Computational Hydraulics
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Lecture 10
Partial Differential Equation: IBVP

Welcome to this lecture number 10 of the course computational hydraulics. We are in the module number 2, numerical methods. And in this particular lecture we will be covering unit 6, partial differential equation IVBP that is initial boundary value problem.

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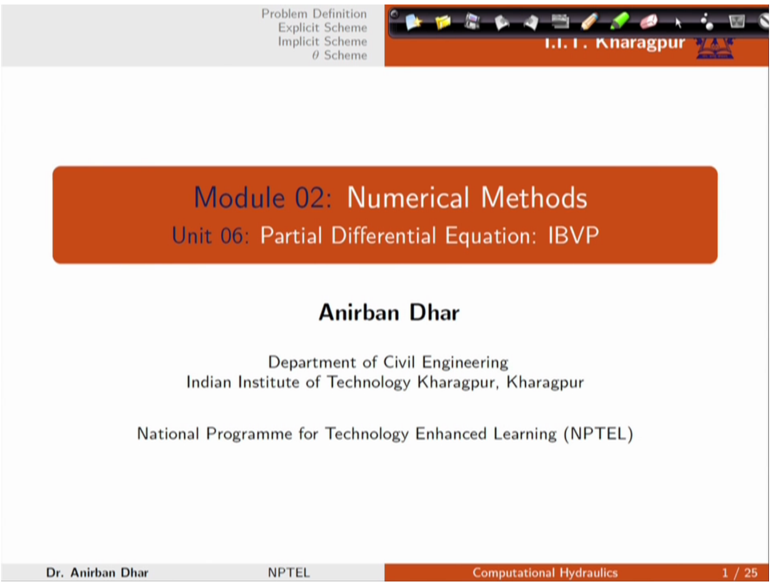

What is the learning objective for this particular unit? At the end of this unit lecture students will be able to discretize the spatial and temporal derivative of single valued multi-dimensional function using finite difference approximations. Also they will be able to derive the algebraic form using discretized partial differential equation, initial condition and boundary conditions.

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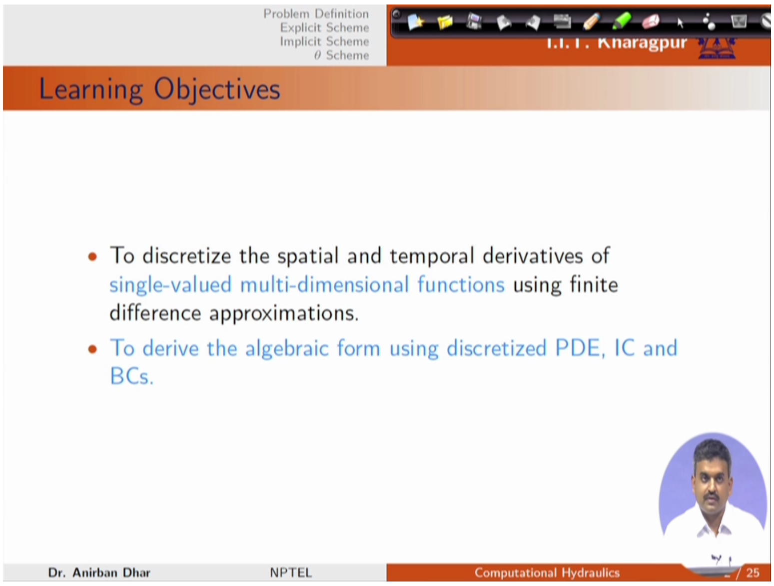

Let us consider our general equation. In our previous lecture we have considered only this term and source sink term for modelling the boundary value problem. Where phi is the general variable and in that case we have considered a two-dimensional system where this tensor is having four values or two directions.

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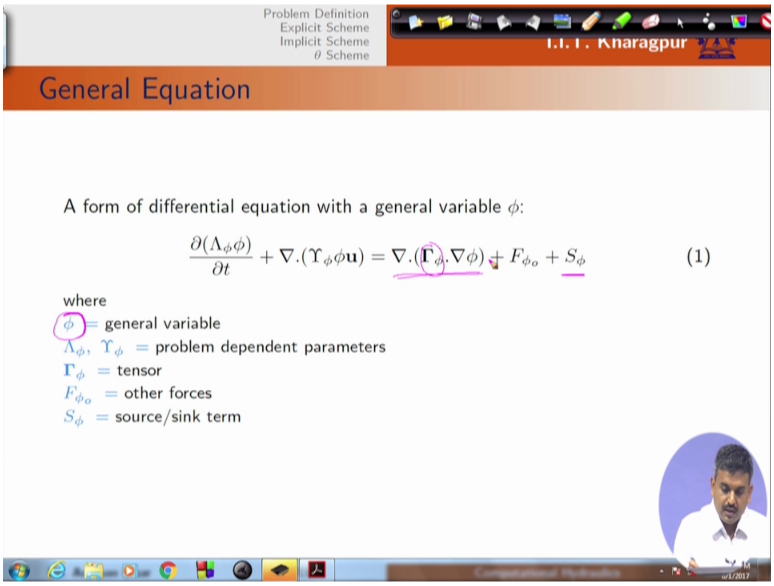

In this particular lecture we will introduce another term which is the temporal term. So we will have two spatial directions x, y and one temporal direction that is t.

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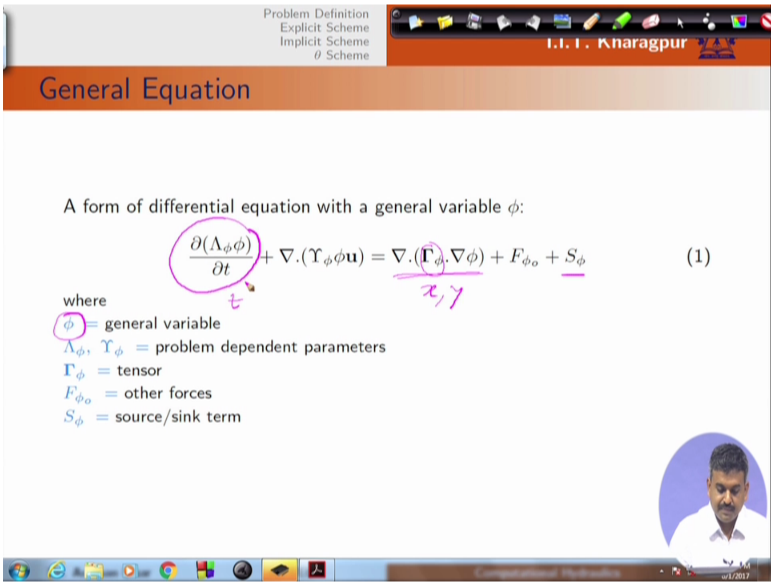

So with this information we can simplify this situation with temporal term and we can write this lambda phi let us consider that this is not varying with time. So we can write it as coefficient del phi by del t gamma X del 2 phi del x2gamma Y del 2 phi del y2 and this is again source sink term. And this is valid for omega that is for our interior domain of the problem.

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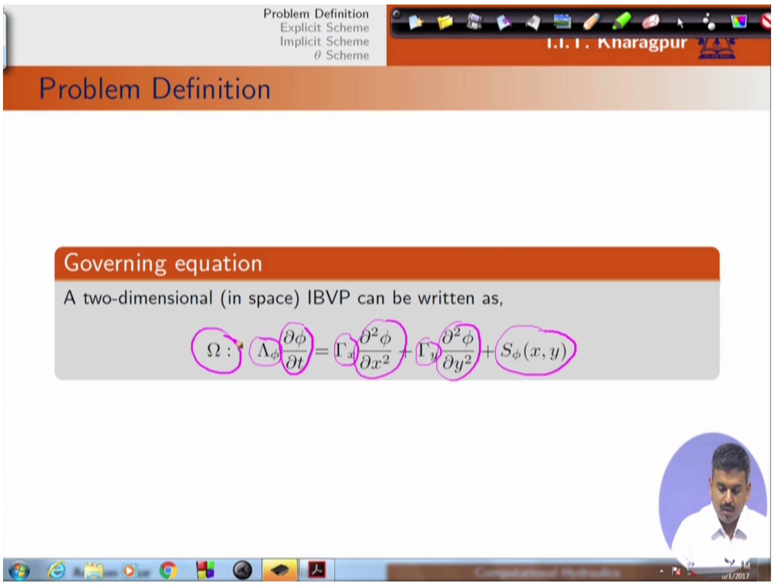

We are considering two dimensional system where on top we have Neumann conditions, in bottom we have Neumann condition and left we have Dirichlet, right we have Dirichlet kind of condition. With this information we can define the boundary condition. However for initial boundary value problem we need to specify the initial condition.This is two-dimensional in space.

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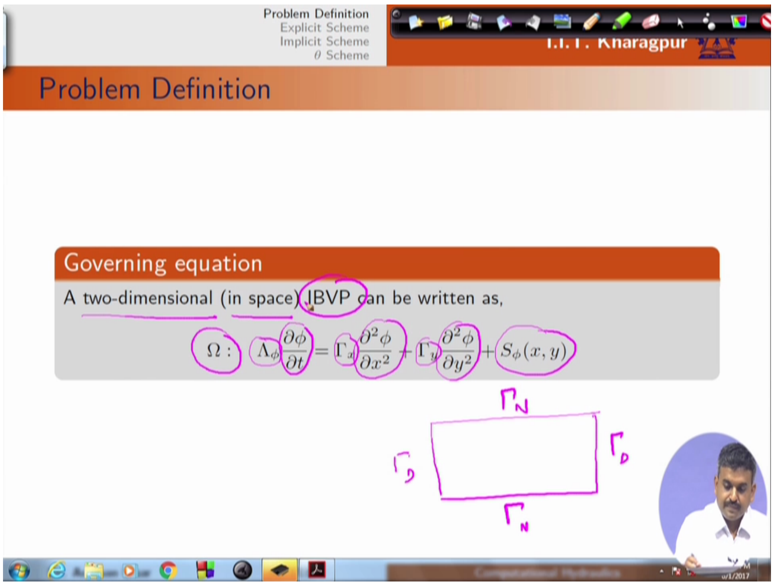

So with this information we need to define this initial condition. Initial condition that is valid for x, y within the domain and zero is t equals to zero, that is initial time level. So this phi 0 is varying with x and y only. Boundary condition on left hand side that is x is equal to zero, we have this phi 1. Right hand side, lx y, t phi 2. On top and bottom this is at the bottom, for all x, y is equal to zero this value is zero. Again we have this x, y is equal to ly that is top boundary. We have specified value is equal to zero.

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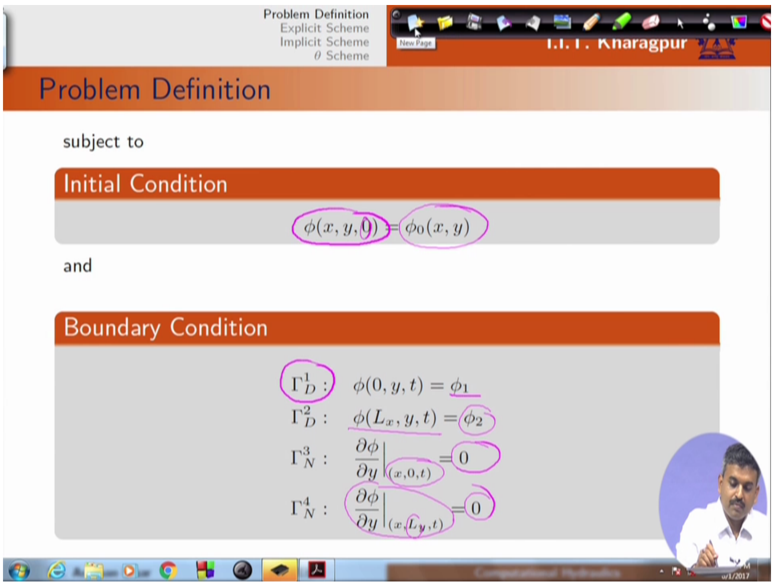

So we have completely defined the problem with our governing equation, initial condition and boundary condition. Now we need to discretize this system with finite difference approximation. So first of all if we consider domain discretization then we can define this lx and ly with del y and del x as space intervals, in y and x directions.With this for a general location ij these are the neighboring points. That meansI j minus 1, I minus 1 j, I minus 1 j, Ij plus 1.

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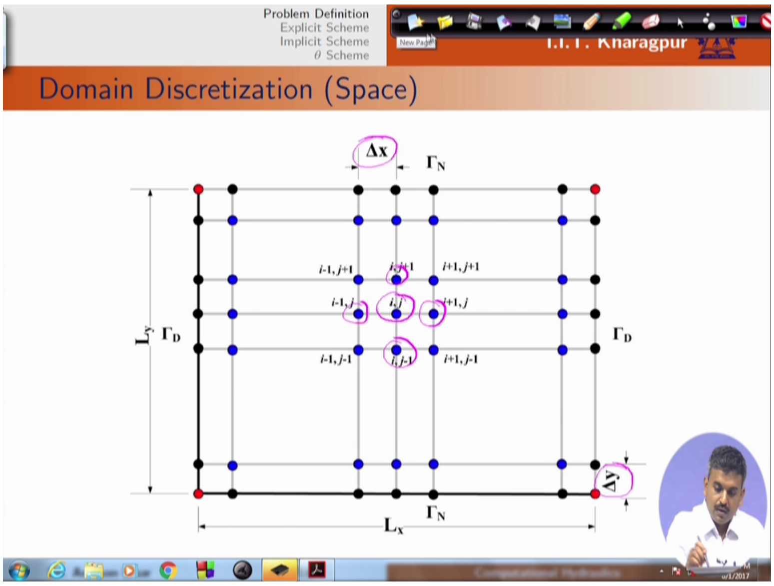

With this we need to introduce the temporal dimension. This is only spatial dimension discretization and this is valid for boundary value problem. And in this case also we can utilize it. However we need one extra dimension to discretize the IVBP problem. So with extra dimension we can define it, another axis t. This is x, y and this is our t direction, that is time direction. So at each time level we need to discretize our domain in spatial direction.

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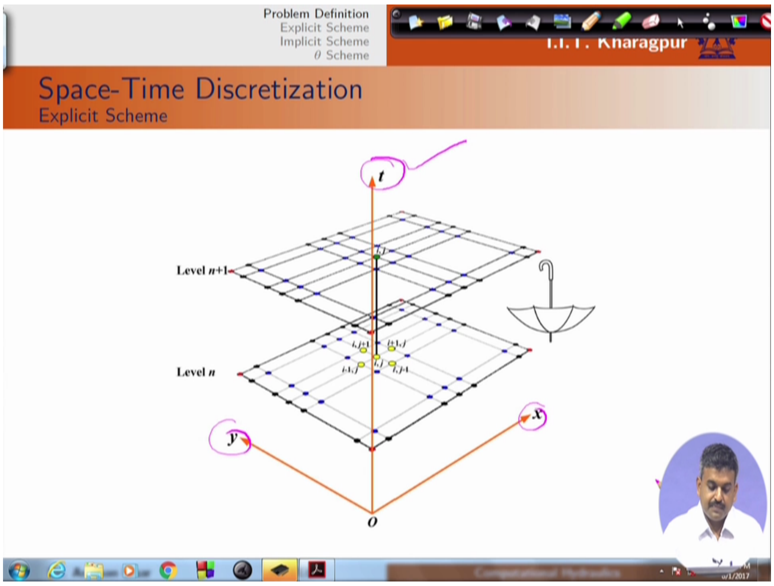

So this is level n this is level n plus1 and we are in explicit scheme. What is this explicit? In explicit scheme we are considering the spatial values for neighboring cells and the neighboring points or nodes and the nodes for which we are discretizing at the present time level or nth level and we consider only the n plus 1 level value for the central or the point which is under consideration.

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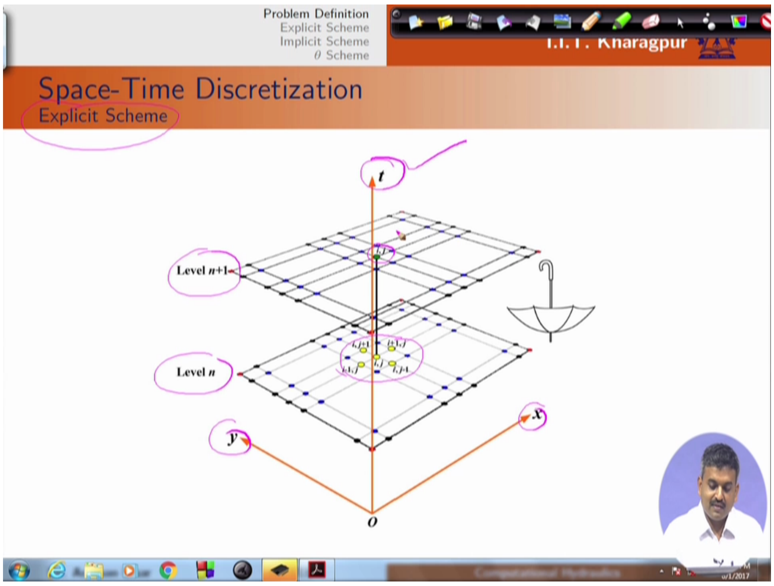

This is like inverted umbrella kind of situation where wehave this portion is considering the future time level value and in inverted umbrella this portion is considering the present time level spatial discretization.

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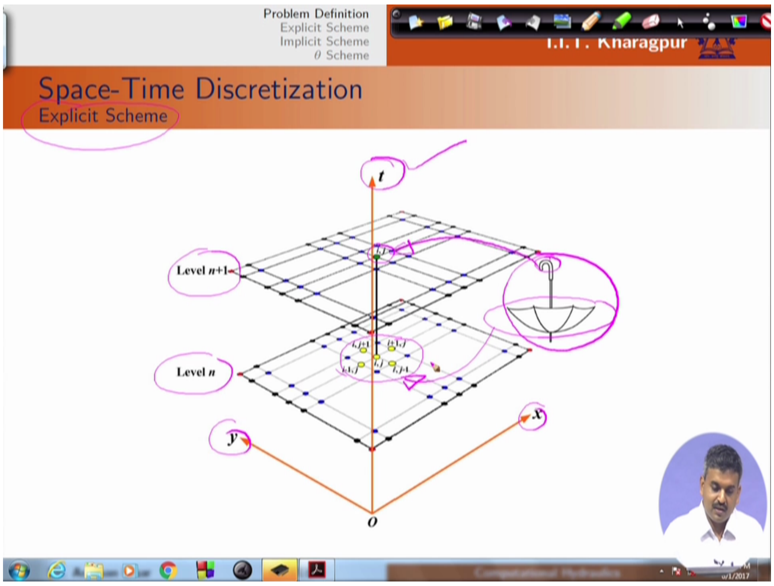

With this information if we start discretizing our governing equation then we are discretizing at the present time level. So that's why n ij. N is for time level ij for x and y direction. So all are at n ij level. This source sink term, we need to consider at n ij level.

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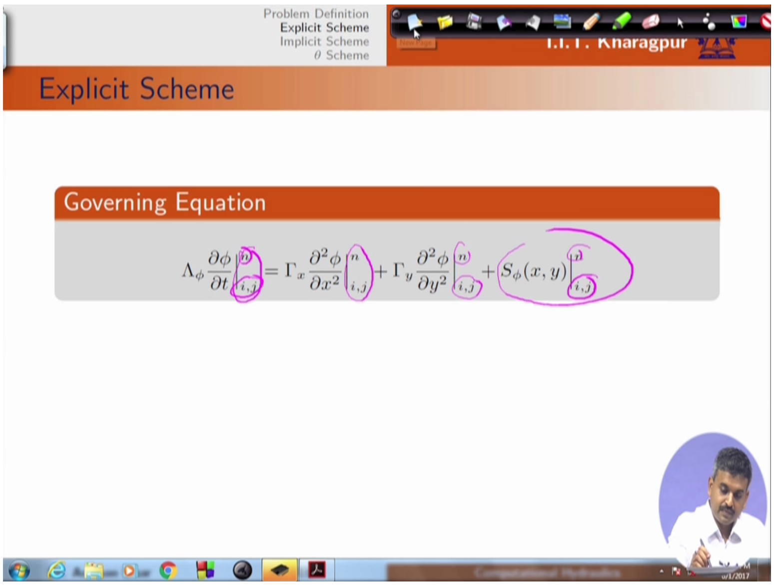

With this if we discretize our temporal derivative then we can see that this is forward difference in time and we have discretization or truncation error of order delta t.

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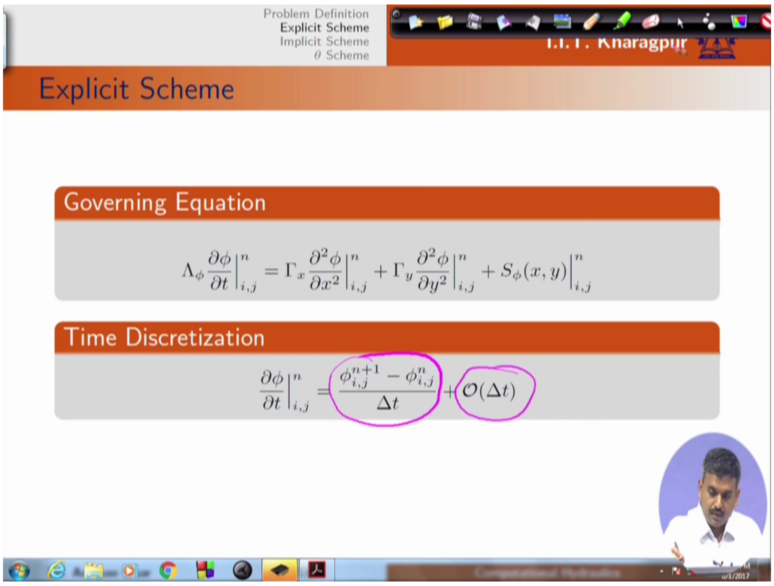

In explicit scheme the space derivatives are discretize at the present time level. So now we need to consider the space derivatives. One, two these two are space derivatives. So this should be discretized at the present time level or level n.

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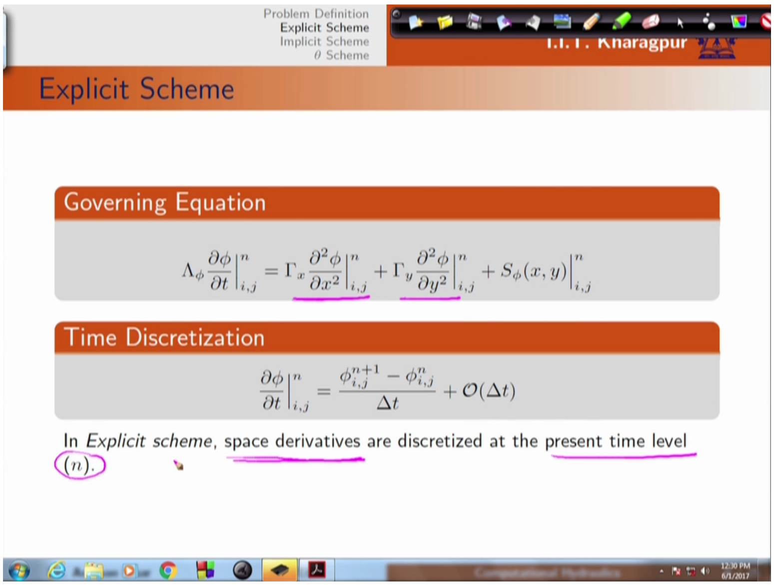

So space discretization for first del 2 phi del x 2, we are considering phi I minus 1 jn, 2 phi ijn and phi I plus 1 jn. So we are only changing the index for I keeping other index as constant. So in this case we have second order accuracy for the truncation error in del x.

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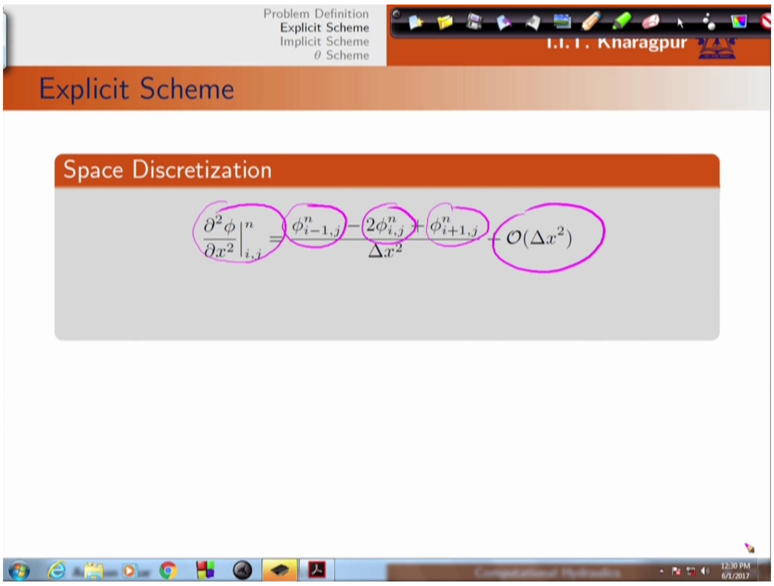

If we consider the y direction, similarly we will have second order accuracy in y direction.

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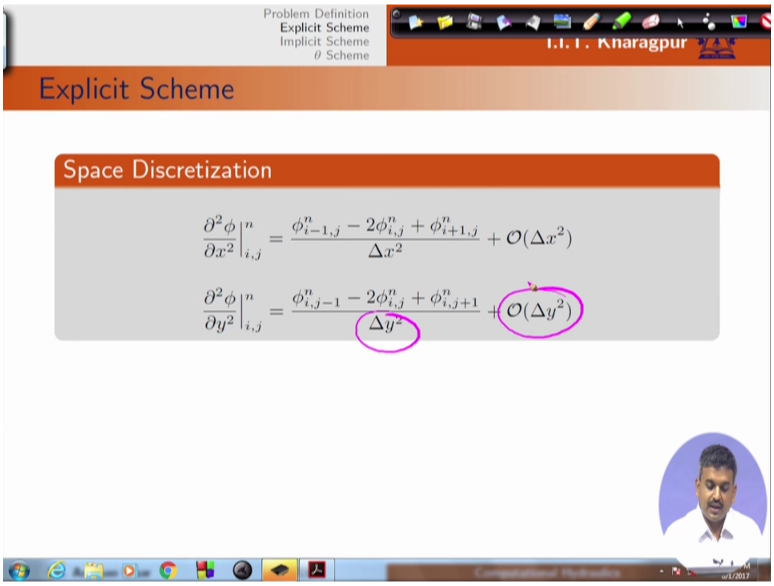

Now we can integrate all these derivatives to construct the discretized form of the governing equation. So we have this lambda phi, gamma x, gamma y these values are as coefficients. Interestingly in this case n plus 1 level value is unknown and whatever value is available at present time step that are denoted with level n. So we have only one term which is unknown, I have marked it with this cross and for other terms values you are known.

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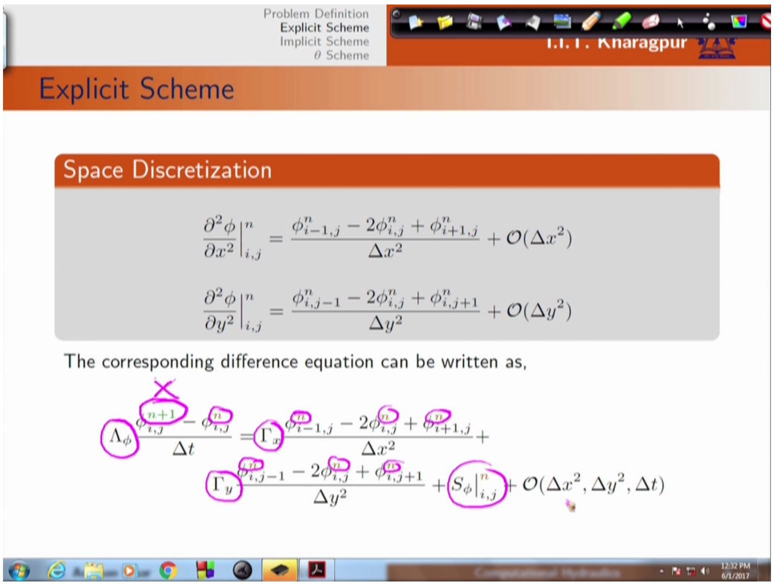

In this case we can see that overall truncation error for this scheme is del x square, del y square, because of two spatial derivatives. Second order special derivative for phi in x and y directions. And del t, this is coming from the temporal discretization of the time derivative.

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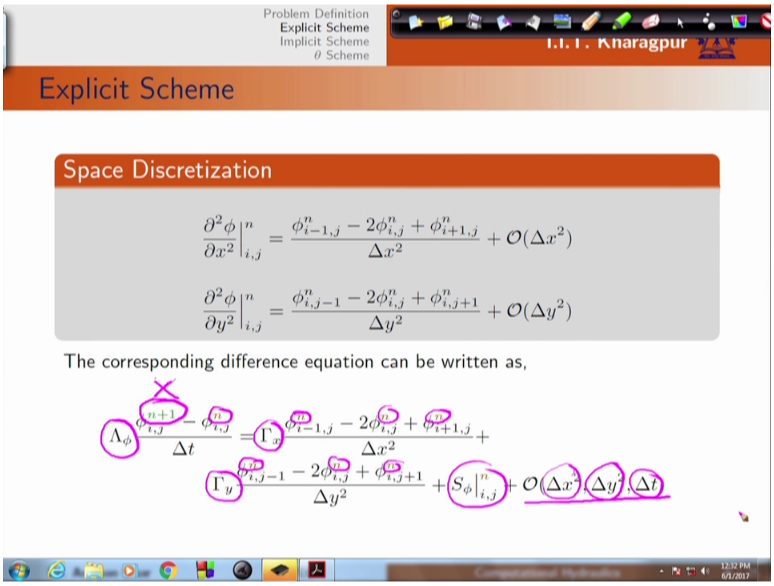

With this if we start simplifying the equation we can write it like this only unknown term in the left inside others are known. So we can write it in simplified form with this notation. Alpha x, where alpha x is gamma x delta t, this lambda phi del x square, alpha y is gamma y del t lambda phi del y square. So with this if we write the right hand side so we have all values known. So now we can try to get the information about the future time level value with the available values of the present time level.

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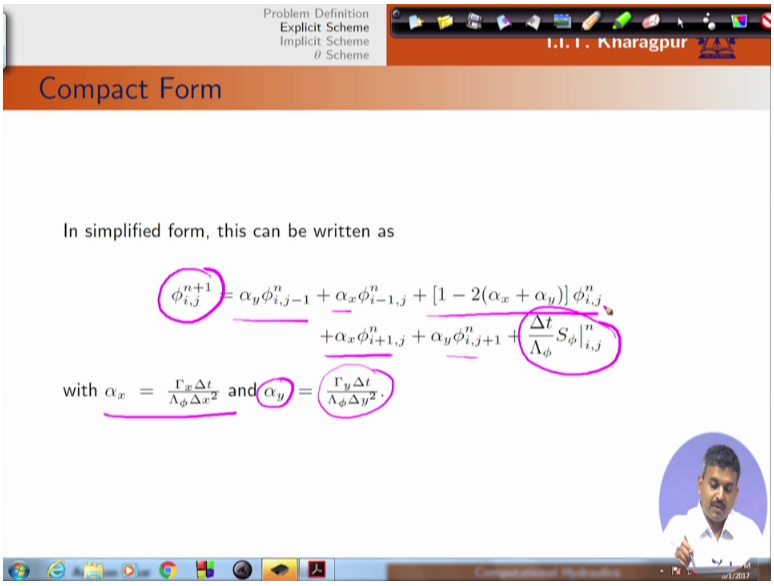

If we use our single index notation again we can transform this whole system into single index. With this we can write it as phi L n plus 1, where this j minus 1 will be L minus m plus 1, I minus 1 j that will be L minus 1. Ij is L, I plus 1 j is L plus 1, L plus m plus 1 is for ij plus 1. So with this information again this is ij is L. So with this information we can get the discretized form of our governing equation with single index notation.

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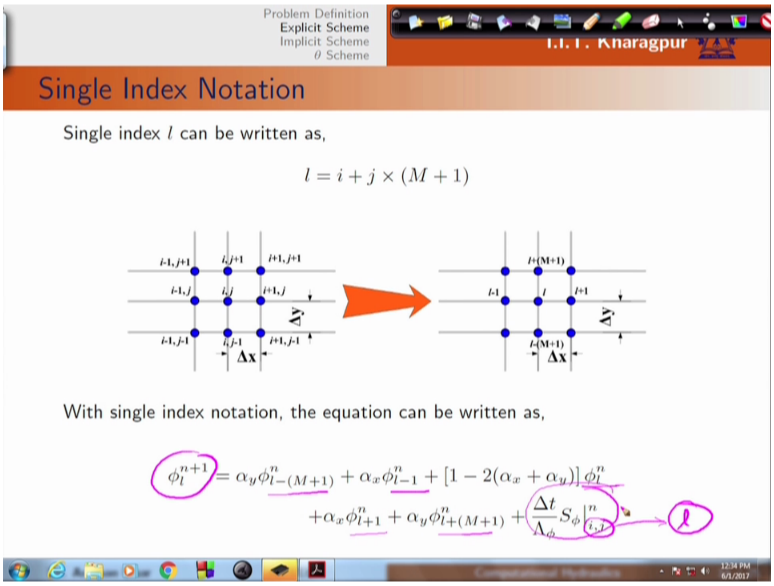

Now we need to see how to get the solution with this explicit scheme. So what are the standard steps for this explicit schemes? Because we need time stepping algorithm. What is this time stepping algorithm? In time stepping algorithm we define the values with change in time in terms of our governing equation and rewrite it in algorithm format. In this case what are the things that are available? We have lambda phi, gamma x, gamma y, s phi, del x, del t del y, del t, phi n at time step n. That means we have n equals to zero.

That means initially we have initial condition available. And at the starting of anytime level, our phi n value is specified. So with this what is the result required? In this case we need updated value of n plus 1 at time n plus 1.

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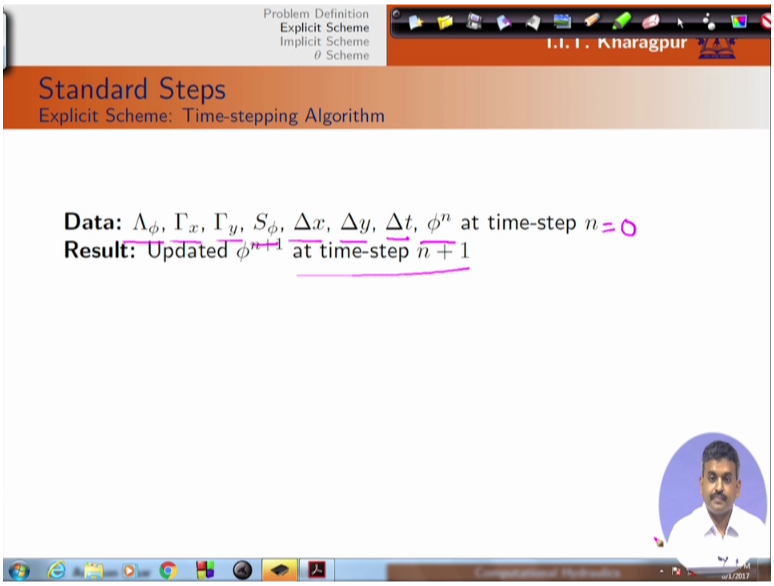

Now we can write this in algorithm format. This is for ij or lth point. And for any time level to predict the future time level we need to run this algorithm. That means t less than n time do. So first we need to solve the governing equation for interior points. For all interior points we need to get the phi L n plus 1 level value with initial condition.

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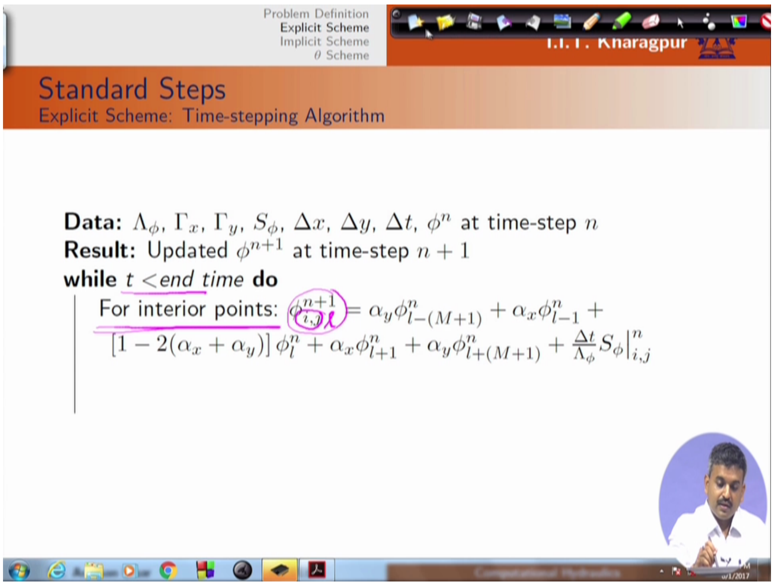

Then we need to solve the boundary conditions at boundary points. And then we need to update the time level to n plus 1 for calculation of the next time step or next time level value. So this one loop or two loop that will run until we reach our end time level. So in this explicit scheme important thing is that very first we need to solve the interior then we need to get the information about the boundary points. That is sequential. And then we need to update this time level from n to n plus 1.

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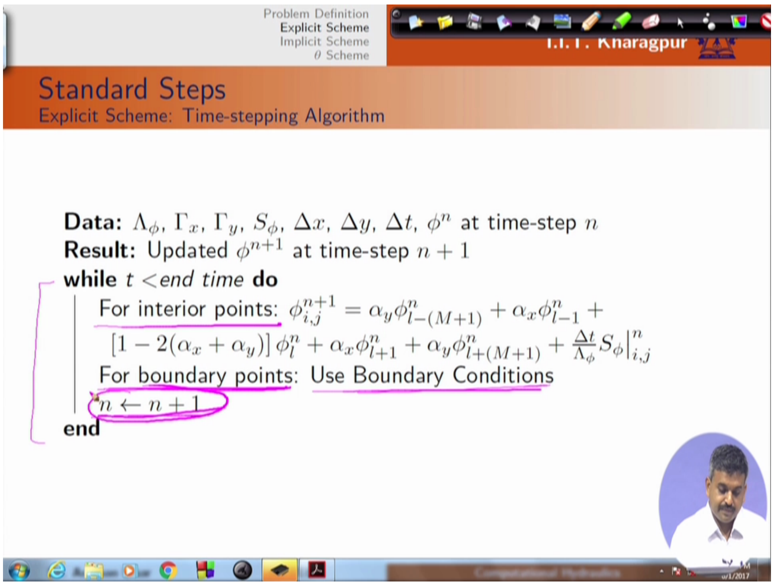

Now let us consider implicit scheme. What is this implicit? In implicit scheme we have n plus 1 level value, nth level value. So nth level value, we are utilizing only one point that is the central point or central node. For others we will be utilizing the future time level value which is unknown. This is like straight umbrella.

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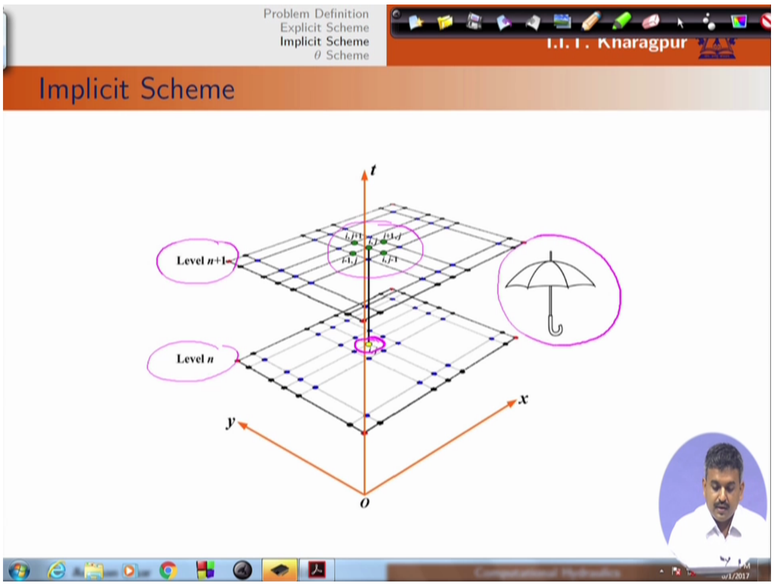

And if we discretize our equation, we need to discretize it at future time level value that is n plus 1, n plus 1.

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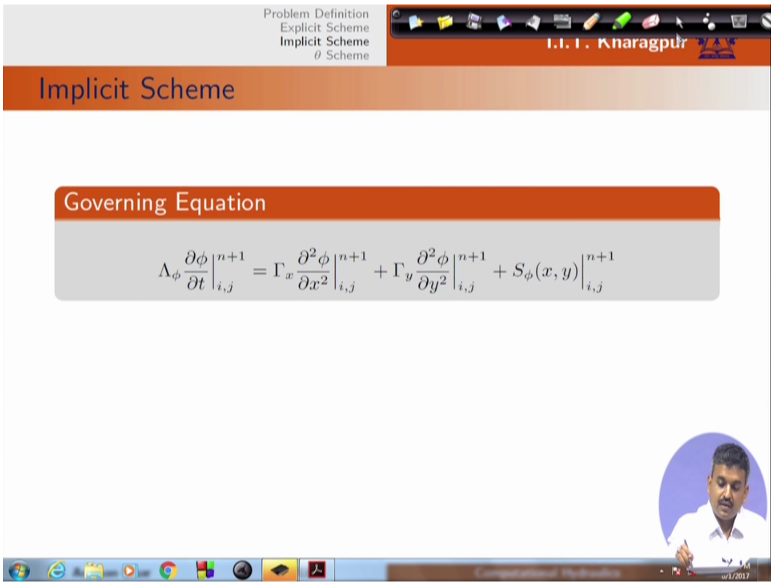

With this time discretization, this is backward in time because we are considering this thing.Again we have first order truncation error for time.

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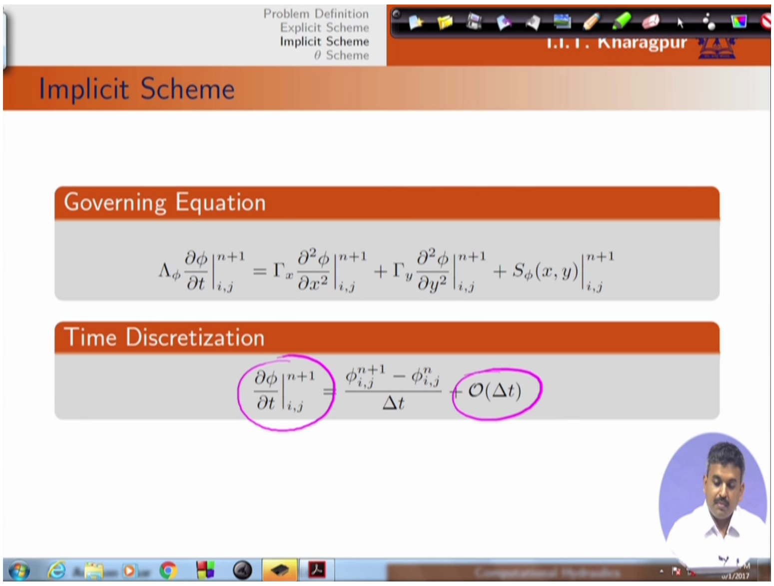

Implicit scheme, space derivative are discretize at the future time level n plus 1. So now if we discretize the space derivatives del 2 phi del x2, del 2 phi del y2, we should discretize it at future time level.

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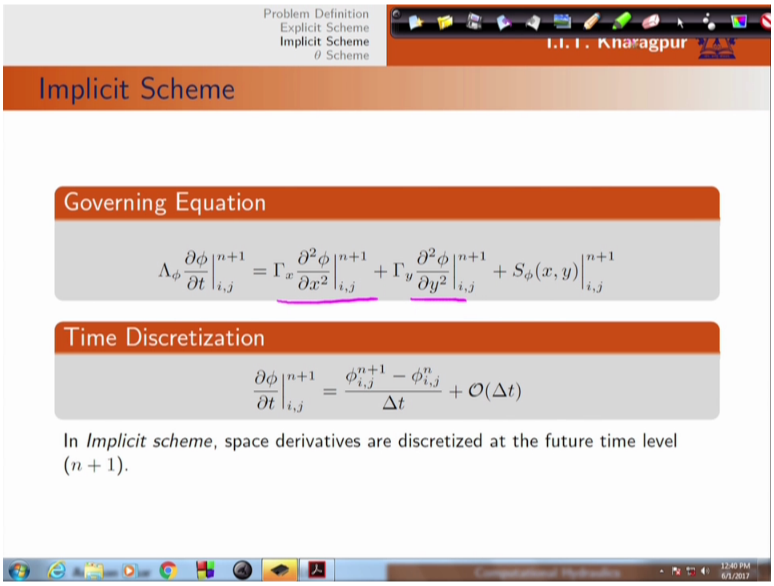

So space derivatives, these are discretize as n plus 1 level, similarly for y direction.

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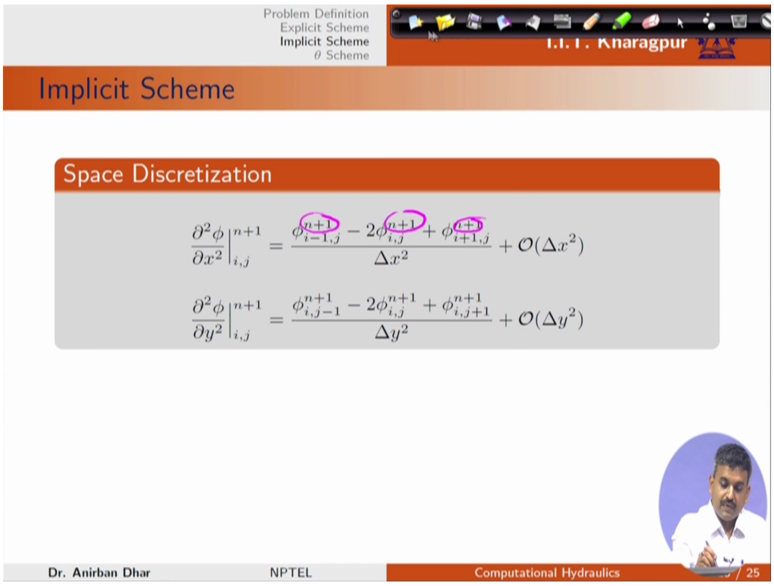

Now we can write our governing equation in discretized form. In this case only unknown thing is or known thing is n level values. Also this s phi source sink term that may be defined for a particular system. So one, two, three, four, five, six, seven these terms are unknown.

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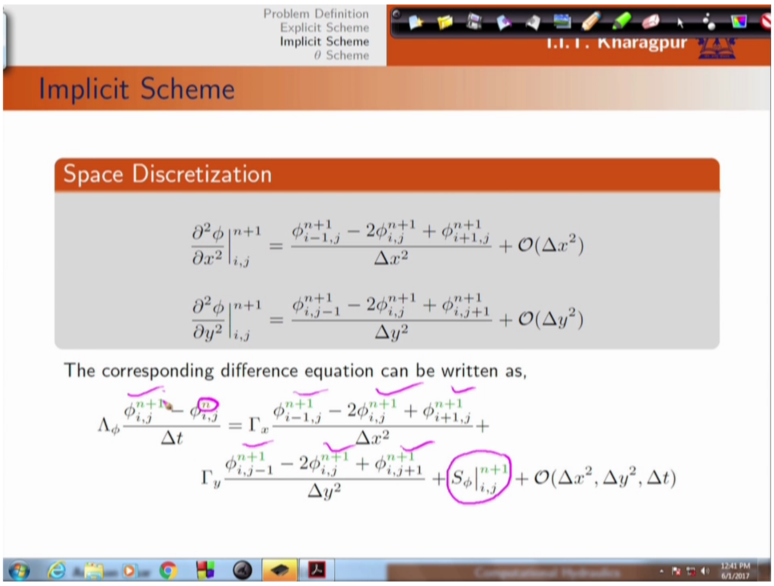

Although this phi ij n plus 1, if we combine with this term and this term that will we be counted as single term.

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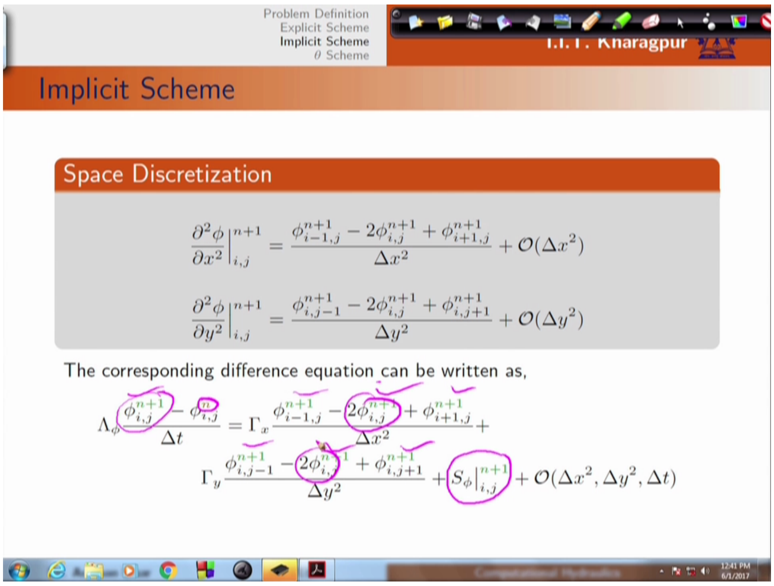

So we have one, two, three, four and five unknown values for this particular governing equation.

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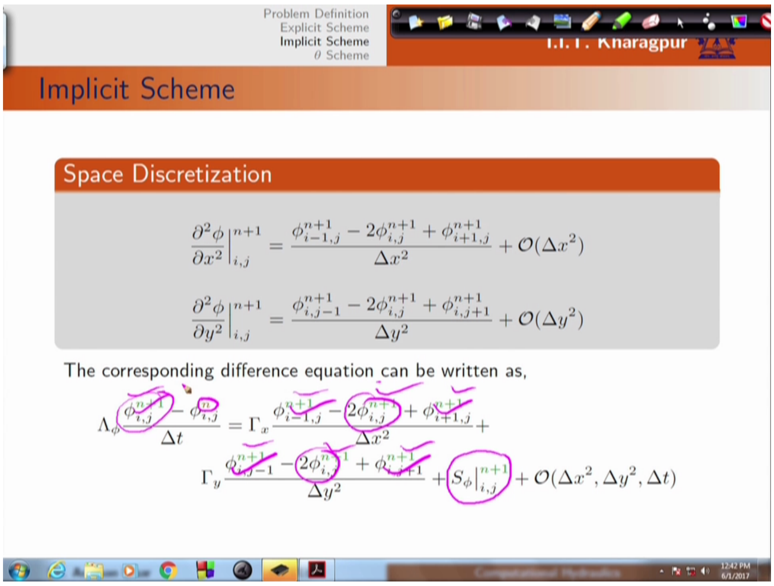

And overall accuracy of the scheme is del x square, del y square in time, this is del t.

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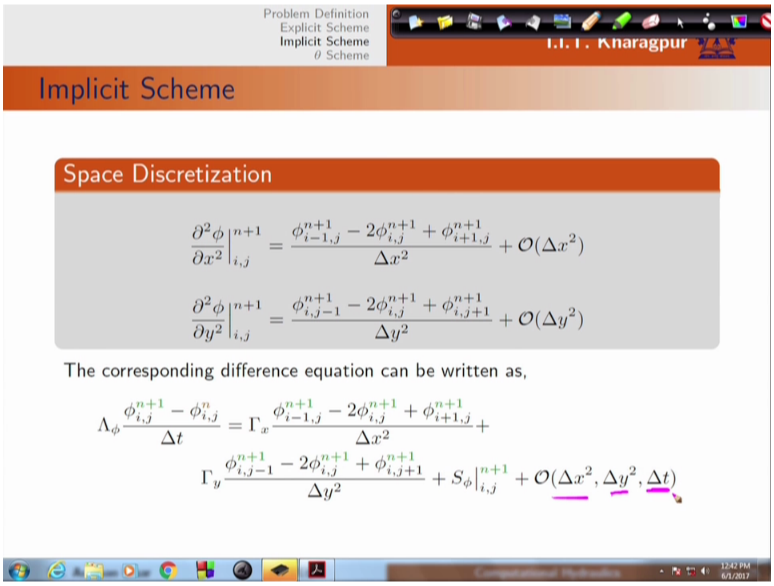

In compact form again we can write it like this with alpha x and alpha y. In this case these are unknown values.One, two, three, four and five unknown values. And right hand side we have known value available. With this information we can construct the matrix form using a single index notation.

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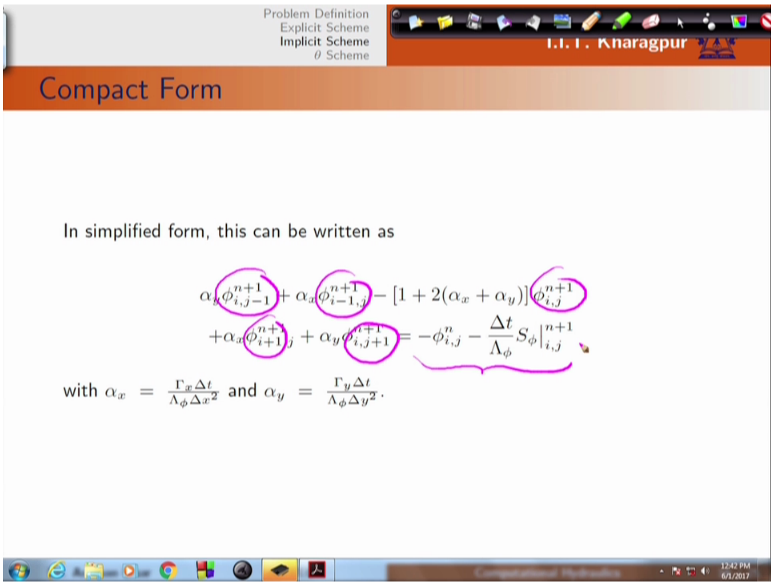

For single index notation again we need to convert it with L minus n plus 1, L minus 1, L, L plus 1, L plus n plus 1 format.

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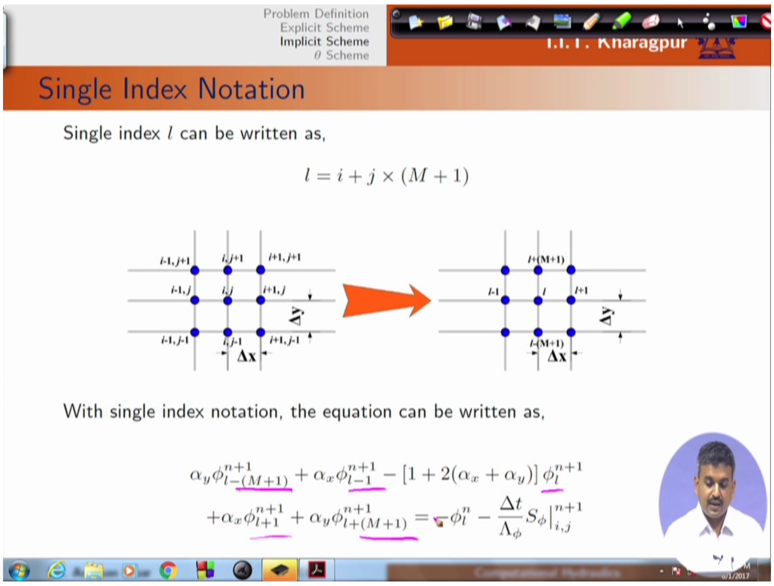

In this case these values are unknown, so matrix form is essential for this problem.

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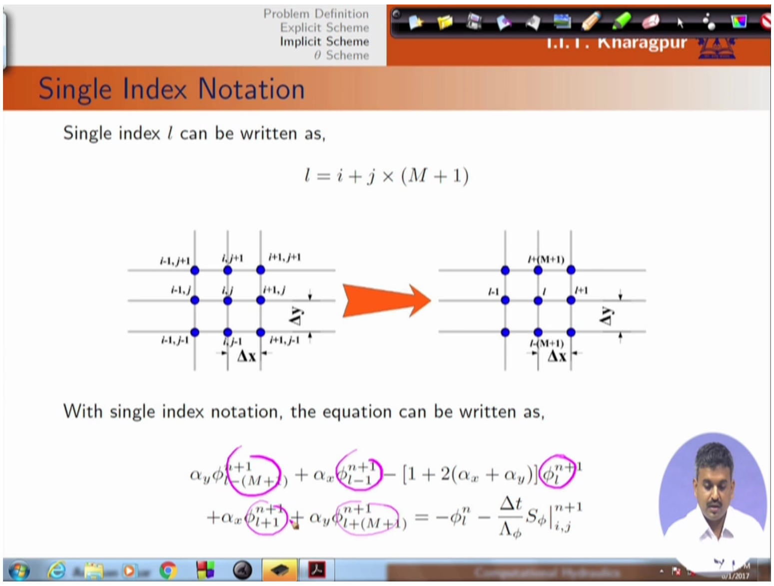

So if you want to solve this implicit scheme, again we need to define this time stepping algorithm for this one. So data for this one is lambda phi, gamma x, gamma y, s phi, del x, del t, del y and del t, phi n at time step n. So with this if we proceed what result is required. Result is updated value of phi n plus 1 at time step n plus 1. In this case we need to run the while loop, t less than n time.

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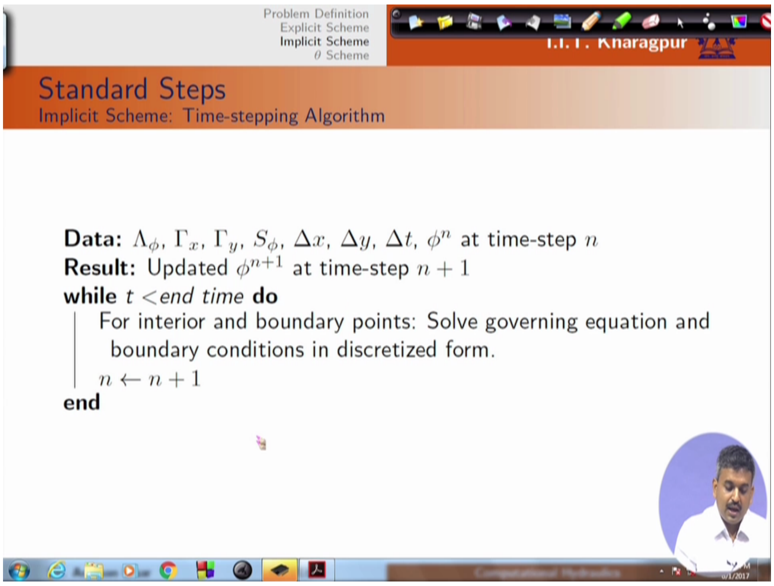

In explicit scheme we have observed that first we need to solve the interior points using governing equation, then boundary points using boundary condition. In this case we need to solve interior points and (bo) boundary points simultaneously. So solve governing equation and boundary condition in discretized form. So simultaneously we need to solve the governing equation and boundary condition. Then we need to update the time level to get the future time level value or present time level value for the next time step.

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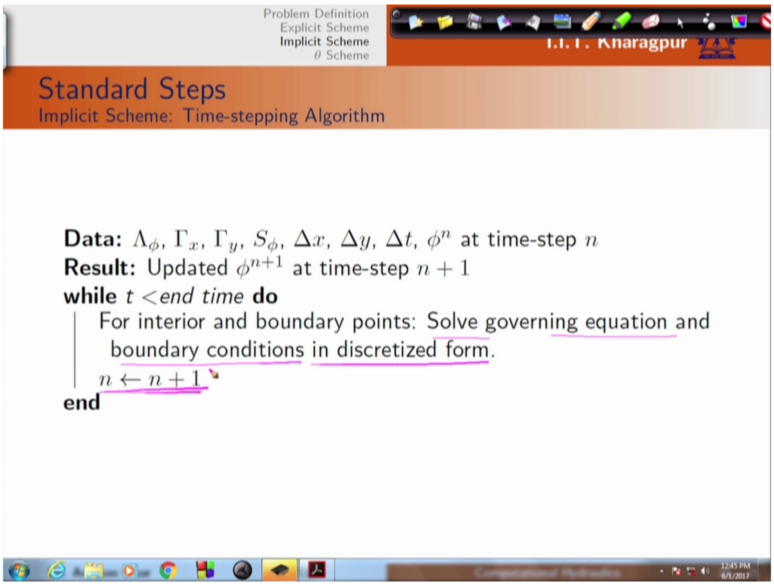

Now we need to define the theta scheme. Theta scheme we consider some intermediate time step.In that intermediate time step first step is explicit in nature. Explicit means we are defining at nth level. Then we define our implicit state. Implicit state is defined at n plus one level.

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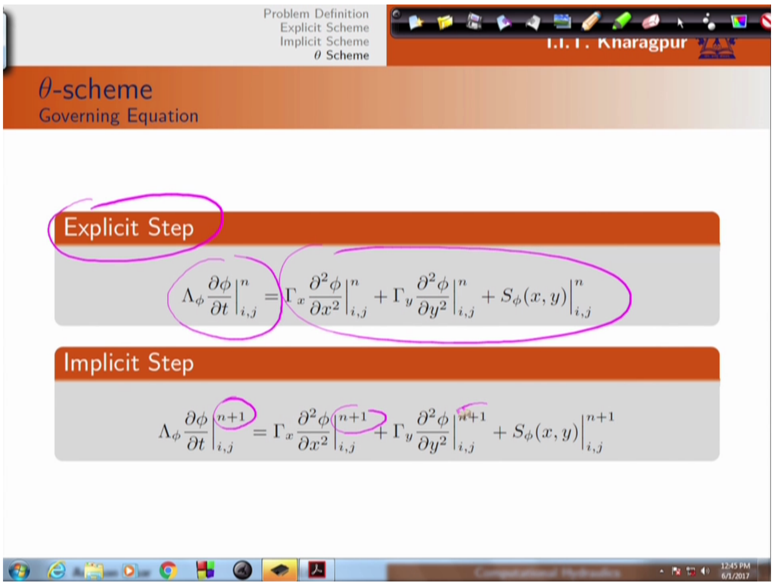

So with this we can write the explicit step for n plus theta level. Right hand side all values are known at nth line level. And implicit step we have all unknown at n plus 1 level and n plus theta is known from the first step.

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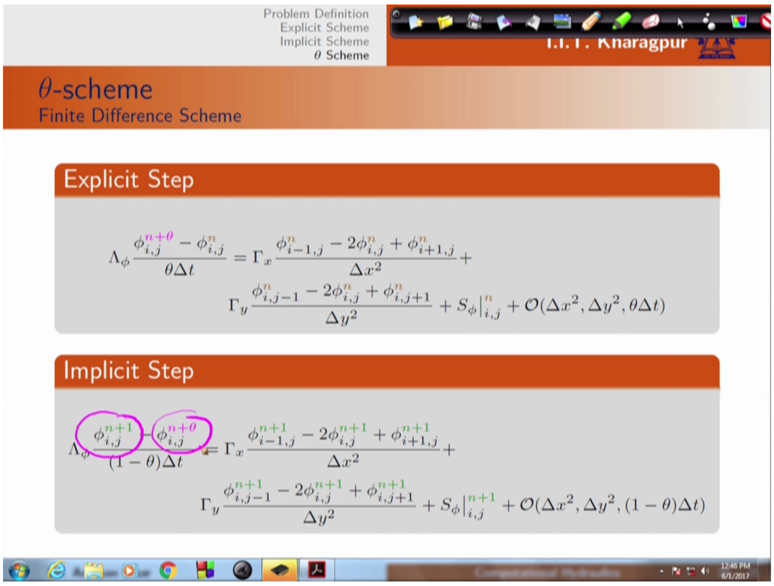

If we combine these two steps then we can write it as ij n plus 1 minus phi ij n and this is some kind of weighted average theta,1 minus theta for x direction. Again theta,1 minus theta for y direction.

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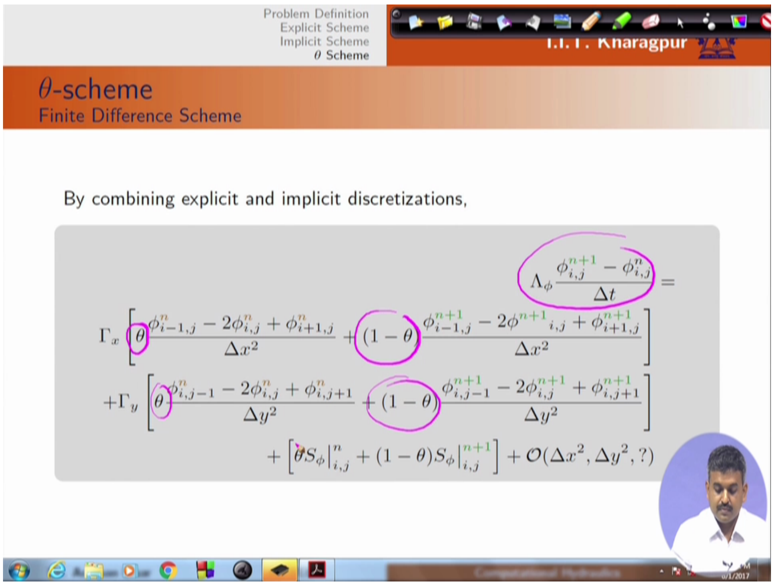

In this case important point is what will be the truncation error order for del t or time? If we write it for explicit step it is in terms of n plus theta. We can write it. Again for implicit step we can write it for n plus theta and if we combine these two we can get the time derivative here.

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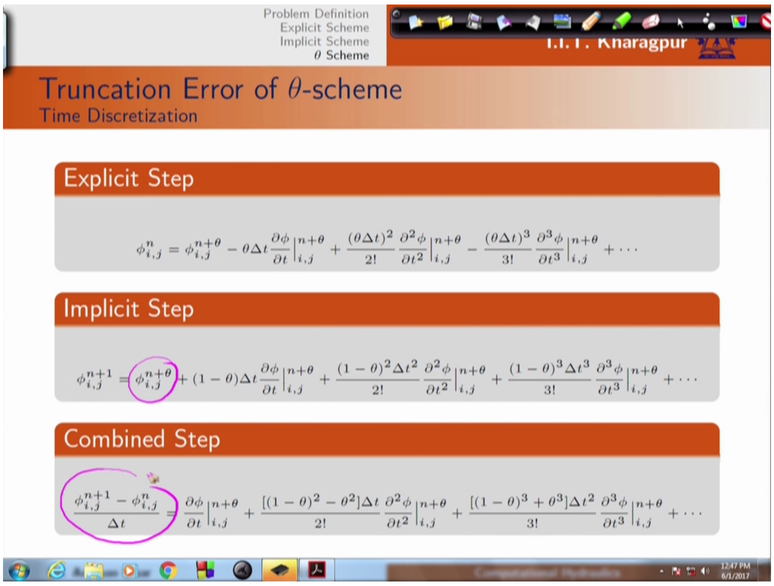

And interestingly this is, del phi, del t.This is at n plus theta level. And we have this 1 minus theta square minus theta square term.

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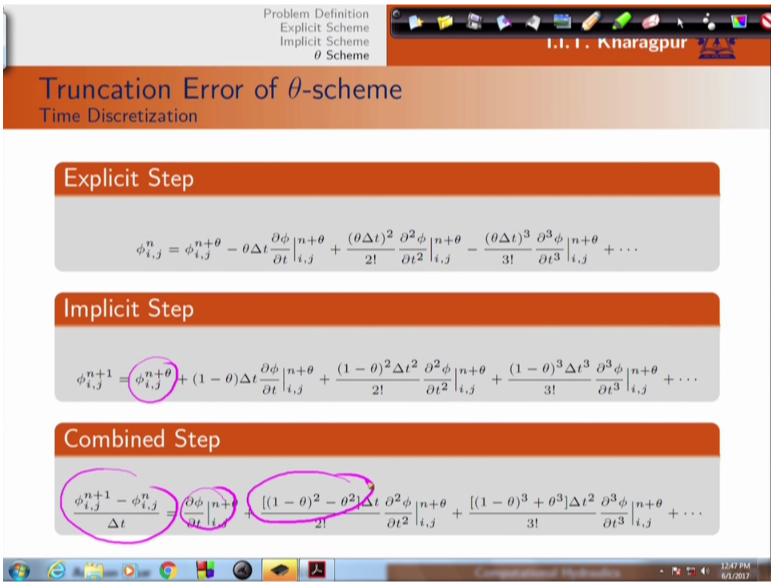

Now if we consider that theta equal to point 5, obviously this term will be zero and truncation error of the scheme will be of the order del x square, del y square, del t square.

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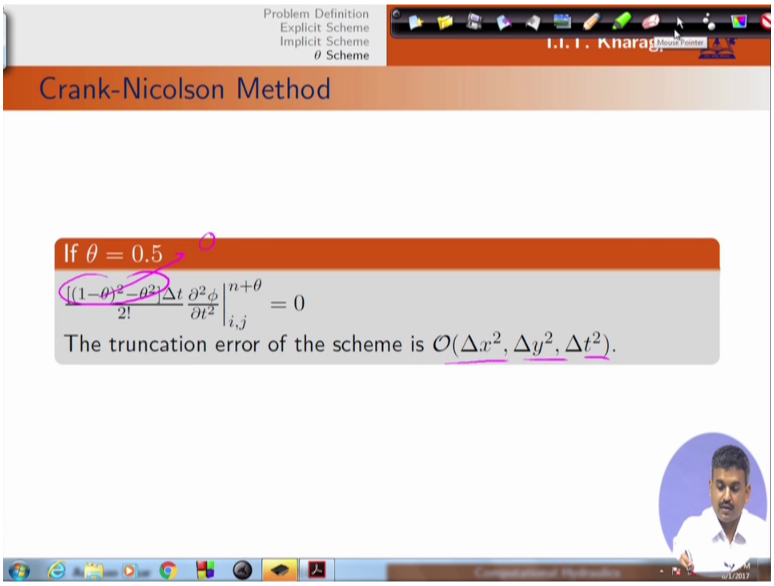

Essentially in this case we are getting second order accuracy in time. The scheme is known as Crank Nicolson scheme.

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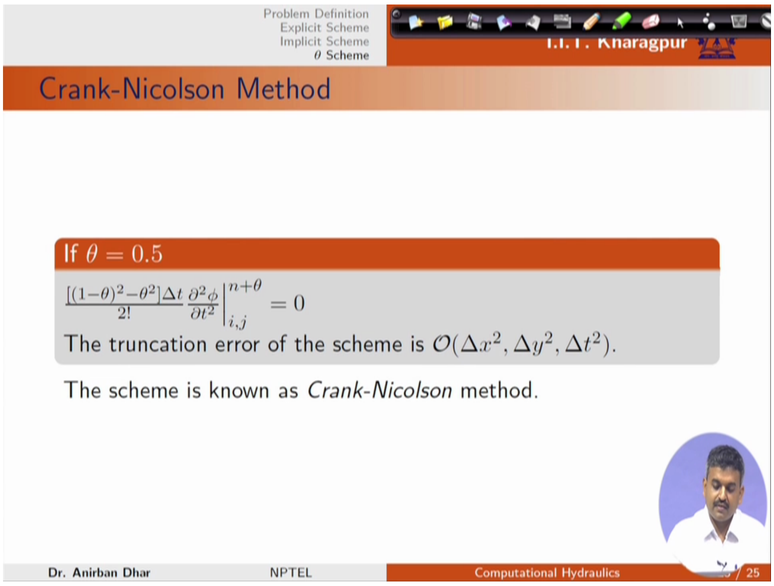

If we see the space time discretization for Crank Nicolson. It's the combination of explicit scheme and implicit scheme.

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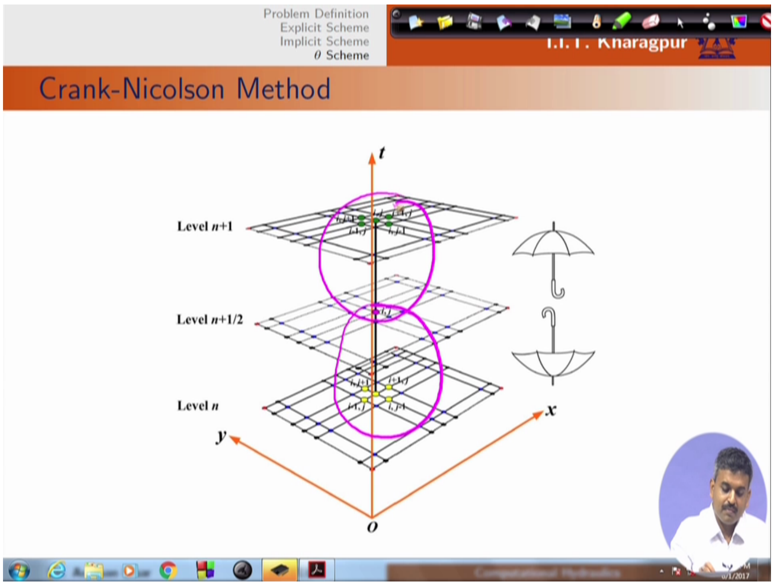

This is n plus half level because theta equals to half. And first step is inverted umbrella and second step is with straight umbrella.

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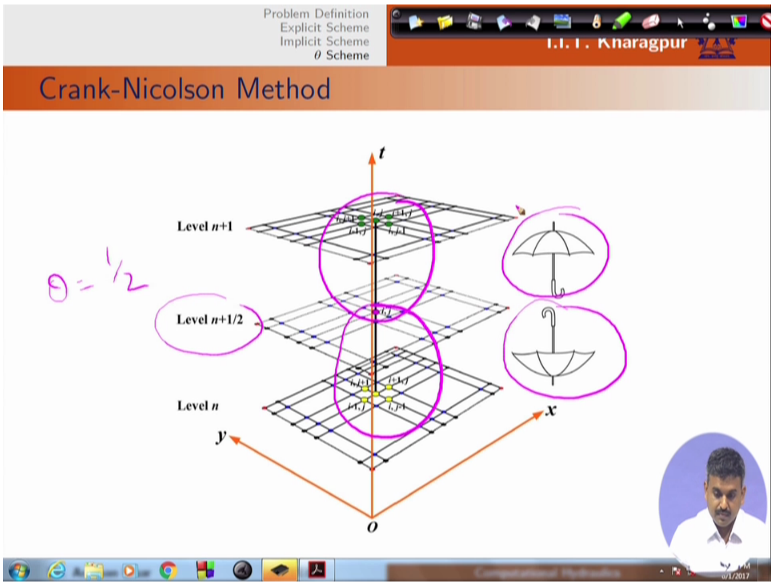

With this we can discretize our governing equation and we can solve using Crank Nicolson scheme. In Crank Nicolson scheme if we see our governing equation in discretized form that is actually can be solved using the implicit algorithm that we have discussed. Because inCrank Nicolson scheme we have unknown values available on the left hand side and right hand side also. Thank you.