Probability Methods in Civil Engineering

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Module No. # 01
Lecture No. # 28
Functions of Multiple Random Variables (Contd.)

Hello and welcome to this tenth lecture of this module.You know that in this module, we arediscussing about this multiple random variable, and it is, in lastcouple of classes, we are discussing about the functions ofmultiple random variable. So farwe have discussed about one function oftwo random variables andtheir general cases and their application to differentproblems we have seen, and specially in the last class, we have seen thatone example on the Civil Engineering problem that, how one function, how to determine the p d f ofthe functions of therandom variable.

So, today's class basically,we will be, we will be, going through the if there are more than one, one, function. So, that means, the known random variables are say x and y and there are two more new functions are there say z or wbased on thisearlier random variable x and y. So, we will be interested to know that what is the joint density of these new functions, that is, z and w. So, now, you know that whatever we have discussed earlier that once we know their joint density, and from there, we can you caneven obtain whatever we need,for example, the marginal densities, we can obtain,we can obtain their conditional density and all.

So, the focus of our today's class isto discuss about the, their density or rather the joint density of two functions of thetwo random variables, and of course,we will take this one from this 2 by 2 case to this general case form this n numbers of variables to m numbers of newfunctions. So, that we will generalize later.

But to start withtoday's class,we will take one more example from thisone function of out of tworandom variables and we will take this problem in such a way that there are more than one root. If you recall thatcouple of class before, we discussed thatwhenever we are having more than one root, then we have totake a summation of for all the possible roots and that type of one example we will take, which is again theonly a single function out of the tworandomvariable. So, we will start with that and then we will go to these two functions of two random variables.

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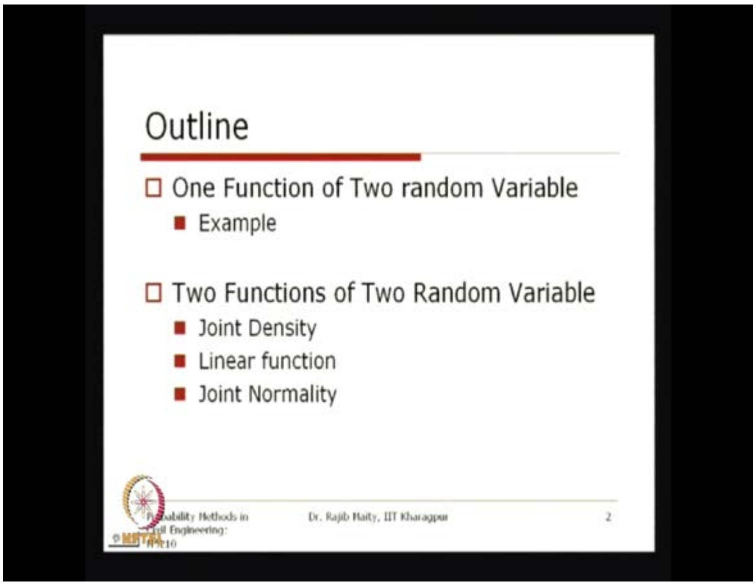

So, ouroutline is this one, one,function of two random variables. So, we will put one example, and after that,we will go to thatthe two functions of two random variablesand we will discuss their joint density, linear function, joint normality.This is important because, so, whenmost of the cases or many cases, I should saythatthere are random variables which arejointly normal. Joint Gaussian distribution is observed, and if they are having twonew functions, then what is their distribution properties and all. So, to start with as I was telling that,we will just take one, one, more example of this one function of two random variables.

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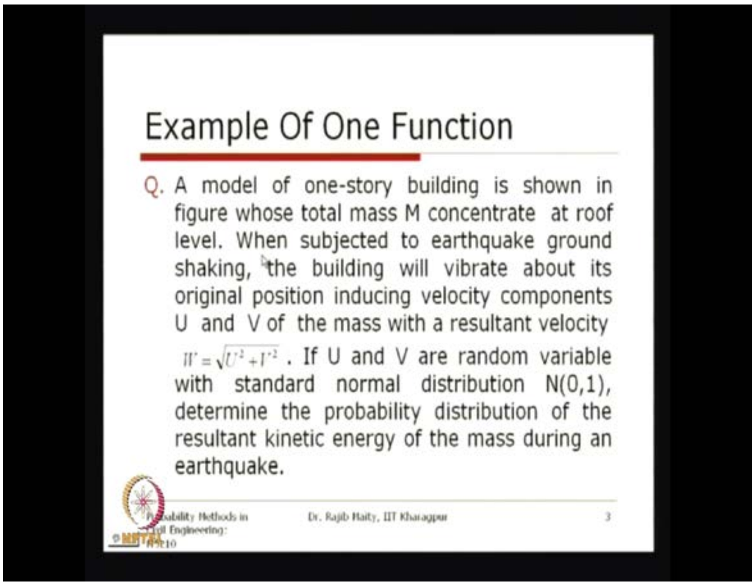

And this example is based on the, based on the, kinetic energy that is being stored in a structure during a earthquaketime. So, the, you know that during the, during the earthquake, generally, the, a structurevibrates and vibrates and we can have their velocity component in two orthogonal direction and their resultant direction, resultant velocity we can obtain and we may be interested to know what is the property of that,of that, resultresultant kinetic energies.

So, based on that,we will be taking a simplified structure, and here, the one, one, storied simplified structure has been obtained and in that structurethere. So, the idealized case you know that, if you havethatthe general way of analyzing, this earthquake force is a we generally assume that concentrated mass at the, at the floor level, and based on that, weobtain some force at that level and we calculate what is thekinetic energy. So, here,we will betaking one such problem for one idealized one storage structure. So, this problem statement is that a model of one-story building isshown in the figure. So, let us see the figure first.

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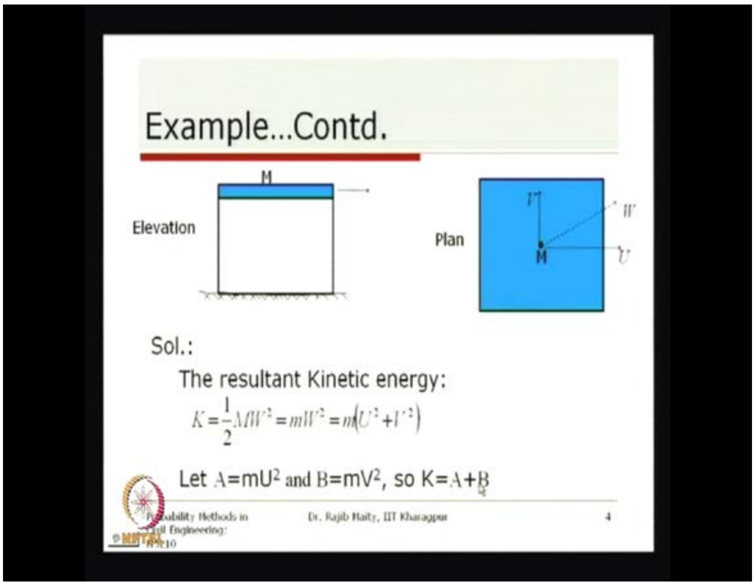

So, this is the one storiedbuilding, and as I was telling that mass M is concentrated at its first floor level. So, whose total mass M concentrated at the rooflevel, and when subjected to the earthquake ground shaking, the building willvibrate about its originalposition inducing the velocity components U and V. So, this U and V are intwo orthogonal directionsof the mass with the resultant velocity. You know that if the U and V are the velocity component in two orthogonal direction, then theirresultant velocity will be denoted as W and W is equals to square root of U square plus V square.

Now, if U and V are the random variable with standard normal distribution, that is, it is a normal distribution with mean 0 and standard deviation 1. So, we know that how this, how this distribution functionis. So, both of them, that is, U and V aredistributedis the standard normal distribution they follow.

So, our, so, what we have to determine is that probability distribution of the resultant kinetic energy. Now, this so, if this is the force, then we have to first know that whatshould be the kinetic energy, andthen,we,we will be able to determine what is this probability distribution. The information that we know that the marginal density, that is the density of the U and V are known to us. We just standard normal distribution. So, determine the probability distribution of the resultant kinetic energy of the mass during an earthquake.

So, as we are justexplain in the problem that this is the mass that is concentrated at thefirst floor level, and if we just see it from the top, so this is the component of this velocity U and this is the component of velocity V and their resultant is W and which is you know thatthe square root of u square plus V square.

Now, if we see their kinetic energy, then this can beexplained; this can be denoted as K and this can be finally written that this m U square plus V square. So, basically, weare having two random variable which are, for, for which, the distribution is the standard normal distribution and we are just adding them up after squaring them. So, U and V we know. So, what is the distribution of the U square plus V V square, where U and V both arestandard normal distribution? That is thebasicquestion here. So, you see that if we just know denote that Aequals to m U square and B is equals to m V square, so we can write the totalresultant kinetic energy is equals to A plus B.

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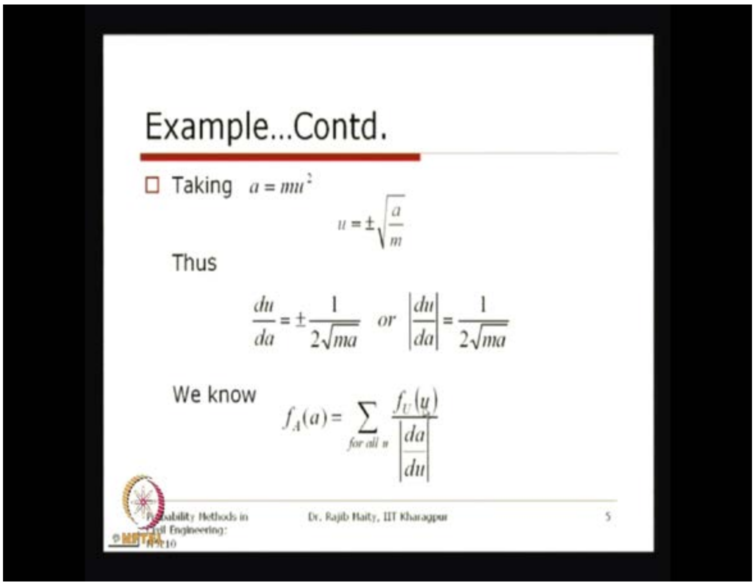

So, now, we will see what is theirdistribution of the A and B first.Now, this one if we just take the specific value of the random variables, we can write that a equals to m u square. So, that u will havethe plus minus square root a by m. Now, you see that here we are getting the two roots.

So, we have to whatever thedistribution that we will get, we have to get it for the both this, both this roots, and now, therewe have discussed it earlier how to get theirdistribution of theirfunctions. So, we have toget their Jacobian, that is, d u by d a and; obviously, the absolute value of this, absolute value of this derivative. So, this is the d u d a isif we justdo this derivation, it will come that 1 by 2 square root m a and itsmode that is absolute value is the positive root of this one. So, that is 1 by 2 square root m a.

Now, we know that for this, if we want to know what is the distribution of this a, when we know that distribution of this u, then we have tomultiply it by its Jacobeanor divide by its inverse of this one, that is, d a d u of this absolute value and we have to do it for the all the roots that is for all e u. Now, here, there are two roots that we, as you can see, and we also know what is the distributionof this u which is a standard normal distribution.

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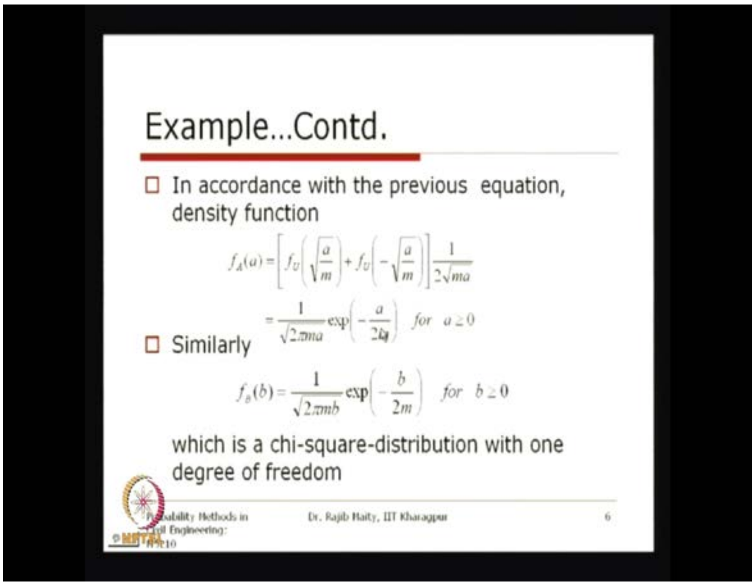

So, if we just put this standard normal distribution in place of the, in place of thedistribution ofu, then what we will get is this, that is, f A of ais equals to thatthe, this is a, this is a first rootwhich is the positive of this square root a by m, and the second root is that negative of thatsquare root that is a by m. So, these twothat is multiplied by its Jacobean 1 by 2 square root m a, andafter somealgebraicrearrangement,we will get that 1 by square root 2 pi m a exponential minus a bytwice m for a greater than equal to 0.

So, so, this is the distribution of a that we gotah fromwhat we know that. So, here, what we have to do is that may beone morestep could have been shown here is that, this is the f U of this component meansthat you are, you are putting their standard normal distribution. You know that how to, what is the form of the standard normal distribution, and so, it will be just that 1 by square root of pi because the sigma is the 1, then exponential ofminus half of this one because the mean is 0 and standard deviation is 1. So, if we put both these thingsinto placeswith thisrespective, thiscomponent that will one is the square root a by m and other one is this minus square root of a by m, and then, if we add them, then we will get this form.

Andfor the b also the expression is same, that is, now this capital B is equals to, capital B is equals to m u squarem v square, v square, and v is also having the samestandard normal distribution. So, from there, if we want to know what is the distribution of thisb; then bdistribution also will be the similar one with this parameter change asb. So, it will be 1 by square root of 2 pi m b exponential minus b by twice m forb greater than equal to 0. So, which is a chi-square-distributionwith one degree offreedom that we knowthis distribution, and now, if we want to know what is the distribution for the k and this k you know that k equals toa plusb.

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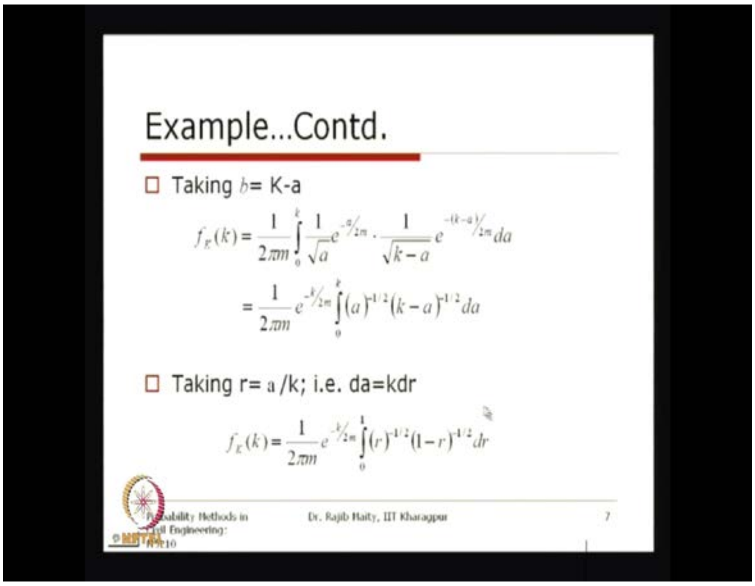

So, k equals to a plus b and that can bere arranged, that is, b equals to k minus a, and from these addition in the lastclass also we discussed thatlast class or that last to last class that a plus b if one function, then how to get the distribution of that one. So, we are using the same thing. It can be expressed in the two different way - one is that you just explain the function in terms of only one variable either a or b anddo this integration for the entire range forthevariable that you havekept.

So, here, so, one such is this. So, here, we are keeping that variable aa here. So, this, so, this is the distribution of this, of this,a and this is the distribution of the b expressed in terms of k minus a. So, now, we are just eliminating out the b. So, we will do this integration for the entire range of this a. So, that is why it is 0 tok andthisit is integration with respect to a. Now, if we do this integration, then we will get that 1 by 1 by 2 pi m exponential of minus k by 2 m integration 0 to k a power minus half k minus a power minus half d a.

Now, so, this integration if you want to do, that is, the closed form; integration is not available, but if you justah replace this a by k by r, so thisa by k, thenit can be expressed that this integration limit will be 0 to 1 and it will be r power minus half and 1 minus r power minus half d r.

Now, this is nothing but a, nothing but a, beta function, beta function, with parameter half and this also half. So, beta half comma half this is a,this is a, function which also can be expressed in terms of this gammafunction and gamma functionfew values few standard values we know. So, this integration component we can get from thatgamma function which isexplained here.

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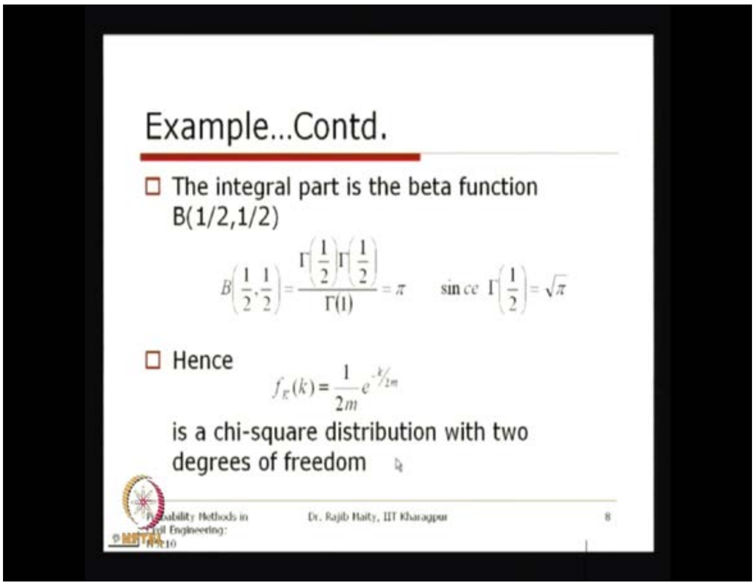

So, thisbetahalf,half, is equals to this gamma halfmultiplied by gamma half bygamma 1. So, which is now equals topi. You know that this gamma half is equals to square root pi.

So, if weare, so, we are getting this beta half, half is pi. So, basically so this integration is the value of this integration is piand that pi and this will cancel.

So, this final form of this kthat distribution of thisrandom variable k will be your 1 by 2 m pi cancellede power minus k by twice m. Now, you know that this also a chi-square distribution with two degrees of freedom which isnothing but the distribution of the total kinetic energy inducedduring the earthquake time, earthquake time.

So, nowmeans, nowthe, once we know thisdistribution, obviously this support is k is you know, it isforgot to mention here that this k is greater than equal to 0. So, now, once we get this support and this will benowwhatever the property that we know, that we can, that we can obtain and that we have discussedin earlier classesas well once the distribution is known all other properties can be assessed.

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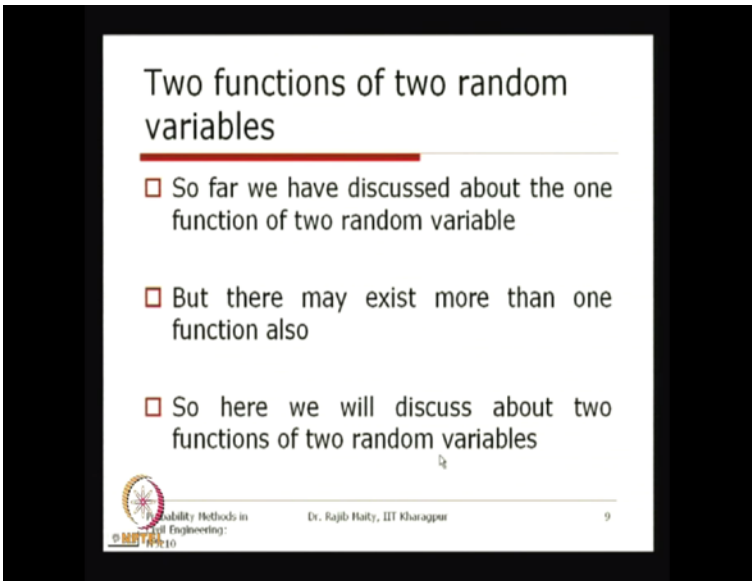

So, now,we will go to the two function as I was telling thatsometimeswe need toget to, we need to know what is the, there could be it is possible to have; that means, two morefunctions out of the whatever the, whatever the random variable that we are having. So, if we are now talking about more than onenew random variable. So, you know that this functions are also random variable. So, if you are talking about that more than onenew random variable, then we will be interested to know what is therejoint density.

So, and also you know that if we know the joint density, then from the joint density, we can get their marginal's or whatever thedistribution or the their properties we can, we can, assess. So, that is why so far we have discussed one, one function, and here,we will see some one more interesting point is that, sometimes even though we are,we are, looking for a singlefunction and it will be, soon that it will be evensometimes easier to introduce a dummyfunction means a another new function of your own choice and follow the procedure to obtain the join density of the two newrandom variable, and then, you get that marginal density of the random variable that is of interest to you.

So, sometimes this procedure is found to bemathematicallymoreeasierthan to get the distribution of a single function out of two random variable. That we will discuss. So, the, our motivation here is that we havediscussed about the one function so far out of two random variables, but there it will be variables, but there may exist more than onefunction also. So, here,we will be discuss about this two, two functions of two random variables and also gradually we will take this same concept to for the, for the, mfunctions out of nrandom variables. So, that will be the general case.

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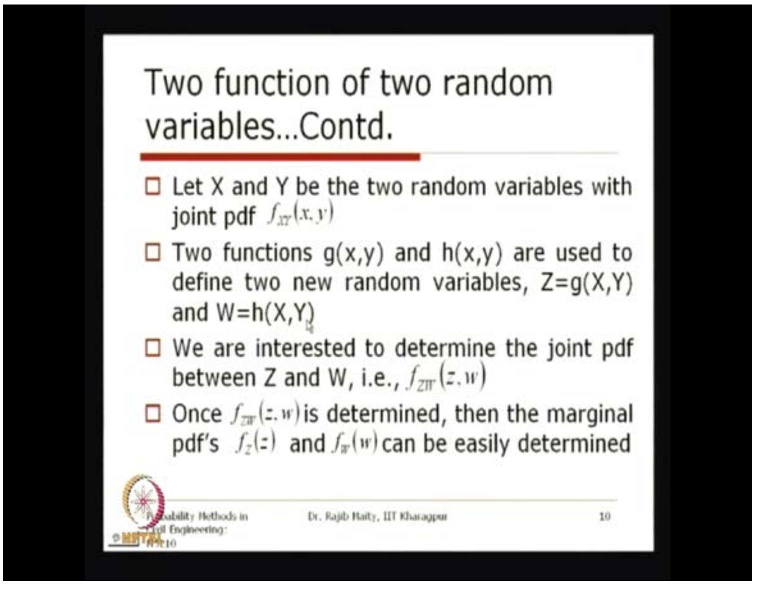

Well, so, what we are having is that two random variables is there. Let that X and Y be thetwo random variables withthe joint p d f f x y x y. So, we have two random variables and their joint p d f is known to us; that means, we knowwhatever the properties their marginal distribution everything we know if we know their jointdistribution. So, this joint distributionof this two random variable X and Y is known.

Now, there are two functions - one is that g x y and other one is h x y are used to define the new random variables. So, you know that these functions are alsothe, also a random variable only. Suppose that we are denotingthat g x y as the z and this h x y as the w. So, what we are interested is, to determine the joint density or the joint p d fbetween this Z and W, that is, f z w z wexpress in terms of theirlower case letters.

So, once we know this one, then you know that as, as we have started thatthe joint p d f of x and y is known means we know their whatever the properties we need to know. Similarly, if we can obtain this joint density of this, of their functions, thenwhatever we need to know we willbe knowing.For example, theirmarginal p d f's of this Z and W. This f zexpressed in terms of z and f w expressed in terms of wcan also be easily determined from the theory that we have discussed earlier. So, now, we are interested to know what is that the joint density of f z w from the joint density of f x y, wherex y is my original random variable and Z and W are the functions of thisoriginal random variable X Y.

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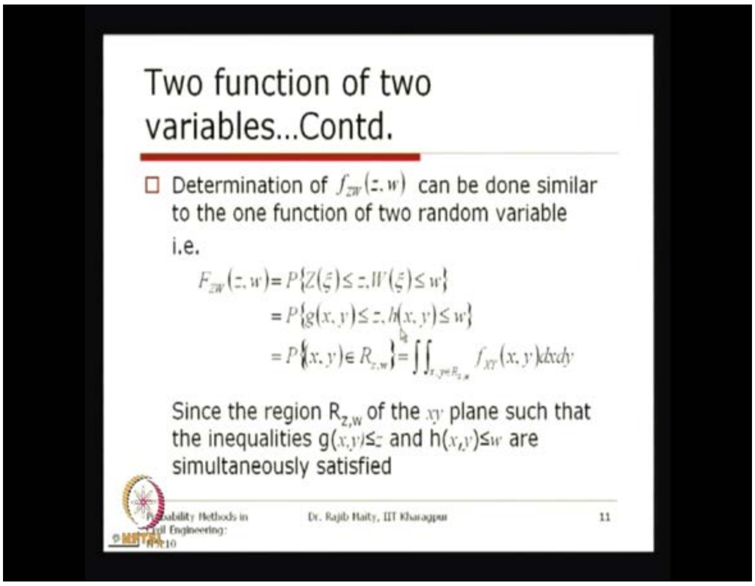

Now, the determination of this joint density f z w can be done similar to the one function of two random variables. Basically, the, keeping the fundamental concept same, what we can express is that their cumulative density, that is, f z w express in terms of z and w can be express at that probability that this, that is, z for a specificoutcomeofin the,in the, sample space, it should be less than equals to z and w should be less than equals to w. So, this comma you know that. That also we discussed earlier also. This comma here indicates that the simultaneous occurrence of both thisevents.

So, in whatever cases, the, both these events occur. Simultaneously we will get the probability and that probability is nothing but the cumulativeprobability ofthe jointdistribution between z and w.Now, this one can be again, again, express that, you know that this z is the function of this g x y. So, this can be expressedin terms of g x y which is less than equal to z and this can be expressed as h of x y less than equal to w. So, basically what isexpress is that this x y. So, that we are looking for a subset x y which belongs to a specificset R denoted by r z w. So, this is the probability where this condition, this simultaneous occurrence of these events is satisfied.

So, once we know this region that means I have to integrate the joint density of the x and y over that region only. So, this is the integrationas there are two random variables. This is the double integration over the region of this R z w, R z w, where this condition are this, this, joint occurrence is satisfied.

So, what is this region R z w? So, since the region R z w of this x y plane, such that the inequalities this g x y less than equals to z and h x y less than equals to w; that means, these two events are simultaneously satisfied. So, this simultaneously word is important here as I, as I, was telling that both this event should be satisfied simultaneously. So, that that is what it means the joint occurrence and the then only we will get what is the joint distribution. So, this is straight forward; straight forward in the sense that this is the fundamental concept where evenwhile discussing about the one function out of two random variable.

So, this fundamental concept was same.Only thing is that there the resultingfunction was a one-dimensional means that only for the single random variable. Here, we are going from one-two dimensional plane to another two-dimensional plane. So, it is map from the one surface to the another surface. The first surface is the x yconsist of this original two random variable and the othersurface that we are talking about is this z w which is mapped from this x y through thisfunctional dependence g x y and h x y.

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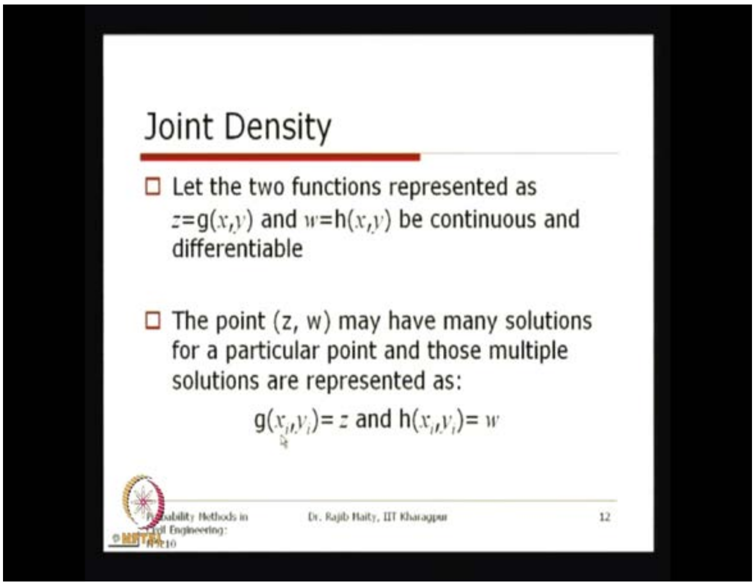

So, now, let the two functionsrepresented as this z equals to g x y and w equals to h x y be continuous anddifferentiable. So, if we justgoing on this with this assumption, so basically what we are talking about these random variables are the continuous randomvariable, and whatever the function that we have taken that is differentiable at every u point.

Now, the point z w may have many solutions fora particular point and those multiple solutions are represented as that g x i y i is equals to z and h x i y i is equals to w. So, now, for the, so, it can happen that from the surfacefrom the plane z w. For one combination, we could have, we could have multiplesolution means multiplerepresentative point on the x y plane and those are denoted by this x i y i.

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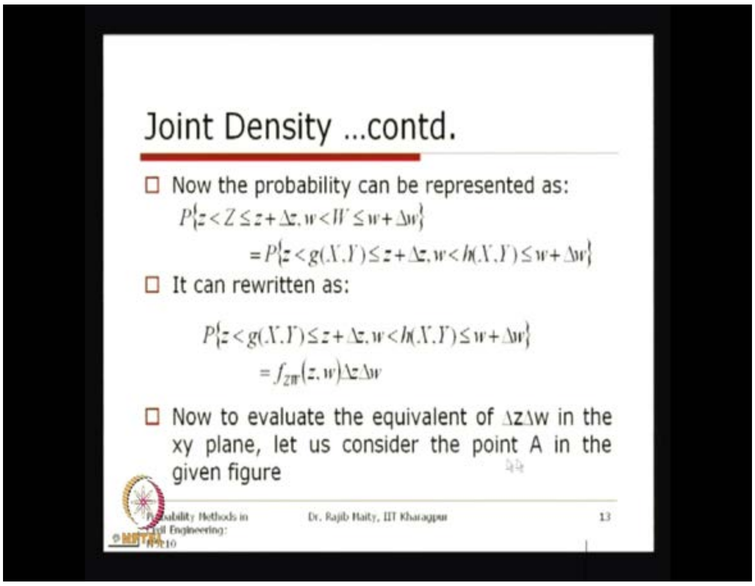

So, ifthat is the case, now from the concept of thisprobability, that we can be, we can express that the probability of Z from a very small region of this delta z, that is, Z lying betweenz and z plus delta z. So, remember that this capital letter as I was telling I told earlier also that this capital one is the random variable and the lower caseletters are the specific values. So, this random variable z in between that z and z plus delta zand simultaneous occurrence of this W in betweenw plusw and w plus delta w which can be also, which can be also written as this. In place of this z, we can write that thatfunction g and this w can be replaced by the function h. So, if these values are in between thissmall area of this delta z and this deltaw.

It can also be written as thesefunction is equals to nothing but their the density, the probability density at that point; that point means the z w multiplied by that elemental area. Area means here we are talking about the two random variable that,that is why it is area, so thatdensity multiplied by this small elemental area. So, that will be the total probability. Now, see the difference here from thesingle random variable to thetwo random variables here, is the single random variable. Whenever we are discussing, we are discussing about the probability density multiplied by a small elemental length over the a axis represented, represented, by the random variable.

So, that the area itself is giving you theprobability, but here, when you are we are taking it to, to the two-dimensional case, that means I have to consider a volume that is a density multiplied by the small area below thatdensity curve. So, that will give the probability. Similarly, you can even extend this one to this even for thethree random variables, four random variables. So, that will be some hyper plane and the hyper volumeto represent thatprobability.

So, here, for the two-dimensional case, the density multiplied by the small a area which is theprobability of this joint occurrenceof this two event. Now, to evaluatethe equivalent of thisarea, that is, this delta z anddelta z multiplied bydeltaw which is the,which is the, areaon the plane represented by the new random variable, that is, z and w. Now, this area I have to find out what is the equivalent area on the x y plane that is the original random variable.So, let us consider a point a in the given figure.

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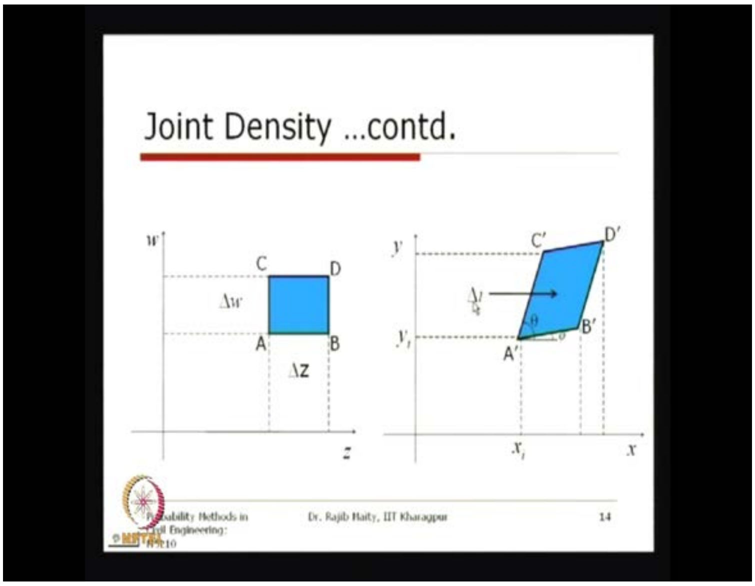

So, here, you can see that this is theplane which is by, made by this z and w. So, here, the point that we are talking about the A and A is the, is thecoordinate having that z and w. Now, we have considered a small elemental areameans surrounded by this A B D and C and this A B is your delta z and this A C is your delta w.

Now, we are interested to know so thatsome equivalent area or equivalent or the representative of the same area in this z w plane on the x y plane. So, you know that this will be obviously be distorted and that equivalent area if we represent at this atrepresented by this A prime B prime D prime and C prime and this total area as we are talking about is the delta I.

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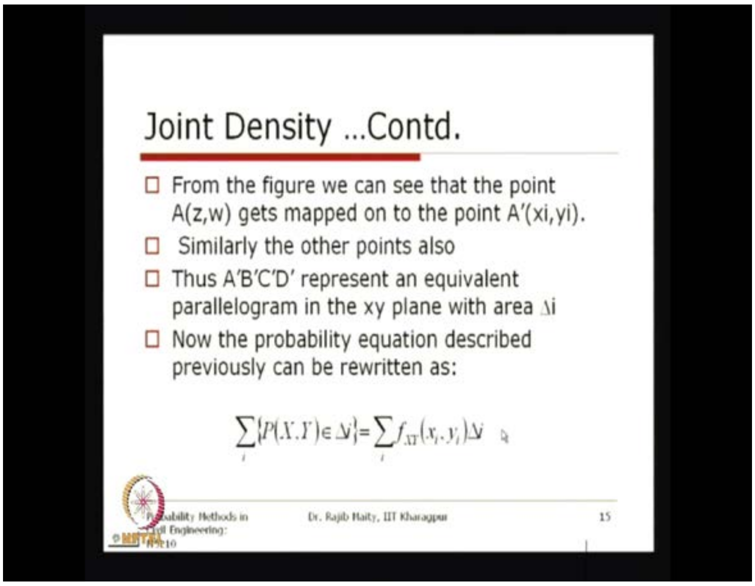

Then, we can express that from thefigure, we can see that the point A z w gets mapped on to the point A prime x i, y i. So, this a is mapped to the point a primewhose coordinates is x i, y i.Now, similarly other points also means other thatB prime C prime D prime are also will be a, will be mapped in a similar way. So, thusthe A prime B prime C prime D prime representequivalent parallelogram in the x y plane with the area delta I what we have just now shown in the, in the, in the, figure.

So, now, the probability equation describe previously can be a, can be, rewritten as thatthis summation of this all this, for this all this representative area, that is, if it is belong that P X Y, the combinationbelongs to this. That, that representative area delta I should be the summation of the, for the all thisrepresentative point x i y i multiplied by theirelemental area. So, this is the joint density of the x y between x and y and this is the area over which we are interested to know how much probability is there. So, we have to just multiply that area and its density.

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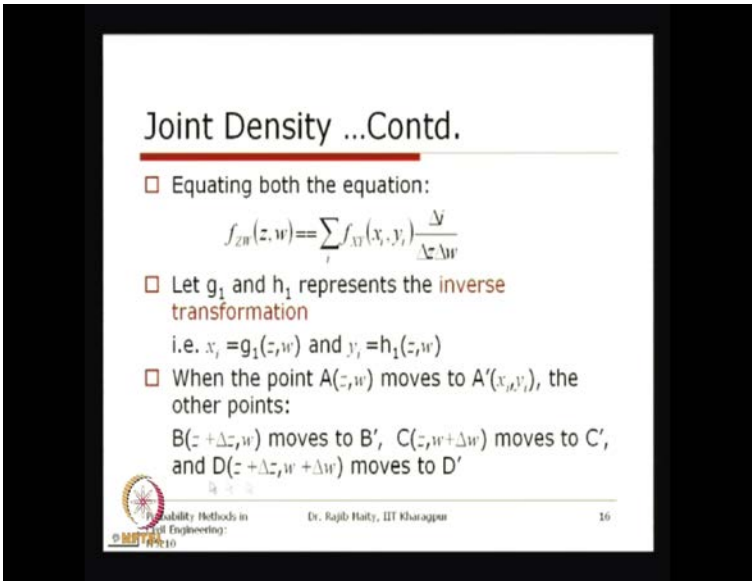

Now, if we equate the both the equations now from the whateverthe equation that you got from here and the equation that we have represented here. So, both we are, both we are referring to the same probability and express one in thez w plane; other one is the aon the, on the a a x y plane. So, you can equate these two as the, as the, probability that we are looking for is the,is the, sameprobability. So, these two are now equated to get that f z wis equals tosingle equal toof thisfunction, and so, thisdelta z delta w is taken here as ain a, in thatdenominator.

Now, let that this g one and h one represent the inverse transformation. So, inverse transformation means that, we have that z equals to g x y. So, now, we are just representing it that in terms of thex, we arerepresenting in terms of the z w. Just like that we are looking for the solutionfor this x and y. So, this is so, whatever the function through h, it is represented we are just denotingah as the g 1 which is nothing but the inversefunction to represent the x, and similarly, for the inverse function of the h isnothing but the h 1.

When the point A z w moves to this A prime x i y i, the other points B z plusdelta z, w moves to B prime;C z, w plus delta w will moves to C prime and D. I think it will be better explain form this fifthfigure directly.This is as I was just telling that a is having thecoordinate z w. This is mapped through a prime.Now, what is the coordinate of this B? B coordinate is nothing but it isz plus deltaz and w and coordinate of C will bez, w plus delta w and coordinate of d will bez plus delta z, w plus delta w. So, these points are being mapped to this A prime,B prime,C prime and D prime. So, now, as these are mapped like this, I think the map will be the better word instead of this moves.It will be mapped through thispoint.

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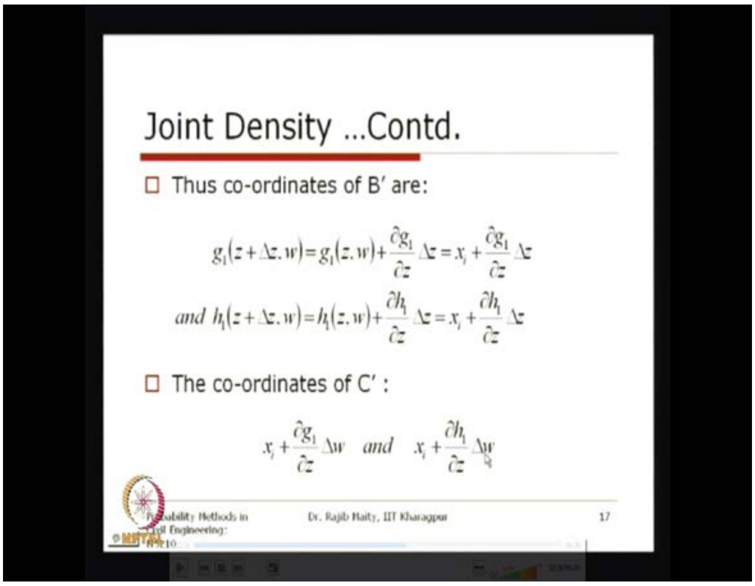

Then,what we can write that the coordinates of the B prime can be written as this g. There is a inverse function g 1 h plus delta z, w is equals tog 1 z w plus its derivative at thatpoint multiplied by what is the change; that means, the delta z. So, we know this one from this fundamentaltheory of thiscalculus. So, this can be again as you know the, this is the x 1 plus dg 1 bydou g 1 dou z multiplied by theirvalue of the change, that is, delta z. Similarly, this h 1 z plusdelta z comma w can be represented by x i plus delta dou h 1 dou z multiplied by this delta z. The coordinates of C primewill be now this one that similarly that this is for the B prime this is for the C prime x i plusdou g 1 dou z delta w and x i plusdou h 1 dou z delta w.

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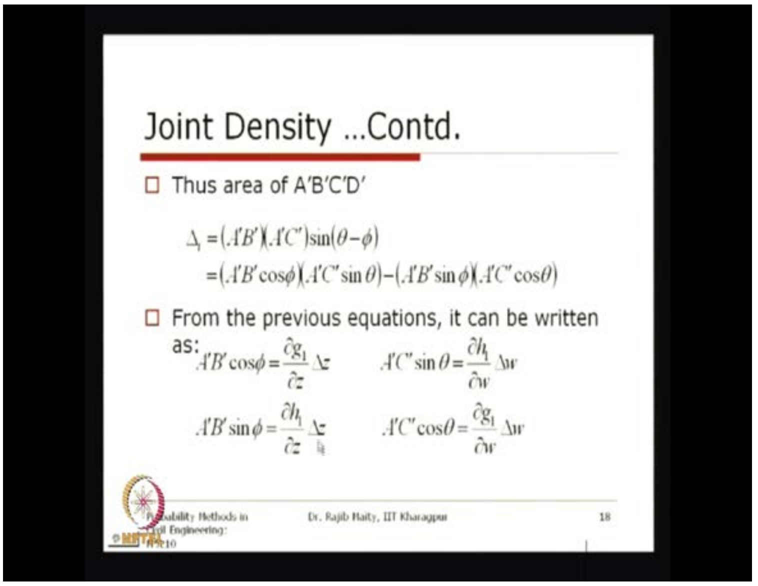

Similarly, that, so, thus the total area, that the area that we are talking about in thisnew, inthis x y plane is that A prime B prime C prime D prime is nothing but the A prime B prime multiplied by A prime C prime sign of theta minus phi. So, there is a kind ofdistortion that you are takingthrough this. So, this A prime B prime cosphi so after thistrigonometricalrearrangement, we can get that.

So, this A prime B prime cos phi is nothing but dou g 1 dou z multiplied by delta z, and similarly, the other points can also be represented by thisderivativetheir rate of change multiplied by their amount of total small change. So, this can be, againthisfigure can be referred to basically we are,we are, what we areobtaining is the total area, total area, total representative areathat is on the plane x y. So, these are thevalues we got, and if we just put it back to this equation,we will get what is the total area delta i.

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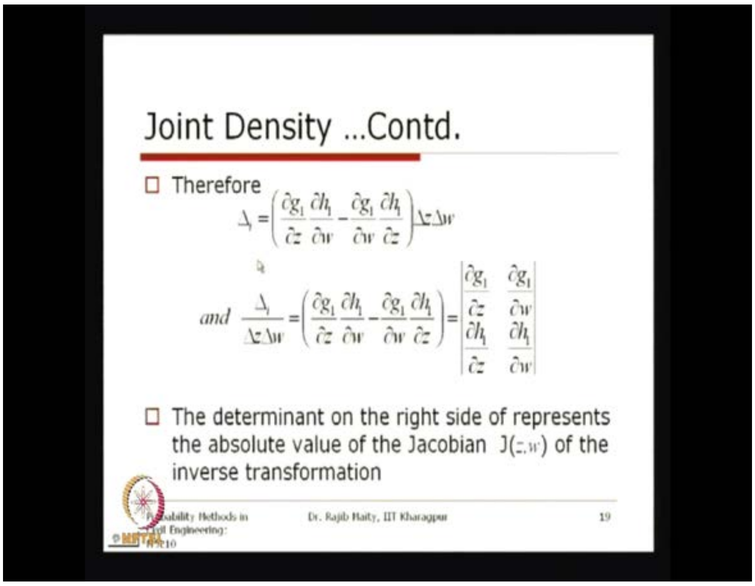

This is exactly what is donehereand we got this form. Now, what we are interested is now is that this delta i by delta z delta w. So, this is the, this is the point that we got here. So, this one we just want to, want to obtain. So, this expression now can bedefined through this one, that is, this will be cancelled and this dou g 1 which is the inverse function of this gdou g 1 dou z dou h 1 dou w dou g 1 dou w dou h 1 dou z.

Now, this can also be expressed in terms of the determinant of the matrix of this one. So, this multiplied by this minus this multiplied by this. So, this is the, now this determinant, if you see, this is the Jacobeanof this, of thisJacobean represented by the z w. Now, if we recall when we are, when weareinterestedto know the single function, then also we got the similarJacobean consist of this only one element and that one element can be seen here as this one by one this one. So, dou g 1 dou z, so, which was the inverse function, inverse function, of that one and this determinant we use in case of thisin this single random variable.

Now, when it is becoming that two randomvariables, two functions, then we are getting a 2 by 2 matrix for which the determinant is nothing but its Jacobean. Similarly, if there are three, three functions, then obviously, if there are three functions and three random variables, this will be your 3 by 3 matrix for which the determinants needs to be a evaluated which is nothing but the Jacobean of, of the threerandom variables. So, here, we are just talking about two random variables.

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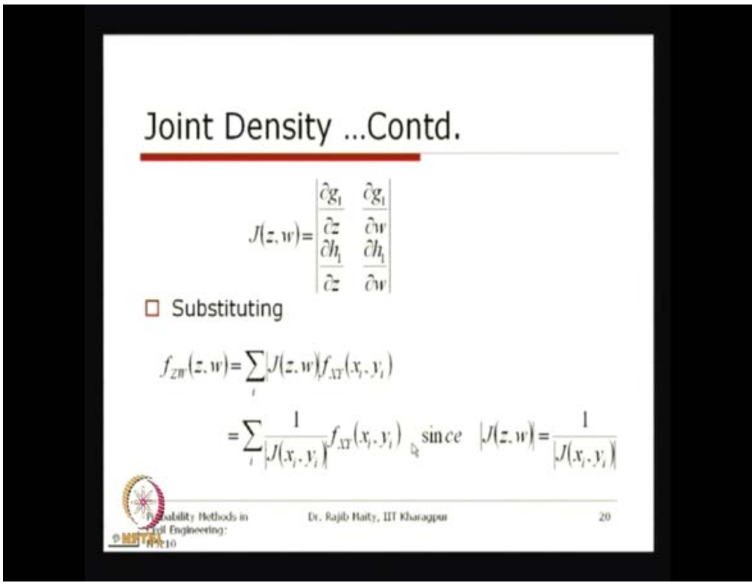

So, I, we got that this is theJacobean, and again from thesame equation where we are justputting that is delta i divided by delta z multiplied by delta w that is replaced by this Jacobean. Now, to get thatah close form, form. So, this is the joint density between this zw which is nothing but the, thisjoint density of this x y multiplied by its Jacobean and it is summing up for the all the possible solution, all the possible roots, and this one obviously we have to express in terms of this z and wthat this x y, the joint density between x and y. So, this is the final form if we are having the two functions out of two random variables.

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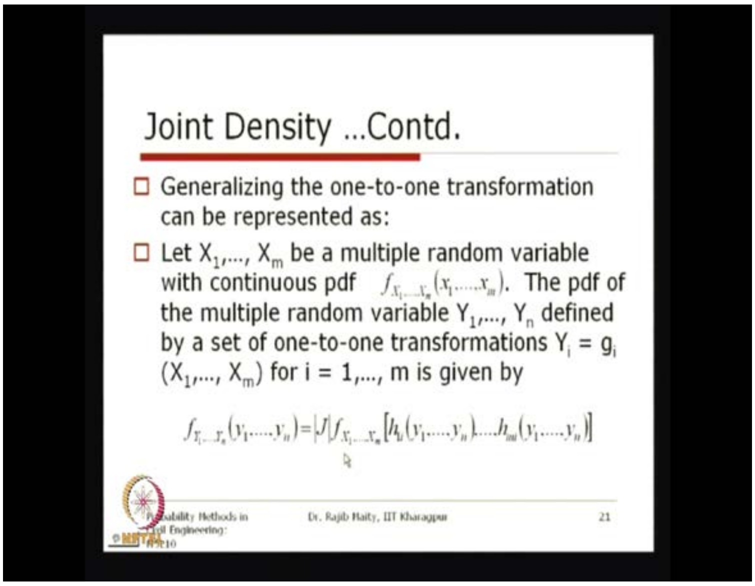

Now, if we want to generalize, that is, one to one transformation, so, remember that what we forgot here. To mention that, this is a, this is their only for when we are talking thatthere are some one to onetransformation, then what will happen?The, so, one to one transformation means there will be only one root, and that one when we are talking about that one to one root, so that means there is no need of this summation that this f z w will be equals to the Jacobean multiplied by theirthat or joint density of the original random variable express in terms of z and w. That is important.

So, here, so, iflet that x 1 x 2 x m be the multiple random variable with a continuous p p d fand the continuous joint, joint, p d f which is f of x 1 x 2 x m express in terms of x 1 x 2 up to x m and the p d f of the multiple random variable y 1 y 2 up to y m defined by the set of one to one transformation.Say that y 1 equals to g 1. This g i, this is the functiong i of this x 1 x 2 x m for i equals to 1 to m and this is given by again from the sameprinciple, that is, this new joint density is equals to the Jacobean multiplied by the originaljoint density function express in terms of their that y 1 y 2 y mthat Jacobean look like this.

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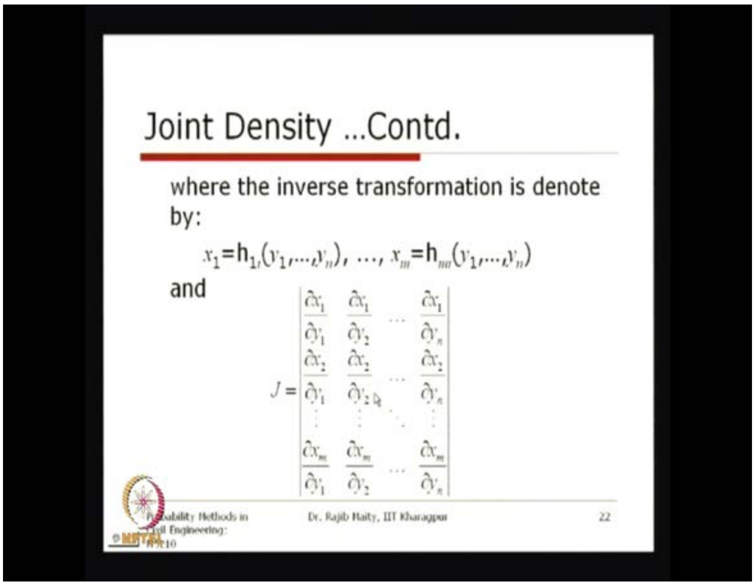

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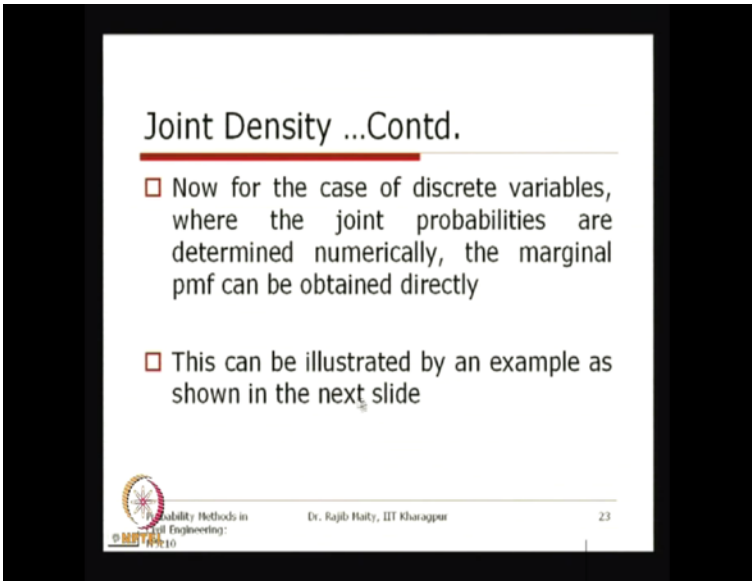

So, these are the x 1 x 2 x 3 are the original random variable and y 1 y 2 are their function and this is the jointJacobean and this Jacobean has been multiplied the, that absolute value of that Jacobean. Now, if wetake the case of this discrete variable, that is, where the joint, joint, probabilities are the determined numerically and it is concentrated at some specific point.

The marginal p m f can be obtaineddirectly, and how we do it? It will better explain in terms of one example, and after we justdiscuss this example of this, of this, discrete random variable that is to p m f. After that,we will take itfor the case whateverin case of the continuous random variable what we have discussed now. Now, for this p m f, how we can do it? We have taken oneexample and we will see that, that case means the point to point basis we will just evaluate what is their, what is the distribution.

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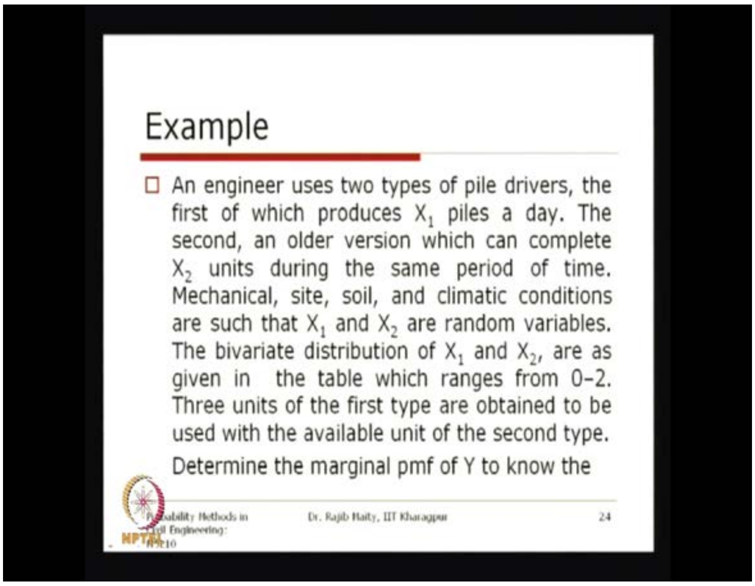

So, we have taken one example here.An engineeruse usestwo types ofpile drivers, thefirst of which produce x 1 pile a day, and the second and second one is an older version which cancomplete x 2 unit during the same period of time. Mechanical conditions, site conditions, soil conditions, climatic conditions, all these conditions can influence on this x 1 and x 2 so that this x 1 and x 2 are treated as the random variable.

Thebivariate distribution of thisx 1 between the x 1 and x 2 joint p m f are given in the tableand which ranges from this 0 to 2. So, 0 to 2, two units. So, x 1 can take the values 012 and x 2 also can take the values 012 and their joint p m f is shownin the next slide. What we are interested? The three units of the first type are obtained to be used with the available unit of the second type.Determine the marginal p m f of y to knowthe distribution of the random variable y equals to 3 x 1 plus x 2.

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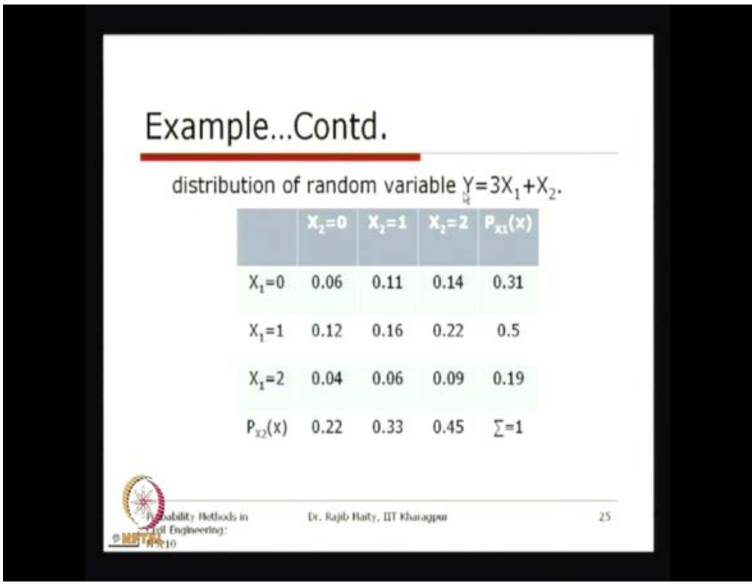

So, ourfunction here is that 3 x 1 plus x 2, andwhat is supplied is that there joint density is givenhere, that is, when x 1 equals to 0 and x 2 is also equals to 0, it is 0.06. When x 1 equals to 0 and x 2 equals to 1, it is 0.11.Like this 3 by 3 matrix is given here, and you know this last column is the marginal of this x 1 and this is the marginal of the x 2. You know that total summation will be equals to 1.

So, now to know the distribution, the marginal distribution of thisy, what we can do easily because this is only that 3 by 3 matrix only, what we can do is that we will take this specificvalue what the y can take. So, if a, if x 1 is, x 1 is 0 and x 2 is 0, then the y also will be 0 and the maximum value of the y will be obtain when this x 2 and x 1 both are maximum, that is, the x 1 equals to 2 and x 2 equals to 2; that means, the y will be 3 into 2 plus 2, so, 8. So, y can take a value from 0123 up to 8. So, thesetotal nine discrete valuescan be taken by y. And what we are interested to know? What is the probability distribution that is the p m f for the y.

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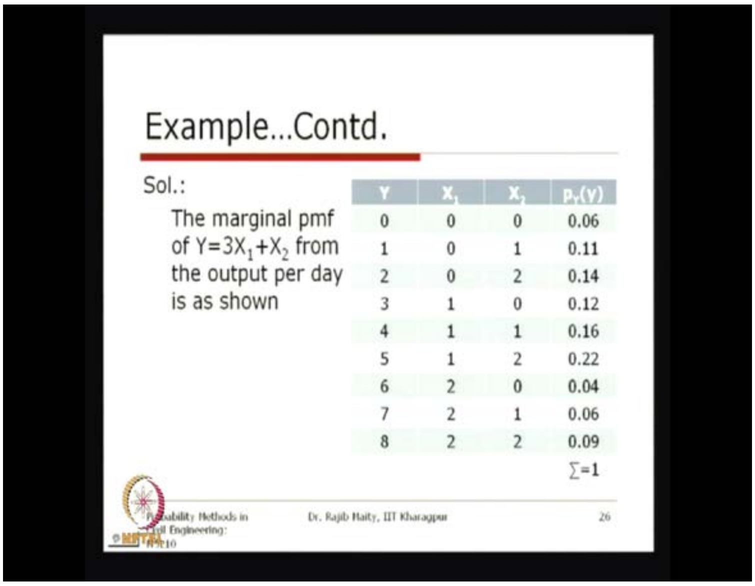

And now, if we just take it by their marginal p m f of they just 1 by 1, that is, when the y equals to 0, that means the x 1 equals to 0 and x 2 equals to 0, we know that their joint density is 0.6.Similarly, for the all such combination of this x 1 and x 2, we can, we know what is their joint probability and we know what is the value of the y.

So, when we are,when we are, interested to know what is the marginal distribution of y, so these values we should see, that is, y equals to 0, y equals to 1, y equals to 2 up to y equals to 8, and for this, what is their probability; for this, which is shown in the last column. So, this is the, this is the, p m f for the y and y can take nine discrete values and their probabilities corresponding probabilitiesare shown in the last column of this table. So, this is for the discrete case where this is a very small problem, where we can just pick up the corresponding values directly.

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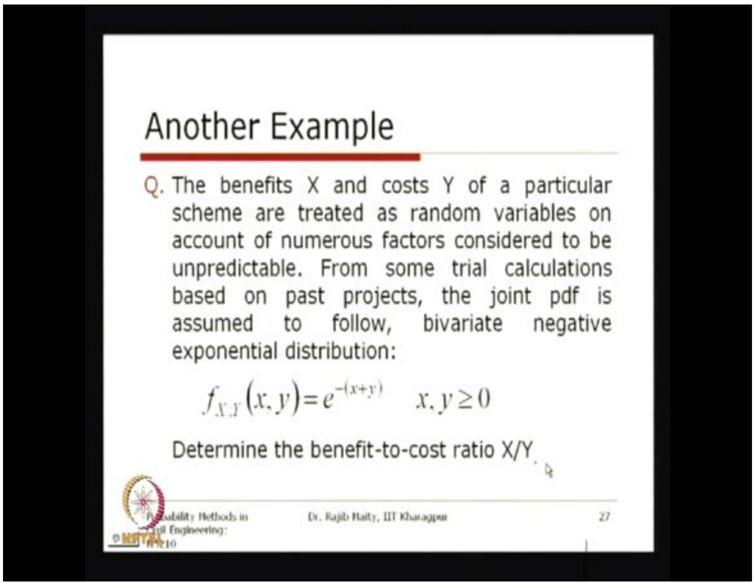

Now, if we take one problem which iswhere the,where the, random variables are continuous and we know their joint distribution and we want to know the joint distribution of the other two random variable which are the function of this original random variable. Once this example is taken through thiscost benefit ratio of the projectof some civil engineering project is the benefits x and the cost y of a particular scheme are treated as to be the random variable on the account of the numerous factors considered to be unpredictable.

From the trial calculation based on the first projects and the joint p d fis assume to follow a bivariate negative exponential distribution which is represented by this that f x yx y is equals to e power minus x plus y, where this x and y both are greater than equal to 0, and what we have to determine?Determine the density, probability density cost to benefit ratio, so, x by y. So, this one is the benefit is x and thisy is the cost. Their joint density is this express in this form and we are interested to know what is theprobabilitydensity of this benefit to cost ratio, that is, x by y. Now, see here, there, there, are two random variables - x and y, and one their joint distribution is given and we are looking for one function here.

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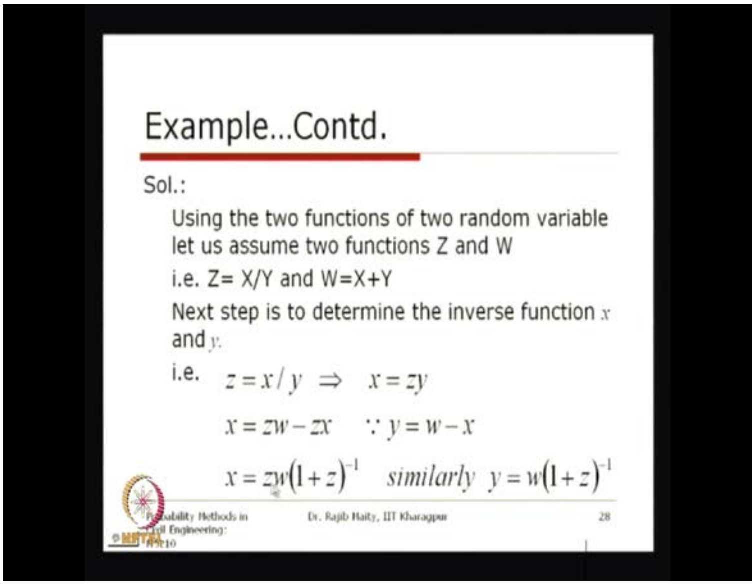

Now, as I wastelling before I start these two, two, functions. Sometimes this can also be done the way that we have done is there if there is only one function isasked for that. Whatever we have done earlier, that can be aused that theory to get what is the, what is their ratio, but we will just see through this example that, if we introduce another new randomvariable which may be not of our interest so far as the problem is concerned, but the problem can be solved very easily, because we will first find out their joint density, and from the joint density,we will find out what is the marginal distribution. This is what is explained through this problem is that.

So, one function, one function is the z equals to x by yand the other one we have just introduced is that w equals to x plusy. So, sometime the solution aremeans the joint, joint, density will depend on what is the new function that we have introduced, but so far as the marginal isconcerned, that marginal of this z that will remain same. So, generally a very simple function is assumed as this, as this function so far to make theit is calculationmake the calculation simple.

So, the next step is to determine the inversefunction of x and y, that is, z equals to x by y. So, x equals to z y and x equals toz w minus z x, that is, y equals to, y equals to w minus x. So, x equals to z winto 1 plus z minus 1. Similarly, y is equals tow into 1 plus z y. So, what we have done? These are the two functions are there and we are, what we are doing is, there we are getting the inverse function first because we know that we have to get this inverse function to get its Jacobean, Jacobean.

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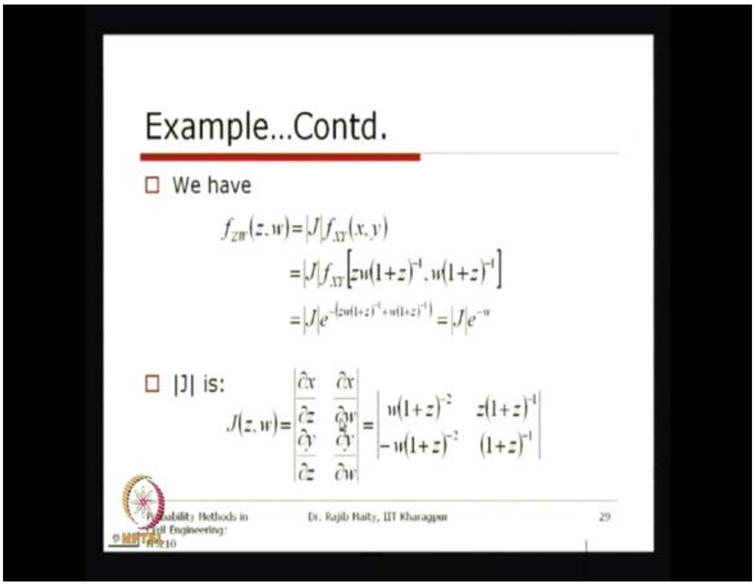

So, once we get this one, then we cancalculate what is theirJacobean. This is the final form that we have to obtain, that is, f z w is equals to theirsJacobean multiplied by the original density, and so, this original, original density when we are talking, we have to express it in terms of the z and w that I mentions.

So, this in place of the x, we are writing thatz w 1 plusz inverse and w into 1 plus z inverse. So, which are getting from thisx and y here and this is multiplied by this Jacobean and this joint density also we know that e power minus x plus y. So, this one plus this one which we can simplify to get that z equals to e power minus w. So, what is pending is that we just want to know what is this Jacobean and Jacobean is you know that this is adou xdou z anddeltaxdelta w and this Jacobean if we justsolve,we will get this form.

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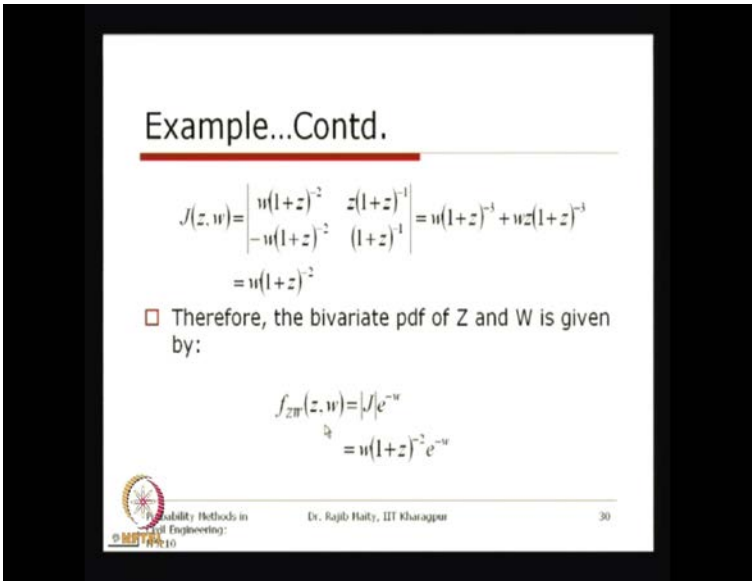

And after doing this calculation,we will get.Finally, what we will get is the w into 1 plusz power minus 2. So, this is the Jacobean so that, so that, joint distribution will be itsJacobean exponential minus w, so, w into 1 plus z power minus 2 e power minus w. So, this is the joint density between z and w.

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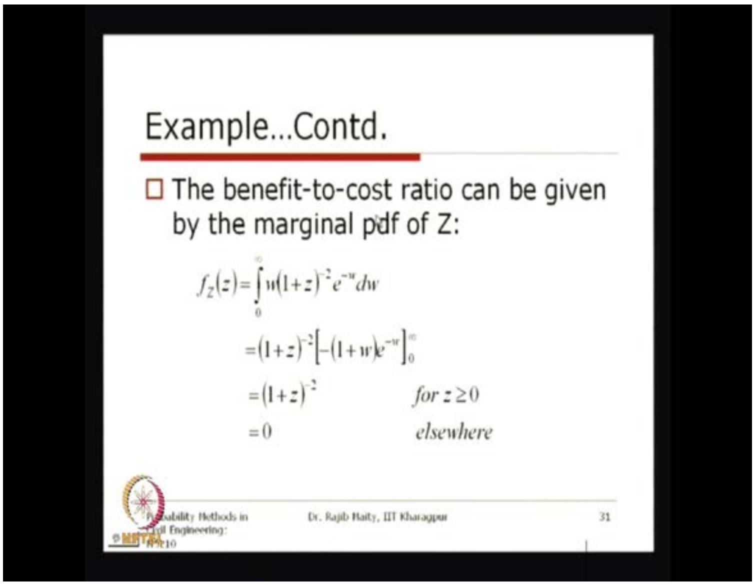

Now we are we want to know what is the, what is the, density of this z only becausez is the benefit to cost ratio. So, we have to calculate its marginal. Now, when we are calculate thismarginal that means,the other variable has to be marginaled out by integrating over its entire range which is 0 to infinity. So, when we are expressing this one, here we are, we are,suppose to write that this z, w is should be greater their support. So, z, w is greater than equal to 0. I hope it was mentioned earlier this form like this that x, y is greater than equal to 0 this is its support.

So, we are just integrating out that w to get the marginal density ofthat z, and this isnothing but if we just do this integration,we will get the form as that 1 plus z power minus 2 for z is greater than equal to 0, and otherwise, it is 0. So, this marginal density of z equals to 1 plusz powerminus 2. So, this is thedensity of this cost to benefit, that benefit to cost ratio that is x by y.

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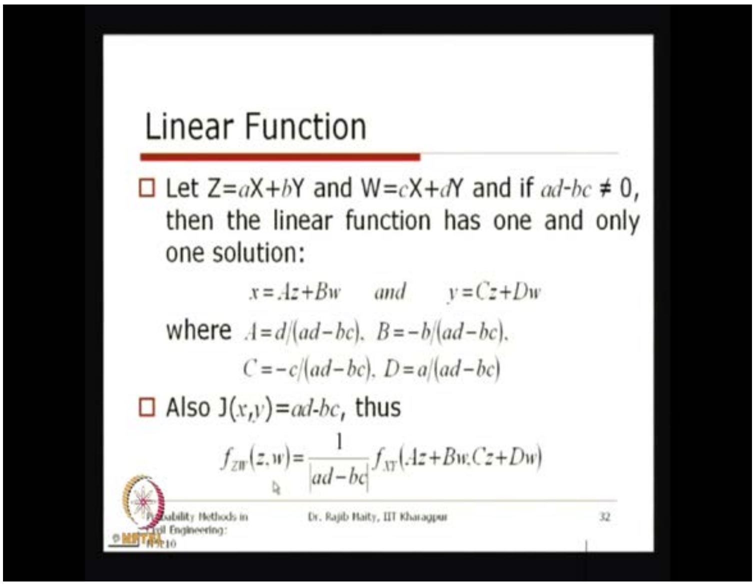

Now, we will take up somespecial cases wherethatthrough this two,two function. The first one is thatlinear, linear functions only. Now, linear functions means when there are, we areconsidering to adjust multiplying it by someconstant and justthere we are adding them up, this is the,this is the, linear function, and we will just see thatfor a linear function, how thefinal form looks like and also we will take upone thing of this joint normality. If the original distribution is the is join normal distribution, then what will happen to that, to that, to that, resulting joint distribution.

So, first this is the linear, linear function. Let that z equals to a x plus b y and w equals to c x plus d y, and if thata d minus b c is not equal to 0, this is one condition because this a d what is the coefficient of x and coefficient of y minus the multiplication of the coefficient of y and coefficient of c. If this is not equal to 0,we will see in a minute why this is one requirement. Then the linear function has one and only one solution and this solution is that x equals to a z plus b w and y equals to c z plus d w, where this capital A B C D are the constant which can be expressed in terms of the lower case of this constant theA B C D through this expression.

This is just basically what we are doing? We are gettingtheir inverse function. So, once we know this inverse, one inverse function, and also we know that what is the Jacobean of this x y and this Jacobean of the x y will be we can just express that dou x dou y which is a d minus b c.

So, thus this joint density is that z w is equals to 1 minus 1 by a d minus b c and f x y a z plus b w, c z plus d w. Remember here,here, we are using this Jacobean of x y.We can use the Jacobean of thiszw also be depending on where we are putting this 1 by a,1 by this Jacobean that we aredoing. In the earlier example, what we did is the, we just multiplied with the Jacobean and thatJacobean wasbetween the function, that is, the z and w. So, based on that where we areputting this.

So, now, if we know these functions, so, if it is a linear function, we know their coefficient; we can easily calculate what is this capital A B C D and we canexpress in terms of their z and w and we will, and we will divide it by thisfunction this Jacobean and that will give the joint density here. Now, you see, it is as this is in the denominator. So, if this is 0, then we will not getthe solution which will be infinity. So, this if this is not equal to 0,then we can get their,their, solution at that, what is the joint density between z w, z and w.

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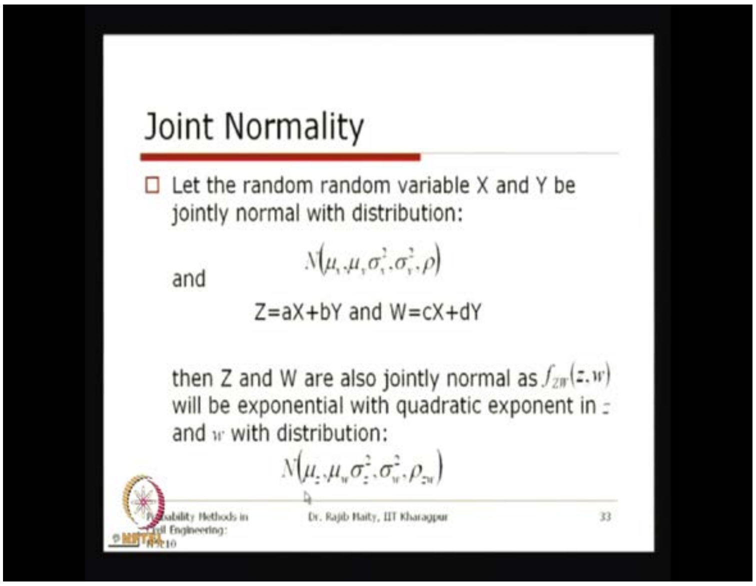

Now, the joint normality let the random variablex and y be the jointly normal with the distribution. You know that it is mu x mu y sigma x sigma y and rho. So, this mu x is the mean of thisrandom variable x; mu y is the mean of the random variable y. Sigma x's square is thevariance of the random variable x and sigma x sigma y square is the variance of the random variable y and rho is thecorrelation coefficient between x and y. There will be a comma here between mu y and sigma x's square.

So, this is a jointnormal distribution and the functions that we are talking about the linear function again that a x plus b y and w equals to c x plus dy. If wehave this function, then we can solve it to know that what is its jointdistribution; then it can be shown that this z and w are also jointly normal as this f z w will be exponential and the quadratic exponent in this z and w with the distribution n mu w, mu, sorry, mu z mu w sigma z square sigma w square rho z w. So, this is the correlation between z and w variance ofw. This is variance of z. This ismean of w; this is mean of z; obviously, again one comma will be there between mu w and sigma z square.

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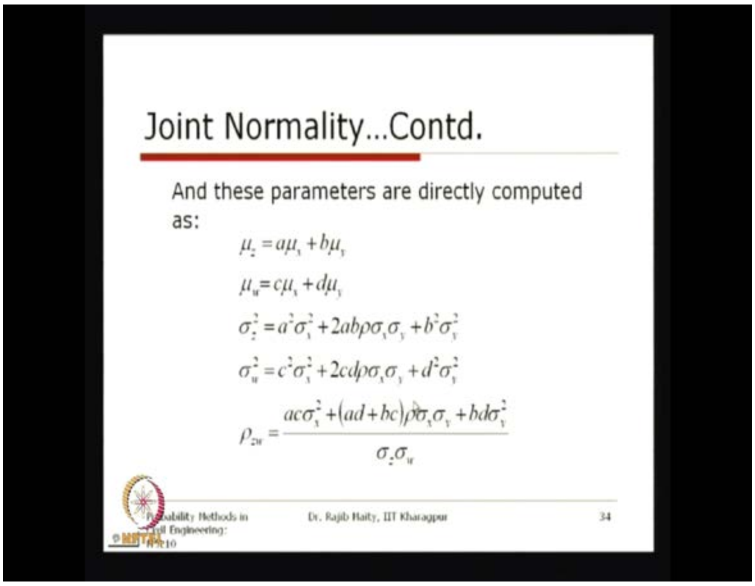

And to know that, what is this mu, z and mu w? If we just solve this one we can say that, thismuz is equals toa mu x plus b muy. Similarly means it is just you can see that here, the way we are just adding, it is very easy to remember that this a multiplied by this mean of this andb multiplied by the mean ofmean of this. So, this is basically we are taking the expectation of the both the side.

So, this is the mean or this is the mean of this w and this is the sigma, sigma z square. The variance of this z is that you know that this is multiplied by thissquare of the coefficient a square sigma x’s square b square sigma y square multiplied by a factor. If they arenot independent, then this 2 a b rho times theirproduct of their standard deviation.Similarly, for this sigma w square is expressed through this one and their rho z wcan also be expressed through this expression using thecorrelation coefficient between x and y here.

So, we will take up a problem on this one maybe in the next class on this joint normality. In many cases, there are some random variables where we can say that these are jointlynormally distributed and we know theirsome linear functions we have to use for some of the Civil Engineering problem.

And if we know this expression, then we can easily get what is the joint distribution of their, of the new random variables there. In the next class,we will be,we will take up some as I told some of the problems on this one that the theory, that we discussed in the last, last onthese two random variable, may be one or two problems. We have already discussedmay beone or two problem we will take, and after that,we will take you through thatcopulawhich will be used for this multiple random variables to know their joint, joint distribution from their marginal distribution. Thank you.