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Probability Methods in Civil Engineering

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Lecture – 14

Topic

Functions of Single Random Variables

Welcome to this lecture, today we will start a new module and this module is on these functions of random variable. So, why this is important is that, so far we discuss about the different properties of one random variable and their probability distribution, their cumulative distribution function and their different properties. We have seen different standard probability distributions, which are most useful in civil engineering and what we are going to discuss in this module is the functions of random variable.

So, if we are having one random variable and if we can relate that particular random variable to the another one in terms of some functional relationship, then, if I know the distribution properties or the probability distribution of the random variable, then we want to know the properties, the probabilistic properties of the dependent random variable. So for example, it is a $Y = g(X)$ can be treated as to be the one general function. Now, what is known to us is the, all the properties of the x is known and we want know the properties for the y.

So, this is the basic theme of this module and in today's lecture, we will start with the fundamental theorem and that fundamental theorem in the subsequent lecture will be shown that for the different methods for which we can understand the different properties of that function.

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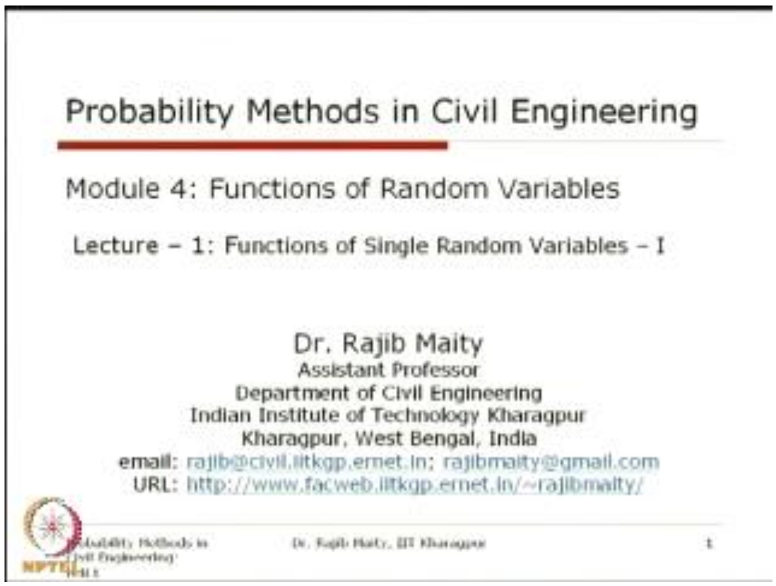

So, our this module name is this, functions of random variable and in this particular lecture, we will discuss about the function of single random variable. So, here the single, what you mean is that, only one random variable is there for as the independent random variable and that is having some functional relationship with the other dependent variable. So that function, once we know that functional form, after knowing that functional form, I want to know their different properties.

Now see, this one is in fact what we can related to that, for example, in the previous lectures, we have seen that normal distribution and from there we have seen the log normal distribution. So that what we have seen is that Y = log X if I take and if I say that this log X, that is the Y is the normal distribution then that X is the log normally distributed. So, now the reverse function will be that $X = e^y$ and if this relationship is known, and I know the what is the function of this normal distribution, then by exploiting the properties that we are going to discuss, then we will know that what should be the properties of this that log normal distribution.

So, here what we mean the property is that, we should first know that probability density functions. So, knowing the density function of one random variable, I want to know the density function of another random variable which is having some functional relationship with this random variable. So, why we are using the word random variable is that we know that this, the random variable is a function that we discuss at that it was initial lectures of this course.

Thus random variable is a function which maps the relation from the outcome of a particular random experiment to some number on the real line. So, that is the functional dependence, while we are defining random variable we know. Now, when we are defining another function based on the random variable, that function is also a random variable. So, our focus today is to know the distribution properties or the density of those functions which is developed based on the based on the known random variable to us.

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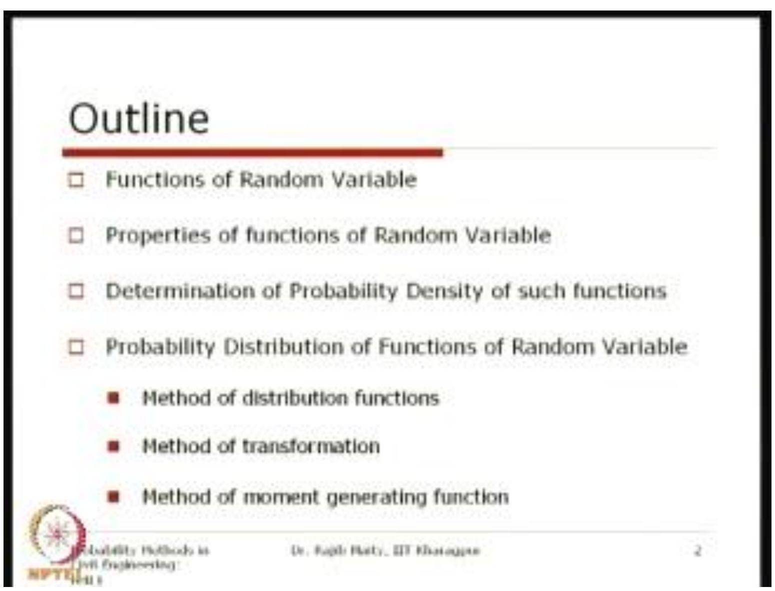

We will go like this; first of all we will discuss the functions of random variable. Basically, the functions should have some characteristics, some should have some properties, before we can say that this is a function of the random variable because that random variable, the functions are also the random variable. So, that conditions should satisfy, we will discuss that one first, and then we will see some fundamental theorem, how we can determine the probability density function of such functions.

There are three different methods, that method of distribution function, method of transformation and method of moment generating functions which are all this three methods are basically based on the fundamental theorem that we are going to discuss now. And also that method of the characteristic functions are there, we will be discuss in this lecture. So, that based on this fundamental theorem and with some with some additional condition, we will see that. So, if you understand that fundamental theorem first, then understanding of this one will be very easier,

Mostly this methods will be covered in the next lecture. Today, we will see, we will understand that fundamental theorem.

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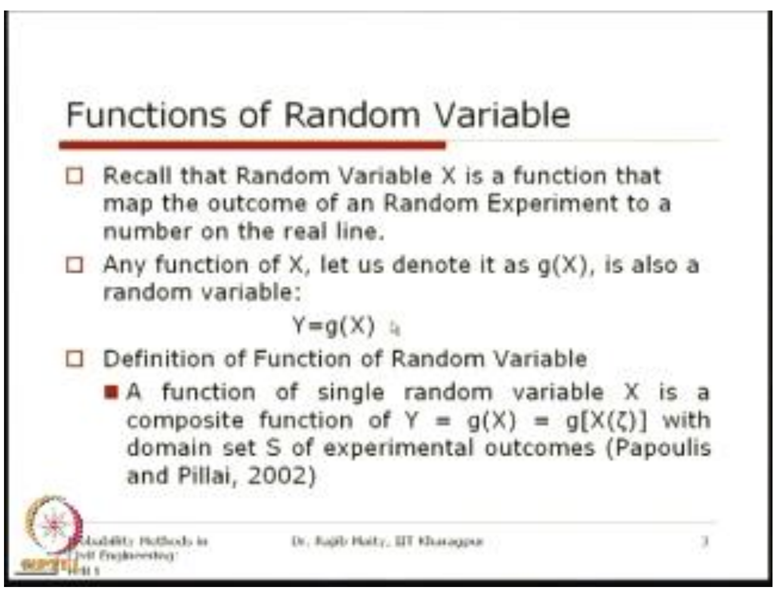

So, first of all what we discuss is that, there any random variable which is denoted as X is a function that map the outcome of a random experiment to a number on the real line. So, random variable itself is a function that we know. Now, this is having a functional relationship with some other. So, now from the X, what we are having is the g(X), now g(X) which is the $Y = g(X)$, now this g(X) is also a random variable.

So, we want to know the properties of this Y, when the properties of this variable X is known to us. So, the definition of this functions of random variable is states like this. A function of single random variable, here we are discussing about the single random variable. Similarly, we will see the properties in the subsequent lecture for this multiple random variable as well, where there will be more than one random variable and their functional relationship with some other random variable.

So here the functions of single random variable X is a composite function of Y equals to g(X) is equals to $g(X)\zeta$ with the domain set S of the experimental outcome. This is taken from Papoulis and Pillai here this ζ is the outcome of that random experiment. Now we know that all the outcome of this random variable is mapped to the real line through this random variable X. So we know so this is generally is going to a really stating a specific number and that number is here so and this g(X) g is a functional form and I want to know that after taking this function what is their properties. So this is the function of that random variable X which is known.

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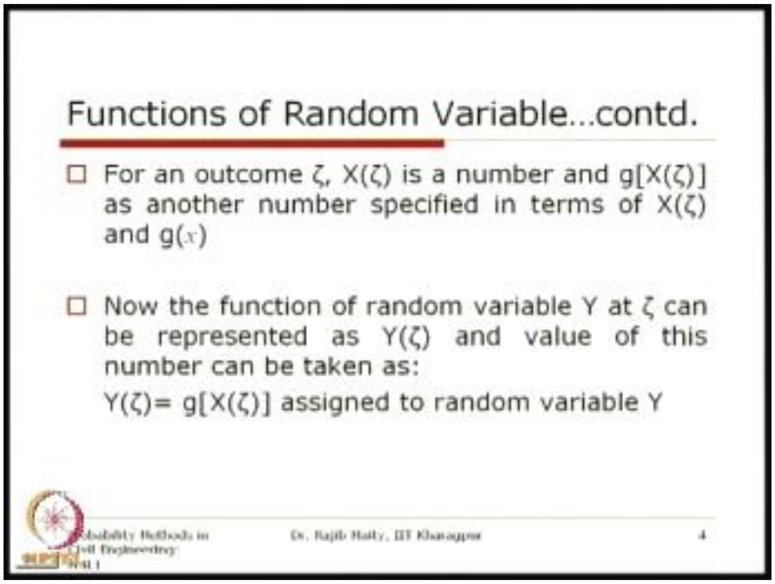

Now, for an outcome ζ so we are taking one specific random experiment and I and just one specific outcome of this ζ is if it is taken, then we know that this X ζ when we are taking that random variable. This is basically is a number which is mapping from this samples space to the to the real lines so this X ζ then is nothing but a number. Now if this X ζ is a number then that this g[(X) ζ] that is after I take that that function that g[(X) ζ] will be the another number which is specified in terms of this that original random variable X ζ and it is that functional form it is that g(X) so if we just see it here so this is that.

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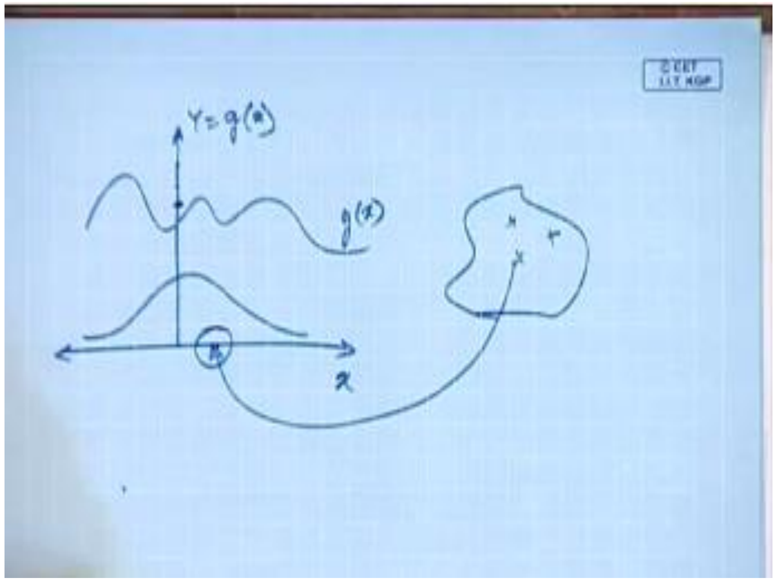

This is that function this is that that random variable access if I just say that this is the x is x that is known to us. Now there can have some so this function can be general like this that which is your which is your g(X) now this function is known to us. So now so for the any anything any number what we are getting is here is that that functional dependence. What we have to see is that, based on this change here in this in this so Y = g(x) so we can just write the is Y = g(x). So, in the a change in that dependent variable I can say that Y. So, what is the what is the change what is the properties that we can invest we get from the original random variable x.

Now from the if I say that I know the original random variable that means what we are what we are what we know is that its distribution whatever maybe that type of this distribution we know. Now for this I have to find out so for each and every possible outcome of this random variable which is nothing but so from the we know that from the sample space each and every outcome of this random variable is mapped to this real line. And so, this from this one based on this functional relationship g(x) and this number, this function is known to us. So, this g(x) is defined is another number specified in terms of this both this functional property as well as this number which is mapping from this experimental outcome to this line. So this is what is specified here.

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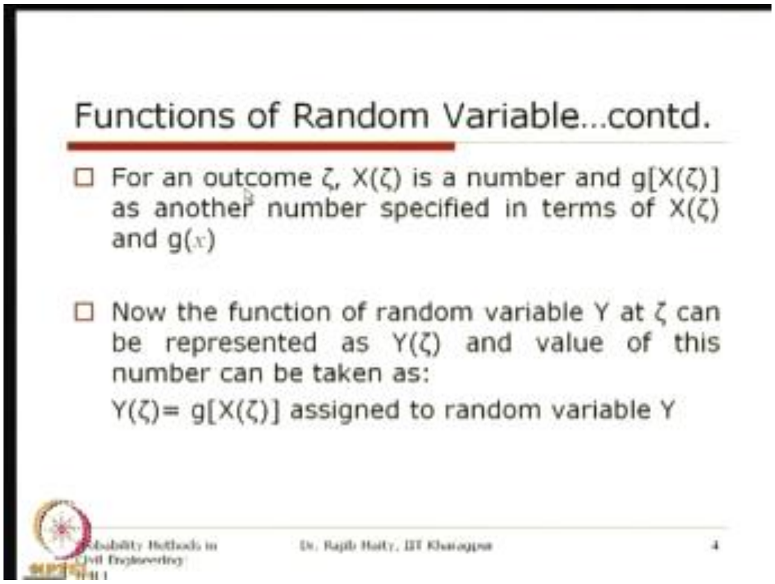

The for an specific outcome ζ, for an outcome ζ X(ζ) is a number and this g(X)ζ, that functional correspondence as another number specified in terms of this X(ζ) and this functional form g(X) ζ. Now the functions of the random variable that is Y at ζ can be represented at Y(ζ) and value of this number can be taken as Y(ζ) is equals to X(ζ), assigned to the random variable Y. Now this is important in the sense.

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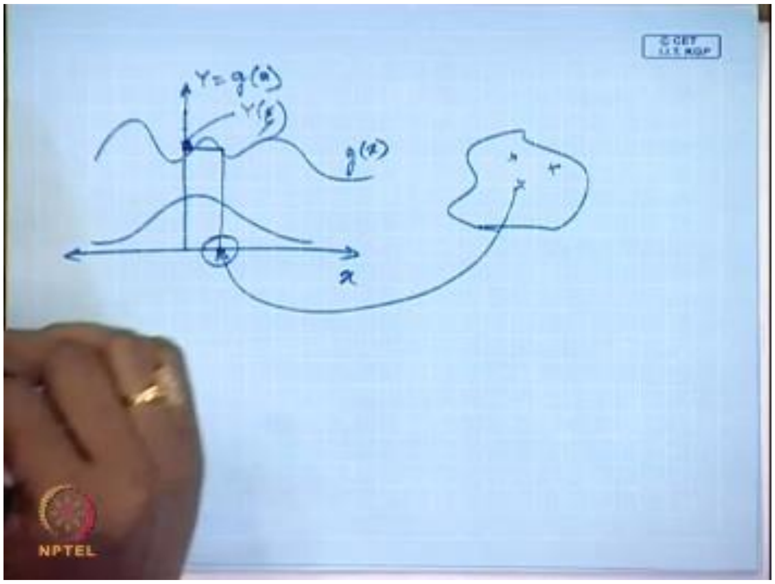

Now what we are what we are doing is that from this experimental outcome. So, this is the specific that outcome ζ that, this is that we are X(ζ) and from here= what we going we are taking this function and to this function g(x) and then we are coming to this X is that is Y = g(x). And this one now is what is that this corresponds to the outcome of this original outcome of this random experiment ζ. So this one is your now what we are saying is that this is your this is your ζ okay

So, from here now we will see the if so may be a very general function is drawn here so there may be some different roots and we will see that if we are having the different roots or more than one roots to be to be in general if we are having the more than one root then how to how to handle this how to handle this property that we will see and that is basically our motivation for this fundamental theorem. Later on we will we will proceed to the specials case where this functions will be defined in such a way that there is only one to one correspondence is there.

So, this one to one correspondence means that whatever the outcome of this original experiment that outcome of this experiment random experiment is there. That is having a particular number that we know and from here when we are going to this function this relationship is just one to one for a specific value of x only one out come only one possible value of Y is there. So, that will be the special case. Now today in this lecture what we are talking about the general case there may be means n number of roots for that for the specific outcome. That we will see today's class.

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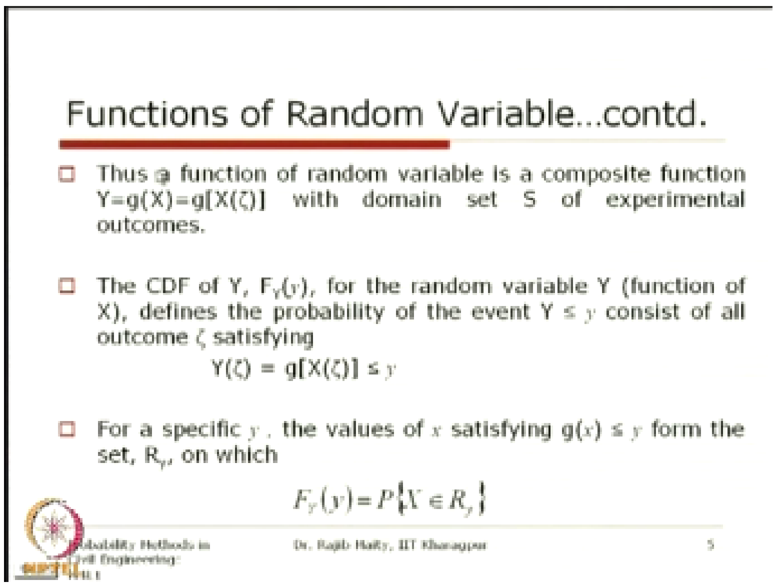

So, thus a function of the random variable is a composite function which is Y = g(X) and this number is g (X) ζ with the domain set S of the experimental outcomes since this domain set S means that original random experiment the outcome of this original random experiment now, so what we have to know is the CDF, that this how this cumulative distribution function of the Y that that function obviously what we are what we are when we are talking about the, when we are talking about the function of random variable that means the all this properties all this functions related to the X is known to us.

So, based on that what we want to know is that this the cumulative distribution function of Y $F_Y(y)$, for the random variable Y which is the function of X defines the probability of the event that Y < y, that means this is the random variable with the specific value less than y, consist of all outcome ζ, which satisfy the condition that these g(X) ζ ≤ y. This is very important and this is, this concept is very important to know that where we are means which value is we should we should classify as this set this event set. This would have been more step forward if we just directly say that is one to one transformation that when we are talking about the general case, then we have to find out those sub set of the X where this condition is satisfied we will see that.

Now, so for a specific value of y the values of the x satisfying the g(x) ≤ y from the set R (y) from the set Ry we are now designating that the set which is satisfying this condition that is this is a functional form g(x) now that g(x) for all possible value of that of that x which occur having this functional transformation will be less than that specific value y.

So if I want to know that what is this one this that cumulative distribution function of Y what to have to do is this is that cumulative distribution of Y. So we have to find out the probability that X belongs to that set R (y) which is which is satisfying this condition that is g(x) is less than is equals to less than equals to that specific value y, where we are the defining that this CDF cumulative distribution function. That we will see in this general case.

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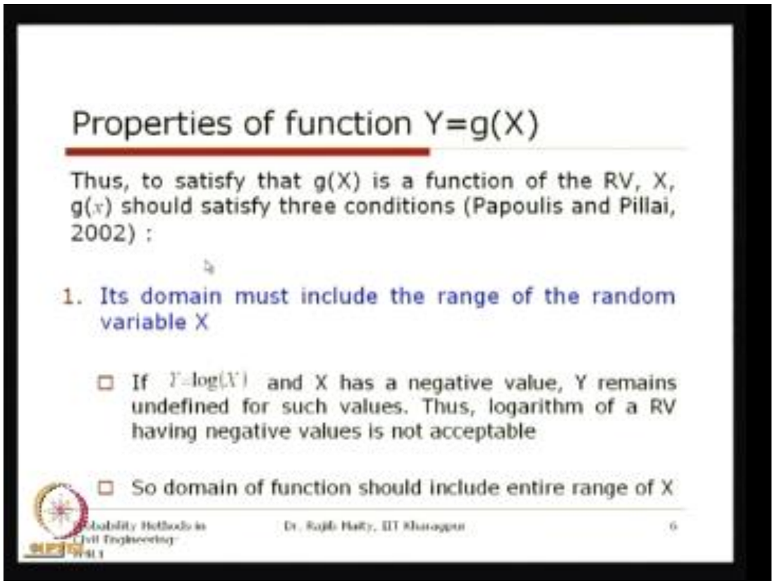

So just keeping this point in mind we can we can say that to satisfy that g(x) is a function of the random variable X this g(x) should satisfy some condition. There are there are three conditions this should be satisfied before we can say because this function itself is a random variable as we as we discuss. So just by knowing that just by just by stating that this it is a random variable. So this should this should satisfy some conditions some three conditions are there that is should that should be satisfied before I say that this is a function of the random variable.

The first condition or the first the first condition is the domain that domain of this g(x) must include the range of the random variable X. So whatever the random whatever the support we generally say that the support of this X is there that should that entire support that entire range of this random variable should be should include the should be included the by the domain of this function g(x).

Let us take one example here that if the function is like this $Y = \log X$ and this X can have the negative values. Now if I say that $Y = \log X$ whenever we are saying the this functional dependence and this is that $Y = \log X$ and while giving the properties of this X and saying that X is a random variable with the range that say for example, the -5 to +5 or some negative values are there for this for this X of for its support.

Then this Y remains undefined for such values. So for this negative zone this obviously this Y cannot be defined. So thus in such cases when this X can take the negative values this logarithm of the random variable is not acceptable, this is also true if I just take the square root. So $Y = \sqrt{x}$ and we are saying that X is having the having the its support towards the negative side as well. Then those functions are not acceptable.

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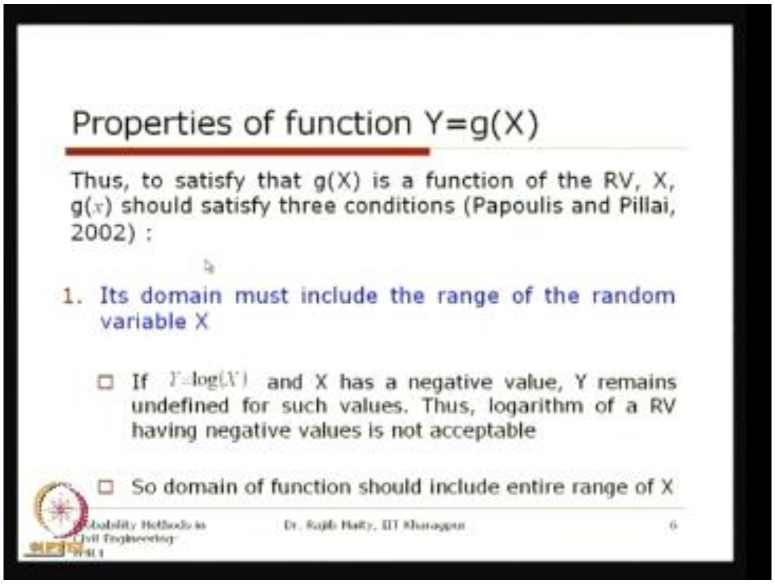

Then in what condition this will this will be acceptable? the if I say that this X is a X is a random variable with its with its lower bounder 0 or some positive number, then I can say that okay fine this functional form in terms of this function of this random variable is acceptable. So the domain of the function should include the entire range of the X that is what the first condition is that so its domain. The function domain of this function g(x) must include the entire range of the random variable X.

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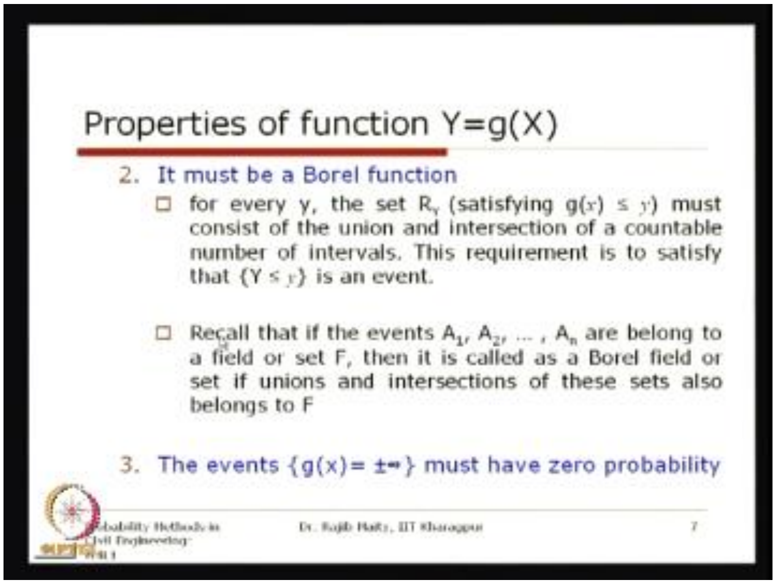

The second condition is that it must be a Borel function. Now you know that when we are talking about this borel function we have defined it in the in to ours initial lecture that.

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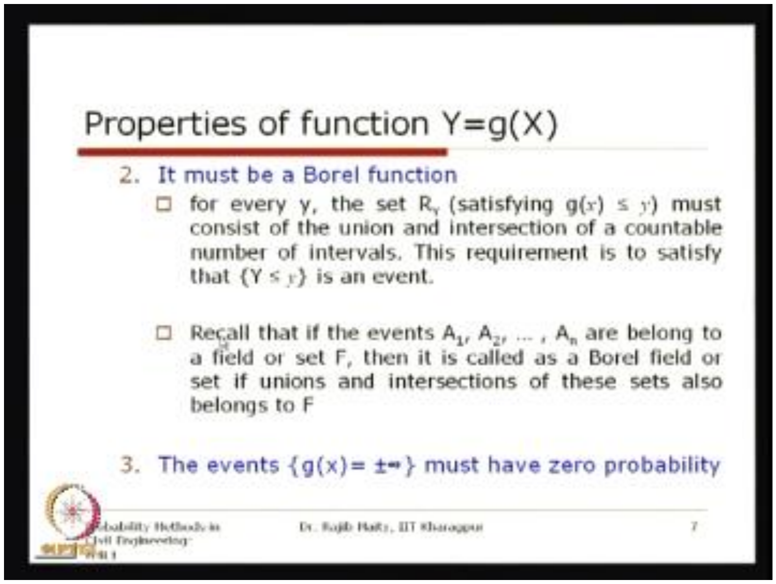

If should be the it should be the that the set its union and the all possible unions and intersections are also inside the inside the set. So if we recall that if the events A1, A2……An belongs to a particular field or set F, then it is called as the borel field or set if the unions and intersection of this sets of this sets means A1, A2….. An also belongs to F, so then we will say that this is this is a borel set.

So the condition the second condition of this function is it must be a borel function this is just to satisfy that this one after taking this function that is also a random variable. So, properties of this random variable should be satisfied, so that is why it comes as a borel function so that we can take that unions and intersection. So which is states the for every y the set R(y) satisfying that $g(x) \le y$.

This must consist of the union and intersection of a countable number of intervals. This requirement is to satisfy that this $Y \le y$ is an is an event, and the last condition which we also have seen that for the random variable is that this random variable that is the g(x) at $\pm\infty$ its probability at ±∞ should be equals to zero. So the event g(x) = ± ∞ must have zero probability. So once this three condition is satisfied.

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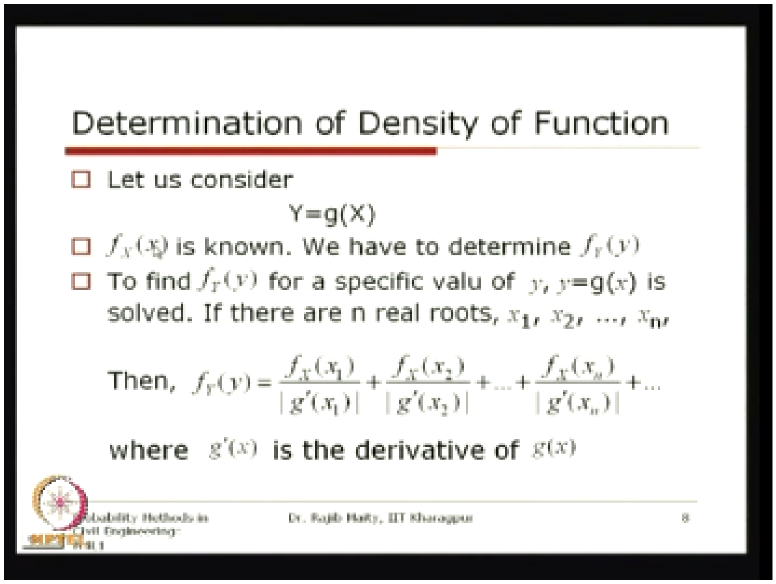

Then that function can be called as this function of a of a random variable and then we will see if this kind of functional correspondence is there between this X and Y then how we can how we can we can get the density function of this random variable Y which is the which is having the functional relation with that X. So obviously the all this properties which is required for this random variable X is known.

Now if I say that this f(x) x, that is the probability density function of the random variable x is known. That means I know everything about this its distribution. So this function is known this density is known for the x and this relationship is known and this function satisfies those three requirements to be a function of random variable. With these things in hand we have to determine the f(y) y which is the probability density function of Y.

Now to find the f(y) y for a specific for a specific value of sorry for this spelling mistake, this value of y then this y = g(x) is solved. Now so while calculating this density function for a particular value of y, I take that particular value of y fitted in this equation that is this y is known. Now this g(x) so X is now is unknown. So we have to find out that what is the value of X now as I was telling that if there is the relationship is said one to one relationship then there will be only one root so one to one relationship means for a specific value of y there is only one value of x and vice versa at here that we are making it the general so this Y = g (x) if there are having the n real roots of this of this equations ay that X1 X 2 Xn. that means what we are telling here is that for this for.

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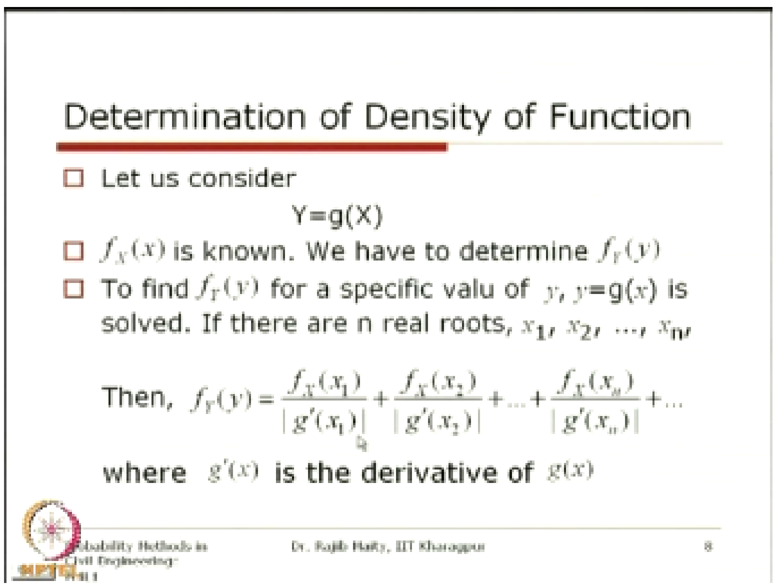

This kind of function if I just take for a specific value of this Y so this is my specific value of Y now I have to get that what are the different points that it can that the root of this of that equation so the first root is here second root is here third fourth so wherever it cuts that that functions we will get that this is and these are the roots so if in general case we say that there are such in such roots are there which are x_1, x $_2$ up to x_n then it can be shown that this f y the density of that of this random variable y is can be stated like this.

So this f x value of this fx at x_1 fx at x_2 these are the that density function of x evaluated at those roots divided by this g prime x_1 it is mode g prime x_2 g prime x_3 so the is the derivative so in the mathematical term we know that g prime x is the derivative of this function x or we you know that this derivative nothing but it is the rate of change of g at that point a x_1 and we are taking its mode so we will just show that whether it is increasing and decreasing it should not it should not matter.

So this fx at x_1 divided by g x1 + similarly for all such roots we have to take the summation of that one will give the density function of y now we will see how this how this relationship is holds good.

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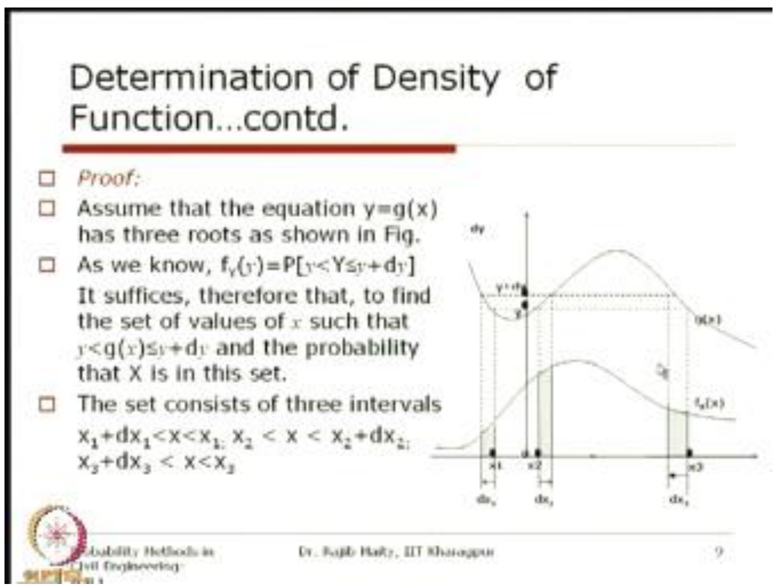

This proof is taken is for the general case and here we have taken those three such roots so basically we are assuming that at a particular value of Y we are getting the three different roots so these three different roots how we can say that for a specific value of Y so this is the location of this Y and if I just go through this one these are the three specific values of the three specific values of that root so this is as it is shown here that x_1 this one is shown as this x_2 and this one as shown it to be the x_3 so it will be more clear if I just redraw it one after another.

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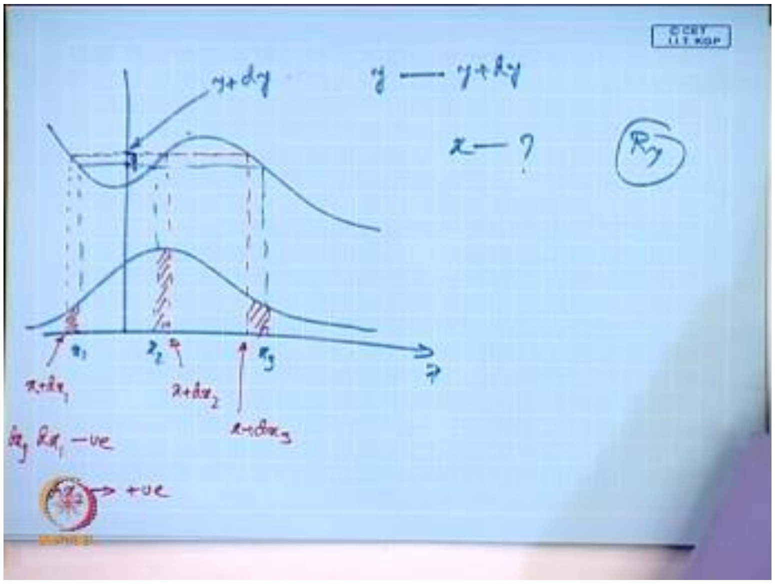

So this is my that y location of y and this is the x and x the CDF that is PDF that is fx affects is known and let us say that this affects of x is like this now for this one this is actually referring what we are first root is the x_1 this is your second root x_2 and this is your third root x_3 similarly depending on this depending on this function there maybe n numbers of roots like this so what we are basically trying to do here is that this Y if I take a small area means small range from Y to say this is that dy so this is the increasing side.

So if I just take that this one is that what we are saying is the Y +dy so what we have to find out that from this y to y + dy what are the ranges that is being covered for this x so if you can identify that then that is basically giving you the set of that Ry that just what we defined gives like that is that is giving you that what is your that set for Ry then we can calculate it is that the density so then add this one what will happen this if I just draw another line here it looks like this.

So this is your what I will say that this is now this location is now your that x plus d x_1 this is your x + d x_2 and this is your x + d x_3 remember that this d x_1 means when it is going from y2 y + dy from this point to this point this is coming from x_1 to this one so this double in space dx_1 is your negative similarly that dx similarly that dx is 3 and this dx_2 is your positive okay now what we have to we have to get here is that these are the set that we are talking about which is covering in this for this y to y1 these are the three areas which is that changing for I mean for the random variable x.

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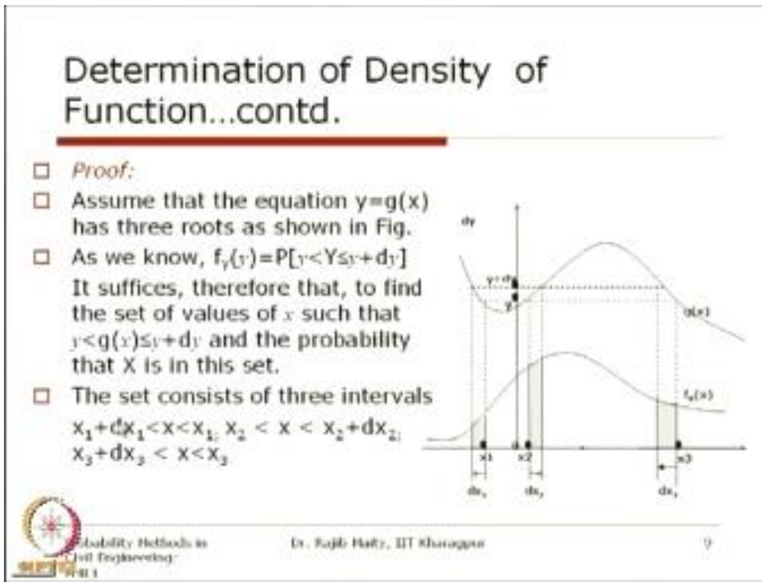

So this is what is explained in this in this in this figure here so now if I want to know that what this f_y y is then this is nothing but the probability of this y to y +d d_y changing to this one now it is suffices therefore that to find the set of the values of x which is my that or y such that this condition is satisfied that is a gx which is nothing but the y is from the y to y + dy and the probability of that X in this set so probability of that set in this in this set of this x means what is here the shaded area here.

And we have shown there that these are the three areas that the probability should be equal to this probability which is the cumulative probability from this we usually form the cumulative density from this y to y + d y now there are this three sets are like this which we have identified and shown in this one the first set consist of this three interval from this x + d x_1 to x_1 this is the first set then the second set is from the x_2 to x_2 + dx_2 and the third set is the x_3 + d x_3 to x_3.

So, as we told that this dx_1 is negative, dx_3 is negative and dx_2 is positive, so that is way how this moment. So, from this y it is going to here so that from y corresponds to x_1 and y+dy corresponds to this point which is, that is why it is dx_1 is negative here, so from coming from this point to this point.

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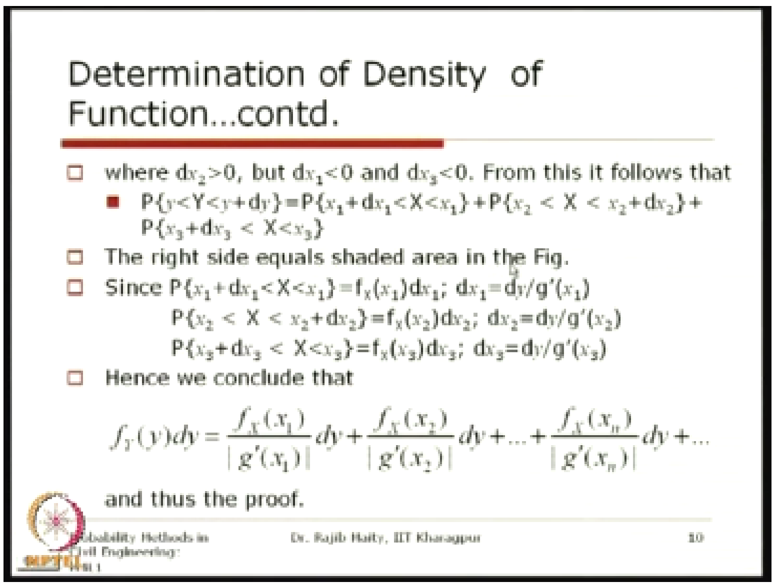

Now, yes these points we just told that which one is positive and which one is negative. So, from this it follows that is probability of this Y in between this zone there y to y+dy should be equal to the probability of this first set which is $x+dx_1$ to x_1 plus the probability from the x_2 to x_2+dx_2 plus the probability x_3 plus dx_3 to x_3. Now, we can extend it if there are n roots, so this kind of n different sets will be there, that you have to add up.

Now this right side of this one, the right side is equals to the shaded area in the figure that just now we have shown that this is the, these are the shaded area of this probability.

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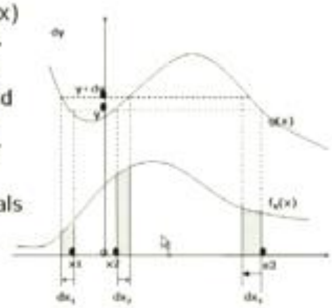
Determination of Density of Function...contd.

- *Proof:*
- Assume that the equation $y=g(x)$ has three roots as shown in Fig.
- As we know, $f_Y(y)=P[y<Y\leq y+dy]$

  It suffices, therefore that, to find the set of values of x such that $y<g(x)\leq y+dy$ and the probability that X is in this set.
- The set consists of three intervals

  $x_1+dx_1<x<x_1;\ x_2<x<x_2+dx_2;\ x_3+dx_3<x<x_3$



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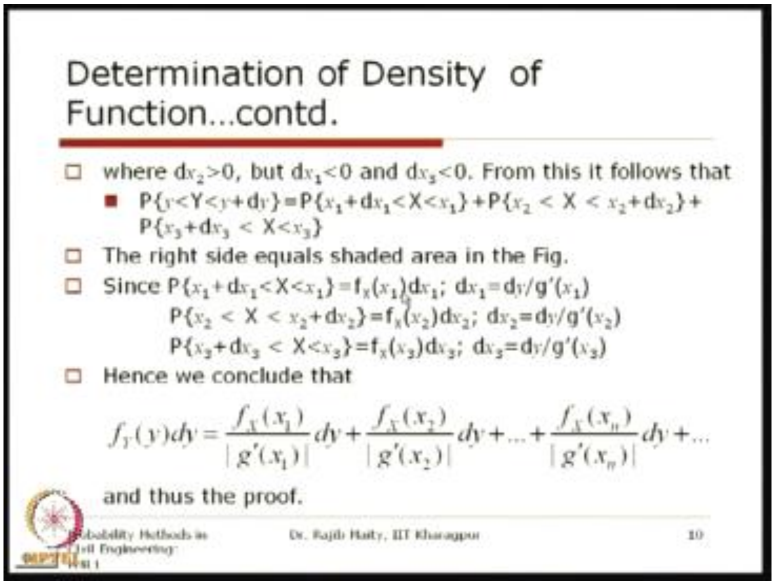

Now, since this probability that is the probability of x_1+dx_1 less than X to x_1, it is the density function at x_1 multiplied by dx_1 that is small change from that point. So, basically this is how we are calculating the area, an area on the PDF, you know that is nothing but the probability, so what if you just refer to this one.

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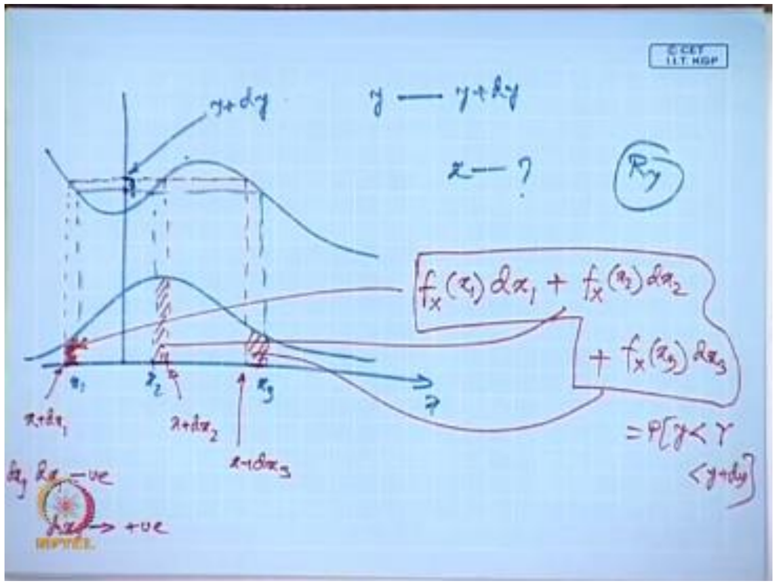

So, this area this area we are talking about, so this area how to calculate this one, so we have to see that what is its height multiplied by this small inferential small area is that this is that f_x, so I am just talking about this area. So, this is that value of this function at that point x_1, this one is multiplied by this dx_1, so dx_1 is this small length so this is basically giving you this area. Similarly if I want to know the area and this area so if I just want to add up, so this area will be then f_x the same function value of the same function at x_2 multiplied by the rate of change.

So, now the multiplied by this small inferential small area which is dx_2 plus this area, now f_x at x_3 now multiplied by this small length dx_3. So, this 3 is the total area of this one which is the probability this should be equal to that probability from this y to y+dy that is y less than Y less than y+dy so that is what is explained here.

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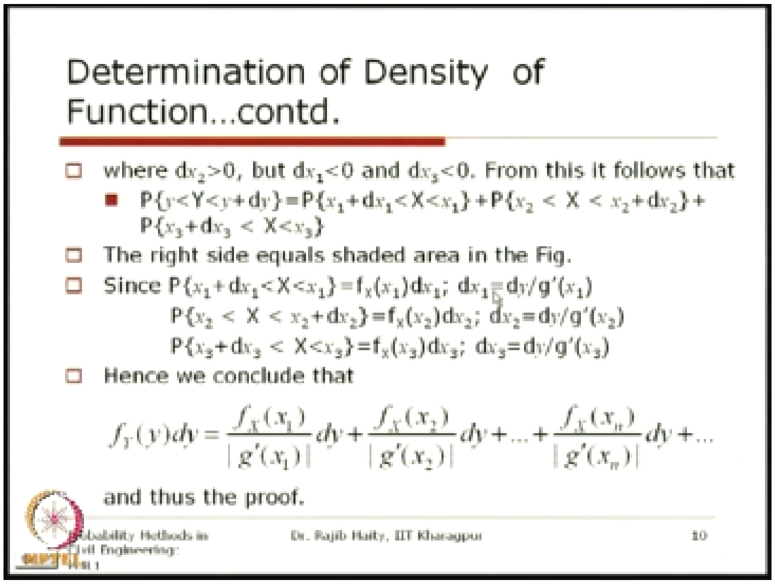

So these are the three different areas and three different areas are shown like this. Now, this one this small, this length can be this dx_1 again can be defined in terms of this, so add this dx_1, this is that dy/ g′ dx_1. So, the rate of, this is the rate of change of that function and this dy is the, what is the small change in that y.

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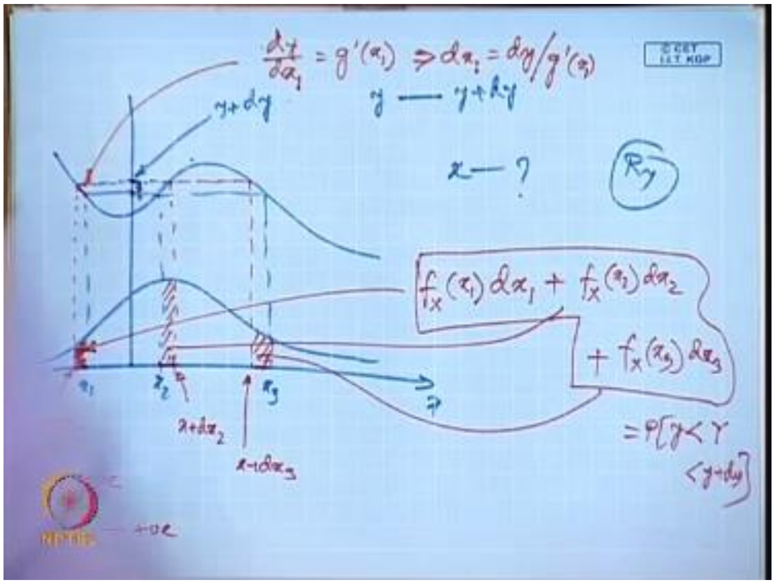

So, this is relating to this one, you can you can refer it to this one, that is this length what we are talking about is that that dy referring to this one if I just say here that this is your dy/dx_1. So, this is this is your nothing but the rate of change of y at x_1, so this is that g′ at x_1. So, this is how this y is changing for this function at that location x_1, and this is simply form followed the from this inferential length where that dx_1 will be equals to dy at x_1, similarly, for this other locations as well.

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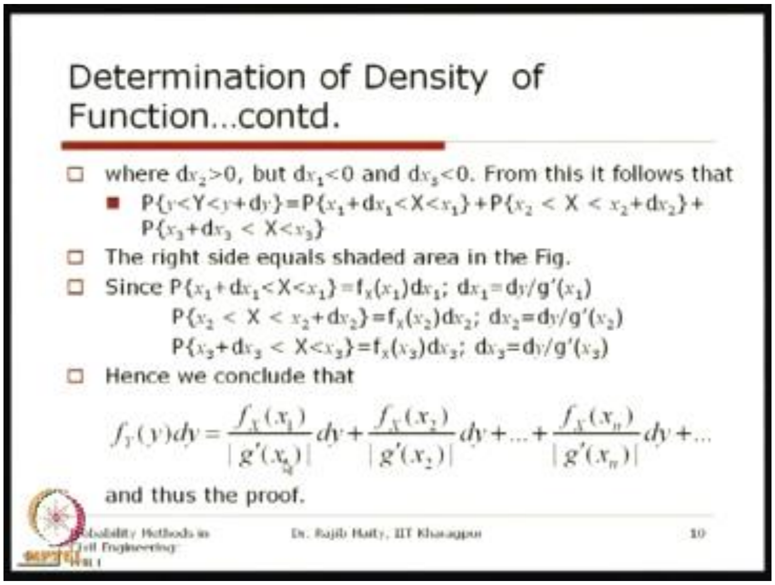

So this three things are shown as the dx_1=dy/ this g′ x_1, similarly dx_2 at x_2 this rate of change of that function at x_2 and this dx_3, it is a rate of change of that function at x3. So, now if we take, if we just take this equation that is this is the probability, this is the total probability for the function y, which is again what is the, this is the probability density multiplied by the small length which is dy. So, this one, this is the density multiplied by this small length is equal to this probability of y_2 y_1, we know that.

Now, from each and every sub set, we can just replace it by this one following this expression that we got that is f_x(x1). So, this dx_1 is replaced by this one, so $df_x(x_1)$ dy by this g′ (x_1). Now, you see that when we are talking about this $g'(x_1)$.

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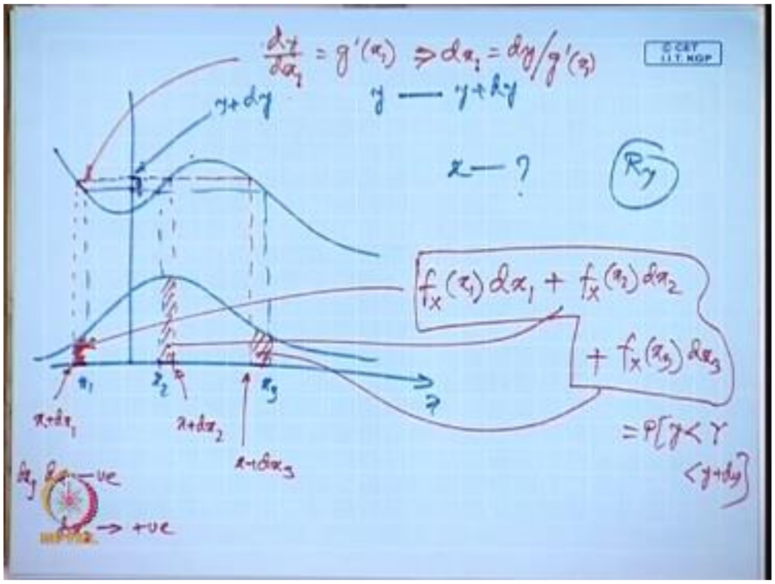

Whether it is going from this direction to that this direction, and this one to this one is does not matter as long as we want to know that total area. So, once we know the total area of this one, then what is happening is that we can take as well that mode just to avoid that if there is something that negative side.

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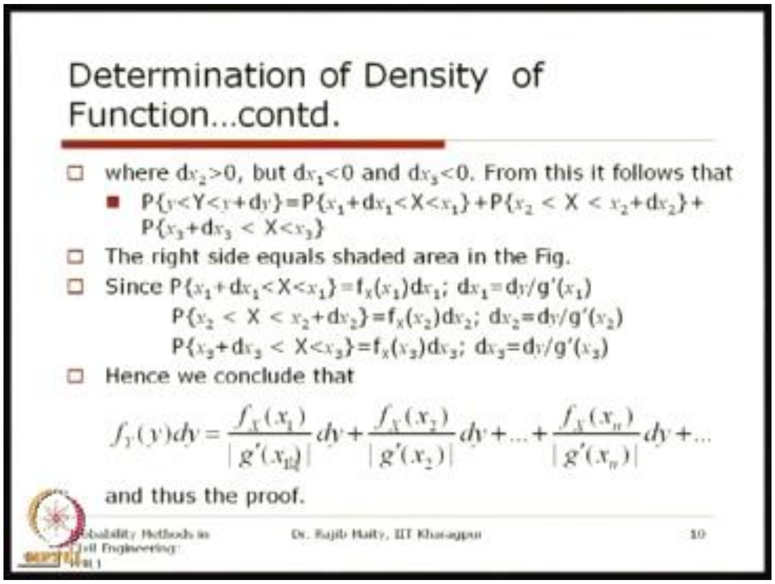

So, that is why this mode is taken for the positive side no effect, for the negative side we will just get the absolute value of that rate of change. Now, so this is for that at x_1 which is the first root and this is similarly coming from the second sub set which is for this at x_2, similarly for if there are as many roots are there, so like this. So, this immediately the next we step we will get that $f_Y(y)$ that is the density function for the $y=f_x(x_1)$ by this means just a summation of these residuals.

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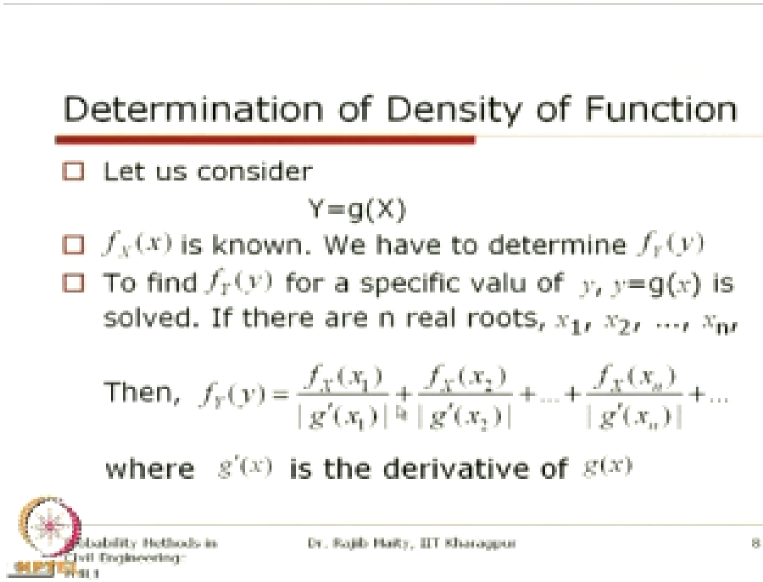

So we are getting this form for this one so this is known as this fundamental theorem for the determination of the density function of y when the density function of the random variable x is known.

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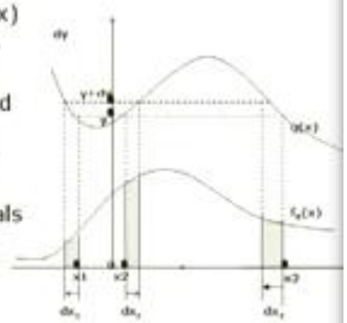
Determination of Density of Function...contd.

- *Proof:*
- Assume that the equation y=g(x) has three roots as shown in Fig.
- As we know, $f_Y(y)=P[y<Y\le y+dy]$

  It suffices, therefore that, to find the set of values of x such that $y<g(x)\le y+dy$ and the probability that X is in this set.
- The set consists of three intervals

  $x_1+dx_1<x<x_1$; $x_2<x<x_2+dx_2$; $x_3+dx_3<x<x_3$



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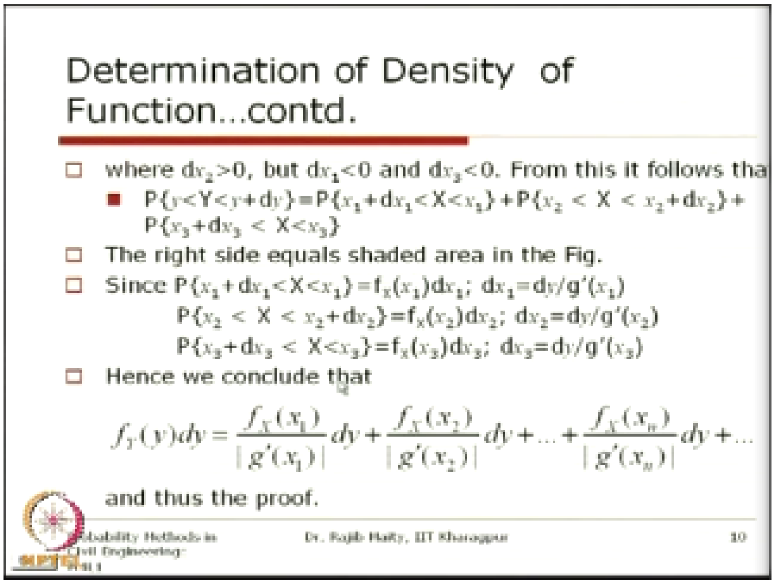

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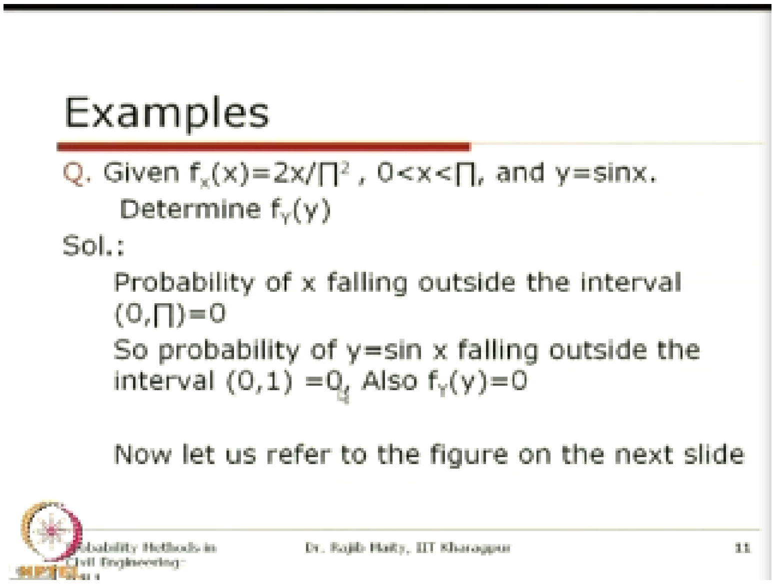

We will just take one small example, this is purely mathematical example, and before I come to the civil engineering example I will show that one. So suppose that here that one function that is that density of a function is given as a $2_x /\pi^2$ so if this function is given, then this it is from 0 to π, this support is from 0 to π, means you can you can also check whether this what are the condition for the, for a function to be a PDF.

So whether his over this supports whether those conditions are satisfied or not, the first condition we know that it should be greater than equal to 0. So as long as it is between 0 to π, all this functions are greater than 0 and you can just follows simple small integration, come 0 to π of this function, we will see that it is coming to one. So this is a valid PDF, but that is not our focus now.

What we want to know is that, what we are interested to know that, if there is a relation like this that is y = sin x. Now, how to know that what is the distribution for this random variable y, so we have to determine that f_Y (y). So this probability of x falling outside the interval that 0 to π is 0. So obviously because we have given this range, the range of this support is support x. So why we are telling is the, this one is that.

If there are some roots while we are calculating the roots if there are some roots outside this zone obviously that we have to discard and you know that sin function is the infinite and if I just take a specific value here. So, it will look like this. I think the diagram is there on the next slide.

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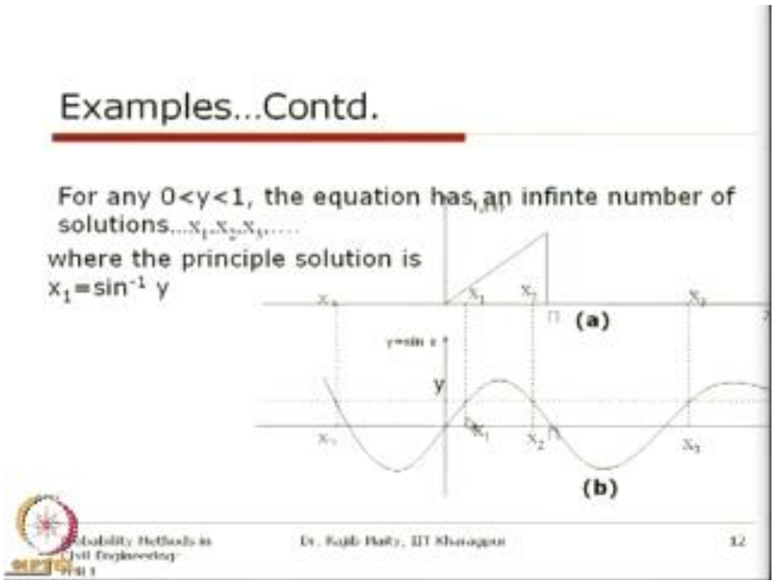

So this is the sin function if we see, and now this one is from 0 to one. Now if I take any value between this 0 to1 y, just remember that the fundamental theorem if I take any value of this y between this 0 and 1. And if I take it if I see its roots and there will be many roots, but we have to consider that only two, only the roots which are in between this range from this0 to 1, over which that x is defined.

Now we have seen that x is defined that x 0 to π, so that there is a possibility that, only two roots x_1 and x_2 should be considered. Now we will go the step by step this one,

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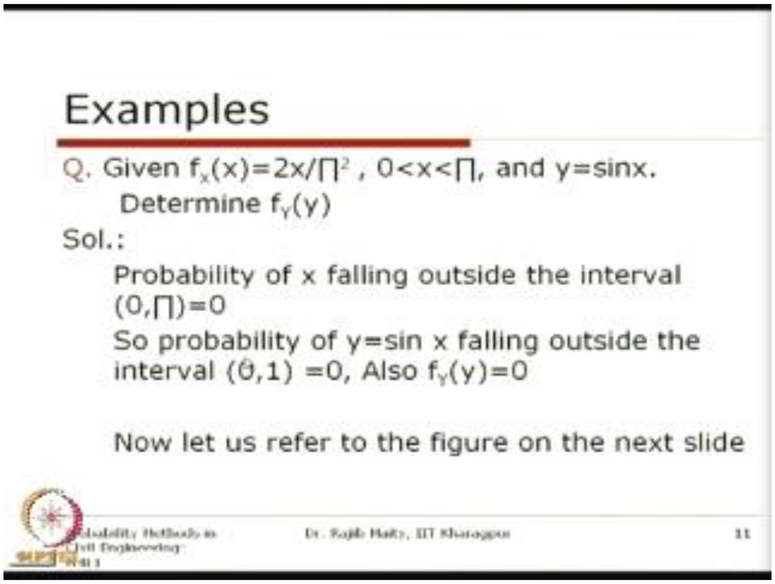

So as this is the range, so the probability of y = sin x falling outside the interval0 1 also f Y is = 0.Now, let us refer to this figure again for any value of y 0to 1 the equation has an infinite number of solutions starting from x_1 x_2 x_3 like this from, it is x minus 1 we are just giving some numbers. So, these are the number of solution where the principle equation is this. Obviously you apply just put one any value of y and this x_1 you will get this solutions will get.

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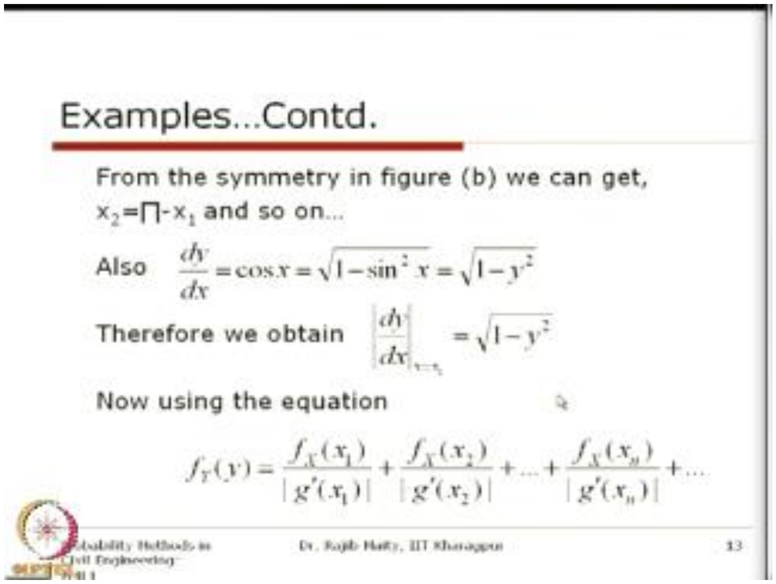

Okay before that, one more things this plot is showing the density function of x that f_x (x) which is a linear function starting from 0 to this, up to this π. So you can see once again here that at 0 f = 0 and at π f_x = 2 / π so this is a linear function this is a linear function between 0 to π. Now so from the symmetry in this in this figure b, that is this one, you know that. Now, we are we are we are considering only this part from 0 to π. So now this is symmetric, now you know that if it is symmetric, then this $x_{2,}$ we can get the x_2 to π -x_1 and so on.

And also that d_y d_x, we have to calculate. Thus d y d x, that is a rate of change of y at any value x is cos x which can be written as the 1 –y^2 sin 1 – $\sin^2$xand sin x is the y. So, there is a functional relationship, so this is √1 – y^2. So therefore at any root any root irrespective of how many roots is there, at any root this mode of this d_y d_x is √1 – y^2.Now we are using the same equation that is developed, that is for this numbers of root this f_y = f_x x_1 g prime x_1 f_x 2 g prime x 2 plus, this if we go on and here there are only two roots.

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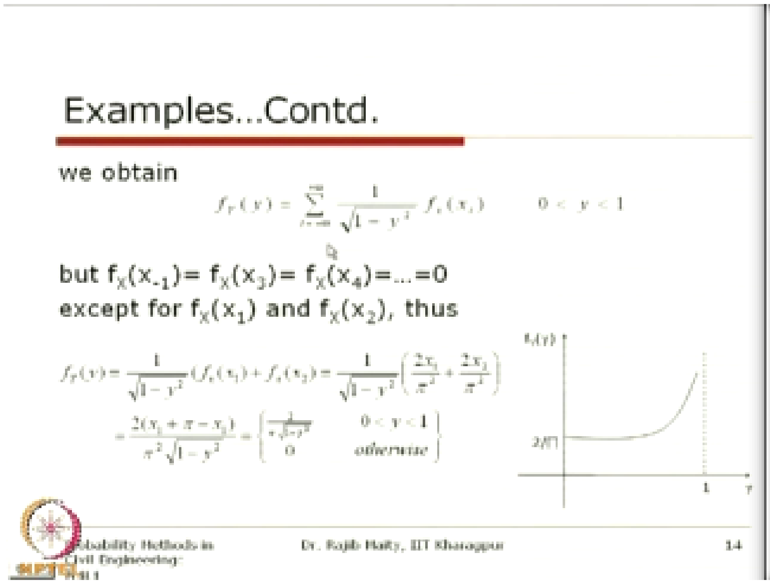

So, this f_y 1 is that the summation of this1 /1 – y^2 , which is the mode of d_x d_y, now and this f_x I which is between this y is between 0 to 1. Now, you see that this f x - 1, so these are just the notation, we have just here is -1 means what you are reference to this one.

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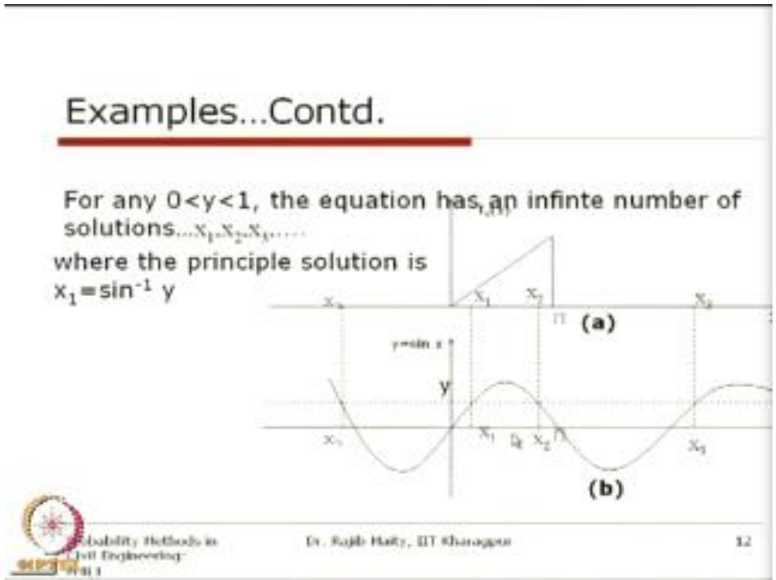

This solution which is which is below which is less than 0 so, this one this point x_1 and this x_3 means the next one, this solution, apart from x_1 and x_2, what we are meaning here. So, this all other apart from the x_1 and x_2 = 0, because these outside the range accept that for this f x 1 and f x 2, so these two things should be added. So, f_x at x_1 and f_x at x_2, that divided by this, because this is the same for both the things. So, to $\sqrt{1- y^2}$, now we just simply put these two functions here, that 2_x 1 $/\pi^2$ 2_x 2 $/\pi^2$. So, 2_x 1 –π- $x_2\pi^2$ $1/\sqrt{y}$, now put this value, and then will get that functional form like this, 2 / π^2 root 1 - y^2. So, these we know that x 2 are we have shown that from the symmetry x_2 is that π - x 1.

So, we these things get canceled and we get the form that 2 by $\pi^2 \sqrt{1 - y^2}$ and this is for the range of this 0 to 1, otherwise it is 0.So, this is the distribution function of this one, so if everything is, if this calculation is as this is also the PDF, that probability density function of another random variable. So, all the condition that we are defining should satisfy by this new PDF as well. So, it should be greater than equal to 0 for the entire range, and the integration over the support should be equals to unity. So, that you can, that we can check yourself that whether the function that you got, whether though that are satisfying are not.

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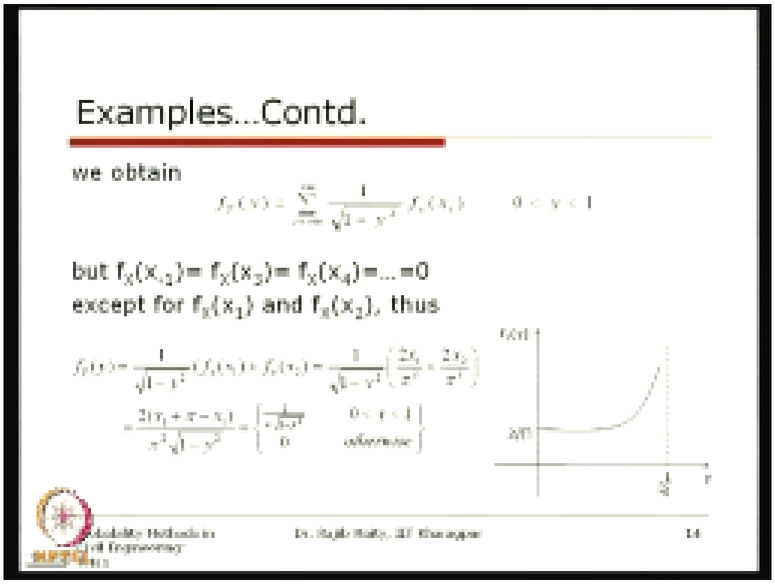

So, this is the presentation of this how this PDF looks like here.

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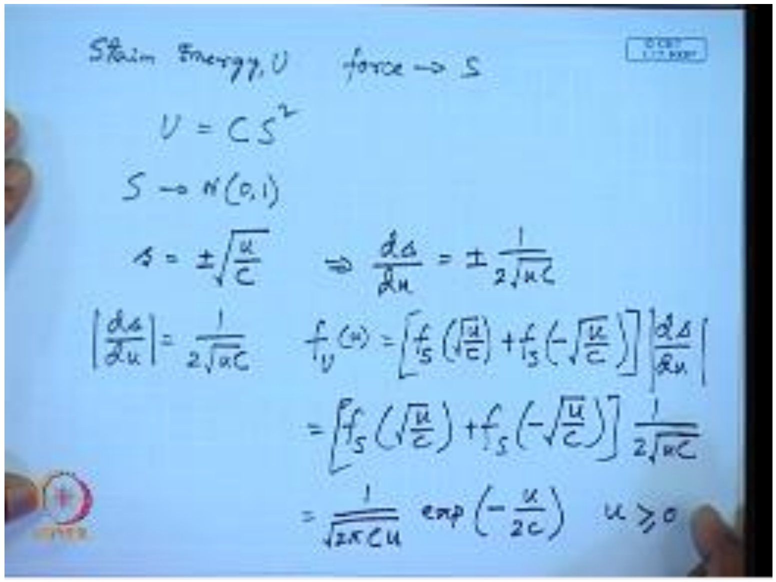

Now, we will take one example here, which is related to the stain energy and in the force, this is denoted as S and this is denoted as U, linearly elastic bar these two variables are related, that is stain energy is equals to this is proportional to the square of this force. Let us say that this is a constant and a square, now this constant basically is related to the property of that that bar, so its length, its cross-sectional area, its modulus of elasticity and all.

Now, if we know the probability distribution of this S whether we can calculate this U that, what we have seen just now, that we will see. Suppose that, this S is having a normal distribution with mean 0 and standardization1, that it is a standard normal variant.

So, then what is the distribution of this U, that we will see in this example. So, as we have seen that these two random variable, that is U and this S are related through this equation U equals to some constant multiplied by S^2. So, we can express that this S, that is now this two are random variable that is why it is capital. So, this s is now can be express that as the two roots, will have one positive and one negative root will be there and this is that u by that constant C. Now, so thus if we from here, we can see that this ds du is equals to, we can do this and we can get that 1 by 2 square root of u that constant C.

So, this is and if we take that modulus of this, that is the modulus of this derivative that is which is also known as the Jacobean is equals to 1/ 2 square root u_C. Now, to get the density function of that new variable which is U, which we can write like this, which should be what we know that at all the roots, we have to find out what is the distribution of the other variable, that is here is the S and add those roots only. So, the one root is that is a square root of u by that constant C and the other root is that your f_S, where it is the other one, that is the root is square root of U by this one. And this multiplied by this absolute value of this derivative ds du, if we just replace this one in this expression, what we will get that, so if we can just write1 / 2 √uc.

Now, we can put this one from this standard normal distribution that is 1 /√2π, and this is that value x and exponential power ¯ half of this square And again this value and after doing those steps in between, we will get that final thing will come like this1 /√2 π, that constant u exponential of ¯ u / 2 c and here, this u is greater than equal to 0, which is a support of this new distribution.

So, we have seen that this S, when it is a standard normal distribution, then the square of this one is giving you a new random variable which is basically a chi ^{2} type distributions with one degrees of freedom and it is support is u is greater than equal to 0.

So, what we have learned in the fundamental theory, we have seen from how to determine the density of this equation from one known random variable and it is, if there is some functional dependence. And we have seen that, what are the conditions should be satisfied to be a function of a random variable and once we know that functions, then how to determine the probability density from the fundamental theorem, we have seen. And in the subsequent lectures, we will see some special cases if there are someone to one relationship, then what will happen and there are some other methods called method of moments, method of which is also similar to the method of characteristic function, we will discuss in the next lecture. Thank you.

Probability Methods in Civil Engineering

End of Lecture 14

Next: "Functions of Random Variables-Different Methods" In Lec 15

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