|_{i,j}^{n+\theta} + \frac{(1-\theta)^2 \Delta t^2}{2!} \frac{\partial^2 \phi}{\partial t^2} \Big|_{i,j}^{n+\theta} + \frac{(1-\theta)^3 \Delta t^3}{3!} \frac{\partial^3 \phi}{\partial t^3} \Big|_{i,j}^{n+\theta} + \dots$$

Combined Step

$$\frac{\phi_{i,j}^{n+1} - \phi_{i,j}^n}{\Delta t} = \frac{\partial \phi}{\partial t} \Big|_{i,j}^{n+\theta} + \frac{[(1-\theta)^2 - \theta^2] \Delta t}{2!} \frac{\partial^2 \phi}{\partial t^2} \Big|_{i,j}^{n+\theta} + \frac{[(1-\theta)^3 - \theta^3] \Delta t^2}{3!} \frac{\partial^3 \phi}{\partial t^3} \Big|_{i,j}^{n+\theta} + \dots$$

And interestingly this is, $\frac{\partial \phi}{\partial t}$, Δt . This is at n plus θ level. And we have this $1 - \theta$ minus θ square minus θ square term.

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Truncation Error of θ -scheme

Time Discretization

Explicit Step

$$\phi_{i,j}^n = \phi_{i,j}^{n+\theta} - \theta \Delta t \frac{\partial \phi}{\partial t} \Big|_{i,j}^{n+\theta} + \frac{(\theta \Delta t)^2}{2!} \frac{\partial^2 \phi}{\partial t^2} \Big|_{i,j}^{n+\theta} - \frac{(\theta \Delta t)^3}{3!} \frac{\partial^3 \phi}{\partial t^3} \Big|_{i,j}^{n+\theta} + \dots$$

Implicit Step

$$\phi_{i,j}^{n+1} = \phi_{i,j}^{n+\theta} + (1-\theta) \Delta t \frac{\partial \phi}{\partial t} \Big|_{i,j}^{n+\theta} + \frac{(1-\theta)^2 \Delta t^2}{2!} \frac{\partial^2 \phi}{\partial t^2} \Big|_{i,j}^{n+\theta} + \frac{(1-\theta)^3 \Delta t^3}{3!} \frac{\partial^3 \phi}{\partial t^3} \Big|_{i,j}^{n+\theta} + \dots$$

Combined Step

$$\frac{\phi_{i,j}^{n+1} - \phi_{i,j}^n}{\Delta t} = \frac{\partial \phi}{\partial t} \Big|_{i,j}^{n+\theta} + \frac{[(1-\theta)^2 - \theta^2] \Delta t}{2!} \frac{\partial^2 \phi}{\partial t^2} \Big|_{i,j}^{n+\theta} + \frac{[(1-\theta)^3 + \theta^3] \Delta t^2}{3!} \frac{\partial^3 \phi}{\partial t^3} \Big|_{i,j}^{n+\theta} + \dots$$

Now if we consider that theta equal to point 5, obviously this term will be zero and truncation error of the scheme will be of the order Δx^2 , Δy^2 , Δt^2 .

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Problem Definition
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Crank-Nicolson Method

If $\theta = 0.5$

$$\frac{[(1-\theta)^2 - \theta^2] \Delta t}{2!} \frac{\partial^2 \phi}{\partial t^2} \Big|_{i,j}^{n+\theta} = 0$$

The truncation error of the scheme is $\mathcal{O}(\Delta x^2, \Delta y^2, \Delta t^2)$.

Essentially in this case we are getting second order accuracy in time. The scheme is known as Crank Nicolson scheme.

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Crank-Nicolson Method

If $\theta = 0.5$

$$\left[\frac{(1-\theta)^2 - \theta^2}{2!} \frac{\Delta t}{\partial t^2} \phi \right]_{i,j}^{n+\theta} = 0$$

The truncation error of the scheme is $\mathcal{O}(\Delta x^2, \Delta y^2, \Delta t^2)$.

The scheme is known as *Crank-Nicolson method*.



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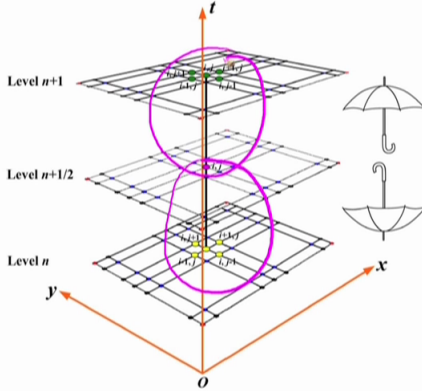
If we see the space time discretization for Crank Nicolson. It's the combination of explicit scheme and implicit scheme.

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Crank-Nicolson Method



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This is n plus half level because theta equals to half. And first step is inverted umbrella and second step is with straight umbrella.

