Probability Methods in Civil Engineering

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Lecture No # 25
MGF of Multivariate RVs and Multivariate Probability Distributions

Hello, welcome to this lecture.Thisis seventh lecture of this current module,multivariate random variable.So far, we have discussed their several properties and you know the last lecture that, we are that we are supposed to cover in this lecture is that moment generating function.So,we will complete that part and after that we willdiscuss something about their different distribution.

So, multiple random variable when we talk it is their joint p d f and joint c d f and if it is discrete then it is joint PMF thatwe know. So,this thing we will be discussing and after that we willgive some standard example of bivariate case means, mostly youknow that when we are discussing different properties. We are discussing in case of the bivariate random variable, where therearetwo random variablesare involve and after that we will also take you through the concept,how it can be extended for the multivariate random variable cases,where more than two random variables are there.

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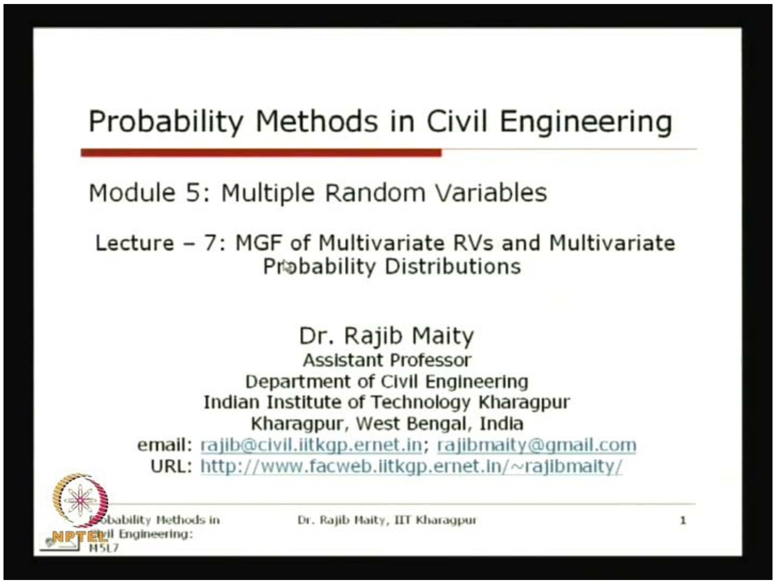

So, todays lecture we will start with the discussion of this moment generating function and this moment generating function we have discuss earlier, in case of the single random variable and as this module is on this multiple random variable.We will discuss thisonefirst, in case of this multiple random variable and after that wewill take different distributionsas well.

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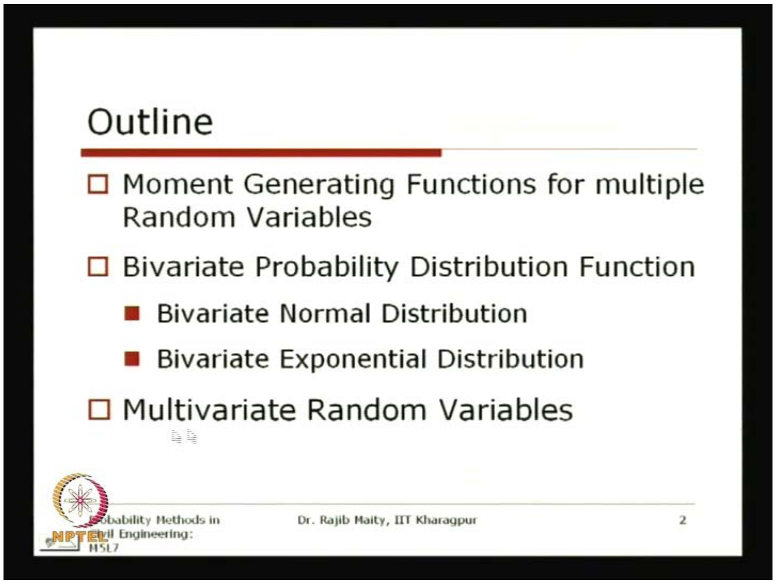

So, outline of our todays lecture is,firstthe moment generating function of multiple randomvariables.We will know that,how in case of first of all in case of two randomvariables, how we can generate that moment generating function and aftergenerating after obtainingits MGF that is moment generating function.We will knowthat their utility thatif you take that rth order derivative.Now when we are talking about the rth order derivative that, is in the context of a single random variable that we have discuss earlier, but here as there aremore thanonerandom variable.

So,there are will be so derivative order also may change for the different random variablesand after getting that derivative, if you evaluate that function at the origin.Thenwe will get that moment that also we willdiscuss, because that is the main focus of thisobtainingthis moment generating function.So,we will get different moments andfor the multiple random variable when you are talking about the moment, when also there is a sense of jointmoment that isfor,if it is bivariate case then it is for the both random variable the order for the different order for the different random variable.

So,we willcome to knowwhat are their moments, so that is the reason, why will learn about this moment generating function. So, after that we willdiscuss about this bivariate probability distribution functions and so far as thisjoint distribution is concern, there are not many examples.So, there aretwothings those are importantis thisare frequentlyusedis this, the bivariate normal distribution and secondoneis bivariate exponential distribution.

This bivariate exponential distribution we have discuss, while the in this module itselfin previous lectures.So, here we will justonce again we will mention it just you complete thisone.So these are thetwobivariate joint p d f and that probability function, probability density function is available. So,these are also of much use in different fields including civil engineering.

But hereonething is important that we also mention the earlier that, getting a joint distribution from there from the marginal distribution is not easy and for example, this bivariate normal distribution case also it is not that, if I get some marginal distribution, we cannot create what is their joint distribution opposite side this true.

Opposite side means that if the if both the random variablesare jointly a Gaussiandistribution then, we can say that theirmarginals are alsoGaussian, but the reverse is not true that also we willdiscuss.

So, what the major part is will be untouched, so far is that how to get the joint distribution.Because, these are the onlytwojoint distribution forms is available.So, therethat the recent technique oncopula that will be taking uptowards theend of this module,that I mentioned earlier.So,for whatever is theexisting effort from thecopula we will discuss and thistwoare the bivariate normal distribution and the bivariate exponential distribution, this we will include in todays lecture.

And after that the property that we are discussing for the case of the bivariate, we will take the same concept for the more thantwoparticularly if I remind that multivariate.So, more thantworandom variable cases we will extendthe same concept.

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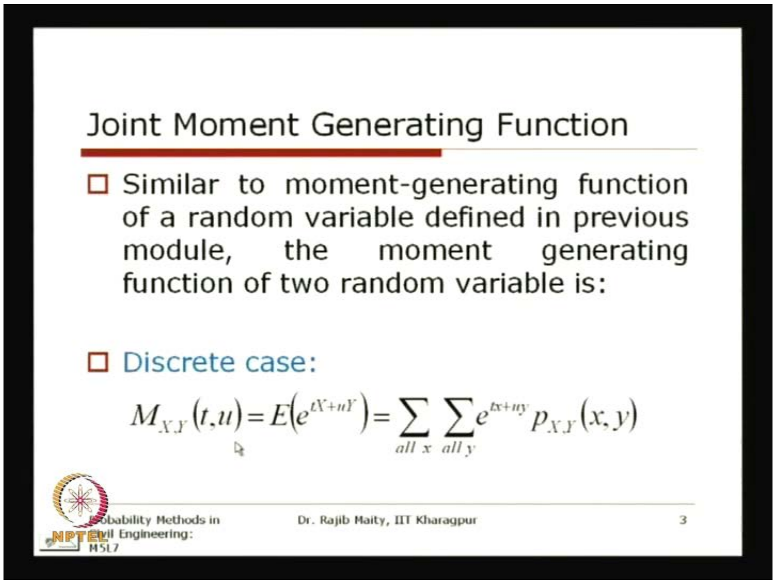

So, start with that joint moment generating function,similar to the moment generating function of a random variable defined in the previous module.The moment generating function of two randomvariables isagain,there aretwo cases.Oneis the discrete and otheroneis that continuous random variable case.

So, in case of the discrete you know, we have discussed for the single random variable.Here, there are two random variable involve then say that x and y. So, for this both this random variable the moment generating function is the expectation of the function of that exponential t x plus u y.

So,if we take this expectation of this function now in the earlier lecture, we have discussed for a function how to get that expectation ofa function.So, that this nothing but this function multiplied by their probability mass function and we haveto sum it over for all x and for all y.So, if we do thisonethe function that we will get that is the moment generating function of the joint PMF between x and y.

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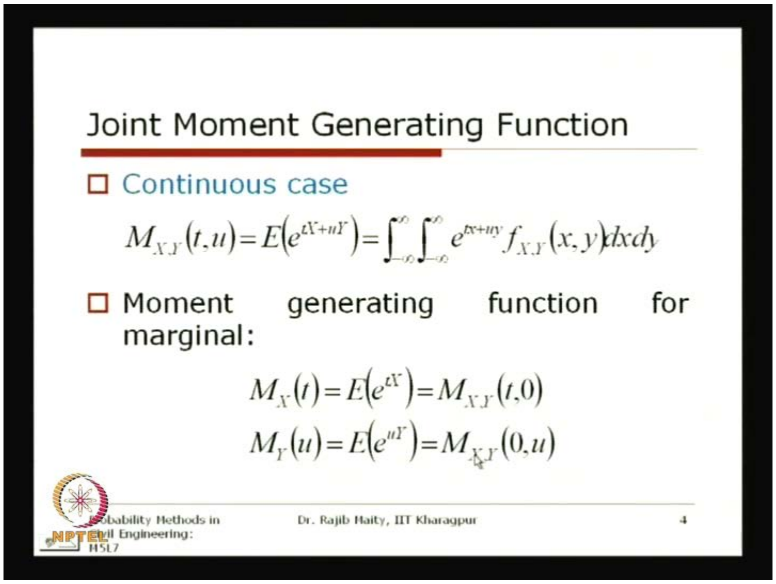

Similarly,in case of the continuous randomvariables if this x and y two random variables,which are continuous then it is also that expectation of thisof the exponential function, which is the exponential of t x plus u y and to get this exponential. We have to do the integration over the entire range of both the random variables.

Here,this it is integrated from the over the entire real line minus infinity plus infinity.This functionexponential of t x plus u y multiplied by their joint p d f x y x y and it is integrated over the entire range.So, thisonewe will get that moment generating function for the continuous random variable.

Now, just if we want to know theirmarginals now, what we have done that, it is just the power that we have use the power of this xand power of the y, so which are here both the cases isone.Now, if you want to know for the marginal andwe know that how to gets that moment generating function of theirmarginals.So, the marginal means you know that, we discuss earlier is that the distribution ofonerandom variable irrespective of the distribution of the otherone.

So, this moment generating moment generating function of the x, if I know then we know that this is the will be the expectation of the exponential t x only.So,now expect that exponential of t x means here, theu y that term was here just it is the y power 0.This component will not be there not y power 0.So, this component will not be there.

So, that u we can say it to be 0.So, not that y power 0 it is u equals to 0.So, now if you put thatu equals to 0,then we can write that the moment generating function of the joint random variable x and y there we have to put that t and for in place of the u,we have put the 0.

So, this is what we are explaining is that so how to get the moment generating function for the marginal that is if we know the moment generating function of the joint random variable then by replacingthe suitable attribute, we can get the moment generating function for the marginals.Similarly, for the y also we have to replace the t by 0 and we have to keep this u in from the joint moment generating function.

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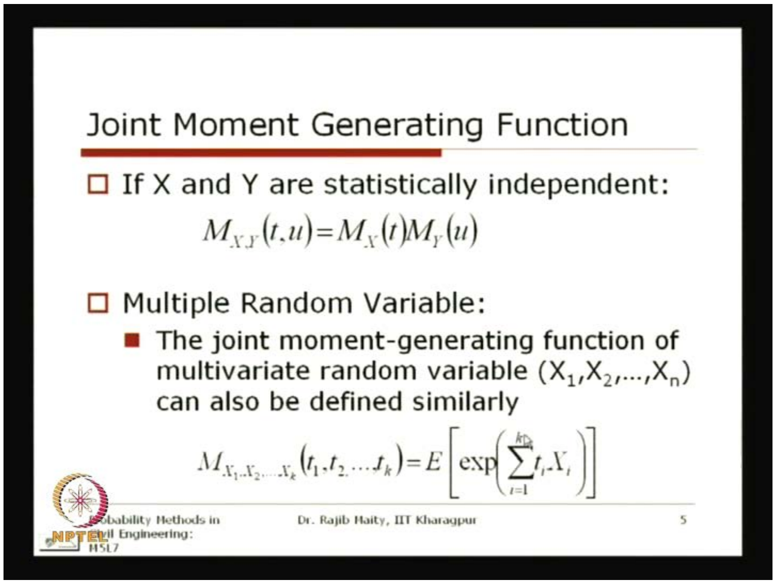

Now, this is also again we have discuss earlier that,in case that x and y are the are statistically independent then so far as the moment generating function is concern, then we can write that this moment generating function of the joint moment generating function should be the product of theirmoment generating function of the marginal of the random variable involved.

So, this m x t is nothing but the moment generating function of x and m y u is the moment generating function of the y. So, both are from the marginal and if we multiply it and if this product is equal to theirjoint marginal.So, joint moment generating function then we can conclude that this is x and y are statistically independent.

Now, for this multiple randomvariable just we have to simply, explain the idea that we have seen is that in case of if there are if there are k random variables, then this for the joint moment generating function between that x 1 x 2 and up to x k then, what we have to take is that there are different these are all means attribute that we are considering.So,this will be the expectation of a function which should be thethat exponential of that t 1 x 1 plust 2 x 2 plus t 3 x 3 up to t k x k.

So, basically from this now,if you just see that this is the for thesingle random variable is a exponential of t x, now for that two randomvariable.We are just adding that that exponentialsare taken for this t x plus u y, if there another random variable that we have to add with some other attribute. Similarly for this k a numbers of random variables isexplained here you have to take the exponential of this function that is tixii where is from 1 to k.

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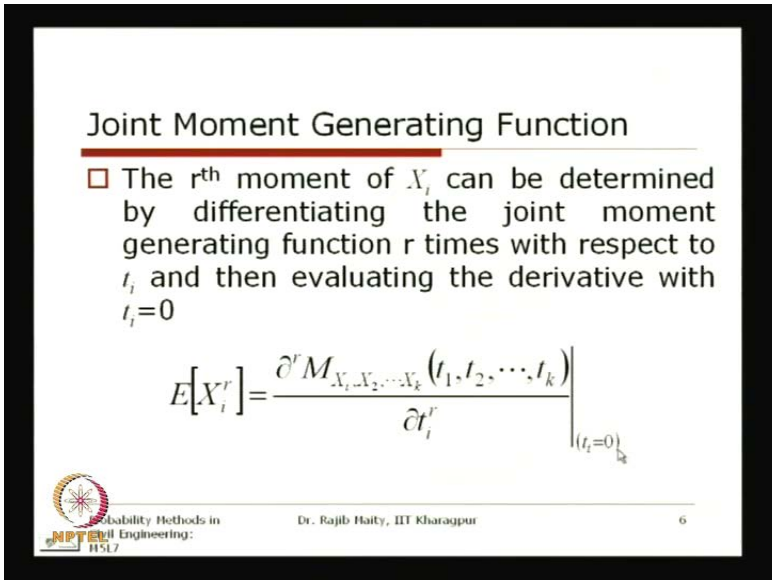

Now, after we get this joint moment generating function, what we generallyget what is the interest that, we have seenin case of this single random variable also. That we have to take some particular order of their derivative and you have to evaluate its value at the origin to get that order moment.

So, here also for the multiple random variable also, we have to get the suitablederivative with respect to the different random variable involved and we have to evaluate thatone at the origin to get their moment for that order.

Now, we will discuss it in general first that, if you are talking about the rth moment of the xi.This xiis what you are talking, whatever therandom variable involve so there are some krandom variables are there.So, anyoneif I want to know and I want to know, what is this rth moment then, what we have to do is that we have to differentiate the joint momentgenerating function r times withrespect to that tiand then evaluating that derivative with tiis equals to 0.

So,the expectation of thisith variableith random variable which isinvolved here power r.So, this moment will be the r th order moment thisoneis the derivative ther th order derivative with respect to that tiwhichoneis taken so it is written at the beginning. So,which is the x 1 butit isnot necessarythat ithas to be put here,so anyrandom variable between this x 1 to x k.

I think the better exponential,you know that it will be x 1 x 2 then xisomewhere in between whereiis I can take any value between 1 and k. So, and with respectso the and it should be derivative it should be taken with respect to the correspondingattribute that is the tihere and after getting this derivative,we have to evaluatethis value of this function at tiequals to 0.

So, this is the generalexpression and it is not only for a single random variable.We can do it for even for this any subset of this random variable.So, just I want to knowthat the expectation of x 1 x 2.Say for example out of these k random variables I just want to pickup x 1 and x 2.So,I have to take thatdouh 2,this moment generating function with respect to that douht 1 douh t 2.So, thattwotimes derivative first time I am doing with respect to the t 1 and then with respect to the t 2 and after getting this derivative, we have to evaluate itat t 1 equals to 0 and t 2 equals to 0.

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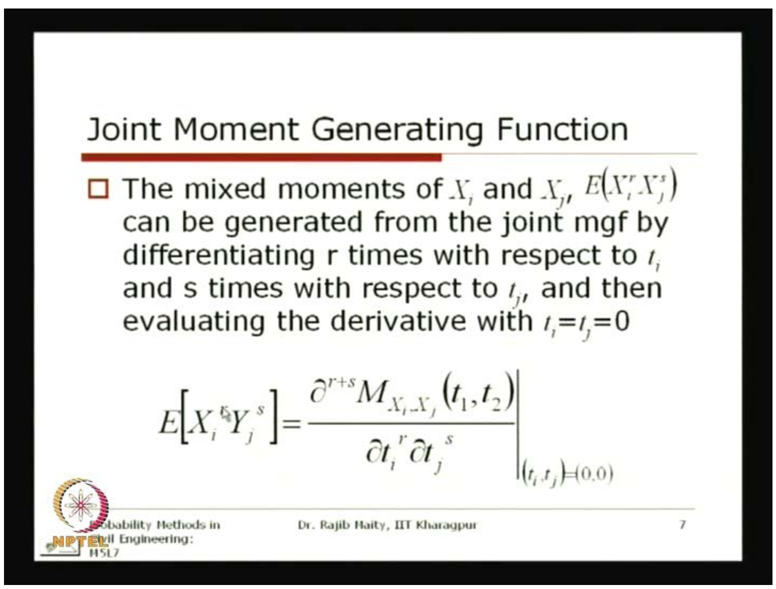

So, this is what we are explaining even in this general case,where there are say that two randomvariables are there the mixed moments.So,you have themixed moments means we are we even can change the order, so that there are two randomvariables thatxiand x j and for thexithe order is the r andxj the order is the s can be generated from the joint moment generating functionby differentiating r times with respect to the ti.

Because, it is xiso r times with respect to the tiand s times with respect to the t j,because this is x j power this says so s times with respect to t j and then evaluating thatthey derivative with tiequals to t j equals to 0.

So,whatisexplained here in case, that there are two random variable.So, we have to doing this derivative with respect tothat the r times with respect to the tiand s timeswith respect to the t j and then we areevaluating that function at tiequals to 0 and t j equals to 0.

So, where will this be useful is that now once, we get this expressionthen I can vary any value of this r whether I if I need that expectation of x. ThenI have to put thatr equals to 1 and s equals to 0,if I want thattheirexpectation of x y then I have to put r equals to 1 s equals to 1 or other way if I want that second moment of y. Then I have toputthat r equals to 0 and s equals to 2.So, in so this general expression canbe used whatever way we want to know.

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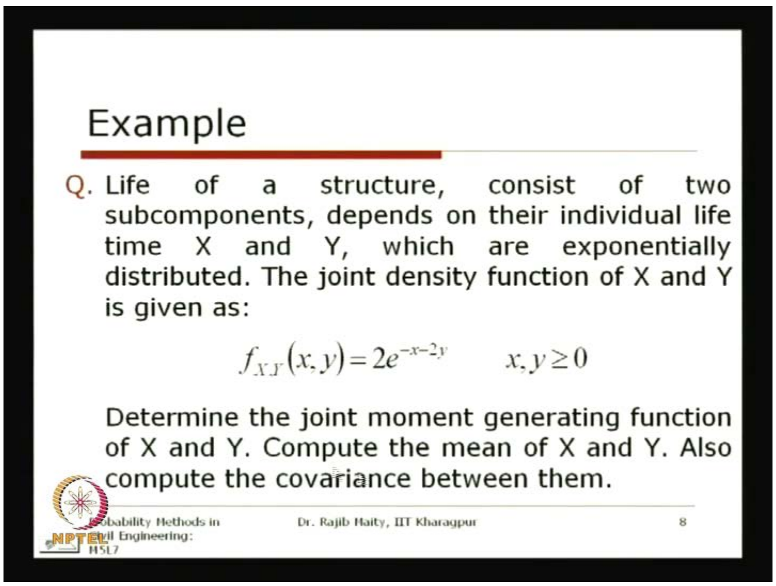

So, whereoneexample you have taken and this example again also taken very carefully, because the properties of this distribution we already know, we have we have seen now whatwearelooking for though this expression though thisproblem is thatthis thing the reason for this taking of this type of problem is that weknow what is this p d f.

And,in fact justby reading the problem itself,we should know that what isthe answer is expected.But, whatever the theory we have discuss with respect to this moment generating function or specifically the joint moment generating function, then we will again apply thatoneto crosscheck that whether what ourbelief is whether this isbeingsatisfied, through this problem or not that will see.

So, problem states herethat this is a life of a structure and this structure the life of thatstructure depends on the various components, say here it is simplified with respect to thetwocomponents only,so just to take this for the discussion problem.So, this life of a structure consist oftwosubcomponents depends on the individual life time say x and y. So, whatever target is that I have to model the life of a structure,that structure havingtwosubcomponents and thetwosubcomponents having their life time.Here both the life time of the both the subcomponents areonerandom variable each.

So, for the first component say the life the random variable denoted for the x for its life time and for the second component the life is denoted for its for a random as a random variable y and both thisoneare exponentially distributed and their joint density function of this x and y is given as this f x y is equals to 2 exponential minus x minus 2 y,where x and y is greater than equal to 0.Determine the joint moment generating function of x and y and then compute the mean of x and y, also compute the covariance between them.

Now before I proceed to solve this problem,why this problem has been taken that I was mentionthat I mentioned justminutes, before is that this function weknow and it can be easily shown that this both this x and y are independent and their joint function is nothing but the product of their individual marginal and marginal for the x is e power minus x and marginal for the y is 2 e power minus 2 y.

Thistype of problem we have taken up earlier.So, I am not going tohow toprove that thatonethat I am not taking at this level, because you can justsee fewlecture previous,where we havesolved this type of problem. So, with thisonethat what you know is that both this x and y are independent.So, if you knowthen the covariance between them obviously will be0, which also we discuss earlier.

Now, you also what we discuss the exponential distribution so e power minus x is the 1 exponential distribution for which is the marginal distribution for the x and the lambda parameter is hereoneand for the y the marginal distribution is 2 e power minus 2y where the lambda parameter is2.

So, that exponential distribution also you know and we also know that the mean of n exponential distribution is 1 by lambda.So, if you know that 1 then the mean of this x will be 1by 1 that means 1 and mean of the y will be 1 by 2 so half.

Now,this things so this things what we will do now for this problem is that, first of all we will determined the joint moment generating function and as we know the answer already, we will just cross-check we take that whether the how we can use this moment generating function to get the respective moments.So, once this answer we can check and then for the any other moment for this joint distribution we can use the same methodology to get that whatever the order of the moment that we need.

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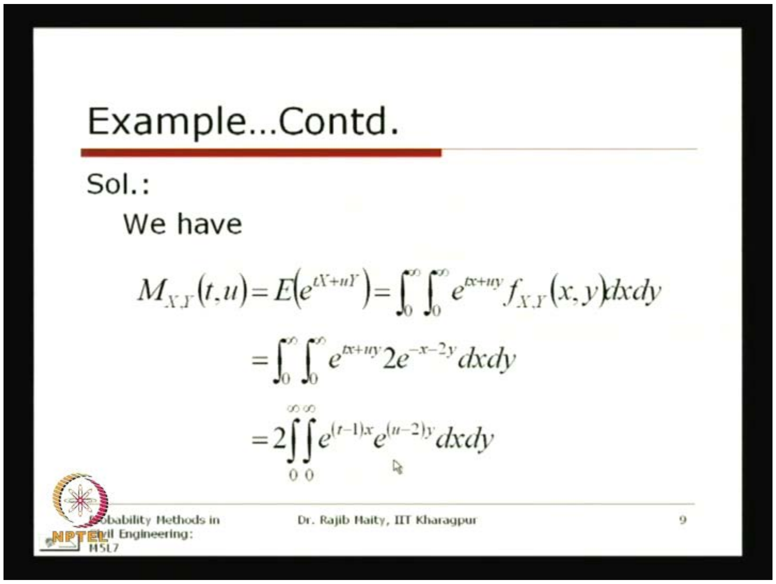

So, we will go for this solution and first what we have to get that we have to get its moment generating function.So, just now what we discuss is this moment generating function is for the two randomvariables x and y is the expectation of the function which is exponential of t x plus u y.

Now, thisonewe can express that te power t x plus u y multiplied by its jointdensity and it should be integrated over the entire support of the random variable, here both the random variables are taking the nonnegativerange.So, it is from 0 to infinity and 0 to infinity so both this x and y.

Now, this joint density we know that is 2 exponential minus x minus 2 y then multiplied withthis e power t xu y. So, this t and u are the variables to which this moment generating functionwill be will be expressed.

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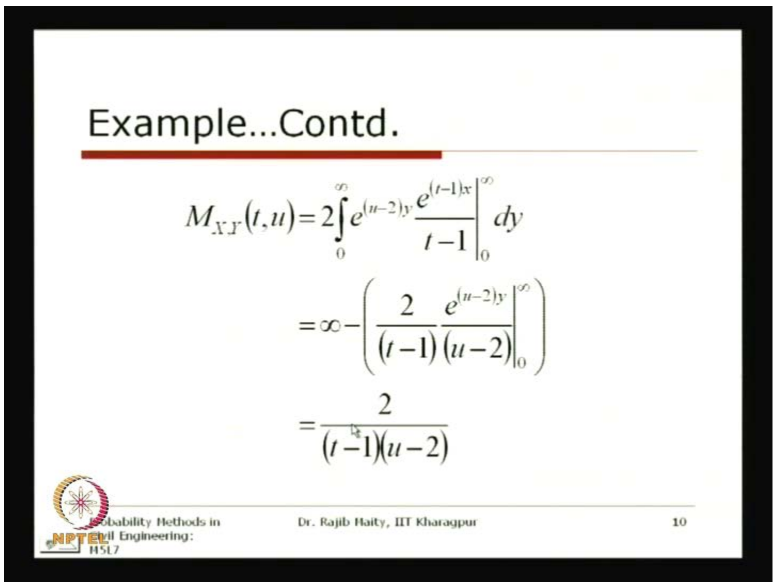

Now,after doing some as usual stepsand doing this integration will get this expression putting that suitable limitup toof course and will get that it is 2 by t minus 1 multiplied by u minus2.So, this is what is shown as to be the moment generating function for the p d f that we that we to, now our next step is to get their mean of x mean of y and its covariance.

Now, to get the mean of x what we have to take is that, we have to take that expectation of x and just now we have shown the shown the step, how to get the mean of their marginal distribution that is we have to put the suitable order, so here it is very specifically mentioned that how we are putting the order.

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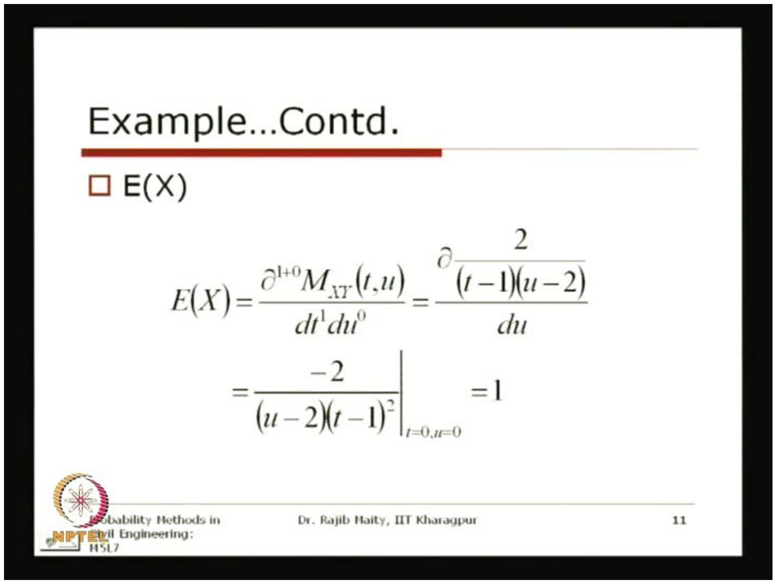

So, expectation of x when you are talking about we are putting that the order of this derivative should be for the u for the t, it should be 1 and for the u it should be 0.So, the basically we have we are doing a first order derivate with respect to u,this will be t so with respect to t, because t is here for this for this x.So, first order derivative of this moment generating function with respect to t this is wrong,this is t. Here, you can see that d t power 1 and d u power 0,so this is d t.

Now, if you do thisoneand at the end what we have to do,we have to put that t equals to 0 u equals to 0 and if we put this 1 with this expression then we are getting it to be 1.Now,we can see that marginal distribution of thexy, then exponential distribution we have seen that 1 and that is e power minus x where that means the lambda equals to 1.So, mean is 1 by lambda that is 1 by 1,which is 1.So, from the moment generating function also we are getting that its mean of the random variable x is 1.

So, now we will get what is the mean ofthis x and we know that before I go to the next slide, we know that that mean should be equals to 2.

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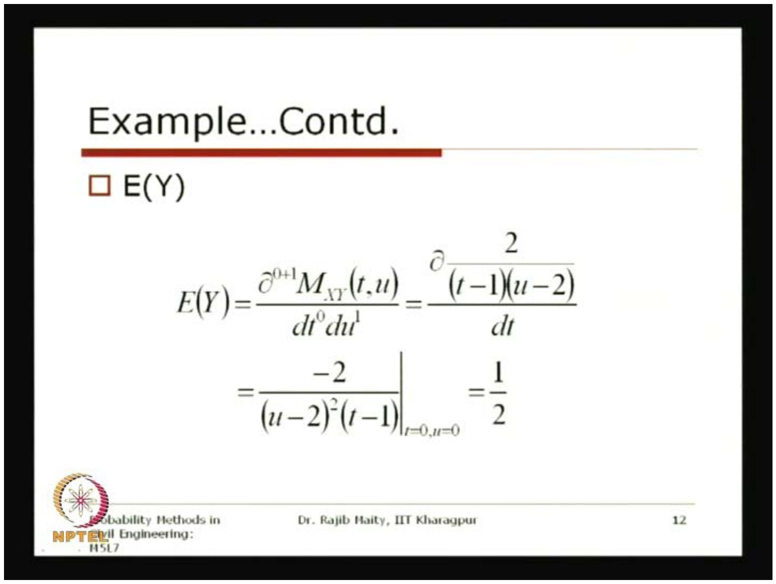

So, how we are putting thisonceagain following the same procedurethat is here we are putting that t equals to 0 and u equals to 1.So, this is again the first order derivativewith respect to u, so again this is just between this should be u.So,here you can see it from here that should be this should be u,we are we are doingthe first order derivative of the moment generating function with respect to u and after doing so what we are gettingis the value at t equals to 0 u equals to 0 and weare getting the value to be half.

So, whichis again as for ourexperience that if the marginal density is 2 t power minus 2 x, that is it is a it is an exponential distributionwith lambda parameter 2 and its means should be 1 by lambda which is equals to 1 by 2, so this is what is there.

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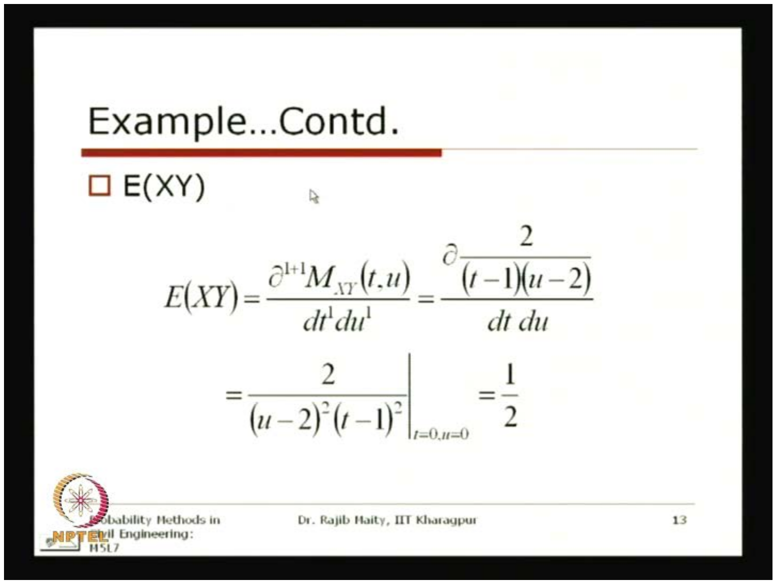

Now, what you have to do we have to calculate its covariance and you know the covariance to get the covariance. First of all, we have to get that expectation of that joint moment that iscovariance is covariance between x and y is equals to expectation x y minus expectation of x multiplied by expectation of y, so that expression we know.

So, if we know that expression we have to first get this quantity x y and thisonewe have done it earlier with respect to the p d f ,in fact all this things that is the expectation of x, expectation of y, expectation of x y we did it with the help of thep d f. But here what you are what we have towedoing we are showing that how we can use the moment generating function.

So, here that expectation of x y,that means here the both that r that notation that weusedminutes before the r equals to 1 and s equals to 1.So, this is that second order derivative with respect to d t and d u both.So, this that expression that isthe derivative of this moment generating functionwith respect to t as well as with respect to u. so, this will be d 2 this will be d 2 here, so 1 plus 1 we can see as this will be d 2.

So,if we do thisderivative when we will get this expression and again we are we will beevaluate this function at t equals to 0 and u equals to 0,which willyields the results to be half.So, expectation ofx y is equals to half.Now, the second on we already know that what is expectation of x and what is expectation of y, the covariance is just this minusproduct of those.

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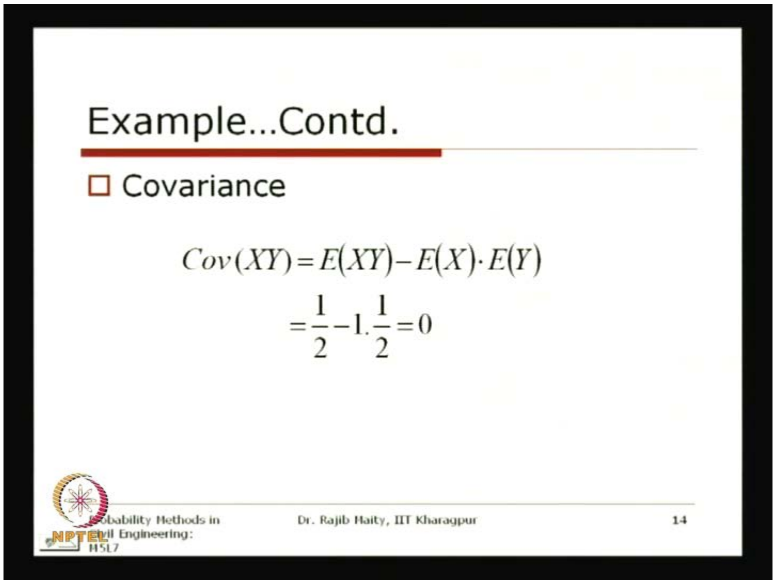

So the covariance is the expectation of x y minus expectation of x into expectation of y,we just simply put whatever we got andwe can see that this covariance is0. Now, covariance is 0 as, I mentioned earlier also again I mentioned,here that the covariance 0 doesnot mean that random variables are independent but opposite is true.

If the random variables are independent covariance is 0.We started this problem we knew that this to random variables x and y are independent.So the expected are what we should get I cannot use the word expected, because this is a statisticalword that we are discussing now.So we knew that covariance will come 0 and that is what we have seen, when we are getting those estimates from the moment generating function directly.

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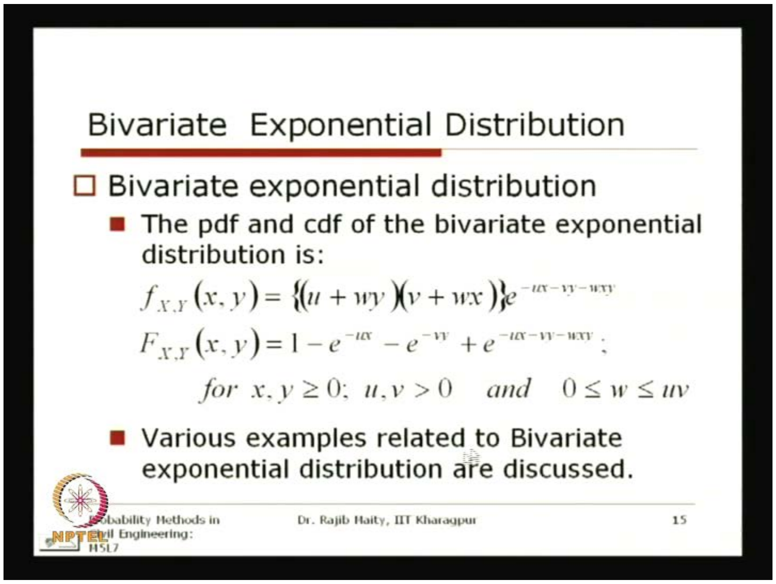

Next, we will take uptwo bivariatestandard distributions which is there and in fact thatbivariate exponential distribution that we are going to discuss now.We have used thisoneearlier also and in case of this bivariate joint distribution, we are having onlytwo randomvariables.If not onlytwothere maybe,but these are thetwothings that is mostly usedthatbivariate exponential distribution and next is the bivariate normal distribution.Thiswe will discuss a little and afterwards we will go for this multivariateoneand again we will come back to this, how to get the joint distributionfrom the marginal through copulalater.

So, now the bivariate exponential distribution herethis is ap d f and c d f of thisbivariate exponential distribution is expressed like this that this is thep d f, you know that f x y x y is equals to that u plus w y multiplied by v plus w x multiplied by exponential of minus u x minus v y minus w x y and if we dotheir if you want to know their c d f that is cumulative distributionfunction.

Then this is equals to 1 minus e power minus u x minus e powerminus exponential minus v y plus exponential minus u x minus uiminus w x y.Now range for this this x and y of the random variables for which therange is greater than equal to 0.Soit cannot take the negative value the support for this distribution is from 0 to infinity.

The parameters, there arethreeparameters we can seeoneis the u v and otherone is w. So, this u and v is straightforward which will which is the nonnegative parameter this is greater than 0.So, far as the w is concern it is having it is limited bytwothings.One is that itshould greaterthan equal to 0 and other 1 is that it should be lower than this u v.We took this problem earlier, just to know that what the possible range of this third parameter w is. Now we can see here ifthe w can take from the 0 to u v so upper limit we have seen the lower limit,if the w is equals to equals to 0 then this component u will not be there.

So, that case basically when we take that two randomvariables x and y are independent, then this component cancels out, so that just now you have seen.So, this w parameteris basically taken care of if there is any means if there is any dependence, between x and y that is being taken care through this w. So we have discussed this problem just now also and earlier also and we have seen their parameters earlier.So, this istheoneexample of this bivariate distribution which is exponential and other one is this bivariate normal distribution.

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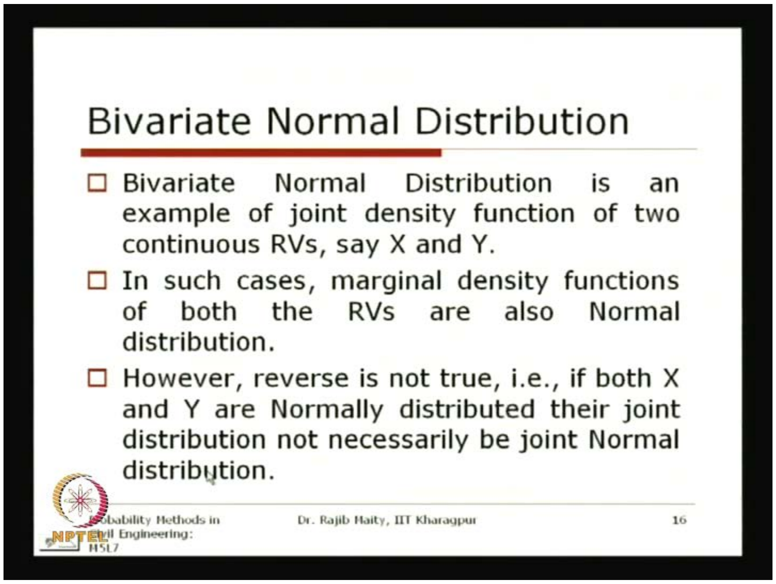

This is having many fields we use this whenever even earlier that, whenever we are talking about this joint distribution.We generally go for this bivariate normal distribution, because this it is grounded on the form theoretical basis and to get for the other distribution, other joint distribution is not that easy and so we have found earlier there is atremendousapplication of this bivariate normal distribution in many research fields including civil engineering.

So, this bivariate normal distribution, when we are talking this is then an example of the joint density function oftwocontinuous random variables say that x and y and in such cases the marginal density function of the both the random variables are also normal distribution.So,you remember that first we are we are stating this and after that we are stating thisstatement, that is first it is if their joint density functionis Gaussian means Gaussianor normal distribution, then we can say that theirmarginalsare also normal distribution.

However,the reverse is not true that is if both x and y are normally distributed their joint distribution not necessarily be joint normal distribution, so this not necessarily be joint normal distribution means thatthe reverse is not true, what we are we have also mention this point earlier as well.

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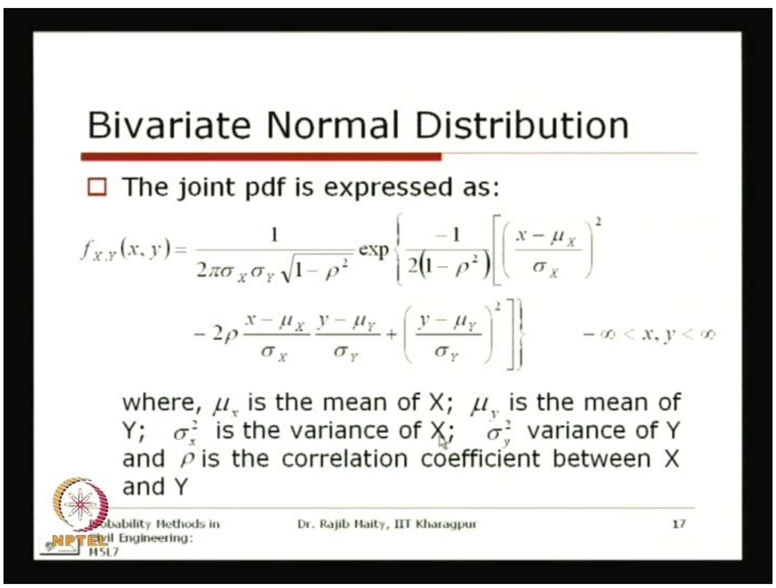

How this p d f is it can be express as like this?So, there are basically fiveparameters for thisone,you know for a normal distribution there are there aretwoparameters,oneis that its mean and anotheroneis that variance.

So, here there are two randomvariables are there so that foronerandomvariable, this will be the mean of 1 random variable variance of that random variable and mean and variance of the other random variable and their joint dependence.So, there are total 5 parameters will be there and this 5 parameters are here you can see that this that sigma x is the standard deviation for the x, sigma y is the standard deviation of the y, this rho is the correlation coefficient that we discuss earlier.So, this rho is the correlation coefficient between x and y, then this mu x is the mean of this random variable x and this mu y is the mean of the random variable y.

Now, just to read out thisoneis that 1 by 2 pi sigma x sigma y square root of 1 minus rho square exponential of there arethree parameter,that there arethreeparts of this what is there inside this exponential.So, firstonethis is taken common that is minus half by this 1 minus rho square after this this is multiplied with thethreedifferentparts.

The firstoneis this x minus mu x by sigma x whole square.We can see here this is written in such a way that you can see now this is also can betaken to the reduced random variable, that is transformed that is x minus mu x by sigma x. So, this is why how it is written just in a minute we will take upthatone.So, this is x minus mu x by sigma x whole square and third component is the y minus mu y by sigmay whole square and second component is their joint again that is2 rho x minus mu x by sigma x multipliedby y minus mu x by sigma y.

Now,you can see here it that if they are independent that is if the x and y are independent then we know that this will be the product of theirmarginals.So, that is the check for this independence, so if they areindependentthen you know the covariance will be 0.Then if the covariance is 0 then correlation coefficient is also will be 0.Youput that correlation coefficient that is rho equals to 0,here then we can easily separate out twodistributions, which are the firstonewill be the normal distribution for x and otheronewill be the normal distribution for y.

So, this we can say so this rho is basically taken care of, if there is any dependence between x and y and also you know that x and y are the support is from the minus infinity to the plus infinity.So, whatever I told it is written here that is mu x is the mean of x mu y is the mean of y sigma x square is the variance of x sigma y square is the variance of y rho is the correlation coefficient between xand y.

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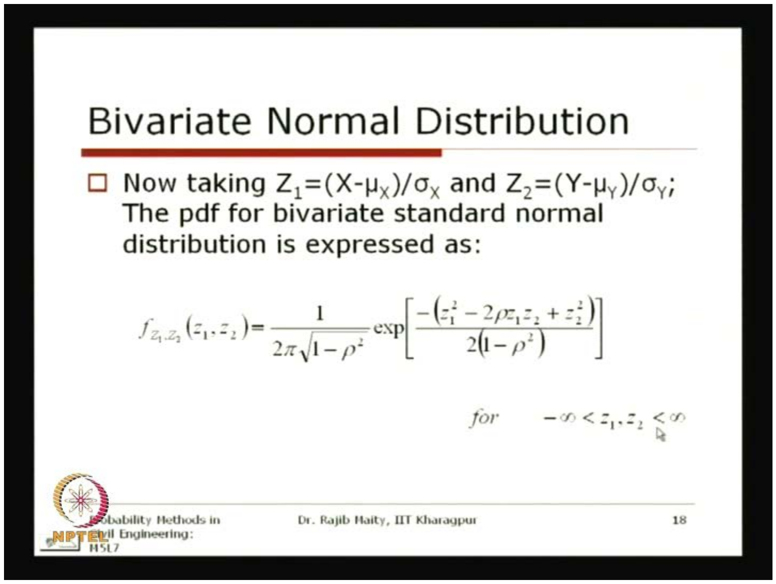

Now, just what I was talking that if you just replace thisonethat replace, means that if I put that another random variable z 1 which is the x 1 minus mu x by sigma x and z 2 is the y minus mu y by sigma y. The p d f of the bivariate standard normal distribution, so earlier we have seen that after this transformation, we are getting the standard normal distribution here also we are gettingthat is again bivariate.

So, this bivariate standard normal distribution if we want to know then for this expression you have to put that z 1.So, this then they are p d f joint will be 1 by 2 pi square root 1 minus rho square exponential of minus z 1 square minus 2 rho z 1 z 2 plus z 2 square divided by 2 into 1 minus rho square. For which again this z 1 z 2 are also not from minus infinity to plus infinity this is from0 to 1 this is from 0 to 1.

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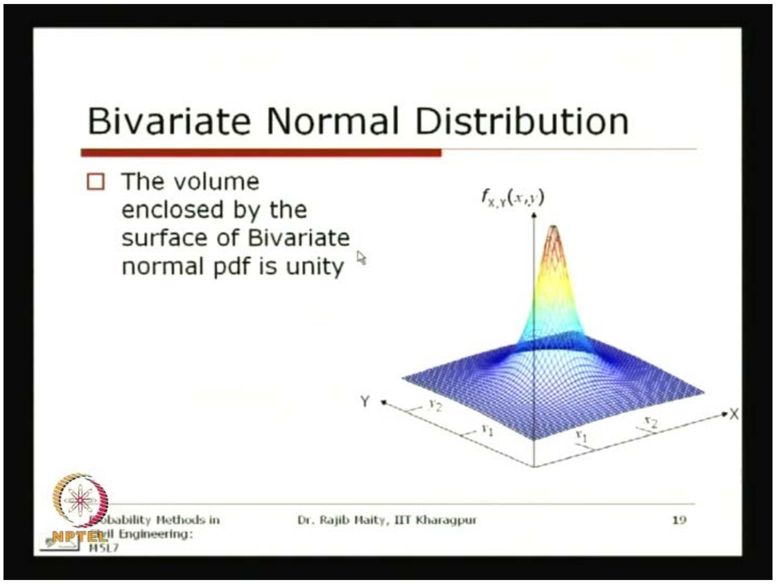

This is in an example, just to view that how the joint distribution is the joint distribution looks like; here we have takenoneexample of this joint normal distribution.We can see that now this is kind of surface andthatis athreedimensional surface and the volume below this surface that is athreedimensional volume and that volume if we just extend the idea of this that, what we have discuss for this single random variable thatarea below the curve for a single random variable is equals to one.

Herein case of the joint, if I take the bivariate case, because this pictorial representation is possible only up to the bivariate case this is x and y. So, this is the kind of surface and there is avolume is there betweenthe x y plane and that surface.

So, what we are talking if you refer to this figure is this is this is your x axis, this is your y axis,so this is your x y plane.So, the volume in between this surface and that x y plane should be equals to 1,just to satisfy what the requirement for a p d f. That is the totalvolume that is between that is covered by this surface is equals to 1.So, volume of enclosed by the surface of bivariate normal p d f is unity is 1 is shown here.

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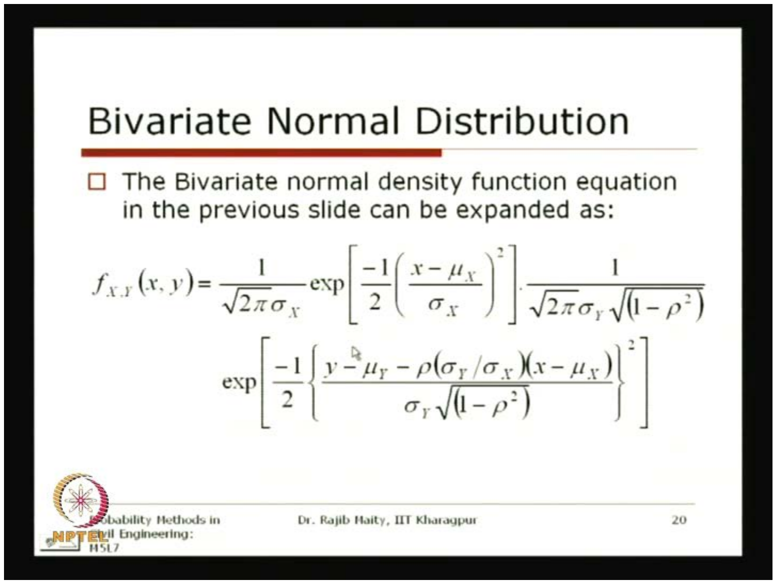

Now, thisonewe will just express this in a similar way just after doing some algebraic manipulation just we will see that why we need this one.So, the same expression so this expression that I am going to tell you now and the joint distribution that is between the x and y are same. What we are doing is that, we are first we are trying to separate out what is the marginal distribution of x. So, marginal distribution of x which is having the mean mu x and variance sigma x square, then we can we know that what is this distribution by this time.

So, if we just want to take out that component out what is this remaining, that is whatcan be done in few steps and this is what is done.So,we first of all what we are writing isthis 1 by square root of 2 pi sigma x exponentialhalf x minus mu x by sigma x square.So, this is nothing but the marginal distribution ofx and this quantity is taken out and what is remaining we are just expressing init here thatone by square root 2 pi sigma y square root of 1 minus rho square.So, we consider this is the outside constant and this is another constant, if we multiply this 2 it is coming that 2 pisigma x sigma y square root 1 minus root square.

So, similarly in thatway so whatever was the inside, we have just taking we have taken thisoneout so what is remaining is noted here.Why this is important, why we are explaining this, why we are expressing this same expression in ain a different way.

We have just to know their what is the conditional distribution and you know that we have discuss for the bivariate case the conditional distribution isexpressed and thishas to be just divided by their marginaldensities.So,that is why the marginal density is taken out.

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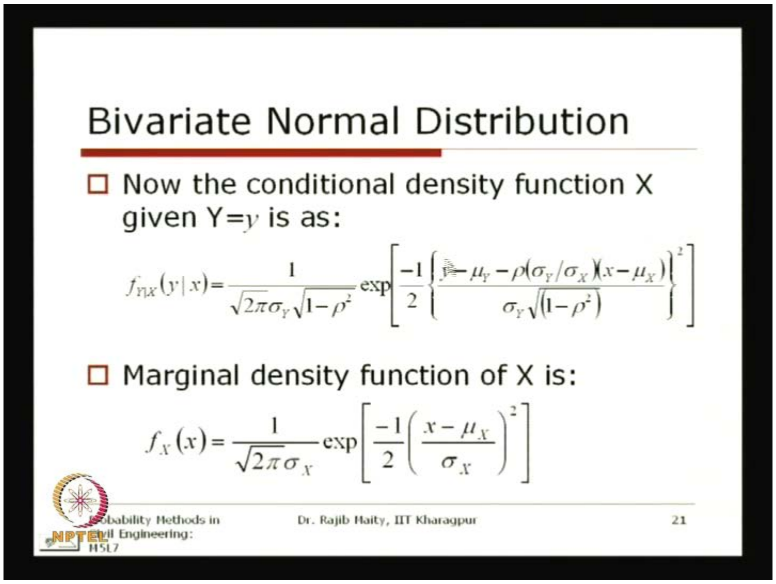

And if I just want to know, what is the conditional density function for this x given y equals to this.So, thisf y given x is equals to so this will be the joint density divided by themarginal density of x.

So, now you can see that this is just said this is wrong so this will be the conditional density function of y given x equals to x. So, this is what is expressed here also so this conditional distribution of y given x is that the joint density divided by the marginal density of x now what we will get.So, this is the marginal density of the x, so this has to be taken out.So, what is remaining here is nothing but the conditional density of y given x.

So, this is what is explained here now what you can see here is that this is also taking the form of a form of a normal distribution as well. How it is taking the form, because you can see that this is square root of 2 pi sigma y this 1 so this pull component, that is the standard deviation that is sigma y multiplied by square root of 1 minus rho square is the standard deviation of theconditional distribution of y given x.

Similarly, exponential half y minus this full quantity is the mean of the conditional distribution of y given x. So, the unconditional mean of y is mu y,now on the conditional mean is mu y minus rho multiplied by sigma y by sigma x multiplied by x minus mu x so we can see that theconditional mean is changing with respect to that it is dependenton the value of the xit is it is here and similarly sigma y 1 minus rho square which is the standard deviation of that conditional distribution.

Marginal density function of x here is so that this is the marginal density function of the x which we have divided it by to get this conditional density.How to get the conditional density, that we discuss earlier you can recall that f y given x is equals to f x y divided by f x, that is that joint divided by the marginal.So this marginal was divided by to get this conditional.

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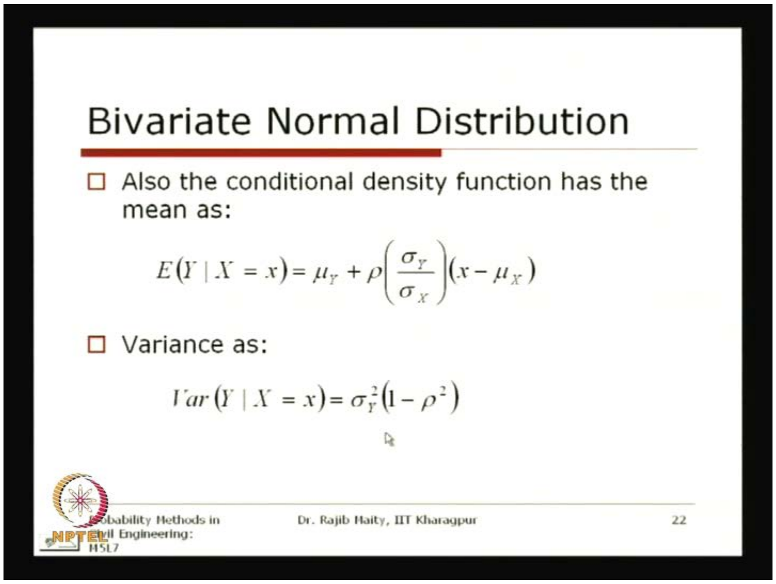

So, once again we will just see what is their conditional mean and conditional variance. So this conditional mean is this that is conditional mean of this y given x equals to x is equals to mu y plus rho multiplied by sigma y by sigma x multiplied by x minus mu x. Just what we have seen it from this expression, this expression is also the Gaussian distribution, that is normal distribution and from this normal distribution form, we can say that this component is the mean of that conditional distribution, which is basically we are taking this minus common.

That is why this sign becomes plus so this mu y plus rho sigma y by sigma x multiplied by x minus mu x, this is the conditional mean, which is shown here and the conditional variance is the sigma y square 1 minus rho square variance we are talking about, because this is the standard deviation so this conditional variance is sigma y square 1 minus rho square.

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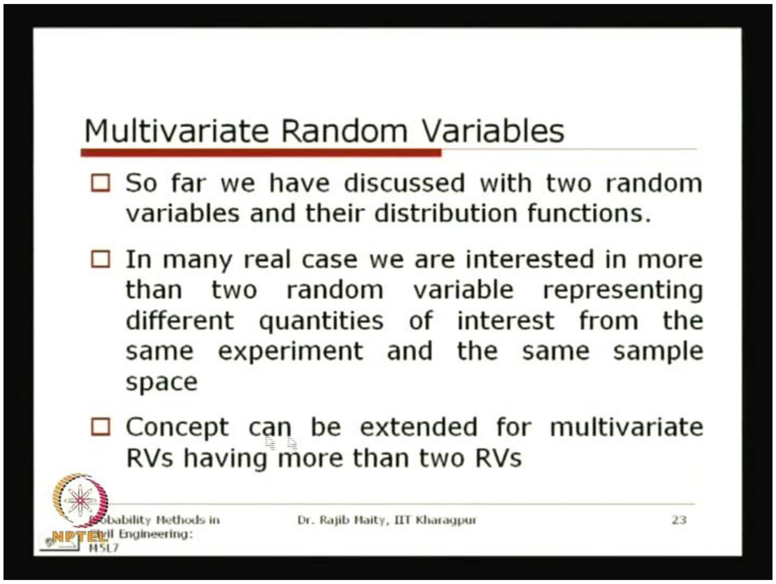

Now, as when we started this lecture that we will takesome of this bivariate concept to this multivariateone,the concept can be easily extended so we will discuss about thatonewith the context of this multivariate random variable having more than two random variables.Basically, we will try to generalize it in terms of that n random variable involved.

So, far we have discussed with the two random variables and their distribution function their different properties and etcetera. We have discuss we have taken of some problems also,but in many real cases we are interested in more than two random variables, representing different quantities ofinterest from the same experiment and the same sample space.

So, what is the volume, what is the totaltraffic volume at the junction, if you want to know then this kind of analysis is there were the number of random variables involved will be more.Similarly the different source of the fresh wateris joining to a reservoir; it is draining to a reservoir.

So, if we just want to know their joint behavior there also this kind of application can be found out.So,now to do thisonethe concept that we have discuss for this bivariate case can be extended for the multivariaterandom variables as well having more than two random variables.

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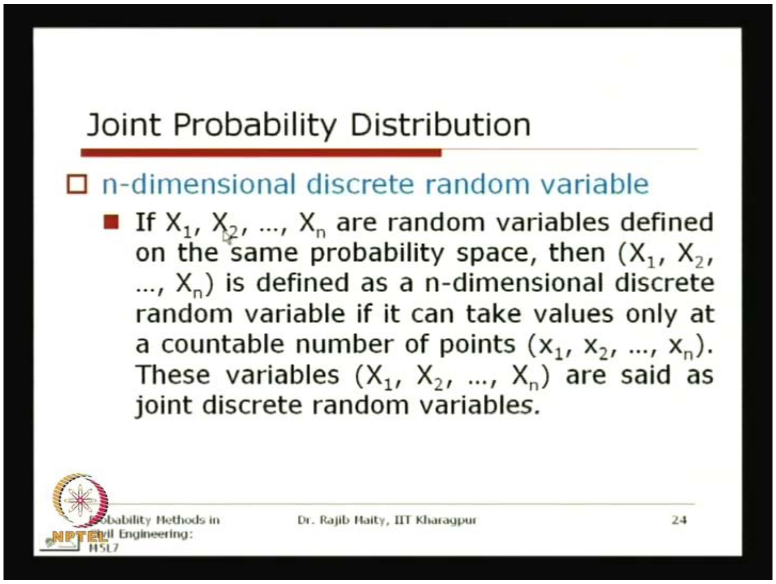

So, now discuss this things in the context of the generalframework that is the n dimensional discrete random variables, if we want to discuss first so this is n dimensional means there are n random variables n triple cases are we have to consider jointly and these are the discrete random variable, incase of discrete randomvariable we are discussing.

So, if x 1,x 2 up to x n of the random variable defined on the same probability space, then x 1,x 2 up to x n is defined as an n dimensional discrete random variable, if it can take values only at a countable number of points x 1,x 2 up to x n.These variables x 1,x 2,x n are said to the joint discrete random variables.

Basically,means what we discuss earlier is that for atwodimensional case, just take x 1 and x 2 there aretwoaxesorthogonal to each other and take some points.So, it can take only few points on that plane made bytwoaxes x 1 and x 2.Now just extend thisoneto this n dimensional space, where there are n axes which are thex 1 x 2 up to x n and take some point which point will be should be denoted by the n coordinates.Now, n coordinates means x 1 equals to this x 2 equals to this x 3 equals this like this up to xn equals to this.

Now, those few those specific points are consist that probability space.So,this if I take the full Ishould not say then volume,because volume is again thethreedimensional.It can go even more thanthreedimensional,so for those specific cases, if the probability mass is concentrated then that is known as to be that multivariate joint discrete random variable.

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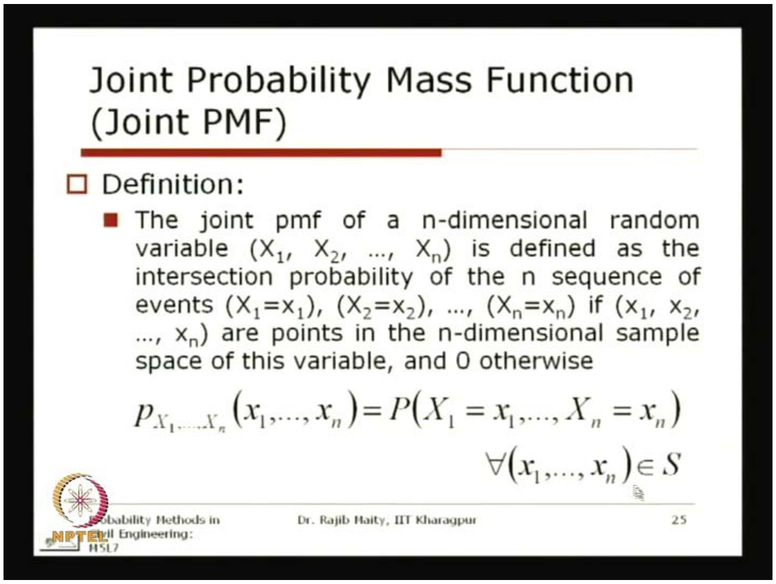

Now, know that what is there probability involve is thisone,so that the joint PMF, because you know this is discrete.So,we have to discuss thisonewith respect to the PMF probability mass function of that n dimensional random variable that is x 1,x 2 up to x n is defined as the intersection probabilities of the n sequence of the of the event.

So, this n sequence of the event is this that is if x 1 equals to x 1 that is now, you can know that these are the random variable this is specific value of the random variable.So, this x 1 takes the specific value x 1 x 2 take specific value x 2 and up to like this up to the n takes this specific value x n. So, for this joint case means all this things are occurring together.So, this is that if that x 1,x 2,x n are the points in the n dimensional sample space of this variable.So, that probability is taken only for those few those specific pointsotherwise it is 0.

Then, that is the joint PMF between the randomvariables x 1,x 2,x n and this is what is shown here that is the probability mass function is defined as that for this x 1,x 2,x n is the probability of the joint occurrence of all this events and this are belongs to that that probability space this outcome the specific outcome joint outcome is belongs to thatsample space.

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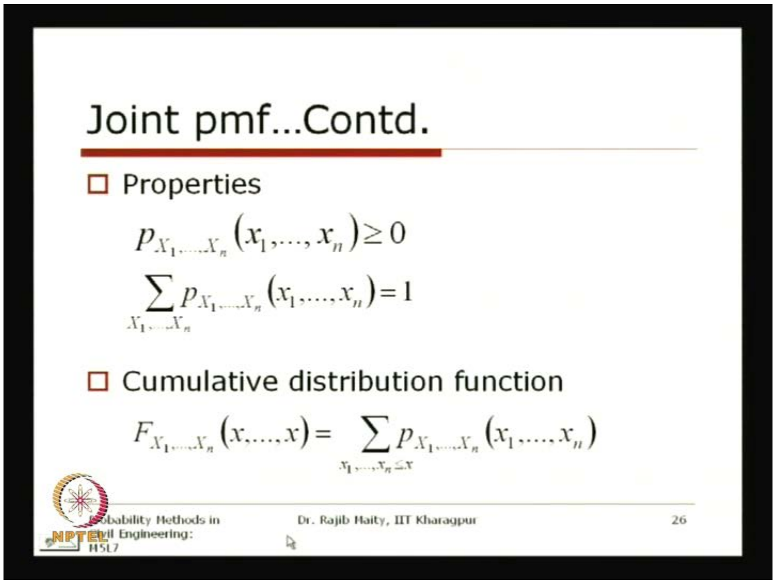

Now properties again extending the same idea that, it should be the total if I take this each and every point.If I take what is the probabilities so that probabilities should be greater than equal to 0,so it should not be then negative number.Again if I just sum up for the all the probability masses for all those feasible points then it should be equals to 1.

These aretwobasic criteria to satisfy the condition for to a validPMF that, we have discussed earlier forthe single random variable.Forthe bivariate random variable, now here it is for this multivariate for the n number of cases.

Keeping the basic conception, these are thetwoproperties to satisfy the condition of a PMF and the cumulative distribution function is that you know that cumulative distribution function here, just I will justtake a minutethat all this attributes now I have taken for this x x 1,x x 2,x nx. Why wehave taken I will just explain in a minute, this could have again is this could have been also that that differentactivity that is for x 1 it is x 1,for x 2 it is x 2 that also it can be done.

But just to show herein this case if we are talking about only a single point of in that probability space, then it is the summation of all the feasible points, those are below this specific number x that is x 1 less than equals to x, x 2 less than equals to x up to this 1 that x n less than equals to x. If I sum up this 1 then I will get the cumulative distribution function for that joint PMF in case of the n a random variables involve in that multivariate random variable case.

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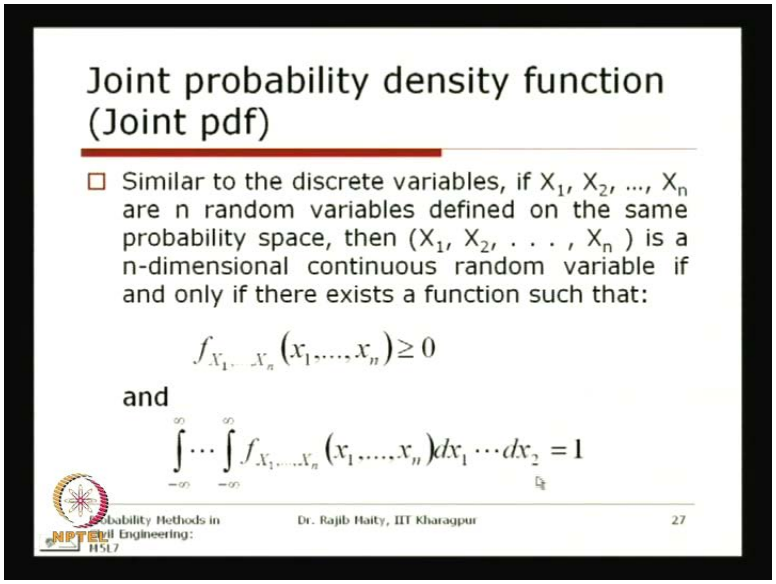

Now for the same is, if I take for this continuous random variables also if this x 1,x 2,x n are the continuous random variable that is similar to this discrete randomvariable.Ifthese are the n random variables defined on the same probability space then x 1,x 2,x n is ann dimensional continuous random variable.

If and only if there exist a function such that these are thetwoconditions again,you know that for the function should be a for all forthe entire range of this random variable should be a non-negative number, should be greater than equal to 0 and their integration over the entire range of each random variable should be equals to 1.

And you know that this should be n, so d x 1 up to in this way up to d x n. So,n numberof this integration in this n different direction of this random variable, so d x1, d x 2 up to this it is d x n so these are thetwoconditionsthat should be satisfied.

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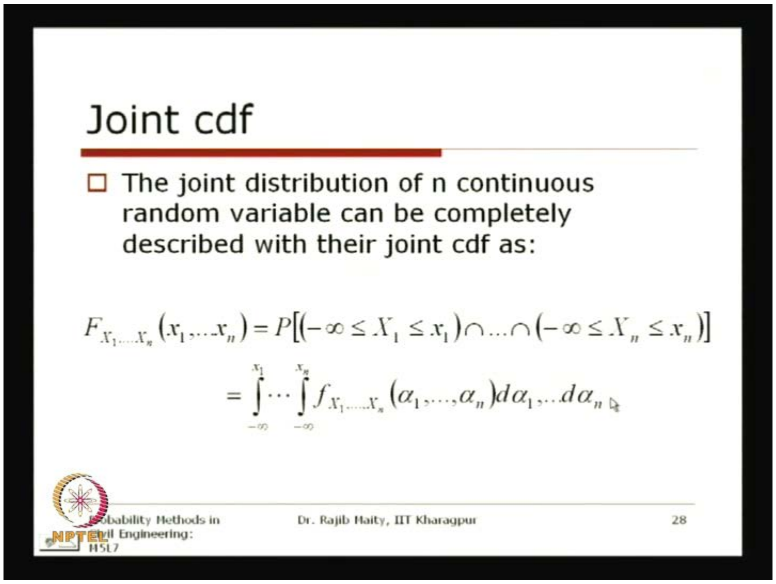

Then joint c d f again, if we are talking of about then here we have just taken as that is a even more general case,for the discrete also that can be the general case,but we have discuss itthein a specific condition.

So, here where we are taking that the x 1 isthe specific value up to this x 1 and likethis, if we want to know what is their cumulative.What is joint cumulative distribution function is that probability of the x 1 from minus infinity to x 1,intersection x 2 from minus infinity to x 2 like this up to x n from the minus infinity to that x n.

So, ifwe take this joint occurrence this intersection means,we know that is the joint occurrence of all this events, then thisoneyou know that.We will get it from this integration that is from minus infinity to x 1 for this d x 1, thishere that is that non variable is represented as alpha 1.So, this d alpha1 like this n times integration up to thisd n.This can go in this way to get this joint c d f for this joint multivariate case.

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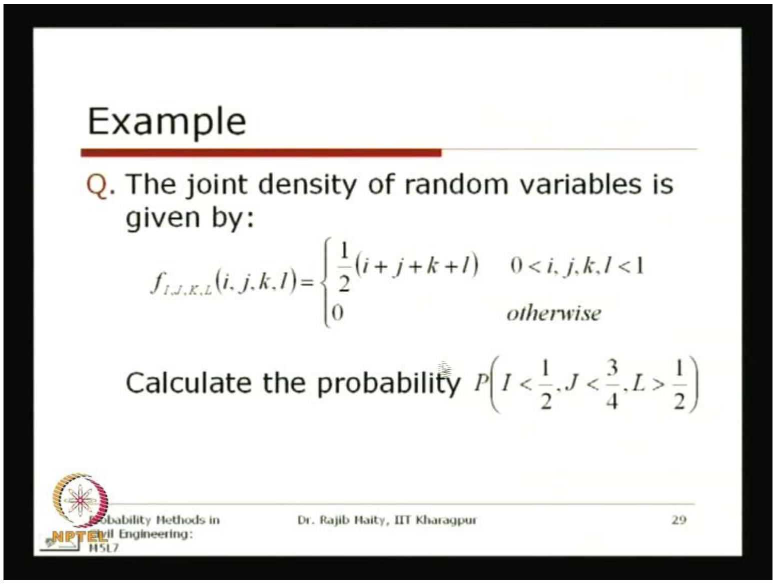

One quick example, we will take this is theirs involve thefourrandom variablesij k and l.This is the p d f is defined by and we have to calculate the probability ofiless than half j less than 3 4 then l greater than half.So there is no so for k it is not defined anything.

So, justthis is you can so here nowonething that we have not that we have not shown here you can check yourself that whether this is a valid p d f or not.Now to check whether this is a valid p d f or not you have to do that integration for the range between given 1 for all the random variablesfourtimes integration,you have to do and you have to put theirintegration limit and check whether the final is coming to be 1 or not.So, here what you aretrying to look at that for this specific space what is the probability.

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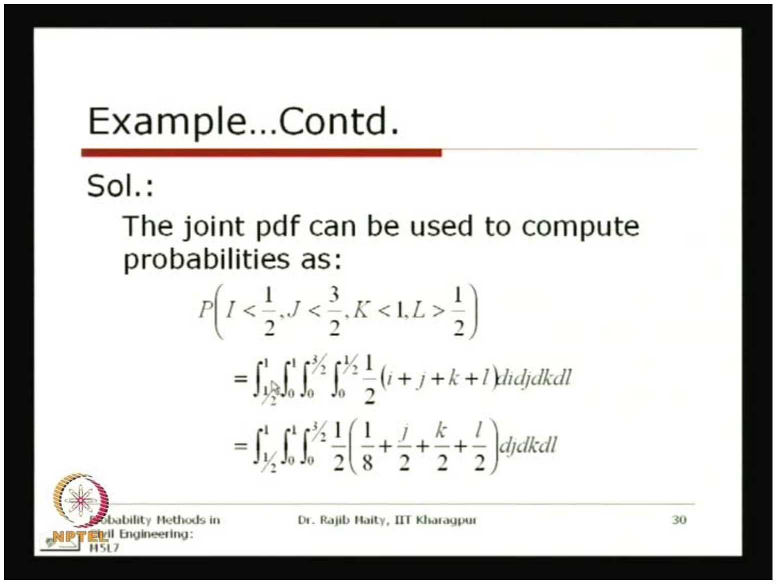

Simply, we have to do the integration for this specific range for this 1,you know that for this half for the j, it is the range is the half to 1 and for theiit is the half to 1 for j this is 0 to1 for this k.For thisiit is 0 to half, for this j this is 0 to 2,third for k it is the entire range 0 to 1 and for the l it is half toone.

Because, we can see that then it is satisfying this limit.So,the inner 1 the first integration is for thisilimit for iwhich is less than half and the lower range always is 0,so it is 0 to half for i.Similarly, we have define this integration limits this is the p d f and we are doing this integration.

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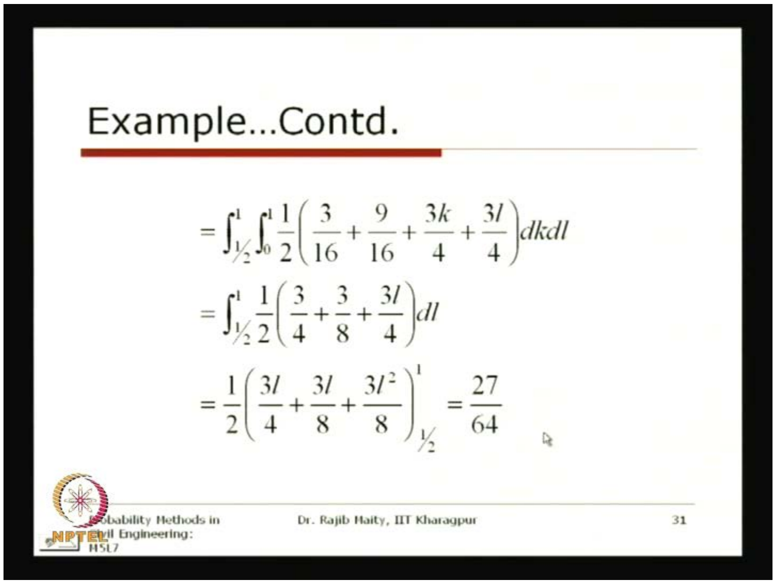

To occur doing this integrationwe will gradually get, what is their joint now what is the joint probability is 27 by 64. So, in this lecture we have discussed about theirjoint moment generating function for multiple random variable.Then we have taken thattwostandard bivariate joint p d f and c d f bivariate exponential distribution and bivariate normal distribution.

After that we have taken the multivariate cases and we have discussedoneproblem as well.So, in the next lecture we will discusssome morethis properties and we will also discuss about the functions of the multivariate random variable case and after that we will again discuss about this multivariatecase to get their joint p d f from the marginal oneusing copula thank you.