INDIAN INSTITUTE
OF
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KHARAGPUR

NPTEL
National Programme
on
Technology Enhanced Learning

Probability Methods in Civil Engineering

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Lecture – 16

Topic

Functions of Random Variables – Different Methods (Contd.)

Welcome to this lecture, today we are taking this third lecture of this module.

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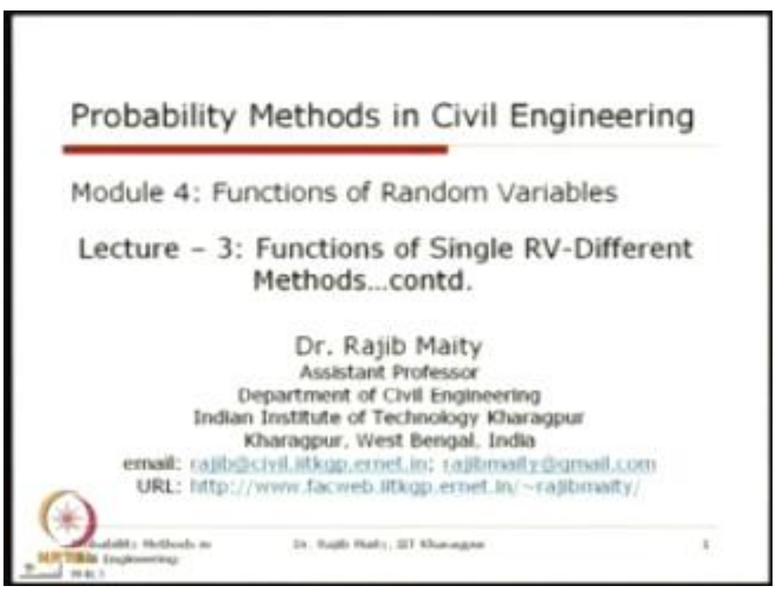

In this module that means the fourth module, we are discussing about the function of functions of random variable. So, basically in the previous modules we have seen the different properties of random variable and their standard distributions we have discussed. Now, from the standard available random variable, if we generate or if we derive some other function of those random variables, we have discussed that those functions are also a random variable.

So, their properties of those functions also we will follow the similar properties of those random variable. So, in this module we are discussing this aspect of this property of this functions of random variables. Now to start with, what we have done is that, we have discussed the fundamental theory, that is how one particular random variable can be linked to the another one through their functional dependence.

So, if I know one random variable which is for which the all the properties, that is the distribution is known and we know its functional relation with another random variable, then for that new random variable that is the function of that, how to get different properties. Now, we know that if you want know the properties of this function, then we have to first of all know what is their probability distribution and that probability distribution to get that in the previous lectures, we have shown some fundamental theorem.

And that fundamental theorem, based on the fundamental theorem, we have, keeping the basic concept same we have derived different methods. So, in this lecture also we will continue two different methods, one method is discuss in the earlier lecture and in this lecture also we will continue with two more different methods. So, our basic goal is to know, what the probability density of this new random variable and once we know this probability density function, then we know that, we can estimate whatever the properties that we need to know, we can estimate that one.

So, the in he in the previous two lectures in this module first we have seen the fundamental theorem and from there we have discuss about the method of distribution function first and in today’s lecture we will discuss about that method of transformation. So, even though we are giving the different names, I repeat that all this methods are based on the same fundamental theorem only with some special assumptions has been taken for different method, that is why the methods looks different.

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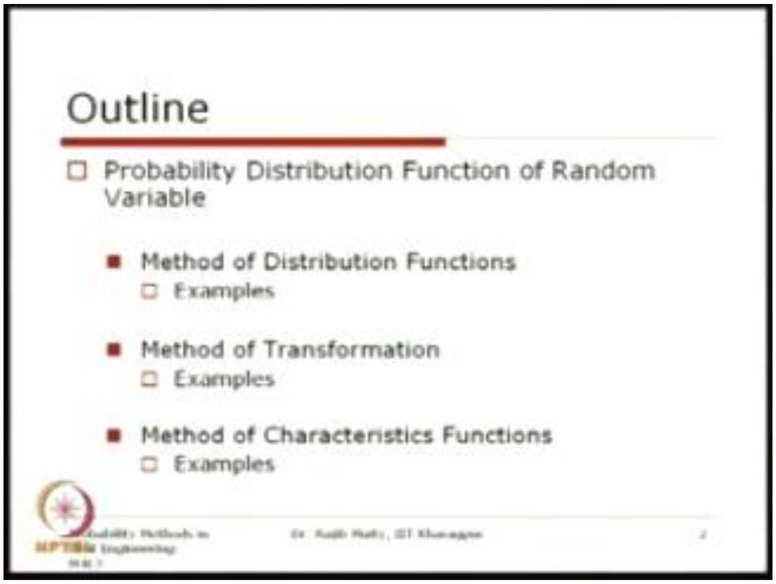

So, we will if I just see the different methods that we have discuss earlier, first in the earlier lecture, we have discuss that method of distribution function with some examples. And today basically we will discuss about this method of transformation with examples. Now, when we are talking about this method of transformation, basically what we are talking that, we are talking about that transformation from one random variable to another one. And here the basic assumption that we are following is that the transformation is known, that is known in the sense that unique transformation is generally assumed that is one to one transformation.

So, for a particular value of the original random variable, there is one and only one value for the new random variable, that is derived random variable. So, this transformation is known as one to one. So, when we are, when we discuss that method of transformation we will see that how this method of transformation is linked, it can be derived from the fundamental theorem for this one to one transformation.

A little bit of discussion was there in the previous lecture as well, so will follow that one is details in this lecture. And finally, there is another method that is known as this method of characteristics function. So, we know that for the standard random variables, if we know the PDF then, we can estimate their characteristics function.

So and we have seen that, usefulness of this characteristics function is that, once we can identify the characteristics function, then using their relationship with the different order of moment, that is first order moment, second order moment, if we can link those things then those, if those moments are known, if those initial, then the properties, their location, their central tendency, their dispersion, so these properties also will be known.

So, now if we know the characteristics function, then from that functional relationship for this derived random variable, we can also get the probability density function for the forth functions of the random variable. So that we will discuss also under the method of characteristics function. So, before we proceed we will quickly see that, what we have seen in the fundamental theorem as well as for this method of distribution function, we will just recall.

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So that it will help us to understand that, what is there in the method of transformation.

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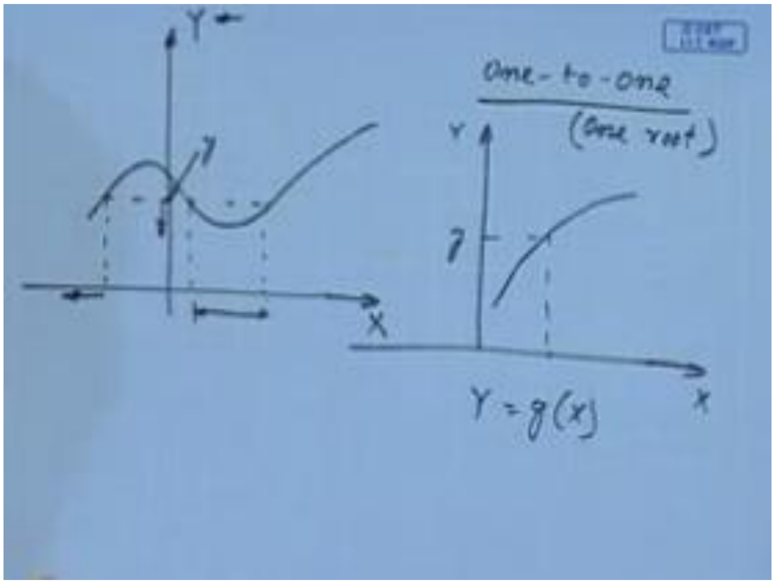

So, what we have seen earlier is that, if this is one random variable, which is the original random variable denoted by x and there is another random variable, which we can derive through some functional relationship with the original random variable x. So, if the functions can have any this type of this type of functional relationship, then for a particular value of y, what we have to do, we have to find out the what is the representative set in this in the original random variable that is your x here.

So, if I know this one, then the probability of this y less than this specific value y, so towards this one, then this corresponds to this zone as well as the zone between this one. So, this thing if you just add, if we get then in this way we can get what is the distribution of this y, for each and every possible value of this new random variable y. Now, when we are talking about this method of transformation, first case that we are considering is that one to one.

So, when we are talking about this one to one transformation that means that the relationship is such that for a specific value of y, there will be only one possible set for this original random variable x can have. So for this one for this kind of relationship from the fundamental theorem we have seen that this will be just only one root so what it is reduced to the fact that it then in the relation to the fundamental theorem is that only one possible root for a specific value of y.

So, with that root with the fundamental theorem reduced to only one only root and using that relationship we will see how we can get the PDF for this new random variable y. Here, we are assuming that functional relationship that g(x) is having only single root for a specific value of y.

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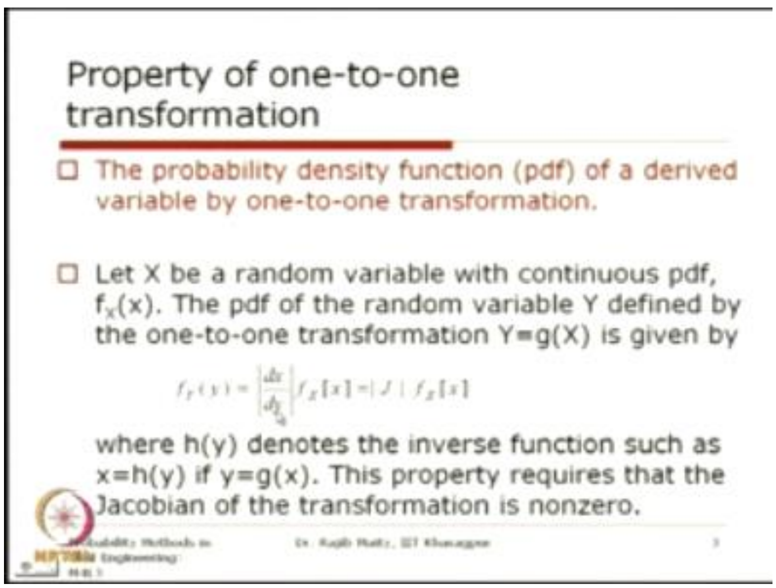

So, here the when so the here the assumption that we are using that we mention is that one to one transformation. So if the x is a random variable with continuous PDF having the PDF is $f_x(x)$. The PDF of the random variable y, which is defined by this one to one transformation this one to one transformation is important that what we are mention here is that y is equals to g(x).So, this is the functional relationship that is having between two random variable x and y and for this random variable.

This distribution is known that is f_x, then from the fundamental theorem, we get that this $f_y(y)$ should be equals to this $f_x(x)$ and multiplying by a factor that dxy, this we discuss in the last lecture as well, that is this one when we are taking this derivative is the we are taking the absolute value and this derivative is known as the Jacobin of this transformation which is denoted as j. So where this denotes the inverse function so now when we are expressing this one that $f_x(x)$ we should express this one in terms of the new random variable y.

So, which is the inverse function such as x equals to h(y), if that y is equals to g(x), so we can also write that x equals to as we as discuss earlier lecture that, x equals to $g^{-1}(y)$, that $g^{-1}(y)$ is here that h(y).So, this property requires that, the Jacobin of the transformation is non zero now this is for the continuous random variable that what we have started with and that is why we are we required that this transformation should be the Jacobin of the transformation should be non zero and we have also discuss that if it is if Jacobin equals to 0 what does it mean so, this means that there will be a spike of the of the probability distribution that is there will be a discrete probability for that zone where this derivative that is a dx dy is equals to 0. So, dx dy equals to 0 means the rate of change of x with respective y is equals to 0 that means.

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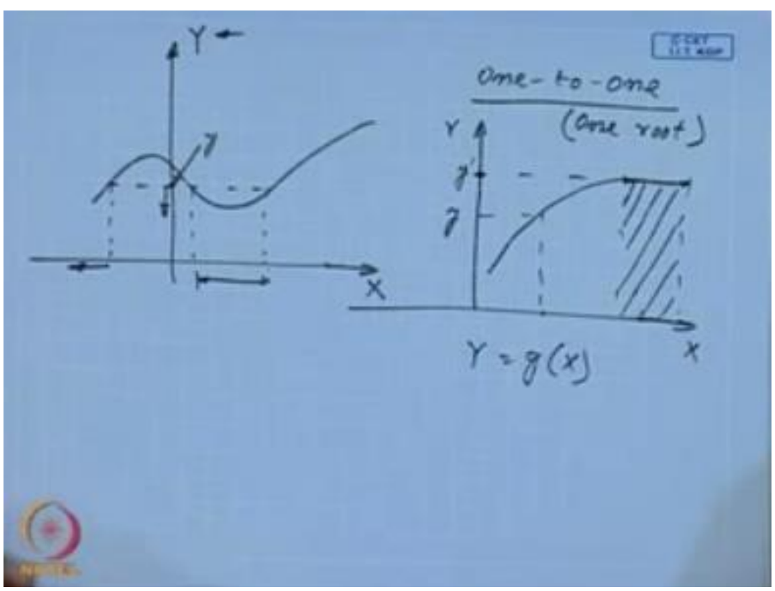

So when So, if the this thing if some functional relationship it becomes constant for this zone then we can say that at this point of this value if this specific value y´ here this rate of change of x with respect to y becomes 0.So, if it is becomes 0 that means that this probability, that is probability of this y is equals to y´ should be the it should be equal to the probability of this total area so that means here this is be this is become a discrete probability at this point some probability mass should be concentrated for this specific value of y. This is what is shown here.

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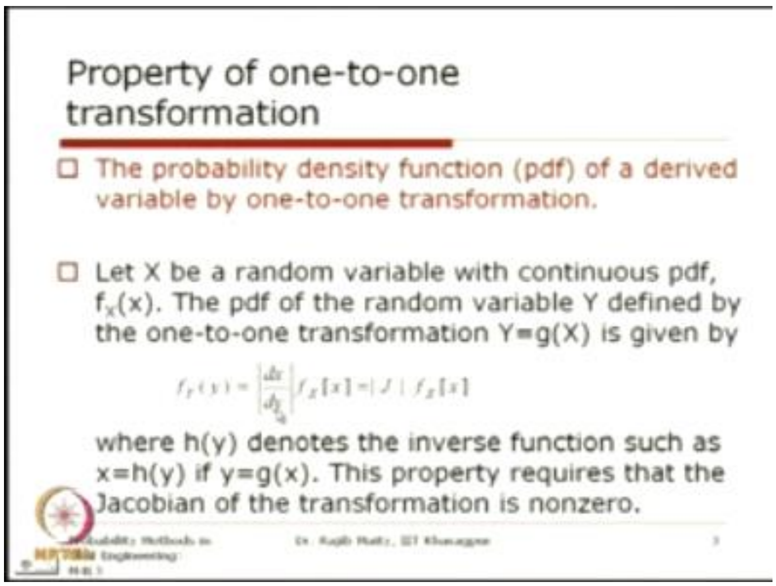

This what explained in this in this statement.

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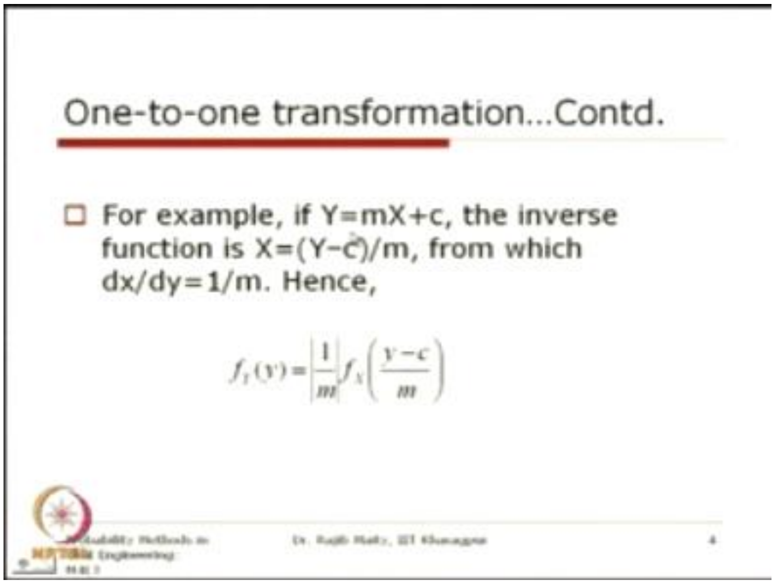

Now, if we take a very a very simple problem if you start with a very simple transformation that is, that y is equals to mX + c and you know that this y is equals to mX + c is a is a transformation of a.

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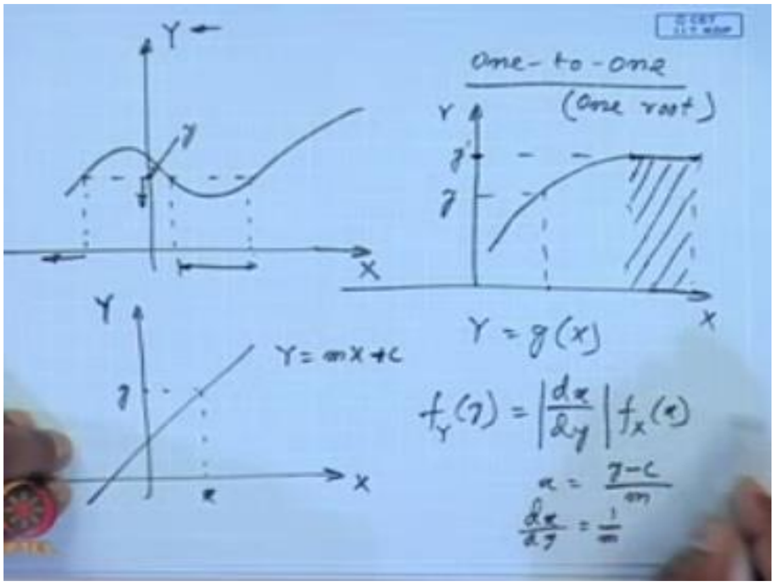

Straight line now this so, the transformation will look will be a will be a straight line like this so where so for this one if this is your x and if this is your y then the relationship is that y = mX + c, which is the equation for a straight line for any specific value of this y, there will be a one and only one value of this earlier that original random variable x. Now we know that if we just want to know what should be there the distribution of this y. So from the equation we get that this $f_y(y)$ that is PDF of this y is the Jacobin dx, dy of this f_x (x).

Now, this x I have to write in terms of this inverse relationship so this x will be equals to your from this relationship can be written that y – c/ m. So, this, so from here also we can also write that dx, dy is equals to your 1/ m, because c / m is constant. So this becomes 1 / m so we can just replace this part here and this one here to get this f_y. So, in this what we will get is.

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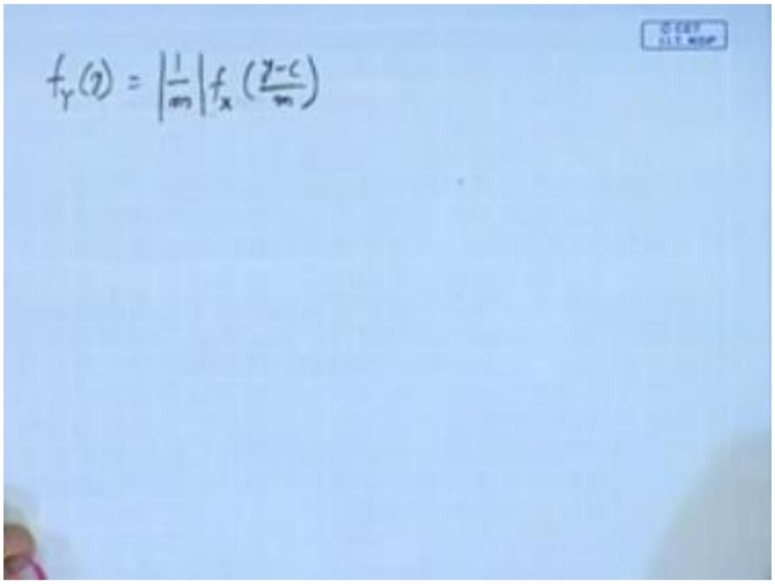

That f_y = 1/ m this should be taken the absolute value depending on what is this the value of this m the slope of this straight line this one multiplied by f_x of this y –c/ m. Now if we know this form, that if you know the PDF of this x will just replace this value and will get the PDF of y, so this expression.

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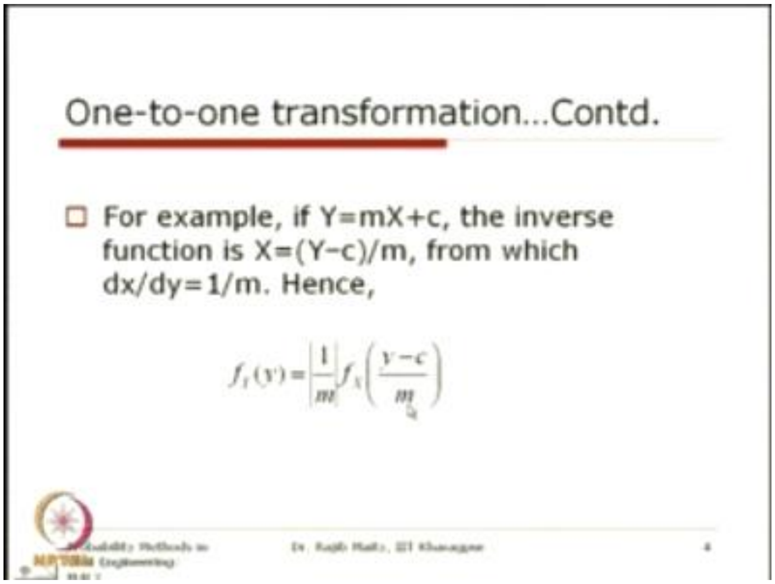

So, this is what is explained here so for a relationship which is a straight line relationship y = mX + c, then the inverse function that we are that we are telling in the last slide it is the y -c / m from which the dx dy is 1 /m thus this $f_y(y)$ = 1 / m absolute value multiplied by $f_x(y)$ - c / m.

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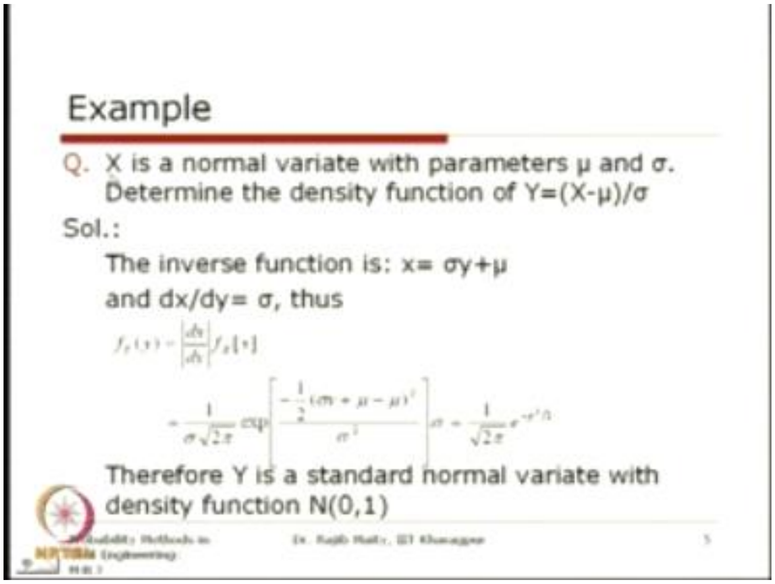

Another example, we also can see is that standard normal distribution and normal distribution. So here if where the x is a random variable which is which the normal distribution is which follow the normal distribution with its parameter μ and σ. So if the relationship now determine the density of the function y, which is having a functional relationship with x is x - μ / σ. Now when we are we have also discuss earlier that why this example is taken here just to show that.

How we can get these one because both this distribution we know that is x as well as y, so x we know because this is standard distribution that we discuss in the last module this is normal distribution or Gaussian distribution with two parameters μ and σ and for A Gaussian distribution if it transfer that random variable through this through this relationship that is x- μ / σ.

We know that this the reduced variety which is also a normal distribution having its mean equals to 0 and standardization equals to 1. So this is nothing but this will follow a normal this will follow a standard normal distribution, standard normal distribution means its mean is 0 andstandardizationis1.So if this one is and we know the distribution for both the random variable x which is a normal distribution and the y which is the standard normal distribution.

So here now the inverse relationship we will see first where this $x = \sigma y + \mu$ this is the inverse relationship and from here we should calculate that Jacobin, Jacobin is your dx dy, so here the dx dy $=\sigma$. So again using that same relationship that is fy(y) is equals to this absolute value of this Jacobin dx dy multiplied by this fx (x) express in terms of the new variable y, express in terms of new variable y. Means we have to place this one this instead of this x we have to use that $\sigma y + \mu$. Now this fx(x) now we know that so we know that for a for a for a normal distribution.

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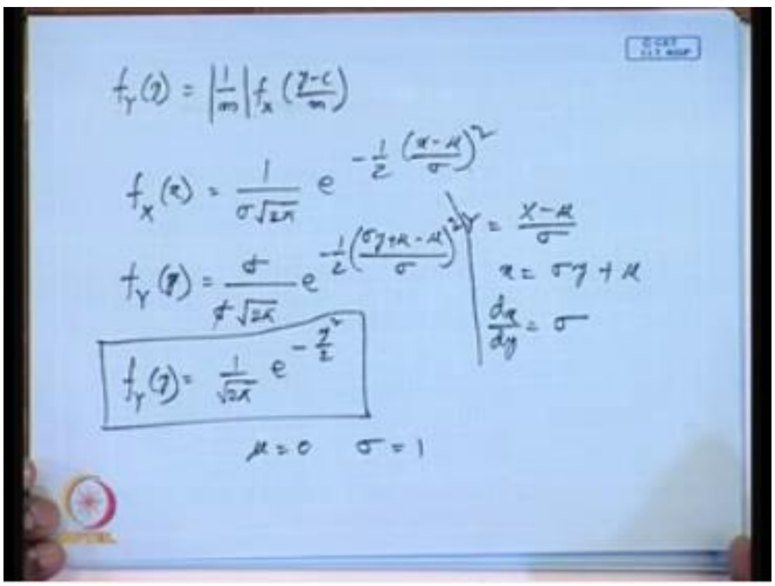

That $fx(x) = 1/\sigma\sqrt{2\pi}\ e^{-1/2(X-\mu/\sigma)2}$. Now so this has to be this has to be replaced by this as so this y if when we are talking about that is a when this relationship that we are talking that $y = x - \mu/\sigma$ and this $x = \sigma y + \mu$ just now we have shown. So here that dx dy is equals to your σ. So this fy is nothing but now that σ we should multiplied that σ that divided by this original $1\sigma\sqrt{2\pi}\ e^{-1/2}$ in place of this x what we are write in is that your $(\sigma y + \mu - \mu / \sigma)^2$

So this is becoming that $1/\sqrt{2\pi}\ e^{-}$. So this sigma cancels so it becomes $y^2/2$. Now so this one is that fy(y) we got. So this is also a normal distribution which is having that μ equals to that mean equals to 0 and sigma that is standard deviation is equals to here 1, so which is nothing but standard normal distribution. We can also before you go to a specific civil engineering problem.

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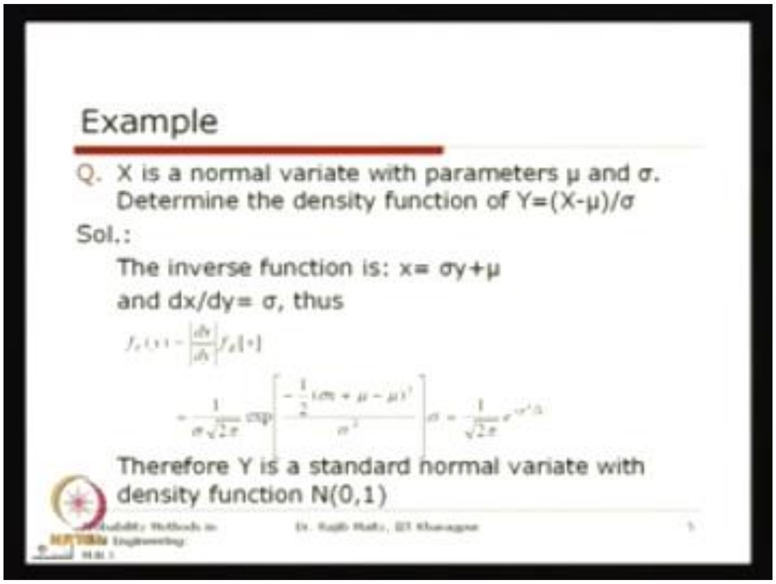

We can also cheek one more relationship this we have discussed earlier is that.

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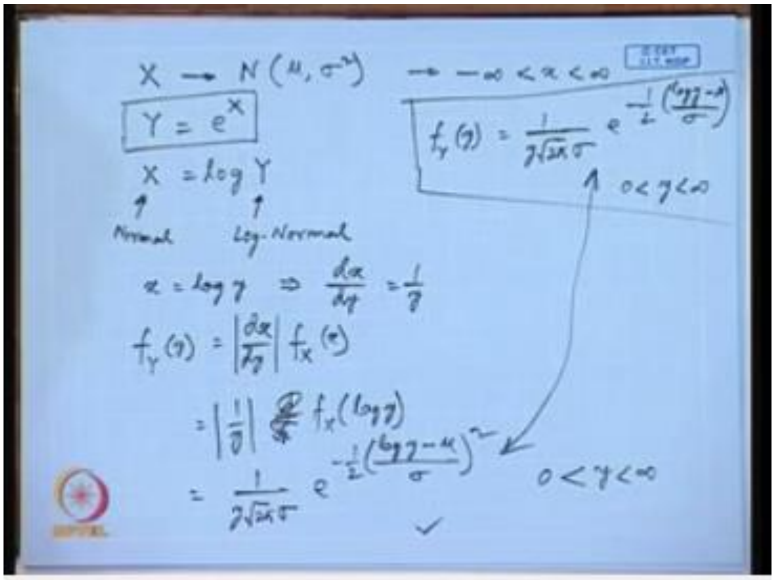

Is that if say that x is a normal distribution with the two parameters σ and say μ and 6 σ with the σ^{2} is the variance for which that you know that support is from the entire real axis - ∞ to +∞. Now if we have a new relationship say that y = ^{ex} if this is the relationship holds then we can also that express that x = log(y). Now if we recall from our from our previous module lecture that when this x is x is normal distribution x is your normal distribution.

That is for a random variable if I take the log it becomes normal distribution that means this y is nothing but your log normal distribution, okay. Now if this one so we know the distribution of x and we also know the distribution of y, so distribution of y means the log normal log normal distribution from our earlier modules lecture we know that for the log normal distribution the distribution looks like this that is 1/√2π σ , σ is out this root exponential -1/2 of this log y - μ / σ

And this one is that 0 to y ∞, so this is a non negative number, so this distribution we have seen earlier. Now if we know this one this distribution the normal distribution if you know that this is the functional relationship that holds between this x and y can we can we obtain this log normal distribution from this normal distribution or not that we will see from this one because this transformation is also one to one transformation.

So now to get this one we have to find out that this the inverse relationship that is x = log y that we have got. So that dx dy equals to your 1/ y now this x is a, So if I just want to know from here that what is this Fy(y) is your Jacobin dx dy and fx(x) . Now from this one so this will be your1/y and multiplied by 1 by this one we know that this is normal distribution √2л so this x now will be replaced by your this log y so I can just write that once more one more step that is f x is of log of y so which is nothing but1 by this y comes √2л σ from this normal distribution e $^{-½}$ log of y – µ so this log of y we are writing in place of this x – σ whole square this zone as we know that this we are taking this relationship this one is valid for this 0 to ∞ as we have taken this relationship.

So this can never come below 0 so the so this is now this one is nothing but is equals to this one so if this kind of relationship holds we can see that if x is the normal distribution y is your y is your log normal log normal distribution which we have got from this one to one transformation method and using this Jacobin of this transformation.

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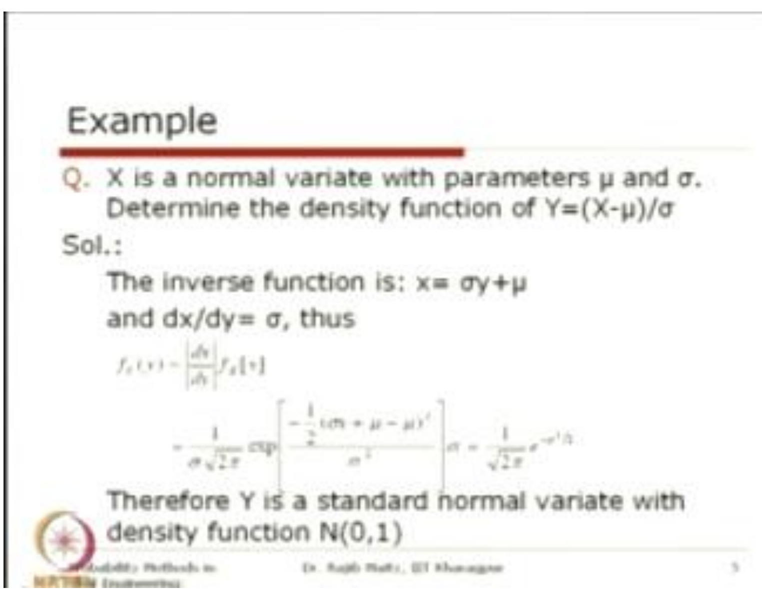

So no so this is the standard so all this distribution just now what we discuss is that normal then standard normal distribution and after that we have discuss that from the normal to this log normal distribution so all this distribution that we now that these are some standard distribution now will take up one example from this civil engineering where the distribution of the original random variable is known so we have to find out that what is the distribution of a derived random variable.

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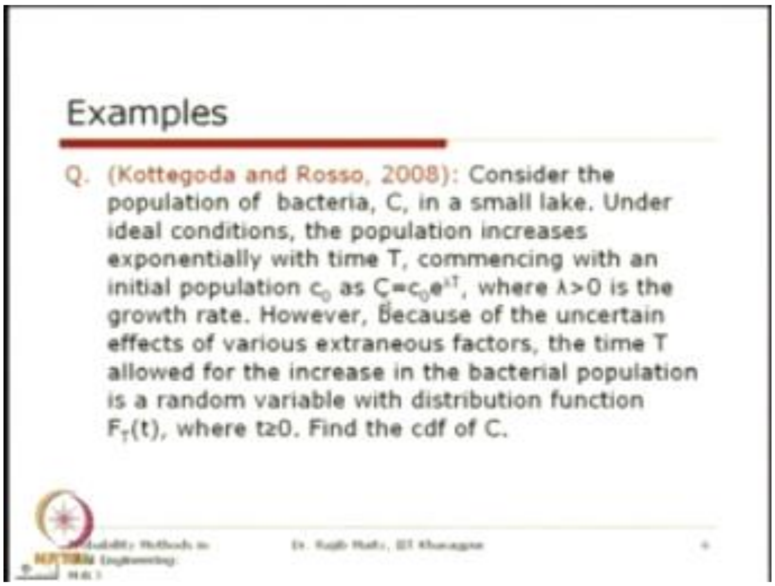

So this problem is taken from the book Kottegoda and Rosso so there are this is on the growth of the population of bacteria in a lake so et us consider that the population of the bacteria that is C that is in terms of concentration in a small lake so this C the population denoted by C under ideal condition the population increases exponentially with time T so as time increases this population of bacteria will also increase commencing with an initial population C_0 so C_0 is the at T = 0the concentration is C_0.

Now, at any time T the C the new concentration is equals to it is increasing exponentially so the relationship is C_0 e λ, T where λ is greater than 0 so λ greater than 0 here means that this one will increase as the time pass that is as T increases this C will also increase so this growth rate however because of uncertain effect of various extraneous factor the time T allowed for the increase in bacterial population is a random variable with the distribution function Ft now this T that is the time that is taken for this increase in the population.

This is we are this is this is we are for from the real condition we have seen this is also a also a random variable that random variable this F_T that is the cumulative distribution function of that one is known some standard distribution is known so if this distribution is known then that we have to find out the CDF of C so basically there is a functional relationship is there this T distribution of this T is known we have to find out what is the distribution for the population of bacteria C at a time at time T o to solve this problem.

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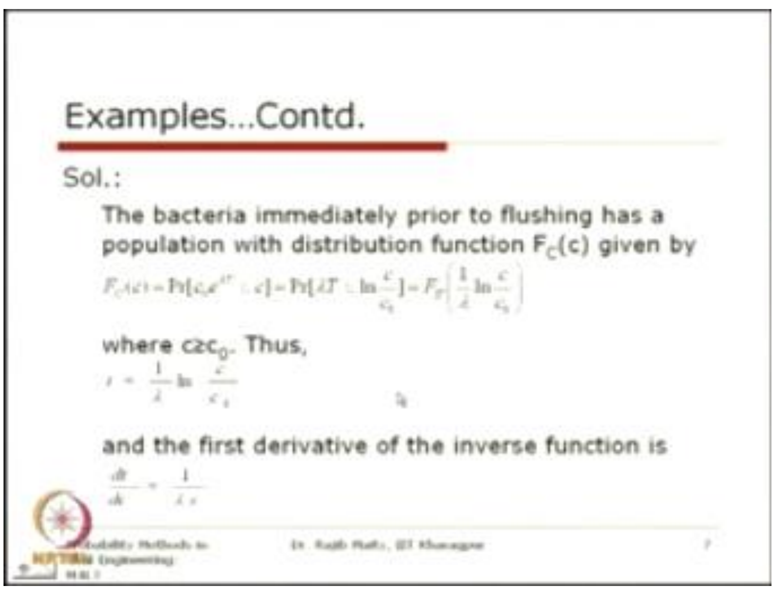

First of all we have to find out that what is this distribution for the concentration the bacteria immediately prior to the flushing have a population with the distribution function F_c (c) so this population which is we are talking about this should be means this should be express in terms of the this is a this is from the from the standard properties of the CDF that we have discuss in earlier lectures that is the F_c c the c is the variable here random variable is C is nothing but the probability of that this concentration at any time.

That is concentration at any time now here is that c naught e $^{\lambda T}$ so this one should be less than as specific value which is nothing but the expression of this cumulative distribution function so now this one after some algebraic transformation that is λT should be is less than equals to log natural C/C_0 which is also further written that is the f t of this so this is T less than = 1 by λ log natural C by C_0 so now when we are writing that probability of T less than1 by λ log natural c by C_0 that means, that is the probability distribution of the cumulative distribution of the random variable T.

And here that 1 by λ log natural C by C_0 has come so now where this C is greater than C_0 so as this C we have seen that it from the initial population this C at any time it is always increase as this λ is greater than 0 thus this T = 1 by λln C/C_0 so this is our that that inverse relationship that we have got hat is given that is $C = C_0 e^{\lambda T}$ so this is our inverse relationship which states that T = 1 by λ log natural C by C_0 c so if you know this one then we can calculate it is derivative that is d t d c which is need for the Jacobin which is equals to so1 by λC so once we know this Jacobin then we have to just get that Ft is.

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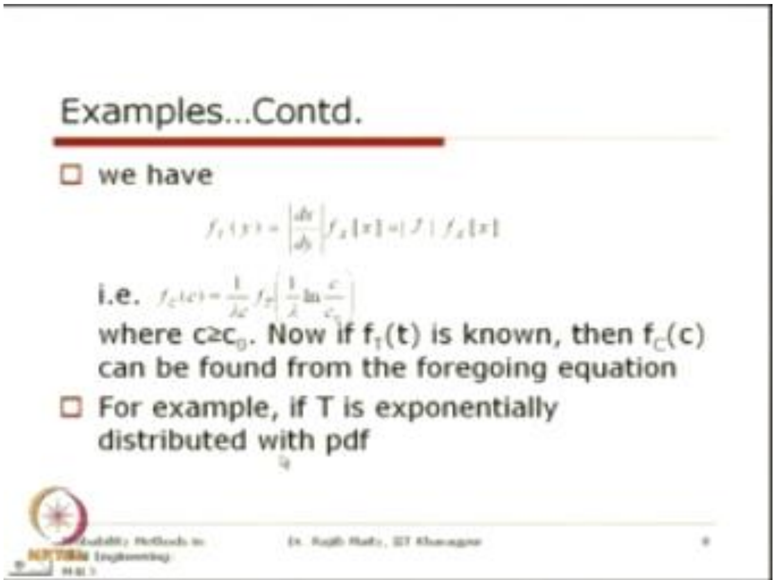

That is also we know that this is the functional relationship that is F_y y the new variable is the Jacob in multiplied by F_x x express in terms of y so this F_c C is nothing but then 1 by λC λis greater than 0 c is also greater than 0 so the absolute value is equals t one by C multiplied by FT 1 by λ l n C by C_0 so this t is express in terms of its inverse relationship where C is greater than C_0 now if this FT is known then F_c can also be found from this equation now what was given in this problem that if from this cumulative distribution of this T is known so as this is known then you can easily get what should be the what should be expression.

For this part now here we can take one specific example that is if this T is exponentially distributed now if this T is exponentially distributed then we have to some assume some we know that how this exponential distribution form is that is.

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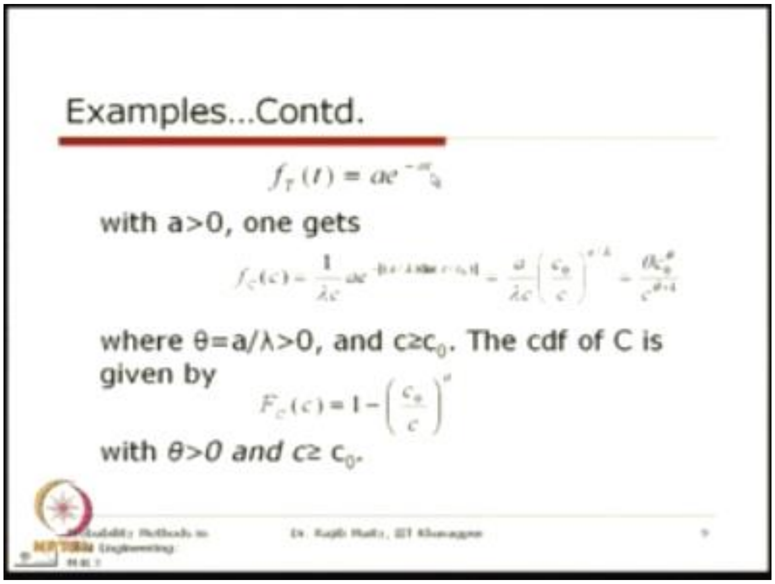

ae^{-t} you recall that from our earlier lectures we generally use this λ as parameter for the exponential distribution, but here the λ is used for some other growth rate. So, we are using some other variable that is a, so this is the form of this exponential distribution this t is the obviously, greater than 0 that is ae^{-at} with this a greater than 0.

Now, So, once you know this one, then this thing can be show f_t is if it is given like this then we are just replacing this value with that expression, that is a/λ log natural c/c_0, so which can be rearrange and express that $a/\lambda c$ $c_0/c^{a/\lambda}$, which is equals to the θ $c_0^{\theta}/c^{\theta+1}$ this θ is now a new variable, which is equals to a/λ which is greater than 0and c is greater than c_0. So, the cdf of C is given by F_c (c) = $1-(c_0/c)^{\theta}$ is, so this is your PDF that is probability density function from here if we integrate, we will get that cumulative distribution function for c, which is $1/1-(c_0/c)^{\theta}$ one and here this θ is greater than 0 and c is greater than c_0.

So, similarly if this f_T follows some other distribution, so this expression we will change accordingly.

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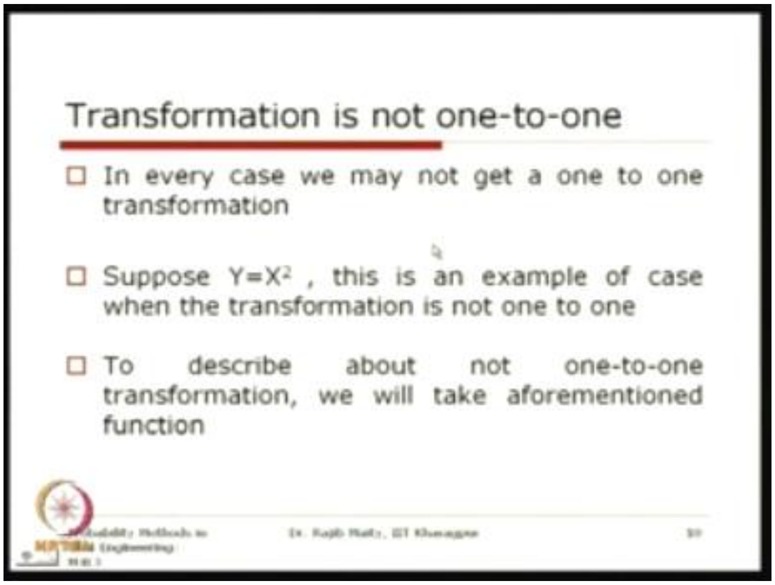

Now, so far what we discuss is that method of transformation is for the transformation is unique that is one to one, now we will see one more case that where the transformation is not one to one then what should we do? So, again we will recall that same the fundamental theorem if the roots are basically roots are more than one then how this transformation changes.

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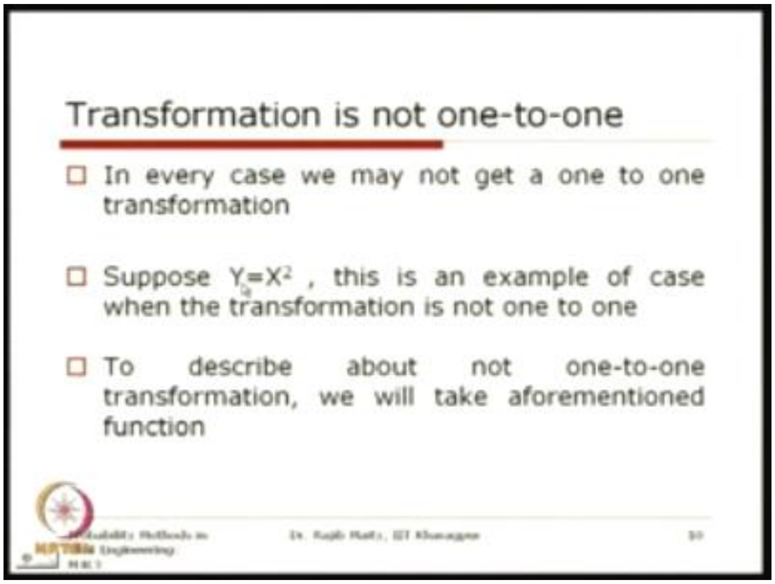

So, here we are taking that the transformation is not one to one, so in every case we may not get a one to one transformation suppose that $Y=X^2$ if this is the relationship, this is an example of the case, when the transformation is not one to one. So, to describe about that not one to one transformation, we will take this as aforementioned function. So, we will take this function that $Y=X^2$, we will define some PDF for this X and we will try to find out what is the PDF of this of this Y.

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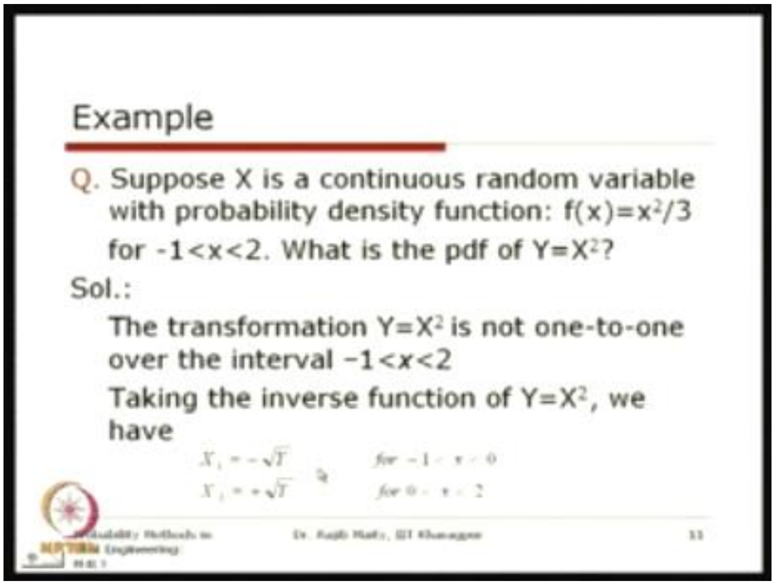

So, here we are taking this example, that is suppose that X is a continuous random variable with the probability density function $f(x)=x^2/3$, for this zone that is -1 x to 2, so -1 to 2 in this zone this density is defined that is $x^2/3$. Now, we can check that whether this integration of this total of this PDF is equals to 1,that is why this by 3 is coming as a normalizing factor. So, this is a PDF and the functional relationship that, we have taken is that $Y=X^2$.

So we know what the PDF of X is and we also know that, this relationship is not one to one then, what should be the distribution of this Y. So, to solve this one that this transformation is not one to one over the interval of this -1 to this 2. So, taking the inverse function, that is $Y=X^2$ that is $X=\pm\sqrt{Y}$, so this x_1 there are two roots we are getting now the $x_1=-\sqrt{Y}$, so this $-\sqrt{Y}$ we will get for this X when it is -1 to 0 and for the x_2 is the $+\sqrt{Y}$ for the region 0 to 2. Now, this can be, so if we just plot this function this will be clearer to express this 1.

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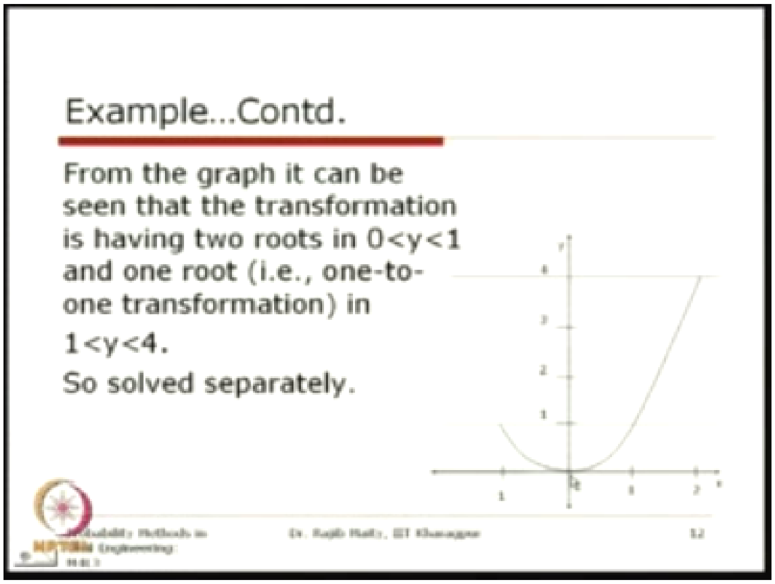

So, here the PDF is shown here from this -1 to this 2 and we have just given two lines here, one is that y=1 and one is that y=4. So, for this zone that is from this x=-1 to 1 the root, if I take the square root of this one. The √Y that minus will give the, this root that is from this -1 to 0. And for this one from 0 to 2 that this zone that there we can take that root of this +√Y that is from this 0 to up to 2.Now, what we can see here that from for this y this variable for this y, there is if we start from 0 to 1, so in this zone there are basically two roots for this relationship.

So, one is on the negative side another one is on the positive side. So, we have to calculate this zone separately that is 0 to1 and for this 1 to 4 in this zone of for the random variable y, for this zone that is1 to 4 then this transformation is one to one. So, from , so we have divided this random variable for this two zones one is from the 0 to 1, where there are two roots exist and another zone is from 1 to 4 where there one root exists.

So, from the graph it can be seen that the transformation is having two roots in the zone 0 to 1 for y and one root that is one to one transformation in the zone 1 to 4, so these two zone is solved separately.

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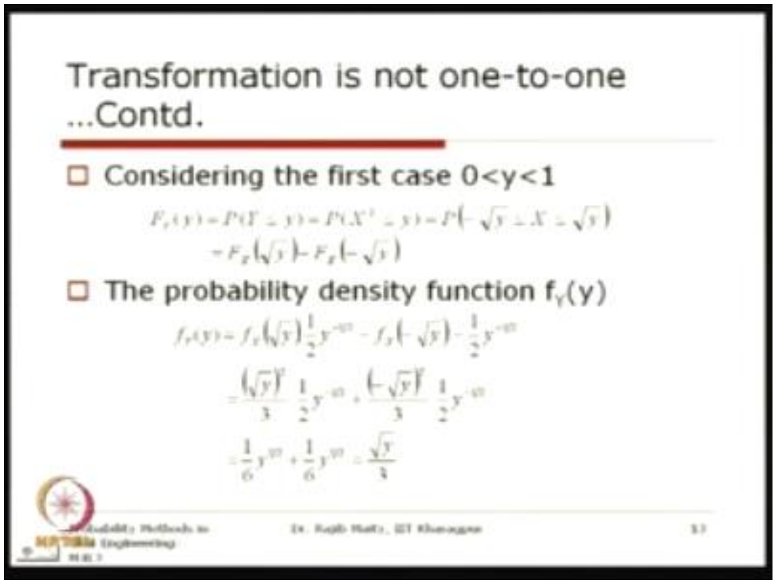

So, the first zone that is when we are taking that y is between 0 and 1 then $f_y(y)$ is equals to probability of y less than y, this specific value should consist of the two zones, that will show that is probability of X^2 is less than equals to y, which is equals to probability of -√Y to the +√Y. So, this $F_x(\sqrt{y})$-$F_x(\sqrt{y})$ so what is there actually how we are getting this one is this one this.

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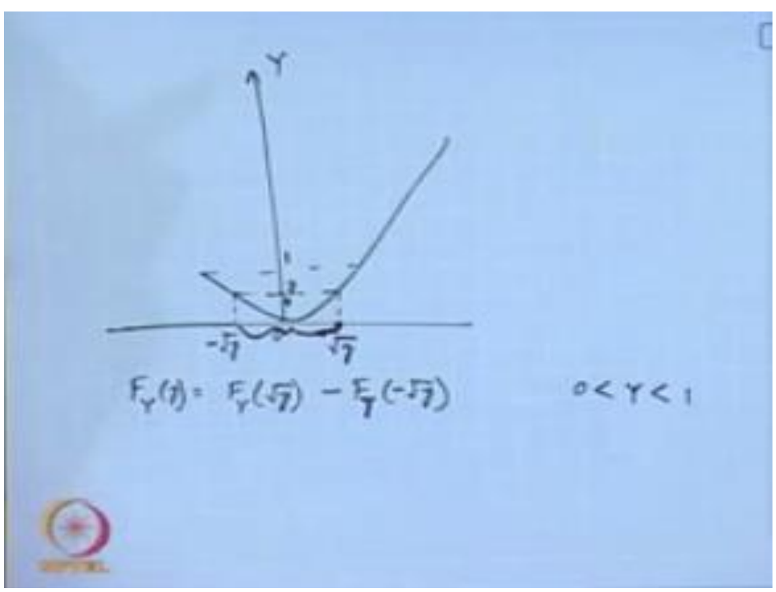

So, for this zone that is from 0 to 1 for any specific value if I take y, then what is this so for this less than y so, this is your random variable y, so for this any specific value that y less then this y then what we are getting is that this 2 zone 1 is which is explain that this is for this side, so this zone that we are we are getting, so now, to get that this zone that is what we need that the square roots of, so this is nothing but, your √y and this is nothing but, that -√y. So, from this 1 this total minus this 1.

So we will get this part, so f y √y – fy of -√y to get this zone which is nothing but, equals to your f y (y) for the zone 0 to 1. So, this one we are getting, which is shown here.

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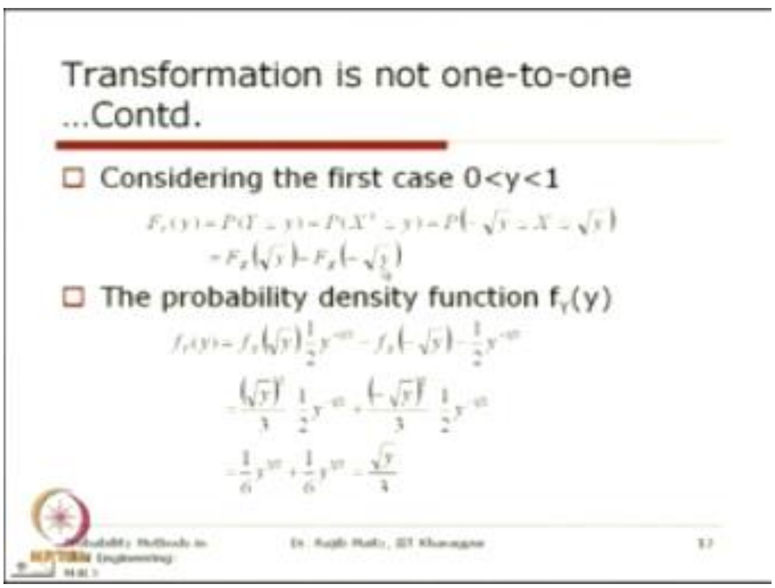

In this relationship, now so this probability density function f y is so, fy we are getting that is in terms of its now we will now do that this one this is express in terms of the root and this is you are the Jacobin of the transformation, so this is half $\sqrt{y}^{-r}$ that is 1 / √y that is that Jacobin. So, this Jacobin is for this zone this is the positive one and this is the negative one, sorry there will be that absolute value symbol here.

So, this -1/2 $\sqrt{y}$ once you take this absolute value of this Jacobin it becomes plus that $1/2y^{-1/2}$ half y power minus half. So, now this f_x $\sqrt{y}$ is express in terms of this, that is this cumulative distribution this f_x is given that is PDF is given to us.

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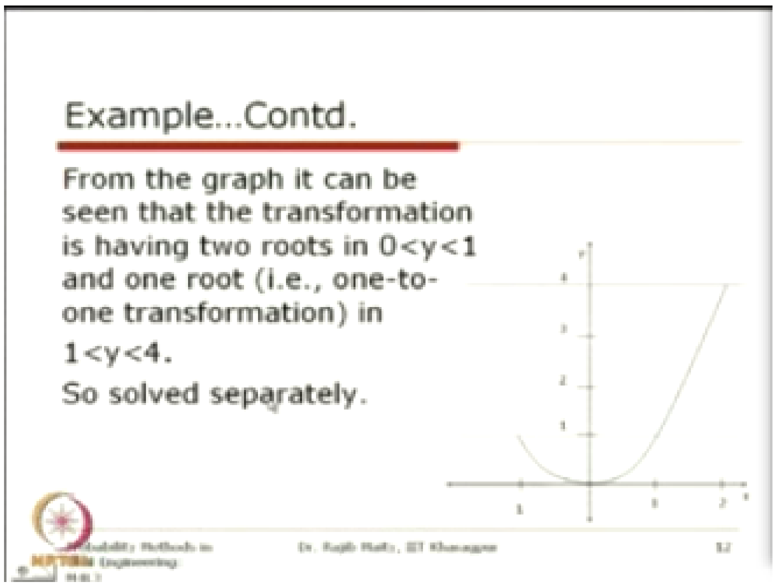

So, $x^2/3$, this $x^2/3$ this x is now replaced by this square root of this y. So, $\sqrt{y}^2/3$ similarly here it is the $-\sqrt{y}^2/3$. So, if we just take this one add this 2, then we get that $\sqrt{y}/3$ so for this zone that is.

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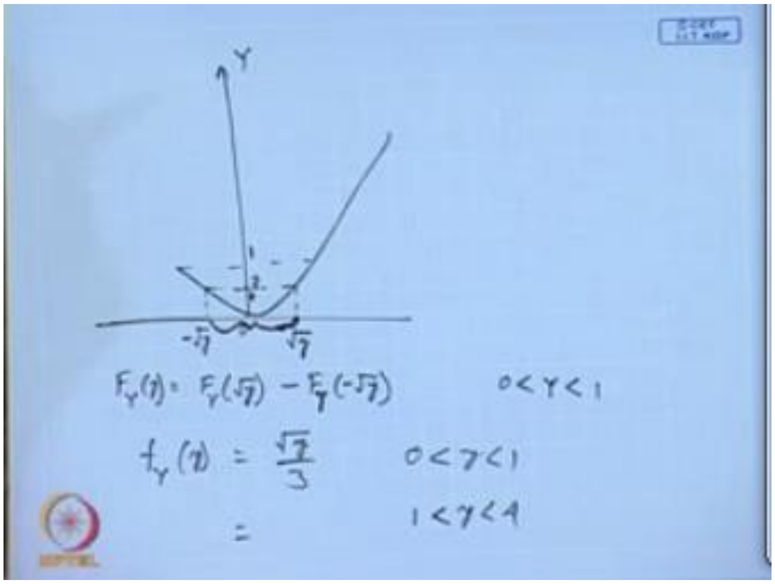

F_y (y) is becoming √y/3 for the zone y this 1, similarly we have to find one more expression for the zone 1 to 4 that that is straight forward because that relationship is one to one as we have discuss.

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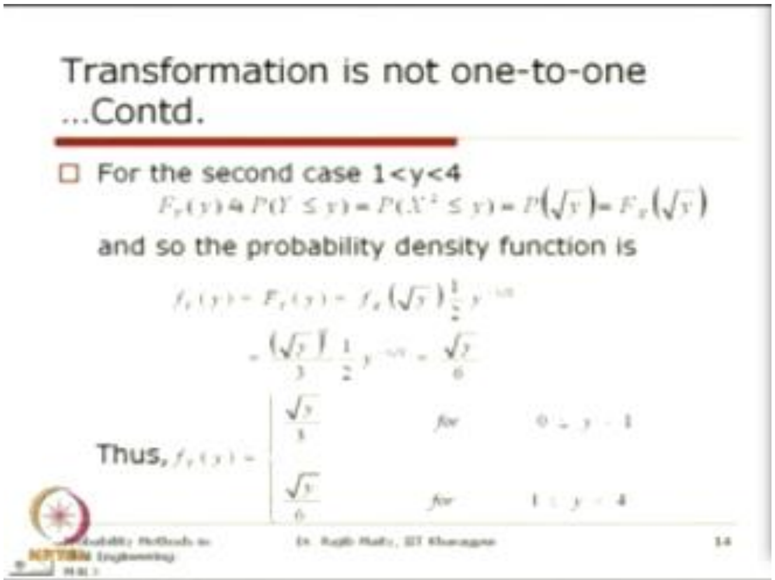

So, for this zone that is for 1 to 4 that fy y is equals to this less than y that is $x^2 < y$, which is P of this √y that is f_x √y. So, the probability function that f_y y is equals to that, this f x of √y single root only the positive root, we are considering multiplied by this absolute value of this Jacobin, which is 1/2 $y^{-1/2}$. So, this Jacobin multiplied now we are expressing this one in terms of the PDF of x that is x^2 / 3 in place of x we are writing √y. So, √y √/3x ½ $y^{-1/2}$ which is equals to √y / 6. So, this expression is for the range1 to 4.

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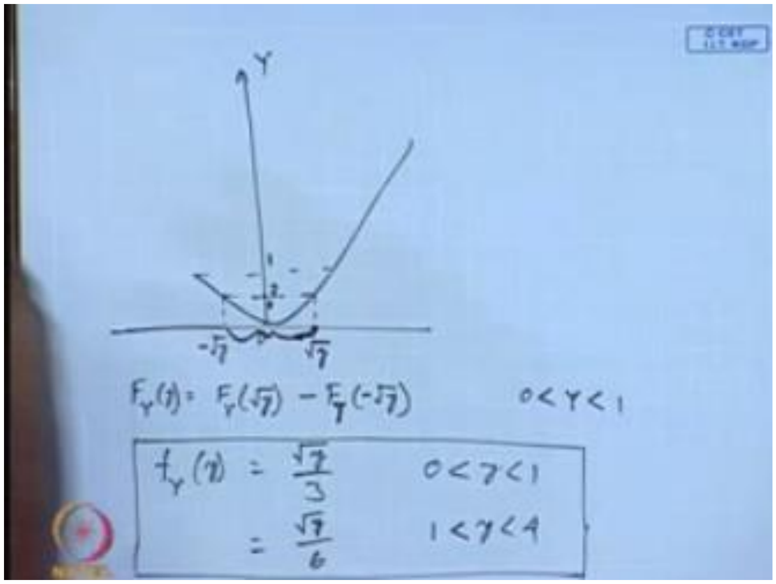

So for thus 1 to 4 what we got that this is √y/6. So, this is the complete statement for the distribution of the y that is f y of y √y / 3 for this zone 0to 1 and √y / 6 for the zone 1 to 4. So, this is this is shown here. So, here also we can once check that if this if this transformation or if this density is correct then we can if we just integrate it for the entire zone that is 0 to 4; and we take this one then also this will this should equals to the unity that you can check to the this expression a valid PDF.

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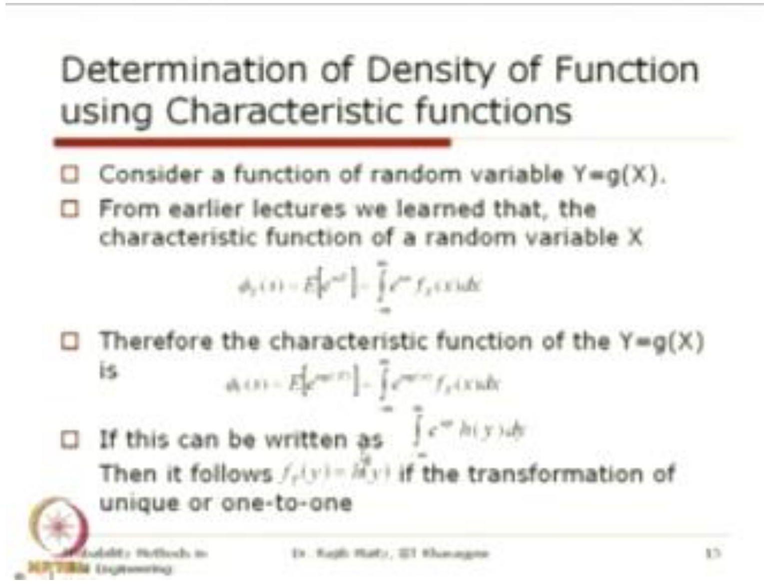

Now, we will discuss about another method, which is the method call that, this method of characteristic function. So, we will determine the relationship we determine we determine the relationship of the PDF of the new variable that is function of the random variable through the characteristic function. Now, this characteristic function we have discuss for a for a variable earlier in the earlier lecture now in this lecture easing that property we will see that how this that property can be utilize, can be used to find out the density of the function of random variable.

Now, we have seen that, if there is a function of random variable like that y =g x, now for these random variable x, that the characteristic function of that random variable x can be shown that, this is f x of s is equals to that expectation of e^{isx} this I is your √-1 that is a complex number s is the variable, that we are using here; and this capital x is your random variable. So, expectation of this quantity is nothing but, the characteristic function that we have explained earlier.

Now, this expectation of a of a function also we have we have discuss earlier that is if. So, this can be express, that is for any function that should be multiplied by the PDF of that random variable and integrate over the entire support to get the form of that form of that characteristic function. So, this is integrated from this minus infinity to plus infinity multiplied by e^{ixs}

multiplied by f x xd x, so this is the expression for the characteristic function and this is obviously, for the continues one and you can see that if it is a discrete random variable then this integration will be replaced by the summation and this f x that is the PDF we will replace by it p m f that, we generally in standard we call that for pmf is for the discrete random variable.

So, this is the characteristic function of a random variable x. Now, now the functional relationship as you have decided that y = g x. Now, for this g x now my goal is to get that characteristic function for this function g x, which is nothing but, equals to another random variable y. So, if I want to get that characteristic function then, that f y s = e^{isy}.

So, this I s y is your random variable here now this y is equals to g x, we are replacing that y in terms of g x, so that exponential of I s g x now this exponential that exponential of i s g x this function if I take the expectation then we will get the characteristic function of that function of the random variable x, which is y,

So from again following the same that definition of this characteristic function. We can take this function and this is multiplied by this the probability density function of this random variable x and if we just integrate it from - ∞ to +∞, then will get the characteristic function. Similarly, again if it is a discrete random variable then this can be replaced by its summation sign.

Now, again if I just one to know if we do not want to express is in terms of x, if we entirely express is in terms of y then this expression can also be expressed as that- ∞ to +∞ e $^{is\ y\ h\ y\ d}$ y now from this two, if you just compare then it follows that fy(y) is your h y if the transformation is unique one to one. So, if we can get this relationship then, this hy that is the inverse that is that that is express in terms of here in this expression cans we can get that PDF of this y.

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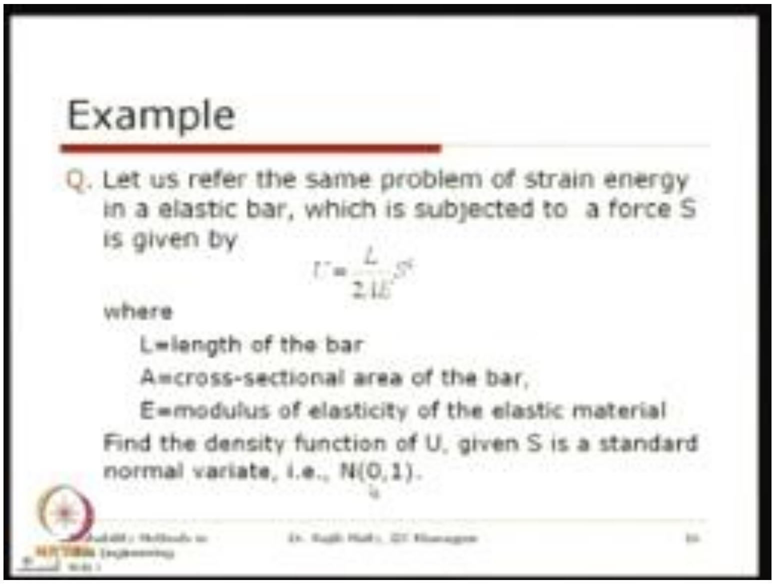

Now, to explain this one we will take one example and this example, we have solved earlier also but using the method of distribution I guess and now we have taken this same example and we will solve this example with the method of characteristic function. So, this one that let us refer the same problem of this strain and energy the problem was on the strain energy of the elastic bar; and that strain energy is generally related to it is force that is under, which the elastic bars subject to.

So, the relationship between this u and this U and this s is like this that U equals to l / 2ae $^{\text{p S square}}$ now, where this l is length of the bar a is the cross sectional area of the bar and e is the modulus of elasticity of the of the material. Now, find the density of the u, given the s is a standard normal standard normal distribution standard normal variety that is the n equals to a normal de normal distributions, which is having mean 0 and standards equals to one now.

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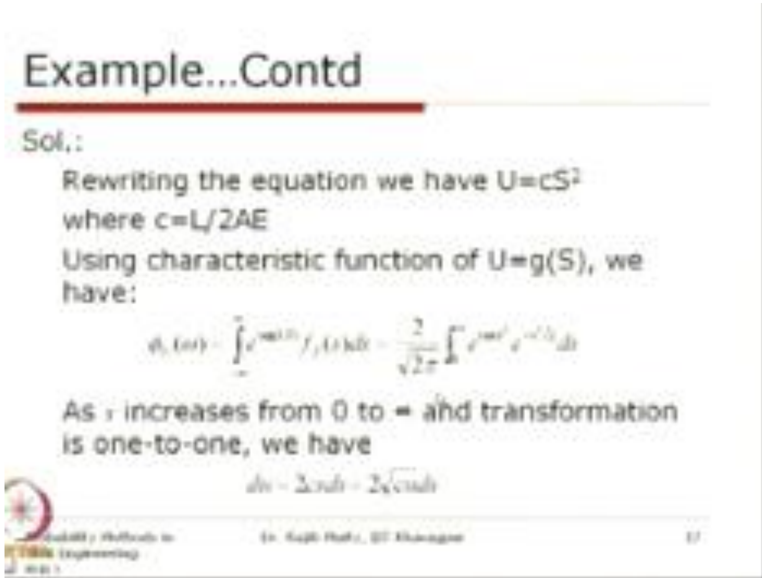

So, this one again we are just clubbing all those constant here through this another constant c. So, the relationship is U is equals to C S^2 using the characteristic function that is U equals to g S, we have that π of u ω is equals to - ∞ to +∞ I ω g s and f s is equals to f s of s ds. Now, this is the characteristic function if you just replace this one this density we know, which is a standard normal distribution, if you replace this one then we will get the 2 by square root of 2π integrate from 0 to ∞.

This is an even function that is why multiplied by 2 and taken from this 0 to ∞ e power is square multiplied by e-s square by 2ds as s increases from 0 to ∞ the transformation is one to one. So, we have that d u equals to 2csds that is 2c $^{2\ \text{root}}$ of cuds.

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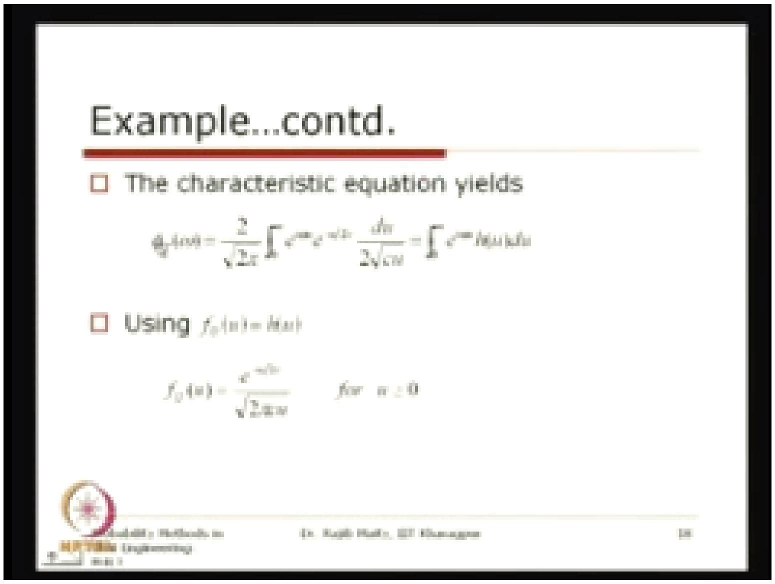

So, the characteristic equation yields that π u ω is equals to 2 by square root 2 πintegration 0 to ∞e power iωu e power - u by 2 cdu / 2 square rots cu. So, we are we are expressing in terms of u here, so which is equals to 0 to ∞ power I ω u hu dy now if we do. So, now if we call if we compare then this fuc that is the density of the random variable is equals to hu here. So, if you just compare this one it comes that f u u is e power -u / 2 c divided by square root 2 π c u, for this u is greater than equal to 0.

So, we got the same distribution that same PDF that you got in the earlier using the method of distribution. So, here also we have seen that using the method of characteristic function we got the same expression as well. So, starting from the fundamental theorem we have discuss that method of distribution method of transformation one to one as well as not one to one as well as not one to one then we discuss that method of characteristics function with examples.

Now, one thing we have to keep in mind that even though this transformation, we have explain particularly in some of the non-linear transformation, which is very common in the civil engineering problems here; that kind of transformation may or may not always in this kind of close form salutation and they are what is more what will be useful in such cases that if we come to know some of the properties of those derived variables that is its expectation that means,

We get the mean variance and it is obviously, if we get the moment generating function. Then we can we know the required information of those the function that is the derived random variable. So, these things we will discuss in the next class how we can without knowing the PDF how we can know those properties of the derived random variable thank you.

Probability Methods in Civil Engineering

End of Lecture 16

Next: "Expectation and Moments of Function of RV" in Lecture 17

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