Probability Methods in Civil Engineering

Prof.Dr. RajibMaity
Department of Civil Engineering
Indian Institute of Technology, Kharagpur

Lecture No. # 29

Introduction to Copulas

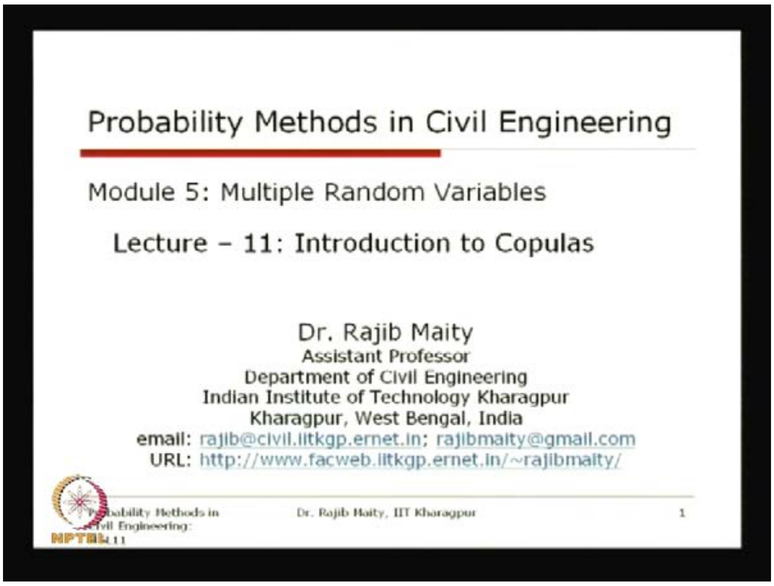
Hello and welcome to thislecture.We are in this multiple random variable module, andthis is our eleventh lecture, and as we are mentioningforsometime in the some previous lecture thatwe will be give some brief introduction on copula and this copulais useful when we are looking for thejoint distribution of multiple random variables.

So, today's lecture,we will be considering abrief introduction on these copulas. This copula theory is relatively newer to thisstatisticaltheory andjust we will go through thatbrief introduction and somespecial class.We will justdiscuss which is known asthe Archimedeancopula,andthroughout this lecture,we will try to communicate that how this theory or how this,this, function, this copulas orsome functions, how this functions can be useful to derive the joint distribution oftwo random variable when their marginal's are known.

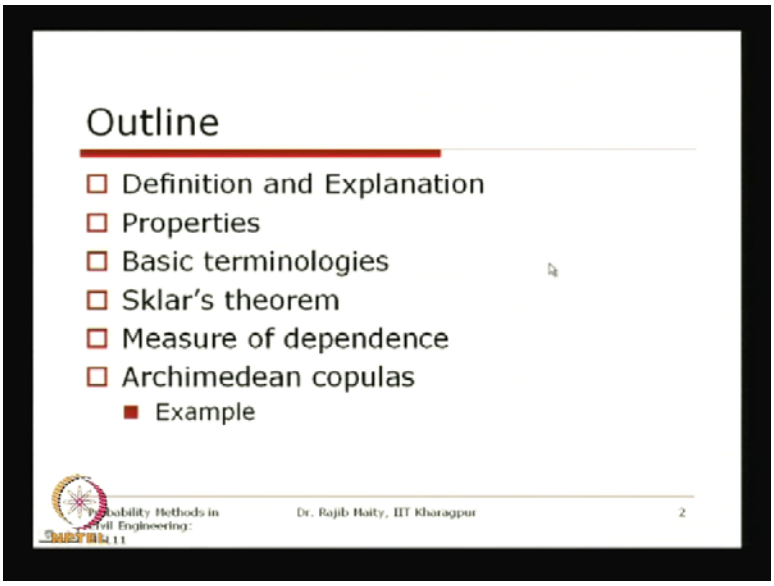
So, you know that we have discussed earlier thateven though the marginal's are known,it may not be that easy to get the join distribution.Only onemeaningful conclusion that we haveseen earlier is that if two random variables arejointly Gaussian or they arejoint normal distribution they follow. Then we can conclude that their marginal's also will be,marginal's also will be normal distributionforor each of the random variables involved, but the reverse is alsonot true,that is,when their marginal's are normal, we cannot conclude that their joint will bethatjoint normal distribution.

So, here we will be introducing some functions, some copula functions, and so, using those functions, we can derive thattheir joint distribution. This function follows someproperties that has to be followed. So, those properties we will be discussing, and after that, we will betaking some of the specific example of thesefunctions and which can be useful andwhat hasrecently found its application to the civil engineering problems.

(Refer Slide Time: 02:33)



(Refer Slide Time: 02:37)



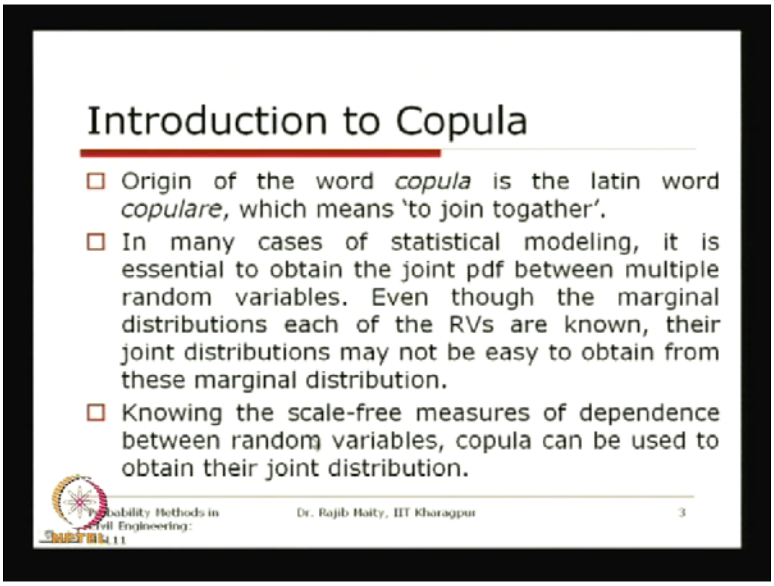
So, our today's lecture is on this introduction to copula andthe outline that we have,that we will follow is that, first,we will go through some basic introduction, and after that,we will see its definition, its meaning and its explanation, how,what is thiscopula function and we will see their properties,the certain properties that it should follow.

First of all to be a valid a copulaand there are some basic terminologies as this is as I told that, this is a newerarea. So, some of the basic terminologies we have to discuss, andwhile discussing this basic terminologies,we will try torelate it to their one-dimensional counter part becausemost of the theories we have discussed for the one-dimensional cases, and so, that as we have discussed it to the one-dimensional cases. Then,now basicallythis introduction that we are talking about like this similar to the earlier lectures.We will first discuss about interms of the two-dimensionalcases, that is, two random variables are involved, and even though,later on we will just show that how this things can be extended to theeven more than tworandom variables involved.

So, so that while discussing the basic terminologies we will just try to relate that what is the counterpart on the,on the, single randomvariable, that is, the properties of the single randomvariabletheir p d f andc d f's.So, that we will see, and after that, there is very,one of the very important theorem which is known as the Sklar'stheorem, and basically this is thebackbone of this full theory from where I can use this function to,to, obtain their,their, jointdistribution of the random variable involved, and that is generally done through some of the measure of dependence and that dependence is generally the scale free measure ofdependence.

So, we will discussalsoon the,on the, issues of this measure of dependence how we can measure thedependence from the population as well as from the sample estimate, that is, from the sample of this data and thenwe will discussspecifically about some on the Archimedean copulas. There are different classes of copulas.For example, the electrical copula, placket copula, but here, sofar themany applicationshas been found in this different problemsof the civil engineering is through the Archimedean copula. So, we will,we will specifically see that what does this Archimedean copula means andsome examples of this Archimedean copula also we will see towards the end of thislecture.

(Refer Slide Time: 05:17)



Well, so, this to start with this word copula, this might be, even the word might be a little bit newerthat,when we are,this copula word is basically aobtained from the Latin word which is known as copulare,cc o p u l a r e. So, this meaning of this word, this is basically a verb and this meaning of this verb is that to join together.So, here, as you know that we arejust joining two marginal distribution oftwo random variables to obtain theirjointprobability density functions or jointc d f.

So, basically we are taking two marginal's and we are joining themto obtain theirjoint distribution. So, that is the name has been taken from that Latin word and this is the background of this word copula.Inmanycases if we see of thisstatistical modeling,then we see thatit is essential to obtain the joint p d fbetweenmultiple random variables else,even though sometimes in the marginal distributions ofeach of the random variable are known.Their joint distribution may not be easy to obtain from these marginal distributions. So, this we have discussed earlier also that we know that what is the, what is the marginaldistribution, but for until and unless, there are very specific cases. We cannotconclude that what should be their joint behavior.

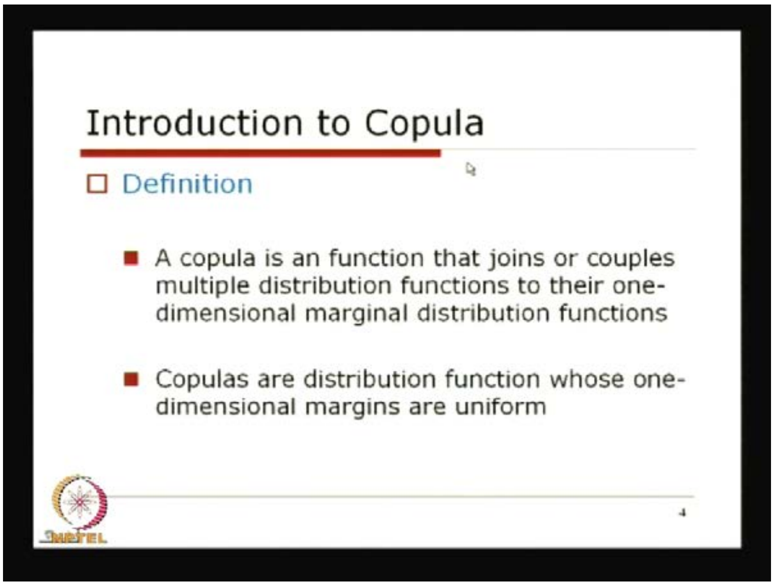
Say for example, if we just say that one distribution is say your one distribution is say that gamma distribution or otherwise the normal distribution, then what should be their joint. So, so far I think if I am not wrong, thenwithout the help of this copula that kind of joint distribution cannot be obtained. So, that while,while, getting their joint distribution also, we have to followsome of the,some of the, measure of their dependence andone of the,the, scale free measure of dependence. The,we will just discuss about what is this scale free meansthat, thatdependence between the random variable. That is one of theimportant parameter that we have to consider before we can go through thatcopula.

Basically, sofar as,now I take some time to discuss on this dependence. Basically, the whatever the measure of dependence in this,in the, in the statistical theory is there that can be broadly divided into two groups - one is thatwhetherthe dependencerather than measure of that dependence is can,can, change if I change arbitrarily the original data to someto some arbitrary scale,scale factor.

So, for example, that we are talking about the dependence between x and y, now, for one of this random variable either x or y, if I just make some changesto some,some, scale changes,say that I make y I i replace y as y square. So, now, the question is whether their dependence will change that the measure of the dependence that we are considering that whether that will change or that will not change.

Depending on these twosituations, this,this, measure of dependence can beclassified into two groups– so, one group is that it is, it is, irrespective ofwhatever the, whatever the, changes that we have made, the scale changes that we have made and other group may changeif we do this type ofchanges. So, the,the, scale free measure of dependence means what we are talking? Ifwe are going for that type of changes, that type,that type of a scale change,then this type of a measure of dependence will not change.So, two suchmeasure of dependence that we will discuss in this lecture is, one is thatthatSpearman’s row and other one is that Kendal’s tau that we will comeafter sometime.Before that, we will seethe definition of,of,this copula.

(Refer Slide Time: 09:32)

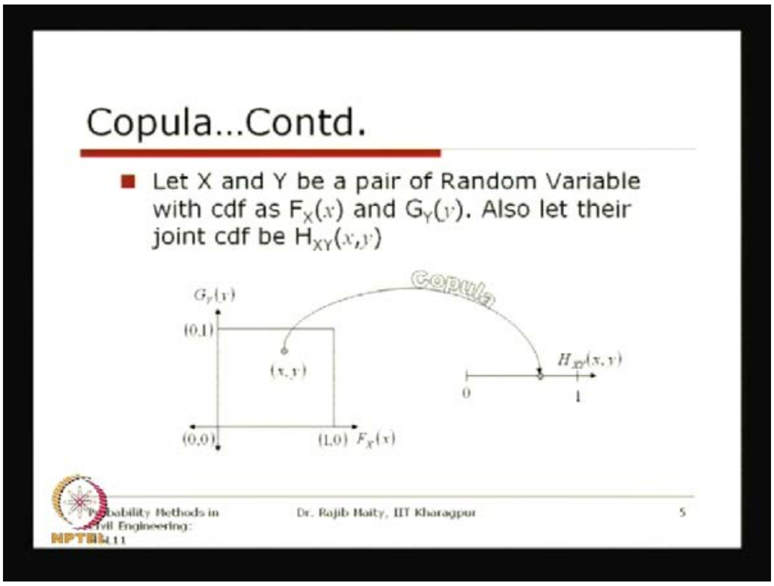


So, the definition of copula - it says that this copula is a function that joins or couples multiple distribution function to their one-dimensional marginal distribution functions. So, so, one-dimensional marginal distribution for each of this random variable involved is available. So, these has to be joined, this has to be coupled together to obtain their joint distribution and this is done through a function and that function isindeed aa. So, that, that, correspondence is,is, 1 function and that function is nothing but what we are talking about is thatcopula. So, and the second is that copula are the distribution functions whose one-dimensional margins or the margins means the marginal distributions are uniform.

So, now, this one-dimensional marginal densities are uniform means when we are taking somerandom variable and ifwe know there,know there,c d f and,and,if so, from the c d f if we go for the reduced variate, you know that c d fchanges fromhas a range from 0 to 1.

So, now, once weknow what is their distribution, what is their cumulative distribution function, and if that cumulative distribution functions properly represents that data, then if we convert that original data set to their through theircumulativedistribution function, then that will converted to the range between 0 and1 and that will become,become, one uniform distribution and that thing can be joined together to get there,to getto obtain that copula that, that, function which is known as copula. So, that is why the copulas are the distribution function whose one-dimensional margins are uniform.

(Refer Slide Time: 11:22)



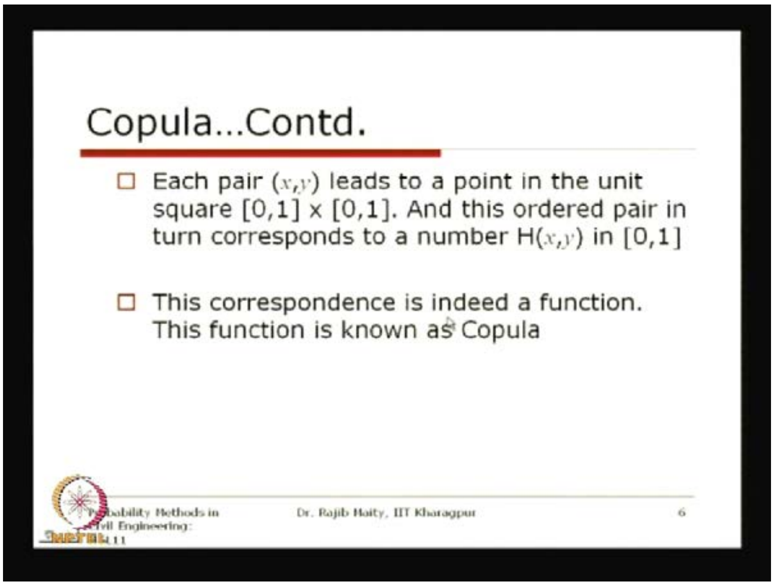
Now, if we see itmore intuitively,say here the examples has been taken for the,the, two random variables only as I was justtelling that,suppose that there are,there are two random variables x and y and as I was telling there their cumulative distribution function that the,for x, the cumulative distribution function is f x of x andthiscumulative distribution function of y is g y y. Now, this f x x, so, this,this, cumulative function can change can take the values from 0 to1.

Similarly, that y also can take the value of this 0 to, to, 1. Now, when we are talking about their joint distribution, soany points on this units square, that is,from00,10,this is 01and this is 1 1. So, within these units square, any point basically represents a pair between the x and y,pair of this x and y.

Now, suppose that this h x yrepresented through this lower case of this letter x y. So, if this is their joint distribution and joint cumulative distribution, then a following the properties of this cumulative distribution, this will also take the values between 0 and 1. So, so, the same,same, combination of this x and y will correspond to one point on thisunits squareover the plane constituted by their,by their,cumulative density function, cumulative distribution function f x x and g y y.

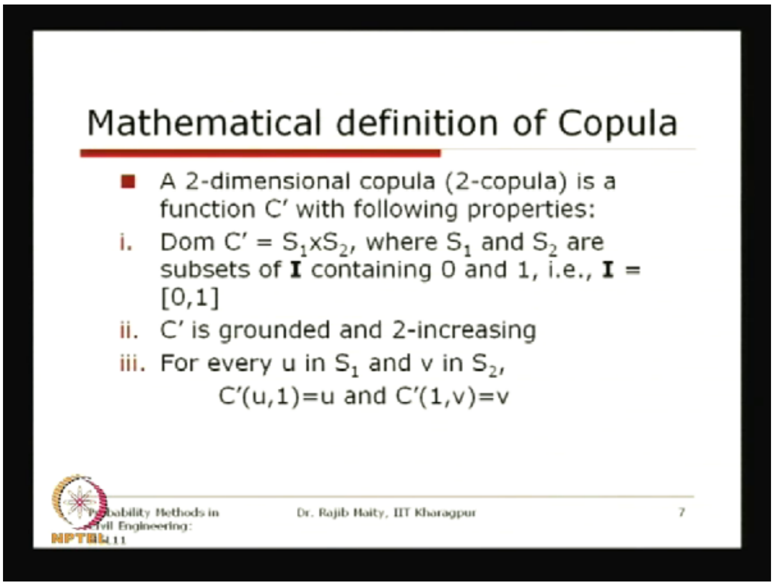
So, this is that point that we are talking about. Again, the same point with the same pairthat x and y, the same pair will correspond to another point over this line from 0 to1 which is their,their, joint distribution that is h x and y. So, so, so thesetwopoints - one is on these units square and other one is on thisline between 0 to 1 basicallyrepresenting thesame pair of this x andx and y. Now, these two points are basically,basically,these two pair basically they corresponds to each otherthrough a function. So, these,these, correspondence the,the, function that makes the correspondence between these two function is what is known as the copula function.

(Refer Slide Time: 14:14)



So, that iswhy that each pair x y leads to a point in the units square 01 by 01as we have shown that on this, on the plainmade by thisf x x andg y y. Here,that we have, we have mentioned. So, this points, so, each pair leads to that point,leads to a particular point on this units square and this ordered pair in turn also corresponds to a, to a, number h x y in that 0 one real line. So, so, this is through that through their joint distribution. Now, this, both these things, both this points as correspondence, sothese correspondence between these twopoints soindeed a function andthis,this, function is,is, nothing but which is known ascopula.

(Refer Slide Time: 15:13)



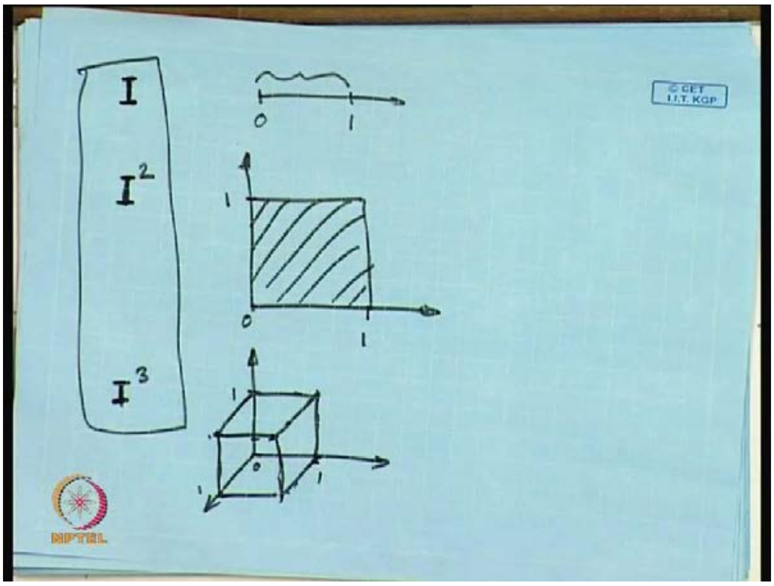
Now,if wego to this mathematical definition of this copula,thenthis,first of all,we will discuss about this,this, two-dimensional cases as we have started. So, these mathematical definitionsalso here we are discussing in terms of this two-dimensional cases first.

So, so, this two-dimensional copula which is alsoshortly written as this two copula is a functionc primewith the following properties. So, this copulas should follow some property and those, there are three properties - the first one is this domain of this c prime. So, domain means hereyou know thatwhen we are talking about a single random variable that time,we have refer to the range of this random variable or the support of the random variable. Here, as it is the minimum is the two-dimensional cases and it can go up to means any dimensional cases.

So, for that function, we are defining is as a, for its entire range, we are just talking about is domain. So, on what domain thatwe have to specify some domain over which that function should be defined. So, the domain of that,domain of that function c prime is s 1 by s 2s 1 cross s 2- where this s 1 ands 2 are the subset of I.This bold phase I containing that 01, that is, I is, I is the line between 0 1.

This is basically what is meant through this mathematical notation is that. So, these domain of this c for a two copula is the subset ofthe units square which is made by these 01 in one direction and other direction also it is 01. So, this I is generally a this bold phase I will be, because we will be using this notation again. So, this bold phase I is generallyretain and with some power and that power is basically indicatingwhat is its dimension.

(Refer Slide Time: 17:30)



So, here if you see, that is, ifwe write that this bold phaseletter I,what isreferred to is that is one line up to 0 to 1. So, these range we are referring to this one. So, now, if that bold phased I we,we, just put that square, that is, in the two-dimensional case, then what we are referring to is that in a two-dimensional space from 0 to 1 and fromagain other direction 0to1. So, we are referring to a square area over this two dimensional case.

Now, if we just, if weput that this bold phase I as cube, then you can see that this what we, what is referred is that is one three-dimensional space where each axis so0 to 1,0 to here 1 and 0 to here 1. So, you can just see that this will be a cube like this. So, this cube is that I see that we are referring to. So, this is,these are the notation what we mean by this by these symbols.

So, here, when we are talking about this domain of this c is that, this is a subset of this,of this, I containing that 01,containing that 01. In that, in this case, that is, I it is that1 to 0 to 1. So, this is that domain of this function, function and means that function means that copula.

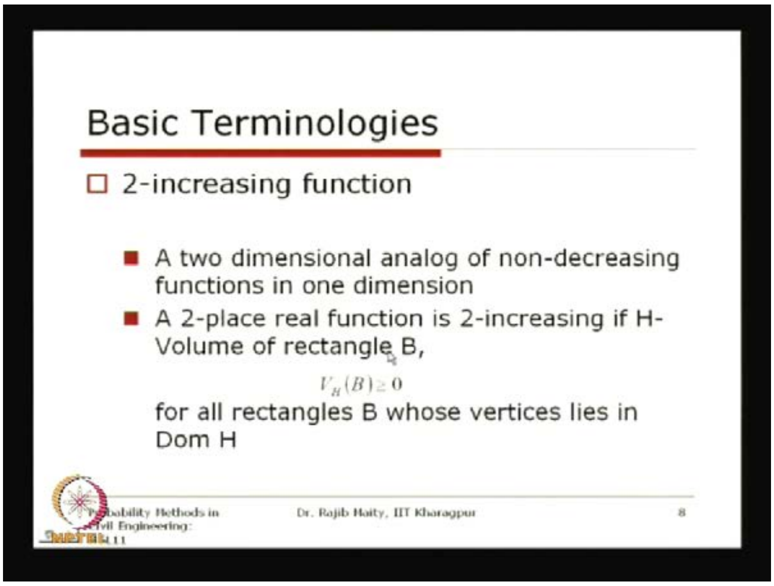
Now, the c prime is that function is grounded and two increasing. So, this grounded and two increasing when we are referring to, basically we are giving some,these two properties that it should follow and it is having some correspondence to thus to the one-dimensional casesalso. That we will take both these properties separatelyto discuss aboutthis, what does this,this, terminologies basically means that, what is this grounded, what is this to increasing. That we will see in a minute.

And the third property that it should follow is that, for every u in s 1 and v in s 2. Now, you can see that there are twothat,values thatweare referring to for the two-dimensional cases - one is in the direction of this s 1which is also between that 0 and 1, and other one is in the direction of thes 2 which is between 0 and 1.

So, this c prime u 1 is equals to u; that means, when the othervariable reached to its maximum point, that is, its 1, then the values of this functions should be equal to the value of theother attribute. Similarly, when u reaches to the maximum value, that meansit is 1, then the value of this function should be the value of the other, other, attribute, that is, v. So, these are the properties. So, this is the maximum.

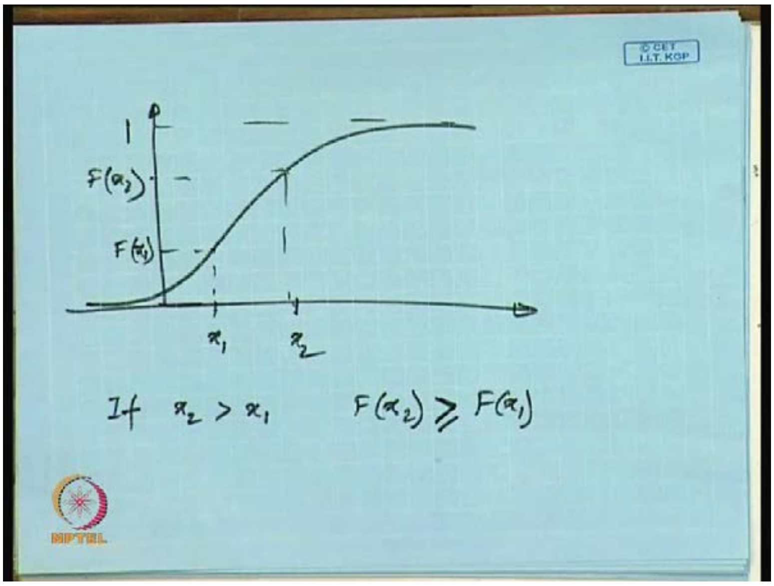
Now, you can see that this the last one what we are talking about the maximum limit that we have, we have, just defined, that is, if one of the attribute is reaches to the maximum, what happens to this function? Now,if one of the function is the minimum one, then what is the value of that function. That is basically is,is, shown in this word that is grounded and another property is that two increasing. So, we will take this grounded a bit later.

(Refer Slide Time: 21:18)



First of all,we will see that what does it mean that two increasing. So, so this is the first that basic terminologies that we are taking if the first one is this 2-increasing,2-increasing function. So, what does this 2-increasing,2- increasing function means? So, a two-dimensional analog of the non decreasing functions is a one dimension. So, this is the first point that we,that we are talking about is thatis,is, that how it is corresponds?What is itscorresponding counterpart in the one-dimensional cases?

(Refer Slide Time: 21:58)



Now, if you see here that for a,for a, one-dimensionalfunction, we have seen for its c d f that it is its starts and its reach up to 1. It can be asymptoticor it may not be asymptotic.It can touch itsentire support. It,it, should from the left extreme of the support, it should start from 0, and to the right extreme of the support, it should reach to 1. So, that isthe range ofthis cc d f.

But the other thing that we have, we have already seen that this function is,is, non decreasing function. So, non decreasing means it should, it should, that if that, if this is your say x 1 and this is your say x 2 and we can say that the x 2 is greater than x 1, then these, then these values that is say f of x 1 and this is x 2 f x 2. Then what we see that,if x 2 is greater than equals is,is, greater than x 1, then that f of x 2 is greater than f of x 1; so, that meansit is an increasing function; it is an strictly increasingfunction, but we should include one equality sign here also just to,to, state that this is a non decreasing function.

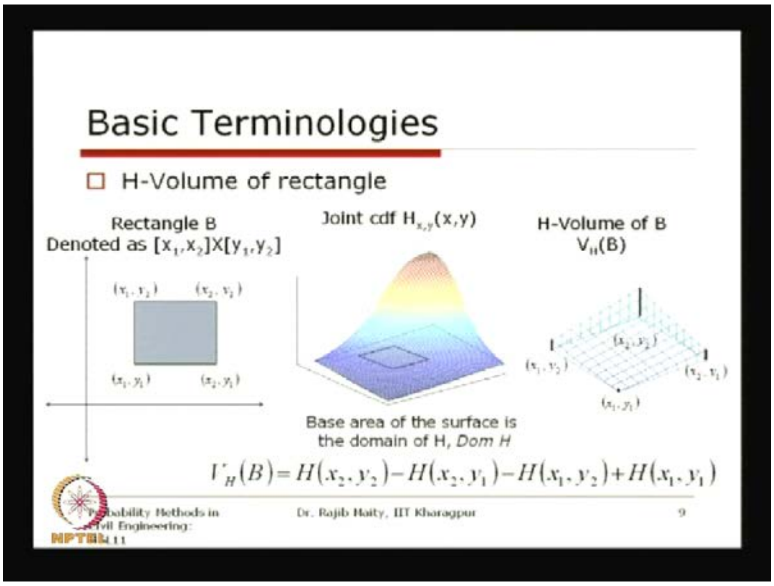
So, it does not mean that whether between these two point, it will increase or not; it can even remain flat for this reason and it can again go, but it will never comecomedown. So, this will be either greater than or it should be, at least it should be equal between these two.The values of these functions atpoint x 2and at point x 1. So, so this is what we have seen in this, in this, one-dimensional cases.

Now, if I want to extend the same property to thetwo-dimensional cases, then what is known as is that,is the, is known as the two increasing functions. So, what does this? So, this is a,this is the correspondence between the one-dimensional cases what we have we have seen earlier, andthe corresponding thing the same, same corresponding property in the two dimensional cases is known as that properties it is 2-increasing,2-increasing functions.

Now,the a two place real function is 2-increasing if h volume of the rectangle b is greater than equal to 0. So, again, here we have got two more new terminologies - one is that h volume of the rectangle b. So, first of all you have to know the, which rectangle we are talking about, and if we know what is the rectangle, what does this h volume means?

So, that we will see and this is for this rectangles b and this should be valid for all rectangles b whose vertices lies in the domain of h. So, we have to know that. So, domain we have seen that it should bethe domain of the c that we know that it iswithin this 01 and 01 is bound, and we have to know thatin this domain if we can identify one rectangle and what is that h volume means and that hvolumes should be greater than equal to 0 tosatisfy the condition that the function is 2-increasing.

(Refer Slide Time: 25:48)



Now, let us discuss about this what is this h volume means. Now, let us see this a, this area. First, that is theone rectangle, that is, whose vertices are thex 1, y 1;this is x 2, y 1; this is x 1,y 2 and this is x 2,y 2, and if we say that this,this, rectangle is in the domain between these 0 and 1, that is the domain of that function, and if from this rectangle also you can say that or if we just specifically mention that x 2 is greater than x 1 andy 2 is greater than y 1. Then this isthis rectangle what we can say in this, in this, form. Now, sothis is the plain by thattwo random variable. Suppose that this side it is x and this side it is y. Now, the same rectangle is again representedhere on that plane. So, this side is your x and this side is your y and this is that rectangle that you can see here.

Now, to see that, what is the hvolume we,we, mean that, for that one,we have to first see that how that how that joint c d f between that x y looks like.So, this is as you know that this should start from 0 and it will go on and reach to a maximum valueto thatreach to a maximum value of one. So, so, this is your that h x y axis and this should be your here 0 and this one, this line it should be written as 1 which is missing in thisfigure, but you can, you can,easily make out that this is the starting point is your 0 and this is going up to a maximum value of this 1. So, this surface that you can see. Suppose that, that is, 1 thatis the joint c d ffunction in a two-dimensional case. So, two-dimensional function that meansthis is a surface.

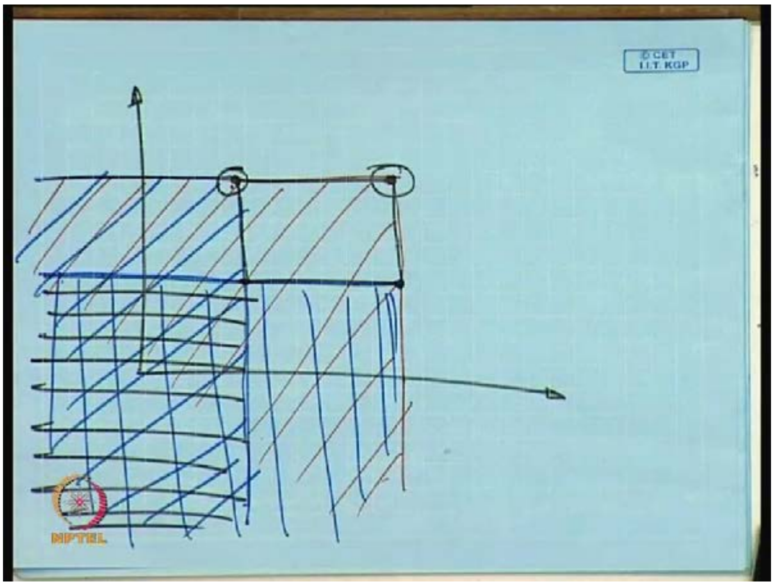
Now, that h volume when we are talking about, soit is just like that one surface is giventhereand you are cutting that surface through this,through thelines of thatrectangle which is shown on the base.Base means on this surface considered by x and y. So, now, if you cut this,this, surface is just like a slicing the cake along this lines. So, what you will get is a volume like this. Now, if you get the volume like this, then what you aregetting? So, you will get fourordinateat this four vertices of the, of the, rectangle shown on this,shown on this ground means, ground means here on the surface.

Now, that value that we are talkingabout is your this h x 2y 2 is nothing but is the what is the ordinate at the pointx 2y 2; that means, at this, at this point, what is the, what is the value of that joint c d f function which is shown by a black vertical line here. So, now, that h volume of this rectangle is this one, this height minus this height minus this 1 plus this,this, height atx 1y 1. So, if we just go back to this original rectangle, then the value of the function at this point minus value of the function at this point minus value of the function at this point plus value of the function at this point. So, this is giving you the h volume of that functionh volume over the rectangle bwhich, for which the vertices are as shown in thisfigure that is x 1y 1,x 2y 1,x 1y 2 and x 2y 2.

Now, if just, if it is,it is not that difficult to even remember that this is what we are taking is that,first we are starting from thisextreme right top corner minus other twodiagonals plus this one, but if you say that why this one it is also, it can also be easilyreferred in terms of the c d f.Basically, what does it mean? The, what is this ordinate here at this x 2y 2 means. If we see that, this is the joint c d f, that is, cumulative distribution function of this one. Then what,what, this ordinate means is nothing but what is the total volume covered by the p d f - probability density function - up to this point.

Now, instead ofconsidering this c d f over this plain, if you just consider what is the c, what is the p d f, there is the probability densityfunction over this one. So, what you will get is that, that, sothatit can be you can see that as a, as a, surface again over this zone, over this area means depending on the support of this two random variable x and y and what is the total volume that is a covered up to this point which is represented by this ordinate when you are talking this,when we are taking this surface as its c d f.

(Refer Slide Time: 31:55)



Now, if I just represent that samerectangle here and if we want tosee that, if, for this rectangle if we want to know, what is the volume over this rectangle? When I am considering a,considering the, their joint p d f over this zone, then what will happen? For this one, for thispoint to get that, what is the volume above this rectangle? Then what I have to do that, I have to first take what is the totalvolume support that,suppose that this is your, the left extreme of the random variable that we are talking about. Then or,or, I can even just say that why to take that. It can be say that minus infinity. Then what I am talking that, I am first taking the full area up to this point. What is the total volume up to this, up to this point? This should be…

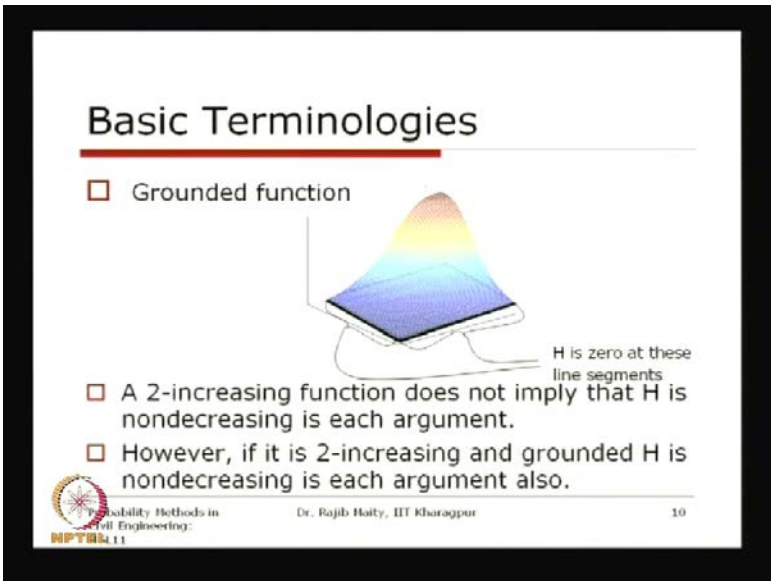
Now, to get only this rectangular area, then I have to deduct this area first. Let us use other colour. So, this total areaminus this area minus this area; that meansnow you see that in,in, in, this process,this area has been deducted twice. So, this area should now be added which is nothing but the total volume up to this point. So, that is why the total volume under the p d f, remember, total volume under the p d f - probability density function - up to this point minus total volume under the p d f up to this point.

So, this area I have first deducted; then this area is again deducted. Then, I have to add this area because this area has been deducted twice while deducting this point and this point, and this is why this, when we are talking about this age volume; that means, that time we are taking that this area minus. So, this,this, ordinate, this ordinate is nothing but the total volume under the p d f that just now what we discussed, so, this one minus this ordinate minus this ordinate plus this 1.

So, this gives you the h volume over this rectangle. Now,now, if I want to say that this function is always increasing in both the direction in, in, the directionin the, when this x 1,when, what I say that that x 2 is greater than x 1 and y 2 is greater than y 1. Then this volume should be the volume below that p d f should be greater than equal to 0. So, that iswhy sothis volume should be greater than equal to 0, which is represented as,as, v h b and that is why the requirementhere is that v h b should be greater than equal to0. So, this is what actually ais the meaning of this 2-increasing function.

But remember one thing that only this condition, that is, h,v h b, that is, h volume of this b if it is 2-increasing, that does not guarantee that each every direction, that is, x and y both the direction that function also will be a non decreasing that is not assured. So, to assured, but we know that the marginal's that the marginal distribution also is a non decreasing function. To assure that, that one what is the othercondition is there that also should be satisfied. Then that other condition is that it should be grounded.

(Refer Slide Time: 35:53)



So, what does that grounded function means? So, that is here;that is shownhere, that is, this grounded function here means that if two increasing functionsdoes not imply that h is non decreasing in each argument; however, if it is 2-increasing and grounded, the h is non decreasing in each argument also. So, these are the afterwe discuss this grounded;

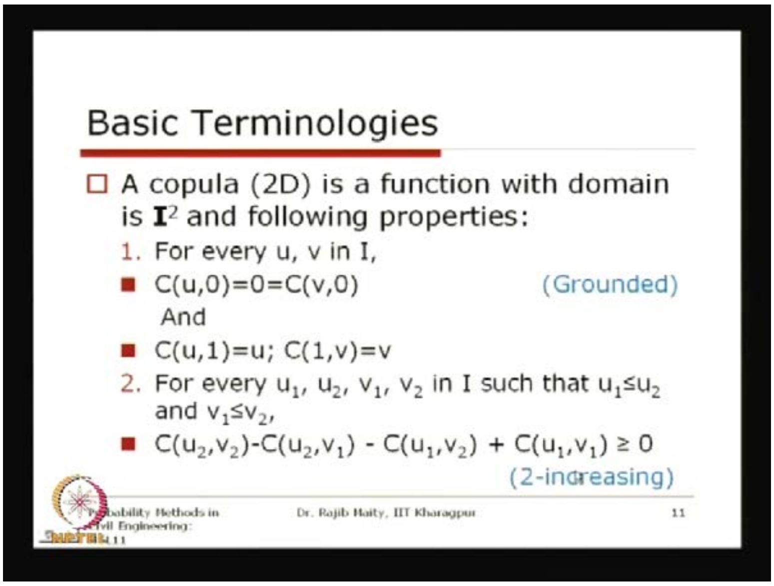
these are the,the,the, conclusion, but before that, whatis this grounded function means. So, you see here this, this, line that we are talking about andthis line is h the value of that function h is 0 at these lines segments.

So, what does this line segment means, that is, when one of the argument is 0, either u or v means which are the reduced variatefor the x and y. If either one of this u and v are 0, then the value of the function is 0 and that is what is known as the grounded function,that is, at both these line segment, the function is starting from the ground. So, now, if I go back to thatone of this, this,this, property, what we discuss here that, if one of the values are reaching to its maximum value, then value of the function is equals to the value of the othersattributes, that is, c prime u 1 is equals to u and c prime 1 v is equals to v.

Now, the,by the word grounded what we mean is that, if one of the argument is 0, then the value of that function equals to, will be equals to 0, that is, c prime 0 one is equals to 0 and c prime0 v is equals to 0. So, irrespective of what is the value of theother one; that means, here I should say that c prime u 0 equals to 0 and c prime 0 v equals tov,sorry, equals to0. So, this is what is meant by this grounded function which is represented through this diagram here over this line segment. This, the value of this functions should be equals to 0.

Now, as I we are just telling that it is simply if we just say that a function is 2-increasing, that does not imply that the in both the argument, both the argument means in both therandom, in the direction of the random variable is does not guarantee that whether they will also be non decreasing or not, but if we say both the properties that it is 2-increasing as well as it is grounded, thenwe can say that it is non decreasing at each argument also. So, which is obvious because their marginal's are also one c d f which is non decreasing. So, thattheir joint function that is the property of the,property of the, copulafunction should be 2-increasing as well as it should be grounded.

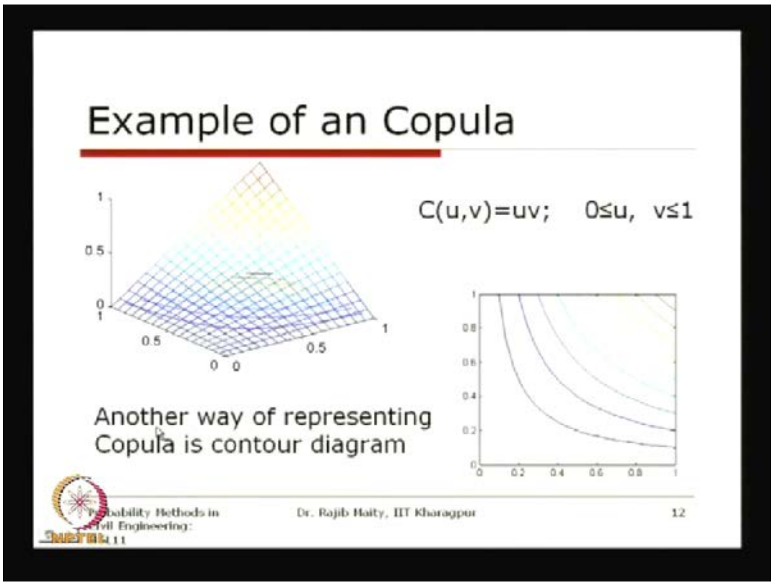
(Refer Slide Time: 39:07)



So, acopula, and here, it is the two-dimensional case is a function with the domainis I square here. Now, I square just a few minutes before I have discussed. So, it is a two-dimensional plainbounded by01 and 01, that is, vertices are00,01,10 and 11. So, it should have the following properties - one is that this is grounded as I told; it is c u 0 equals to 0 and c v 0 equals to 0, and other one is that, when it is reaching one of the argument is reaching to its maximum value, that is, c, u 1 equals to u and c 1 v equals tov.

For every u 1 u 2 v 1 v 2 in I such that u 1 is less than equals to u 2and v 1 is less than equals to v 2, then the copula function at u 2 v 2 minus the value at u 2 v 1 minus the value at u 1 v 2 plus the value atu 1 v 1 should be greater than equal to0, which is nothing but it is ensuring that the copula function is 2-increasing and this one this whole statement is nothing but the h volume.

(Refer Slide Time: 40:34)



So, one example of this copula which is,you knowthatwe use it also earlier, that is, thatindependent.If the two random variables are independent, then we know that their joint distribution is nothing but the product of their individual marginal distribution. So, say that if, so, ahere that so, the one sothat particular copula function will be is equals to your c u v equals to u multiplied by v. So, when we can say that their joint distribution is their product of their individual marginal distribution,then we know that this is 1their,their, joint distribution is the product of theirindividual,individual, marginal distribution.

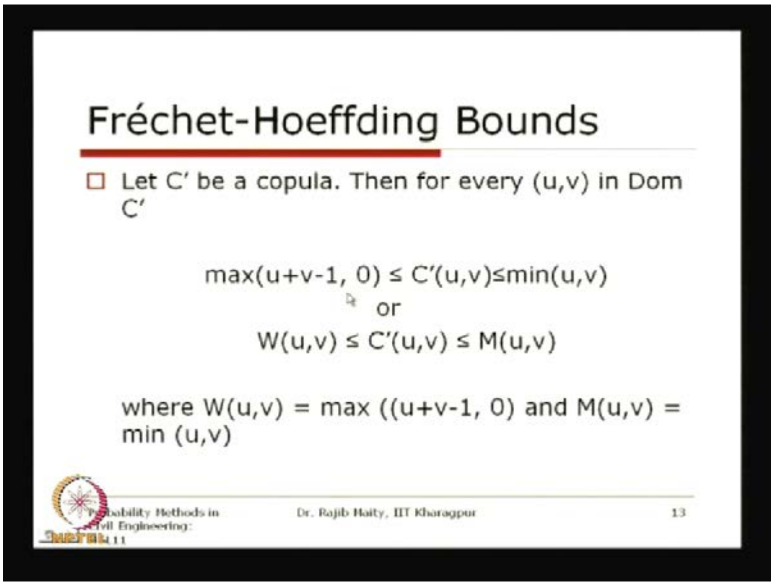
Now, this is also then, this is also one, also one, one, copula and which is known as the independent copula. So, this function is known as the independentcopula where this u range of this u and v both are varying from 0 to 1. So, and if we represent this one how this copula function looks like, it will look like this; it will start. You see that both the functions are,are,it is grounded. Both the functions are starting from thisline segment.Over this line segment, the value is 0 which you can also cross check from here that just,if we put that one value is equals to0, then the value of the function is becomes, value of the function becomes 0.

Then, if one of the function reaches to its maximum value, that is, if u equals to 1 and v is in any value, then the value of the function isv and vice versa. So, here, that is reflected through this line which is a,which is astraight line,45 degree straight line through this plain. This is thatdirection of this copula function; this is one of this random variables. So, this line is a 45 degree line over this plain. Similarly, this line is also a 45 degree line over this plain which is the other random variable verses thatfunction of this copula.

So, at 0.5 and this 0.5 you can see that ifoneis reaching one and other one is 0.5, this value of the function is 0.5. This just for one example and this is true for entire this straight line and which is also can be cross check from,from, here. So, if we just put one value equals to one, the value of the function is the equal to thatthatvalue of the other attribute.

So, this is also a surface. So, this is a copula function which is satisfying the,it is required property and that function looks likethissurface that you can see here and also the another way ofrepresenting the copula is the contour diagram and the contour diagram is shown like this where you can see thatthisdifferent contours are shown to with this different color bar. Here, the color bar is not mentioned here, but you see that this blue one is 0 and this red one is 1. So, in this way, it is gradually increasing; obviously, the copula this values of this contour should be mentioned here,here,to properly represent what is its,what is height above the ground of this surface.

(Refer Slide Time: 44:12)



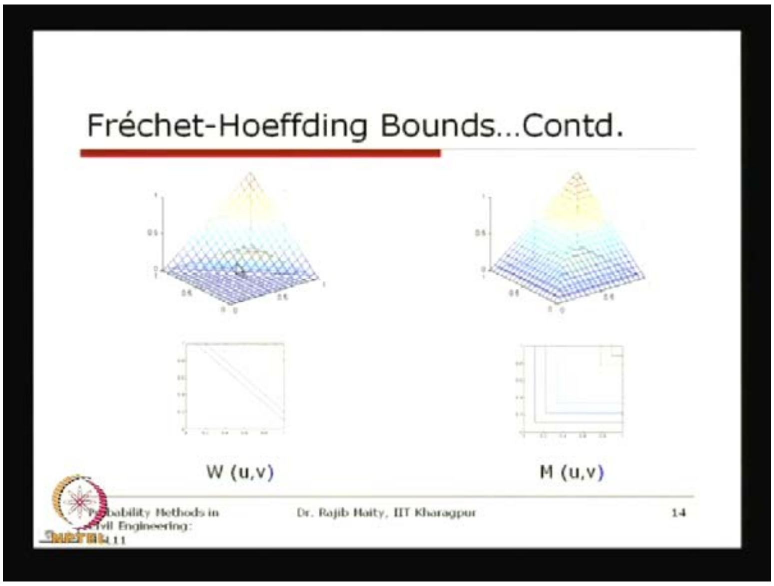
Now, there areone more property,property, related to the bound of this each copula that we shoulddiscuss now. So, this isknown as thatFrechetHoeffding bound - f h bound -

abbreviated form thatthis f h bound it says that the any copula function should be bounded by,by two,two functions and those functions are thatthe minimumthe lower bound is that maximum of u plus v minus 1 comma 0. So, i, so, in this 2, u plus v minus 1 or 0 whichever is maximum. So, this is the lower bound of that function and minimum ofu v this is the value of the,the, upperbound.

So, this is generally represented by thisw u v and this maximum one is represented bythis, the upperlimit. Upper limit is the minimum of u and v; the upper limit is represented by this m, u and v. So, why this is w and why this is m that is no,these are just the notation, but you canjust for the sake that this is the lower bound. That is the lower side the, the, way the word w is written that the lower side is justbounded and the way the m is written, it is upper side is bounded. This is just for the notational sake only. So, this is that your lower bound and this is your upper bound of the copula function.

So, that this lower bound is the maximum of thisu plus v minus 1 and 0and the upper bound is the minimum of u and v. So, how does this, how does this bounds lookslike? That we will see now. So, first we will take this lower bound here.

(Refer Slide Time: 46:12)



So, these are the two surfaces that we havedrawn, is that this is the lower bound and this is the upper bound and these are their corresponding contourdiagram.So, now, you see that whenever we are talking about this lower bound on this floor of this 01 and 01 on this floor if I just draw the diagonal here, then up to that diagonal, the surface is on the ground, and from there, from this diagonal of this base, we are just connecting to this upper side other side corner. So, this is a, this is a,this is a, straight surface.

So, this is the bound the, this surface and this one is thelower bound. This we can, we can also cross check, that is, when we are just talking about say that u equals to 0.5 and v equals to 0.5, that time the value of the surface should be at 0. So, if we just see one or two such values, then u equals to 0.5, v equals to point five which is equals to 1 minus 1 is 0. So, maximum of 0 and 0, sowhich is0. So, in this way, if we take whatever the combination that we take which is coming to this,to the, to the, this side of this diagonal of any combination of this u and v, then we will see that this value will become 0.

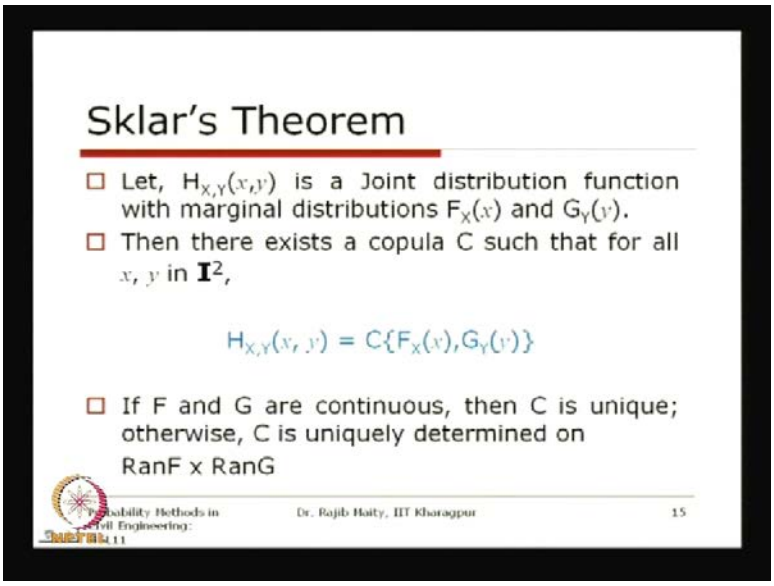
So, which indicates that this surface is basically on the ground up to this diagonal, and after that, it is going and touching this values where this side you can see that this is alsomeans, when one is reaching to this maximum one, the value itself is thatvalue of the otherattribute. So, that, suppose that just put one of this value as 1. So, the v equals to one. So, this 11 is cancel. So, the maximum of of u,0 and you know that u is theu is always greater than equal to 0. So, maximum value is u. So, so that is, sorry. So, that is what is reflected here thatwhen oneattribute is 1, the value of the surface is at the value of the other one, other attribute.

So, this is the lower bound, and similarly, the upper bound also we can see this is the minimum of u and v. So, just if we take any, any value, say that one is say that point five and other oneis say 0. So, this is the minimum of this 2; that means, at this point, the value will be1, and if I take that 0.5 and 0.5 both, so,it is the minimum of both the values. So, it will go and thisheightthere at this point,0.5 0.5 also will be0, which will be more clear through this contour diagram. You can take anycombination of these two and their point will be the, will be the minimum ofthe combination.

So, on this line, the minimum value is always 0.1. So, that is why what of irrespective of the value of this axis, this value is always 0.1.Similarly, in this way, we can just say this is one straight this is just like a half of a pyramid that kind of surface you can see that this is 1 surface of the pyramid and this is another surface of the pyramid. So, that is the way this upper bound isdefined.

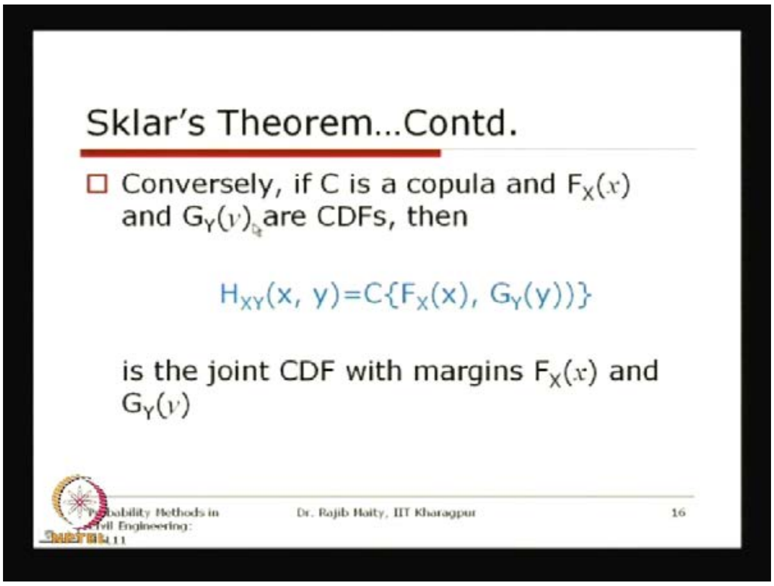
So, what is important here for the copula function is that. So, now, you can see that this is what we are talking about is a lower bound and this is what we are talking about the upper bound. So, any surfacewhich is representing a copula function should lie in between these twosurface. So, basically I can place this surface over this and you can imaginethat there will be a,there will be a, empty space between this two surface and all the copula function should lie in that space and that is what this upperbound and the lower bound of the copula function pictoriallyrepresents.

(Refer Slide Time: 50:20)



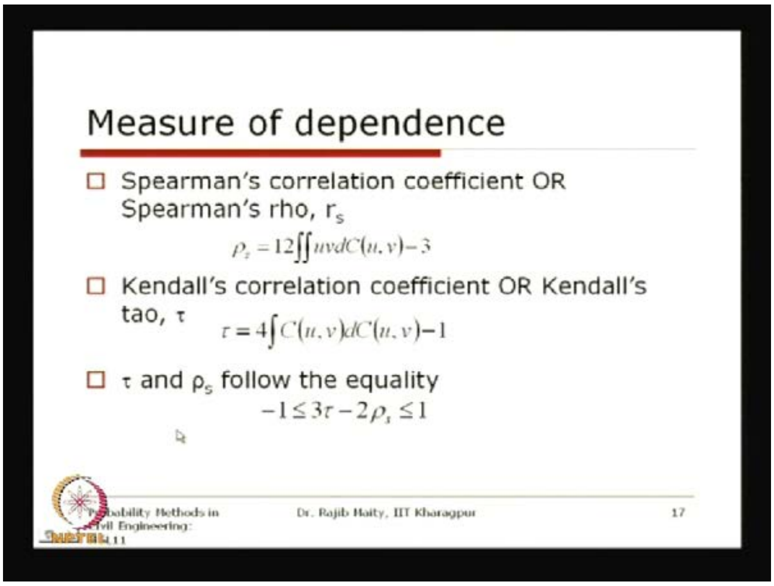
Now, the Sklar theorem which isthe most important theorem in this,theory of this copula, through which we can connect or we can join, we can couple their marginal distribution to have their joint distribution. Let that h x y is a joint distribution function with marginal distribution f x x and g y y.Then there exist a copula c such that for all x y in I square, you know that what is this I square we defined earlier. So, for allx y combination, that h x ythis joint distribution is equal to the, that c of, that is a function of that f x x and g y. If f and g are continuous, then c is unique; otherwise, c is uniquely determined on the range of f and range of y. So, this is consisting that surface over this range, over this area, this should be uniquely determined.

(Refer Slide Time: 51:31)



On the, so, this is the, the, Sklar theorem. In the other, in the converselyalso we can state that if c is a copula and f x and g y are the c d f's of their random variable, obviouslythe x and y, then their joint distribution is equals to the copula function of f x x and g y ofy.So, if we have that f x and g y, then we can get thatthis,through thiscopula function, we can obtain theirjoint distribution which is a, which is h x y represented through this x y. So, this is the joint c c d f of the random variable x y with the margins means the marginal distribution f x x and g y of y.

(Refer Slide Time: 52:20)



Now, as I was telling that we will be takingthe measure of dependence through which basically this copula function should be defined because there is a parameter in the copula function that generally preserve that scale free measure of dependence, sothatthe two scale free measure of dependence we will discuss here now.

Now, the first one is the spearman's correlation coefficient or the spearman's rho,rho s,this should be rho, this is the rho s which is the,through this copula,it is expressed that it is twelve of this double integrationofu v d c. So, with respect to c, that copula function if we integrate this function minus 3, this is the form of thespearman's rhowhich can be derived from this,from thiscopula.

Similarly, the Kendall's Tao, there is the Kendall's Correlation Coefficient, Kendall's Tao is equals to the four times the integration of this copula functionwith respective d c minus 1. So, this is representing that Kendall's Tao, and this Tao and rho sfollow the inequality that this three times Tao minus 2 times rho s this is generally bounded by from this minus 1 to plus 1.

So, even though this functions that we have discussed in terms of this through this copula,copula, function, this 2,this 2major of scale free dependence, but for the, for the real,for the real time,time, series or the data that we are having. If we want to know what is their scale free dependence, then we have to go for some sample estimate, and from the sample,we will first estimate what is the Kendll's Tao, and from that Kendall's Tao,we will first find out what is the parameter of the copula function that we aretaking, and after that, we canobtain what is the complete definition of the copula for the data in hand.

So, in the next class, what we will do?We will just, we will discuss first how to estimate thismeasure ofdependence from the samples and we will discuss a particular class of copula which is known as the Archimedean copula and thatand there we will show that how we can estimate from the sample and get the complete form of this copula function, which can be used to, and ultimately,we will be, we will be deriving theircomplete form of this jointp d f. So, we will discuss these things in the next class. Thank you.