Probability Methods in Civil Engineering

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Lecture No. # 17
Expectation and Moments of Functions of RV

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Module, this fourth module we are discussing different aspect of the functions of random variable. We have discussed earlier that functions of random variable are itself is random variable. And so far in this module we have discussed about the different method to know the probability distribution function of the functions of random variable. Now, if you call that the functions of random variable as a derived variable, then the how to get their properties, their characteristics of different probabilities? That is probability density function and there after that there are different properties of those random variable, those derived random variable that we have discuss.

Now, in this lecture what we will discuss is the expectation and moments of the random variable. So, this expectation and moments of the functions of random variable why it is needed is that? Sometimes when, even though we have discussed different methods of this to derive this derived random variable. But it may not be always possible to get a close form solution for this kind of derived random variables to get their PDF. Now, it is true that if we get their direct PDF that is probability density function then all other properties for example, their mean, their variance, their skewness everything we can obtain from that PDF or that distribution function. But sometimes it has been found that this may not be possible or that is computation is not viable to get that PDF.

So, in such cases if we can get the moments are the expectations of the some initial moments of the derived random variable, then that is will be very useful without knowing it is probability density function the form of it is probability density function. So, the theme of this, theme of today’s lecture is that we will find out that how to get the expectation and moments directly without having it is PDF probability density function.

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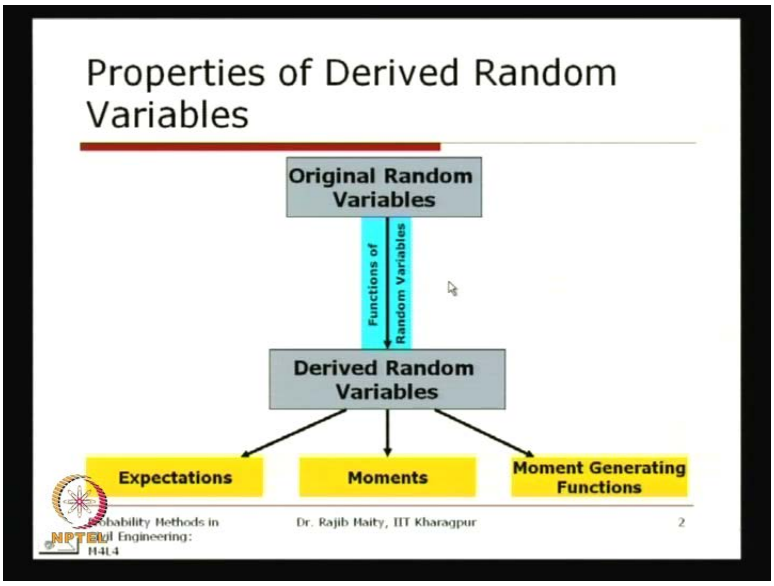

So, this lecture what we will do that? We will do that expectation and moments of these functions of random variable. So, before we do what we are doing is that in this module we are so, there is one original random variable. And we have seen that, this original random variable can have it is some functional correspondence to another variable and that variable is also a random variable. So, this function of this random variable itself is another random variable, which we are calling now a derived random variable. Now, there are so, far in this module as I just now I told that. For this random variable, for this derived random variable we have discuss different methods based on the fundamental theorem that we discuss in the second lecture that how to get that PDF?

So, if we, but sometimes it may not be computationally feasible to get that direct the distribution function. So, what is our interest in today's lecture is that we will know that how to get their expectation their moments or if we get their moment generating function. Then it will be easy to get that almost the complete description of that random variable of the properties of that derived random variable will be known to us. So, we will take one after another maybe the expectation moments we will take first. Then we will go for this moment generating function.

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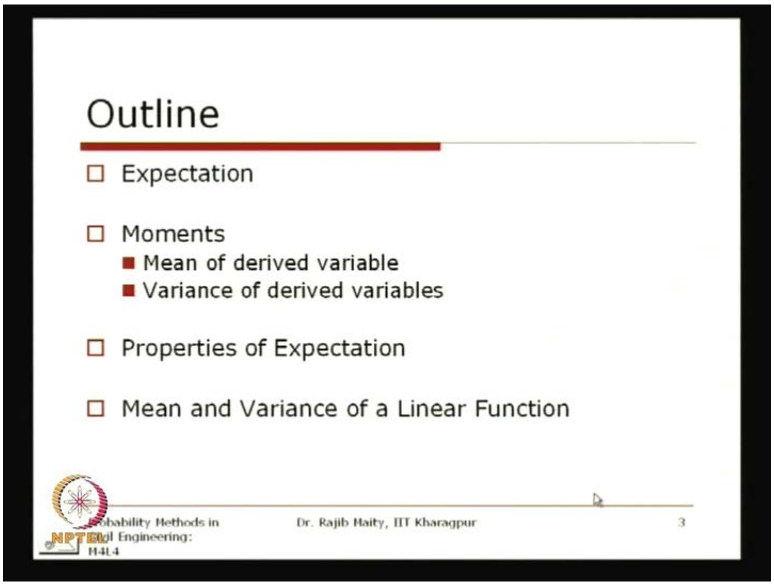

So, today's lecture we will first discuss about the expectation means, these are all the, for the functions of the random variable that is the derived random variable, expectation of a function of random variable, moments of the functions of random variable. They are means once we know there that moments then we can obtain their mean, their variance. Then we will know what the different properties of that expectation are? And then also we will know the mean and variance of a linear function. So, linear functions are mostly applicable to many practical problems. So, the especially we will see that for a linear function what it is mean and, if I expression for the mean and variance relating to the mean and variance of the original random variable.

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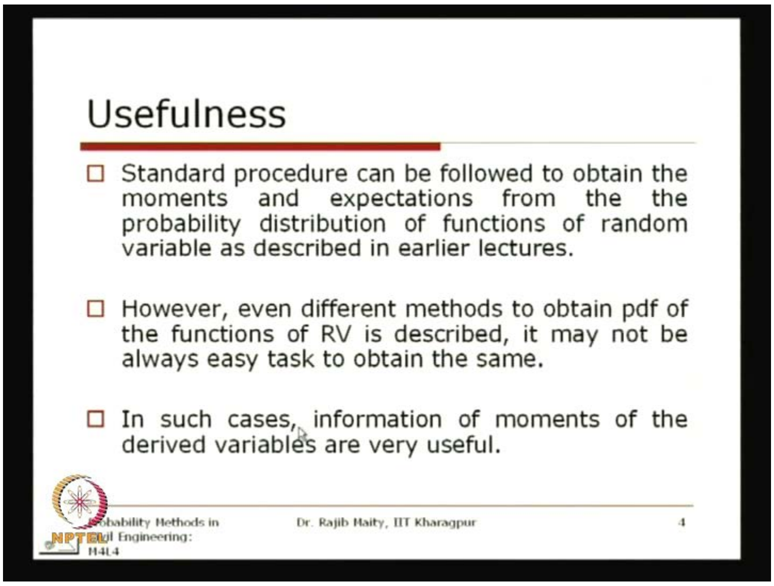

So, this method as I discuss that this method why this things are useful that is the getting the expectation moments and are the moment generating function itself, why it is useful is that the standard procedure that. So, far we have discuss in this in this module, that can be followed to obtain the moments and expectation from the probability distributions functions of the random variable as described in the earlier lecture. So, we have seen that what is the fundamental theorem and based on some specific assumption, what are the different methods to get the probability distribution function? And from that PDF we can get these moments and expectations are whatever the different properties that we need for these functions of random variable, which is the derived random variable.

However, even the different methods to obtain the PDF of the functions of random variable are described. It may not be always easy task to obtain the same and this is in particular. When we see that when this transformation or when this functions is non-linear or as we have discuss earlier that, when the function is having more than one root or more than one root for the new random variable. So, that time it is sometime it may not be that is easy to find out the subset of the x for a specific value of the y that is the function of that x. So, that time... So, special care is needed and so, we have to first of all find out that, why it may become difficult? So, that based on that only we can go for that other analysis.

So, as it is not always easy. So, in such cases the information of this moments and the information of the moments of the derived variable are very useful. So, even though I mention here that particularly in the case of when the function is non-linear that time the generally the root of the function is more than one and that time it creates the problem. But here we will we can also show that, if the function is in case of the function is linear also. Then if, we are interested to know some moments, the first moment or the second moment or the first order or second order statistics. If it is only of our interest, if you do if that the complete description of the PDF is not that necessary then also we can show that if we follow this methodology.

Then it will be much easier in case of the linear I am talking in specific it will be much easier to get those get those statistics first order, second order statistics to this direct method. So, without knowing that PDF in case of both linear as well as non-linear transfer functions of random variable, it will be ea easier. For non-linear function it will be easier to get this one and for the linear function also if the only few statistics are our intention. Then, this method will be much easier compare to the getting of full PDF and then getting the statistics. So, in this in these cases, for we are interested for some moment or the PDF is not that easily obtainable. So, information of the moments of the derived variable is very useful.

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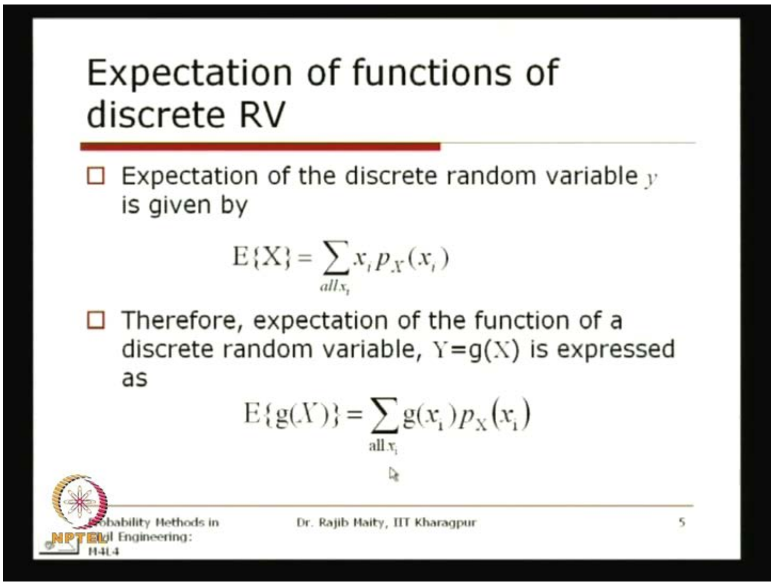

So, the general expression of this expectation of a function, we have discuss earlier and you know that for a single random, how to get that expectation of a random variable? That we have discussed earlier as well. And we will also just show it here for this discrete random variable. You know that expectation of that random variable x is the summation of the x i multiplied by p x x i. So, if this is the summation so, this we have discussed that this is a x is a discrete random variable. Then the probability for a particular value multiplied by that value and summation of all this all such possible outcome of the discrete random variable gives you the expectation of that particular random variable.

Now, similarly if you just extend this one this property that for the functions as well. So, if the function is say that y is equals to g x. So, this is the functional form that g of x is a functional form. Then to know that what is the mean or what is the expectation of this function that is g x? That also can be obtained by this simply replacement of this x on this one in terms of this g x. So, expectation of this g x that is the function of this random variable; obviously, we are talking about the discrete one discrete random variable in this case. That will be the summation of this the value of that function at those specific point, those specific feasible outcome of this random variable, that is x i multiplied by the probabilities. We and summation of for this all such x i will give you the expectation of this g x.

One thing here I want to mention that whatever the theory, that is the function of this random variable. That we are discussing in this lecture it can be an extended to the function of the more than one random variable. But this specific lecture what we are discussing? We are discussing only for the single random variable. So, the function and their functional properties, that we are discussing throughout this module. Not only this lecture throughout this module, are we discussing with respect to the single random variable. So, this same are the equivalent theory is exist for this more than one random variable, which we will be discuss under the in the next module, which is under the multiple random variable.

There we will see that the same thing the expectation again and they are moments of this one. If the function is based on the more than one random variable what will happen, but the basic principle that we are discussing in this lecture. We will remain same in those cases as well. So, here what we have shown that these are the so, these things earlier we know that for a discrete random variable and this is for that function of the discrete random variable. If the function is g x then their expectation can be expressed through this a form.

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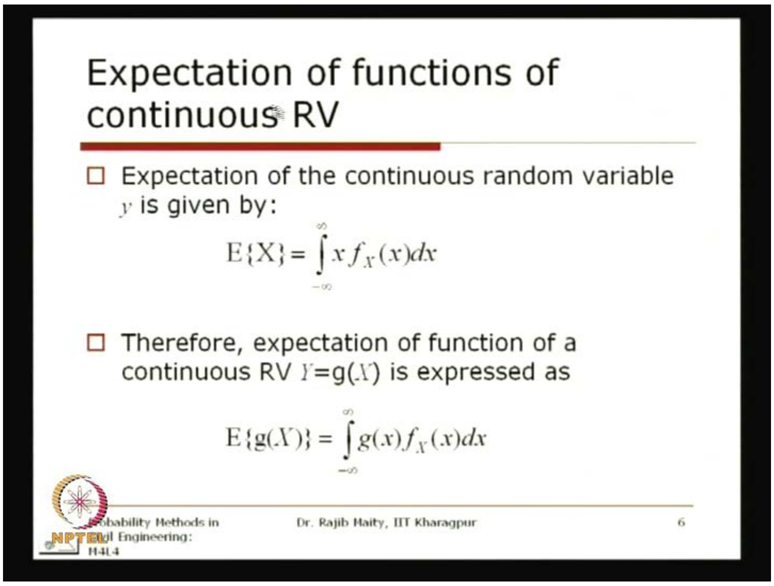

Now, in case of this... So, this is we got the summation, because this discrete random variable. Now, if this random variable is continuous then the expectation of the function of this continuous random variable. So, for a continuous random variable x is given by this will be x. So, is given by this we discuss earlier as well that is the expectation of x is that the multiplication of that variable, that is x multiplied by it is PDF that is f x(x) dx and that is the integrated over the entire support that is minus infinity to plus infinity. So, if you get this form then this nothing, but the expectation. And also we have discussed in the earlier lecture that this expectation is the moment with respect to the origin and we have shown that the analogy of the area in the earlier lecture.

That this expectation can be analogically said that this is where the distance of this, the cg of this area means here the area under this curve that is under this PDF. So, it gives the location of the location of the mean. So, this is expectation of that random variable x. Now, again here similar to the discrete random variable, if we replace this x with it is function that is y is equals to g x. If a new random variable that is the y, which is equals to the g(x). Then we can directly get the expectation of this y, that is g x in this form that is expectation of g x is equals to that this g x multiplied by the PDF of that random variable x and that integration from the over the entire support of this PDF that is minus infinity to plus infinity.

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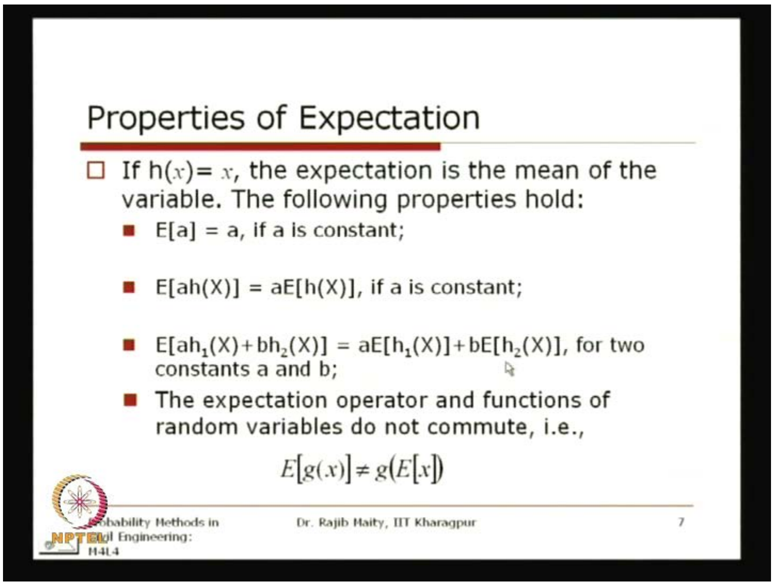

So, now there are some properties of this expectation. Some of them we have discuss earlier also and now we will discuss with a particularly in the view of this functions of this random variable. Now this, if this h x is equals to x that is function itself is only the x that we know that. Incase this case we have discuss earlier that this expectation is the mean of the variable. Now, this mean of the variable means this is the distance from the origin where the c g of the area under that PDF is located. So, that distance there is a distance from the origin to the c g of this. The curve c g of the area under the curve is the distance is the location and that gives you the mean of that random variable.

So, here we can say that function itself is equals to x. Now if, this thing can be generalized to the any functional form any if, any functional form and that we can see that how the properties are varying here. So, earlier we have discussed that if for the expectation of a function, which constant is with respects to that random variable. So, if this is the constant then we know that expectation of any constant is equals to the constant, that is here the expectation of a is equals to a, if a is constant. Now, if the function is h x and if the h x is multiplied by a scalar quantity by a constant that is a. So, then after multiplying that constant with that functional form, if we want to know what is this expectation?

Then this expectation can be express as that constant multiplied by the expectation of the h x. This can be easily followed from that particular form of this equation what we have discuss here now? So, here what you are doing actually is this g x is multiplied by a scalar quantity of this a. So, the expectation of a g x. So, in place of this g x we have to write that a multiplied by g x. Now as this a is constant, that can be easily taken out of this integration and the remaining, what is there inside the integration is nothing, but the expectation of g x. So, that constant can be taken out of this expectation. So, that is why it says that expectation of the function multiplied by a constant is equals to the constant multiplied by the expectation of that function.

Similarly, if there are two functions like this, h 1 x and h 2 x and both are multiplied by some constant a and here it is b. Then first of all this two can separated. So, that if it is the. So, this the additive rules this the this can be express that this expectation of a multiplied by h 1 x plus expectation of b multiplied by h 2 x and again following the earlier rule that is that constant can be taken out.

So, finally, the form takes that a multiplied by expectation of h 1 x plus b multiplied by expectation of h 2 x for two constant here a and b. But one thing that is also important that is the expectation of the operator and the functions of the random variable do not commute. That means so, as it is the constant that we can take out of this expectation, but the functional form. That is the expectation of g(x) is not equal to that function the same functional form for the expectation of x. So, this two things are not equal this is quite obvious, if you just fit this two side to that. Expression that we discuss in the earlier slide so, it can be easily shown that this two case or not equal to each other.

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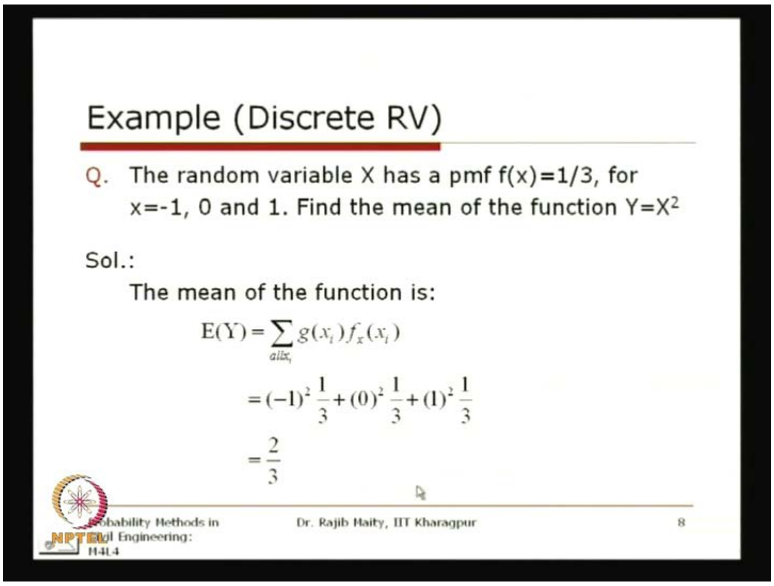

Now, if we take a small example and we have taken this example and we will show that, how the two different ways we can get that mean? So, this is a discrete random variable and this random variable that we are talking here is x and this is having the PMF, probability mass function is equals to 1 by 3 for there are three possible outcome that is, x equals to minus 1, 0 and 1. So, these are all equipping. So, there are three possible outcome and all this three outcomes are means equally possible. So, find the mean of the function y equals to x square.

So, in the earlier lecture now what we can say that earlier lecture generally we have concentrated what is the PDF of this y and that PDF for this one this kind of relationship where it is a square. We have also discussed in the previous lecture that there will be two roots for specific value of y and here as it is discrete value. And you can see that this is symmetric over this 0, 0 that is minus 1 and plus 1. So, here also for a specific value of y that is one there will be two roots that is minus 1 and plus 1. And we know that how to get that expression from the earlier lecture. So, here, if we see that... Here if we see that.

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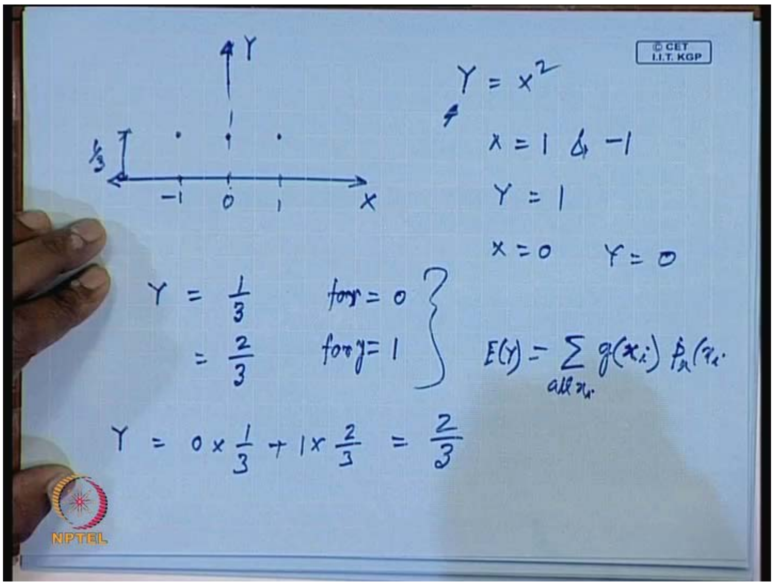

So, this the axis where we are showing that y and this your say a minus 1, this is your say 0 and this is your say 1. So, this are the three possible outcomes and everywhere that probability is just shown here, which is yours. So, these are all 1 by 3. Now, we are taking another functional form that we have seen that y is equals to x square. Now, if you just put it here that is, this 1 is your y then we know that for the. So, as it is square. So, this y is not negative. Now, for that x equals to both that 1 and minus 1 the y is equals to your 1. And for x equals to 0, y is taking a value that is 0. Now, if I want if I follow that our earlier thing that I will first find out what is the PDF of this y or the PMF as it is the discrete random variable. So, what is the PMF of that y and then from the PMF I will calculate what it is mean is?

So, what are the possible outcomes so, y equals to some value will get for y equals to 0 and for y equals to for y equals to 1. So, we have seen that for y equals to 0 the possible set of x that can take is only 0 and at x equals to 0. The probability is 1 by 3 and for y equals to 1 the possible set of this x is 1 and minus 1 both and they are probability mass, for x equals 1 is 1 by 3 and for x equals to minus 1 is also 1 by 3. So, now, if you take this two and put it they are that for the outcome y equals to1 so, we have just add them. So, that y equals to 2 by 3 for y equals to 1. So, this is the complete definition for the PMF of this y.

Now if I want to know what is the mean of this y? You know that this mean for a discrete random variable should be equals to that possible outcome. That is the y multiply 0 multiplied by it is probability mass plus, this outcome multiplied by it is probability mass 2 by 3, which is equals to 2 by 3. So, this is that technique and if, we want to know that direct. If you so, without knowing. So, here in this lecture what we are doing there I do not want to know what is these distribution? I just, I am in just interested to the mean of this y. So, this is what we have discussed.

So, far that we can directly write that functional form, that is g x i multiplied by that your p x of x i and this if we add it off for all x i. Then what we will get is equals to your expectation of y that is the mean of y. So, from this one whether we can get that what is shown in this problem? So, what we have shown that? Now directly if, I want to know what it is means is then what we will get that expectation of y is equals to the summation that g x i of f x i. So, this will beep x i actually as it is PMF. So, but here we have used f x this notation is used. So, fine. So, here that g x i, now the x i can take the all x i means here, the x i is minus 1, 0 and 1.

So, we are just putting that g x i that is y equals to x square. So, minus 1 square at that point what is the probability mass? That is 1 by 3 then 0 square 1 by 3 and 1 square 1 by 3. So, this square is coming from this functional form that is g x i is equals to your x i square. So, those squares are shown here. So, if we just get this one you are also getting this mean to be 2 by 3. So, from both thus thing we have shown the directly without knowing, but without knowing the PMF of this y we are getting it is mean. So, this is what is explained through this very simple problem how we can get the same mean without knowing their PMF?

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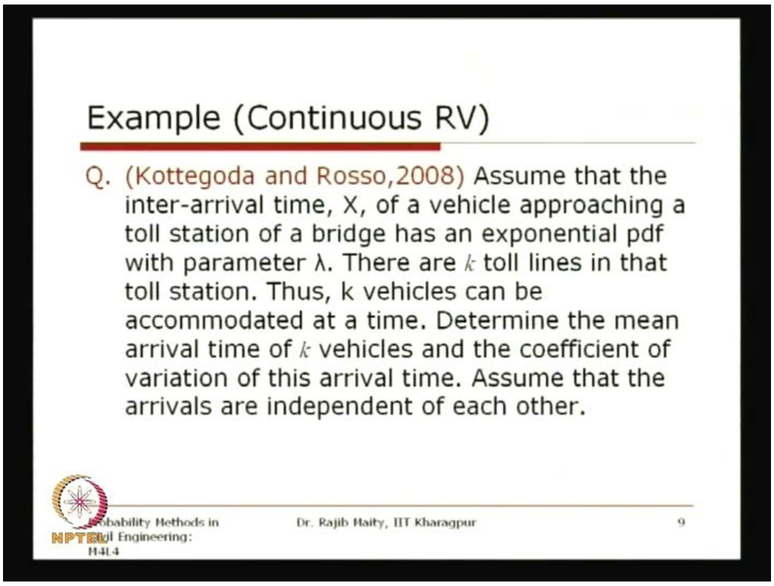

Similarly, if we take another problem, but in this case it is a continuous random variable and this problem is taken from the Kottegoda and Rosso 2008. Now, here it is on a variable between the inter arrival time, on inter arrival time of two vehicles over a highway bridge. Suppose, that there is a toll station and we are interested to know that basically this is a, this kind of problem is used to decide that, how many lines in a toll station is required to optimally handle the traffic volume at that inter section of the road? So, what we have to know is that what is the average inter arrival time and of the vehicles? And how many such things are such the lines of this toll is required?

So, the problem states that assume that the inter arrival time, that is the x here the random variable. That we are using is your that inter arrival time between two vehicles of a vehicle approaching toll station of a bridge has an exponential PDF with parameter lambda. So, this x that inters arrival time following an exponential PDF and the parameter is lambda. So, this exponential PDF we have discuss earlier. So, you know that what that form of the PDF, it will take for the x. Now, there are k toll lines in that toll station thus the k vehicles can be accommodated at a time. Now this is our now is concern that is there are k, if there are k toll lines; that means, the k vehicles can be accommodated at a particular time instant.

So, what we have to determine that we want to know that what is the mean arrival time of k vehicles and the coefficient of variation of this arrival time? So, assume that the arrivals are independent to each other. So, arrivals are independent to each other means so; that means, arrival of a particular vehicle at this time instant does not have any influence on the arrival of this the arrival of the next vehicle to the toll station. So, this the successive arrival times are independent to each other. Now, why we are interested to know the mean of these k vehicles is that, because at that particular station if there are k toll lines. So, there will be k vehicles can accommodated together.

So, we are interested to know that. What is the mean arrival time of the k vehicle at that particular section? So, that we will be, this will be a guidance to determine that k equals to how much should be the optimum for that k. Because we know that average time taken to pass through a toll station with that experience, we can decide that how many such line is needed? So, what should be the comfortable number for that k. So, what we are interested here to know the mean arrival time of the k vehicles. Now, you see the function here, that we are talking about this is one is the first the original random variable is x, now there are k lines, k vehicles can come. So, that x so, this arrival times can be added times and the new the random variable will be obtained and without knowing that PDF. I just want to know what is the mean arrival time of that new random variable, which is the summation of k such x?

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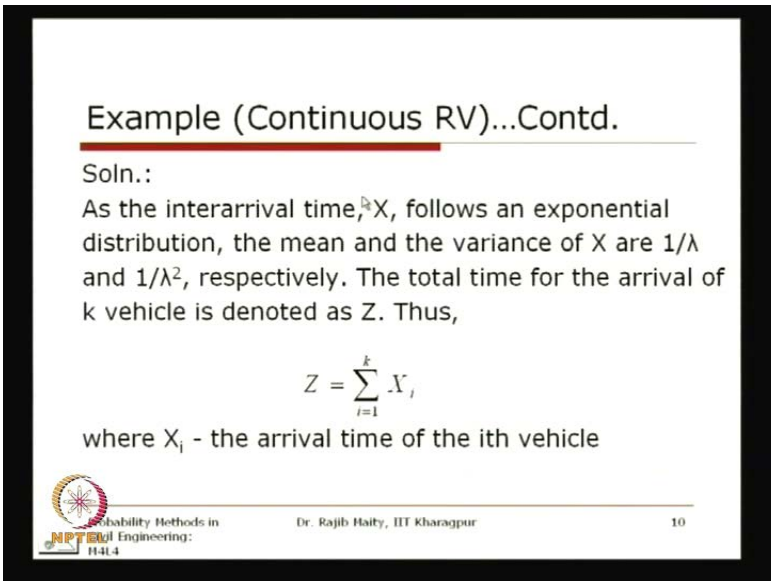

So, that is here, what is discuss as the inter arrival time x follows an exponential distribution. The mean and the variance of x are 1 by lambda and 1 by lambda square, this we have discussed in the last module. That if a particular random variable having the exponential distribution with parameter lambda. It is mean is 1 by lambda and it is variance is 1 by lambda square. Now, the total time for the arrival of the k vehicle. If, it is denoted by z, another random variable. Then, it is straight forward that the z is equals to summation of such x 1, x 2 up to x k so, the summation of this random variable. So, we are getting another new random variable. So, we have to get what is the mean of this z.

Now, if I want to know that mean of this z that can be return that. Now as we have seen that, this x i if it is added inside. Then, we know that can be taken out from that particular property. That we have discuss few slide earlier that this one, if there are some random variable function, if we just adding them up.

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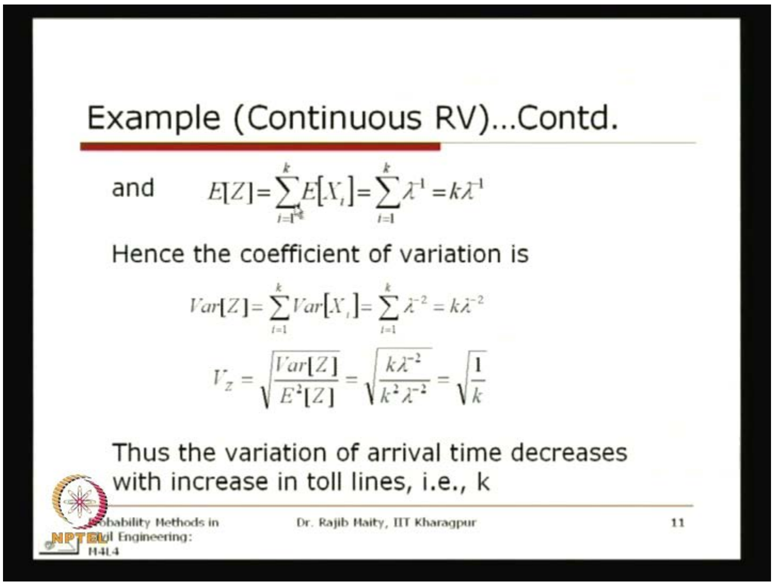

That addition this is the additive rule, following this additive rule. We can just write that this can be the summation of that the expectation of each random variable starting from 1 to k. Now, we know thus expectation of that x, because this follows the exponential distribution. So, this is 1 by lambda. So, this expectation of x i is equals to lambda inverse. So, this lambda inverse should be summed up from i equals to 1 to k. So, there is k such 1 by lambda. So, the expectation of z is equals to k by lambda. So, the mean arrival time of k vehicles to that toll station is k by lambda, where the arrival time of 1 vehicle is equals to 1 by lambda.

Now, one interesting point that we will discuss with this, which is also very important for handling the traffic volume is this. That should, if I now want to know, what is the coefficient of the variation? Then we have to calculate what the variance of the z is? Now this variance of the z is again the summation of this, variance of this individual random variable. Now, this individual random variable variance is 1 by lambda square and that following the same thing that is, it is also become that k by lambda square. Now, if I want to know that the variation that is the coefficient of variation. That we know that this is the coefficient of variation of this new random variable that is the z is equals to the s by mu.

This s by mu means that s is here the standard deviation, which is square root of the variance of this z and this e z square means that is the mean and if. So, this is the means. So, we have taken this full square root that means this is the mean. So, the s by mu that we have discussed earlier, that is the coefficient of variation. So, here variation of the variance of this z is equals to k, k by lambda square and this expectation of z, that square is your k square by lambda square. So, if this two is resulting that 1 by k square root. Now, what happens if k increases? So, if k increases we see that this coefficient of variation decreases thus the, thus the variation of the arrival time decreases with the increase of the toll line that is k. So, why it is important is that.

Now, that if we have decide that they are at a particular toll station there are k number of toll lines are required then that toll lines if the number of toll lines increases then that coefficient of variation. The variation means the variation of the total k vehicles approaching to the toll station that variation will decrease as k increases. So, that we can say that as we increase it then the average time that is the average interval to approaching to a particular toll line will be almost approaching to, almost approaching to uniform. Because the variation of this total k vehicle the time of approaching total k vehicles is reducing. So, this is what that is from what we can conclude, from the coefficient of variation of this new random variable z.

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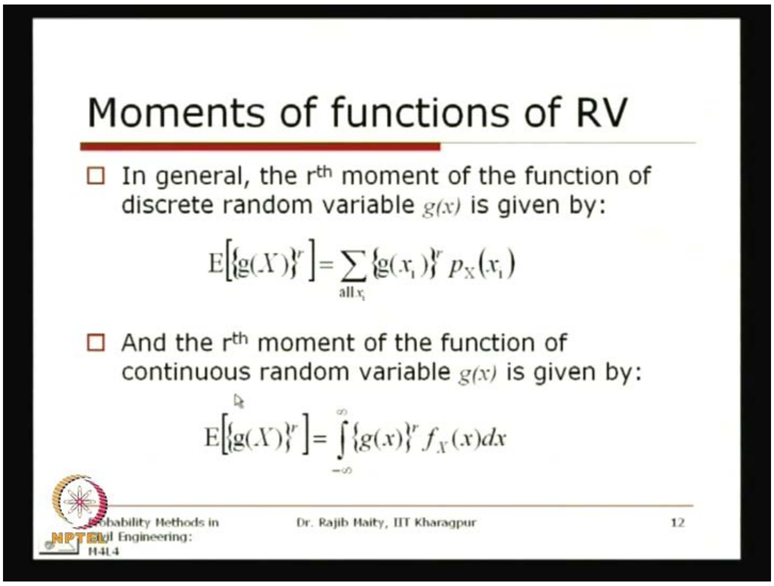

Now, the moments of the functions of random variable again, that we have discuss this moment with respect to a particular random variable. A single random variable here, what we are discussing is that they are function. So, following the same thing from the expectation what we have shown that, from that expectation of x, we have taken it to that expectation of g x. Similarly, here we will take that the same that expectation of a particular variable we will just see how for that the function. So, we can say that this will be the in generally, if we want to state then we will say what is the r the moment of a particular function?

So, that will be the power should be there is the r. So, this is what is explained here for both these discrete as well as this for the continuous. So, for the discrete random variable, the in general form that is the r. The moment of the function of this g x can be express at this g x power r multiplied by their PMF and summation of all possible x i and for the r the moment of the function, of the continuous random variable. That is g x is given by the expectation of this g x power r. That is equals to minus infinity to plus infinity, that is g x power r multiplied by f x x dx. So, this g x power r this g x power r is basically is giving you general form of this r th moment. Now, depending on whether we are interested for the first order of that first moment or second moment or third moment that we will just vary this value of this r.

Now, one thing is a clear that these are all the moments that we are taking with respect to the origin that is from the 0 and you know that these moments are important with respect to the origin for the first moment to find out it is the location of the mean. Now once we have identified the mean generally for the higher order moments. We are taking with respect to the mean from this second moment onwards. The description of that random variable is important, if we take the moment to with respect to the mean. And we know that variance is the second moment with the respect to the mean and we have also discussed that first moment with respect to the mean is 0. Because we are taking it from the same point, where the first moment we got with respect to the origin.

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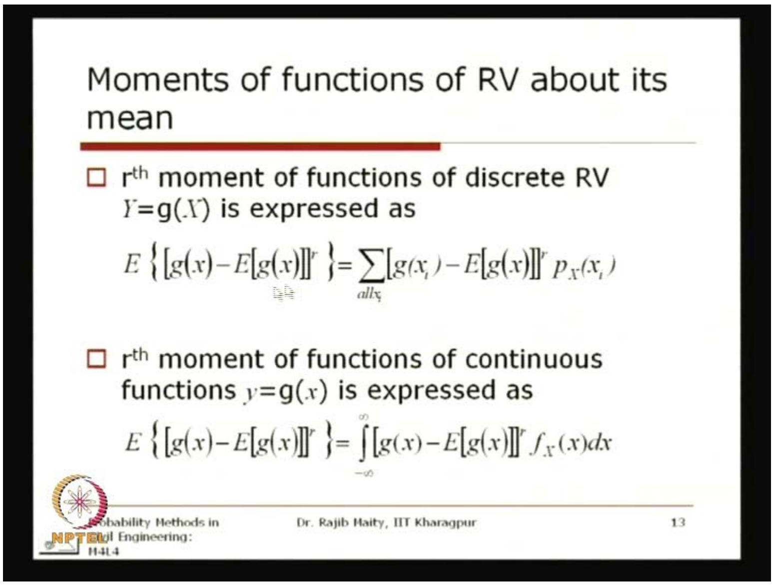

So, here as this general form of this moment we have taken with respect to the origin. Now we will show that if we take it with respect to that with respect to the, with respect to it is mean, that is we know that now the mean is the first order moment. So, that expectation of g x is basically your mean. So, here again the general form in case of the discrete random variable that is the r th moment of the function with respect to the mean, what we have to replace the total function is the g x minus expectation of g x?

So, this is the total form and that power r that is to make it general is equals to the summation of this g x i minus expectation of g x power r multiplied by it is PMF. And this is summing up overall possible x i, we will give you the moment is the r the moment with respect to mean for that discrete random variable. Similarly, for the continuous random variable, if we take then that y is equals to g x is expressed as that expectation of this g x minus that expectation of that first order expectation that is the mean. So, g x minus mean power r and this should be integrated, because this is the continuous random variable. That is g x minus that mean power r multiplied by it is PDF and this integration will give you that r the moment with respect to the mean, that is expectation of g x of that function g x.

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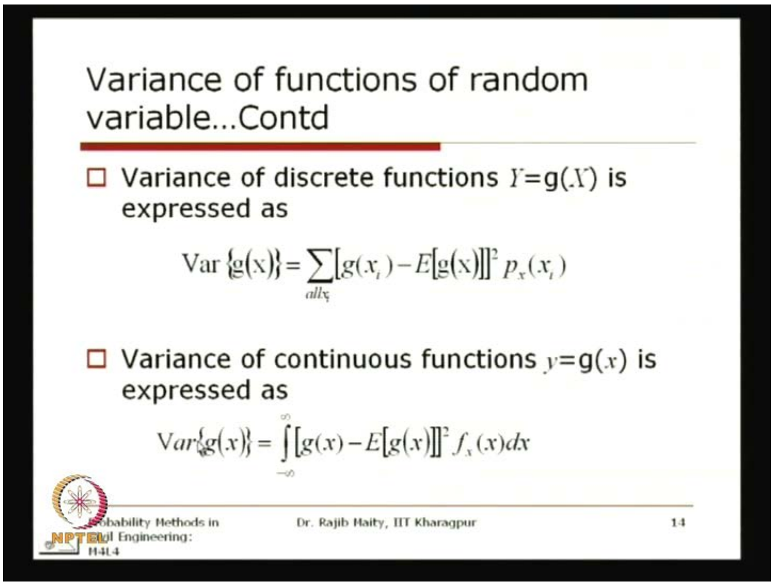

Now, as we have seen that, if we want to know the variance of that function then the variance of the function. If we want to know then we know this is the second order moment with respect to that mean. So, that here that r should be replaced by that 2. So, that variance of a function of that g x is equals to that the second moment with respect to the mean, that is the g x i minus expectation of g x power 2 multiplied by p x x i for all at all an x i. So, this one is giving you the variance for that discrete random for the function of the discrete random variable g x. Now, the variance of the continuous function that is y equals to g x again, we have to put that moment that second order moment we have to take. So, g x minus that mean power 2 that is square multiplied by it is PDF and taking the integration from the entire support minus infinity to plus infinity. We will give you the variance of that function g x.

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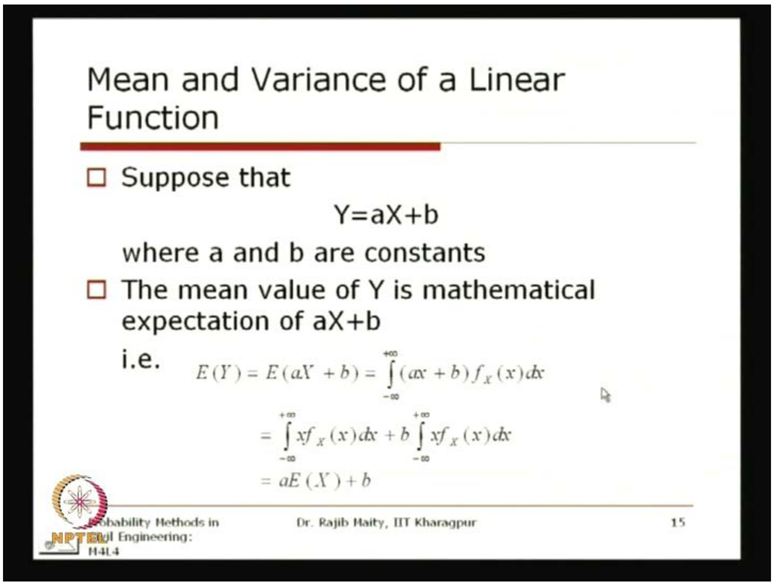

Now using this one if, we just want to see a specific case of the linear function the linear function here. If as, I have started with that thus these things what we are discussing with the respect to single random variable. So, we are taking the linear function as a thus form that y equals to m x plus c. So, in this form if, I just take then we will see that what is each expectation and what is this variance? Then we will see that even though these are the linear function and in such cases following the one to one transformation exemption that getting the PDF is also that processes. We have discuss and PDF also you can obtain, but if we know this form then the getting that. If the first moment that is the mean and the variance is s, is required then following this method this will be much easier to get. What is it is mean and what it is variance even though it is a linear function?

So, here what we are taking is a linear function y equals to a x plus b this a and b are the constant here. So, the properties of this x are known that is expectation of x or the variance of x these is all known. So, now, if I want to know that what is the mean value of this one? This y then the mathematical expectation for this function that is expectation of y is equals to expectation of a x plus b. Now, again following the properties of this expectation that we have followed earlier this can be shown. That this should be plus that is the expectation of a x plus expectation of b as b is constant the expectation of b will be equals to, your this direct constant and this constant will be taken out from the expectation.

So, this can be shown that a multiplied by expectation of x plus b and this can also be shown. If we just follow that the continuous moment that is the expression of the moment for the continuous random variable. That is, this a is functional form of this g x that is a x plus b that multiplied by this PDF dx and integration over entire support from minus infinity to plus infinity. So, we can break this from the integration rule, that this x plus f x dx plus b b f x dx. So, there is some mistake here. So, this will be there will be there will be multiplying factor called a here, because this is coming here and in this case there will only b and there will not be any x.

So, this will be the integration from minus infinity plus infinity of the f x dx. Now integration of this minus infinity to plus infinity of the f x dx, we know from the basic assumption of this PDF that this entire integration is equals to 1. So, this integration this x is a mistake here. So, this integration from this minus infinity to plus infinity of the PDF is equals to b multiplied by 1. So, this is becoming b. So, here again this a is multiplied here. So, this a multiplied by the minus infinity to plus infinity x f x dx. So, now, this x f x dx we know this is the expectation of x. So, this is replaced by this expectation of x and this is your a m.

So, what we are following again we can see that whatever the properties of the expectation. We just have shown few slides back. That is this constant can be taken out and expectation of the constant is a constant. So, following that principle also that expectation of y is equals to the a multiplied by expectation of x plus b.

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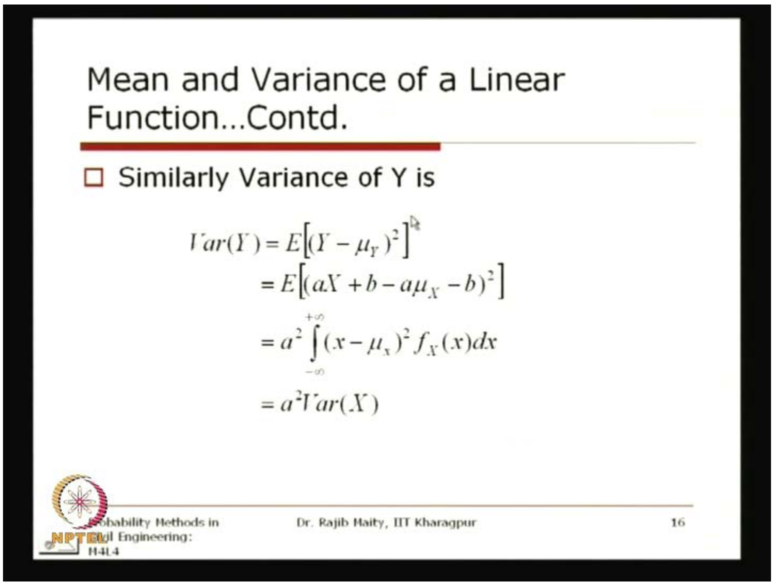

Similarly, for this same linear function if we are, if we want to know what is it is variance? Then the variance of y we know that this will be the second moment with respect to the mean. Now the second moment with respect to the mean when we are talking about this is. So, that this is y minus mean of y, now this mean of y is the expectation of y that, just now we have seen that expectation of y is this. So, this is a of mu x. We can write that this expectation of x is the mu x plus b. So, this is what that mu y is replaced by this a mu x minus b and y is equals to a x plus b. Now, again we can follow this one we can just put this expression that is x minus mu x square f x dx, which is coming.

So, this square is coming out and the other expectation. So, this b, a gets cancelled and this expectation of a mu x this is becoming, this is also becoming. So, that x minus mu x we are that just taking a common. So, x minus mu x will come here, which the square this square is coming here. So, when we are taking out; obviously, a square is coming out of this expectation. So, now, if we express this terms. Now, we know that this is the second order moment for the random variable x with respect to it is means. So, which is nothing, but the variance? So, the variance of y is equals to a square variance of x.

So, if we just summarize in two things that is if the linear function is y equals to a x plus b, then the expectation of y is equals to a multiplied by expectation of x plus b and variance of y is equals to a square variance of a square multiplied by variance of x. So, what we can see is that for the expectation this constant term is getting added and the constants term is getting multiplied whatever it is in the functional form and for the variance. We know that for the constant variance is 0. So, this is becoming 0 and whatever the scalar quantity this is multiplied to the random variable that becomes square, multiplier to this variance of the x.

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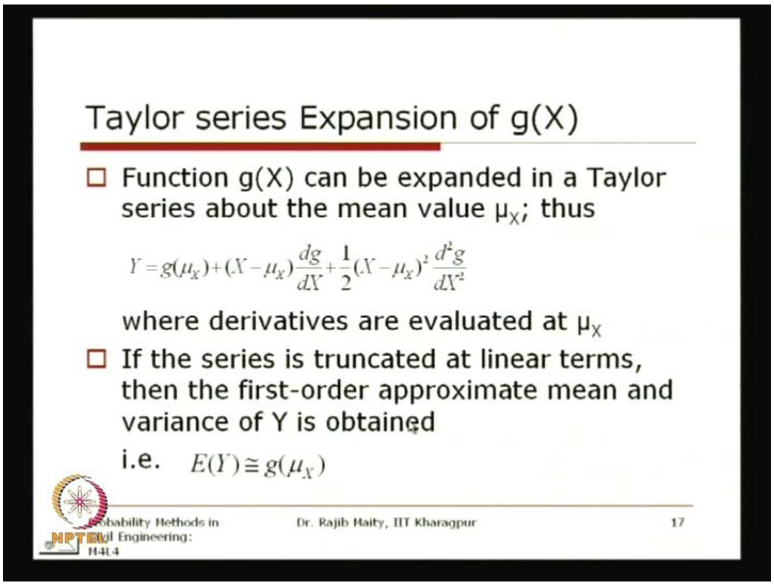

Now, one more thing, which is also important for this is the Taylor series expansion of a function of this g x. Now g x is the y, now the function of g x can also be expanded in a Taylor series about the mean value of mu x. Now you know from this Taylor series expansion, that is y can be express that this functional value at g x. So, we are expanding it about the mean value mu x, which is equals to that g mu x plus x minus mu x multiplied by this first derivative of this of this function with respective x plus half x minus mu x square, second derivative of this one plus it will go up to infinity.

Now, where the derivatives are evaluated? So, these all derivatives are evaluated at mu x that is the mean value of this random variable x. Now if the series is truncated at the linear terms then the first order approximate mean and the variance of the y can be obtained as that expectation of y is equals to function of the same functional form at evaluated at mu x. So, this can be approximated just considering the linear terms. So, which also we can say, we can show that for this linear function, which is exactly same. So, here that when we are taking the expectation of this y, which is equals to know what we are taking that these functional form at mu x. So, which is nothing, but a multiplied by mu x plus b, which is shown here. So, a multiplied by mu x plus b.

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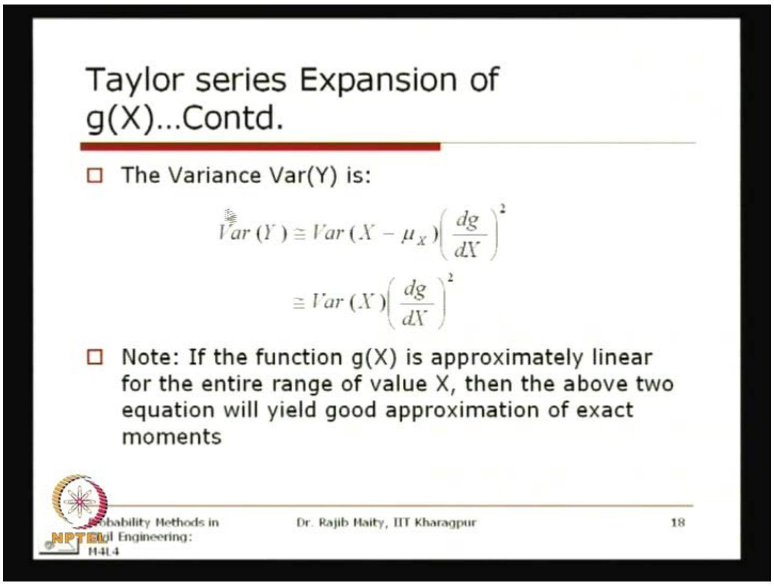

Similarly, if we take the variance of this the variance of y is can be approximated, that variance of x minus mu x multiplied by first derivative whole square. So, this is now the variance of x this d g dx of that square. Now, if we take the first derivative of that functional form that square multiplied by the variance of x. So, for the linear function just now what we discuss is that is the same? See here is the variance of y is equals to if, we take the first derivative of this one that d g dx. Then it will become a. So, that a square. So, that a square multiplied by the variance of x. So, which is nothing, but a square variance of x, which we have also seen if the function of from the Taylor series expansion and approximating up to the linear function.

Note, that if the function g x is approximately linear for the entire range of the value of the x then the above two equations will yield good approximation of the exact moment. So, whatever for the linear function that we have seen this is exactly same? When we are taking the Taylor series expansion up to the linear up to the order so, excluding the higher order. Now, this kind of functional relationship if we say that over the entire range of that function, if you can show that is almost linear, if it is not perfectly linear. Even though it is almost linear over the range then also we can follow this kind of relationship to get a good approximation of this mean and variance.

So, this is what is explained in this one? So, just approximate so, the approximate linear functions also can yield the good approximation of the exact moments. Here one problem how we can solve this one? Let us take that this problem is related to the measuring the length of two rods. What we can do that we can do, we can use two separate methods to measure the length. One is that two rods can be measured separately and what we can do is, we can make the summation of the length of the two rods and the difference of the length of the two rods and after that we can do for these measurements.

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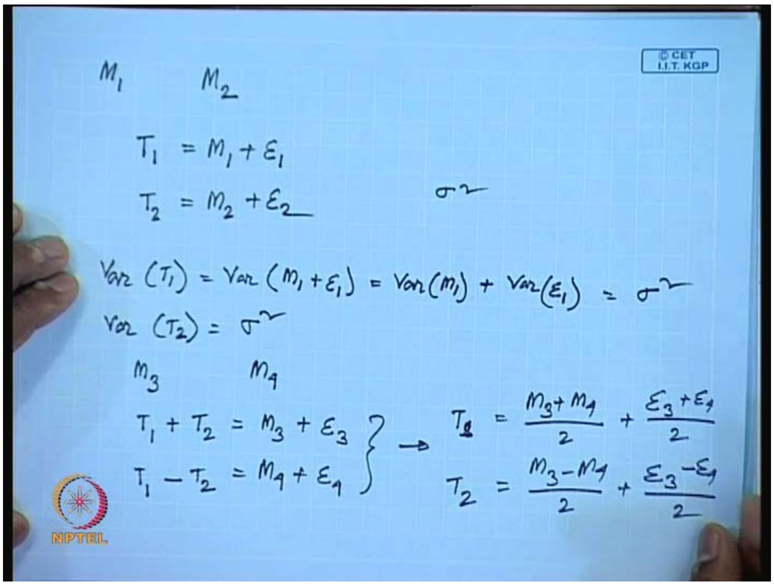

So, what we have to do the problem here is that, which method will be more correct that we have to see. Now suppose that there are two such measurements, one is that is denoted by m 1 and other is denoted by m 2 and their true lengths are represented by this T 1 and this T 2. Obviously, these measurements are support to some errors and that is why it is express of this T 1 is equals to your M 1 plus that epsilon 1, which is the error. Similarly, it is M 2 plus epsilon 2. Now, these are some say these are having some properties statistical properties and which is having some variance and this variance are say that sigma square.

Then, if I want to know what is the variance of their actual of the true length that what is we just get this one T 1 and T 2. Then we can calculate that this variance that T 1 is equals to that variance of M 1 plus epsilon 1, which we can you know from our previous lecture. So, it is variance of M 1 plus variance of epsilon 1 to which obviously, this is a measurement. So, this is a specific value. So, the variance is 0 and this variance is equal to sigma square. So, it is equal to a sigma square. Similarly, if we calculate the variance of this T 2 the second measurement, that the true length of the second rod, which is also giving the same thing it will be also that sigma square.

Now, in the second case when we are measuring the combined length and this difference length if, we just measure and say that those are the M 3 is the summation of those two lengths and M 4 is the difference of these two lengths. Then we can express that this that T 1 the actual length of this first one plus T 2 is equals to your what every our measure that M 3 plus that epsilon 3 and that T 1 minus T 2 should be equals to as is that M 4 plus epsilon 4. Now if we now want to know what is the variance involved in this one? In this one then we have this express this T 1, T 2 in terms of their measurement and from this 2 equation. If we just solve for this T 1 and T 2 we can do it, that we will get that T 1 is equals to your M 3 plus M 4 by 2 plus epsilon 3 plus epsilon 4 by 2. And this is T 1 and that T 2 is equals to M 3 minus M 4 divided by 2 plus epsilon 3 minus epsilon 4 by 2.

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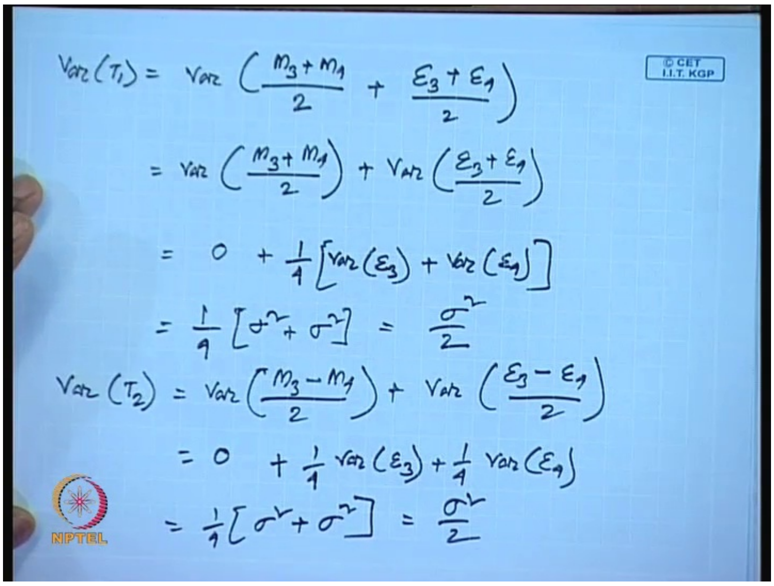

Now, we have to see what is the variance of this T 1 and T 2? Now if we again follow the same this variance then this variance of T 1 will be equals to the variance of M 3 plus M 4 divided by 2 plus epsilon 3 plus epsilon 4 by 2, which is equals to here variance of M 3 plus M 4 divided by 2 plus variance of epsilon 3 plus epsilon 4 by 2, obviously, these are the specific value for the measurement and this will again following the same earlier logic. This will be 0 plus, this variance as we know that now this is a epsilon 3 is a coefficient having half and epsilon 4 is also having half. Then we now that can we take it out this is square that we discuss in previous lectures. So, it will be the 1 by 4 of the variance of epsilon 3 plus variance of epsilon 4, which is equals to 4 plus this variance are same. As we have told that these measurement is having a variance of sigma square. So, it is sigma square plus sigma square, which gives you the sigma square by 2.

Similarly, if we calculate that variance of T 2, which will again come as the variance of this M 3 minus M 4 by 2 plus variance of epsilon 3 minus epsilon 4 by 2. Again, this will be 0 plus, this will be that coefficient. Here is half, which is that 1 by 4 variance of epsilon 3 and this coefficient is minus half to square of that half again that 1 by 4 it is not that minus half. It is not that minus will not come this is square of the coefficient to the minus of square is 1 by 4 variance of epsilon 4, which is again that 1 by 4 sigma square plus sigma square, which is sigma square by 2. So, what we have seen that, in that earlier case.

When you are measuring this to individually we are getting that variance of that thing is coming as sigma square and in the second case it is coming to be the sigma square by 2. So, that measurement accuracy of this second method is better then compare to this first method. So, in this example also we have seen that how we are, when we are interested for a new derived variable. We can calculate their statistics even without knowing their specific probabilistic that probability distribution. So, this one what is essential that is that after we got that derived random variable to calculate the properties of those derived random variable. So, we will take up some more examples related to this line in the coming lectures.