Numerical Methods in Civil Engineering
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Lecture - 40
Integral Equations – III

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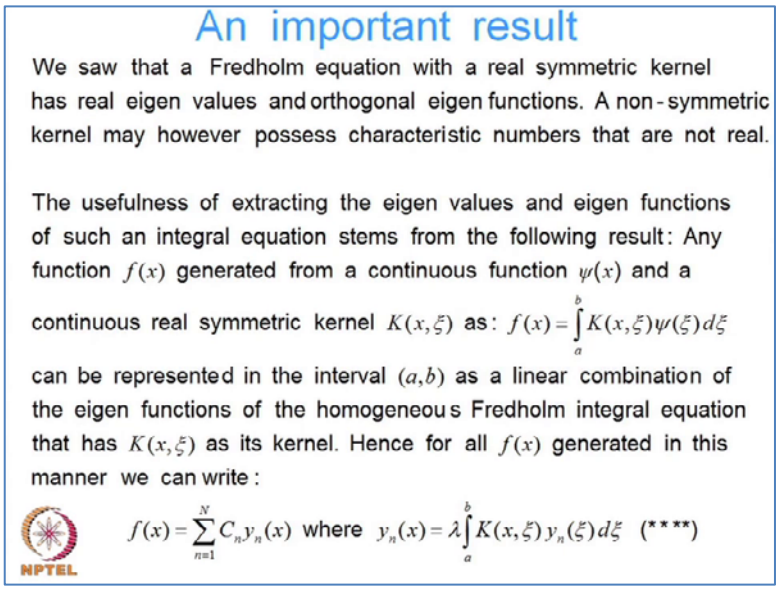

In this final lecture of our series on numerical methods in civil engineering, we will attempt to complete our discussion on integral equations. We will let us recapitulate very important result which we encountered in the last lecture. We saw that a Fredholm equation with real and symmetric kernel and also a continuous kernel, always has Eigen functions that are real, the Eigen values that are real and Eigen functions that are orthogonal, right. We saw that when the kernel is real and symmetric, however when the kernel is not real or it is not symmetric, we have not guaranteed that the Eigen functions, the Eigen values may not be real in that case when the kernel is not symmetric, right. So, that is one we have already encountered in the previous lecture.

Now, we want to talk about another important result which allows us a lot of freedom which basically leads to the usefulness of this Eigen decomposition, finding the reason why we do this, we go to the trouble of finding the Eigen values and Eigen functions will become clear when we see this result. These results basically says that if I have any continuous function psi of x can be for instance that function, if I integrate that function and the kernels for instance I am referring to this integral.

So, I integrate K of x of xi and psi of xi, I integrate that within the limits a and b and suppose, I get a function f of x is equal to integral a to b K of x xi psi of xi and the only requirement is that psi be a continuous function. In that case, I am guaranteed that f of x can be expressed as a linear combination of the Eigen functions of the integral equation, where this is my integral, this is the Eigen value problem and if I solve this problem, I get the Eigen functions y n and I can represent f of x as a linear combination of these Eigen functions, right.

So, this is very important because then only restriction on psi has to be continuous function, right. If psi is a continuous function and I get f of x by operating with psi with the kernel operator, so operate in the sense of a integral equations I mean integrate within the limits a to b, right. I get a function f of x and that of f of x can be expressed as a linear combination of the Eigen functions.

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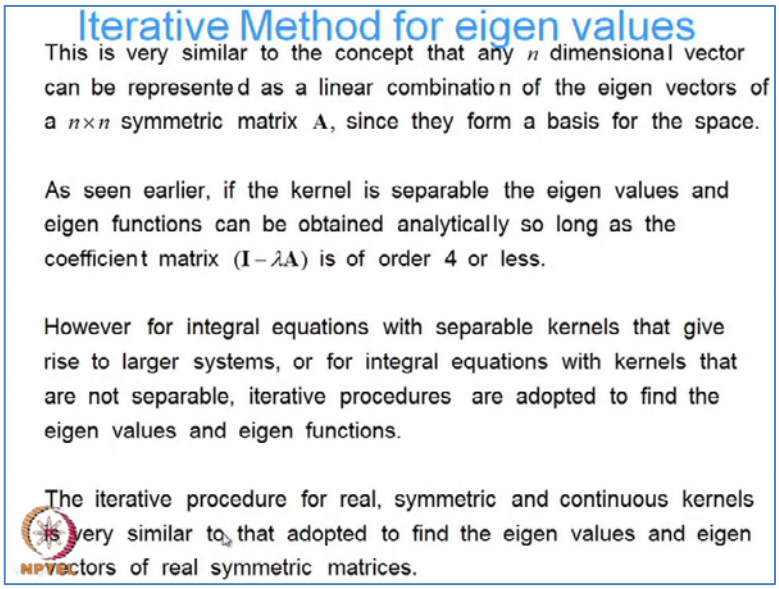

So, this is we have seen before. So, if you have a symmetric, if you have a linear, if you have a linear system and the coefficient matrix is symmetric and if the Eigen values are real and the Eigen vectors are orthogonal and the Eigen vectors form a basis for that space, then in that case, you can represent any vector belonging to that space as a linear combination of those Eigen vectors. Similarly, this function f of x can be expressed as a linear combination of the Eigen function. So, the ideas are very similar.

So, how do we find the Eigen values and Eigen functions? Well, we have seen that for a separable kernel, all we need to do is to solve the characteristic equation. We can write down the characteristic equation and the characteristic equation is the determinant of this I minus lambda a, where the unknown is lambda. So, this is a polynomial in lambda. The determinant of this becomes the polynomial in lambda for a separable kernel, and if that polynomial we get lambda, but if the order is greater than 4 or more, sorry this should be more is we can solve it. So, as long as it is 4 or less, but if it is more than 4, then we cannot get an analytical solution because of more than equation. We know that we cannot solve it analytically.

So, we have to solve it numerically. So, for a separable kernel if the order is more than 4, we have to solve it numerically. However, if the kernel is not separable, then it is hard to find the analytical procedure. So, we have to go ahead and solve it numerically using iterative procedures. Thus, it is very important that we have some idea of how we go about finding the Eigen values and Eigen functions of these integral equations, basically how we use these iterative methods to come up with the Eigen values and Eigen functions.

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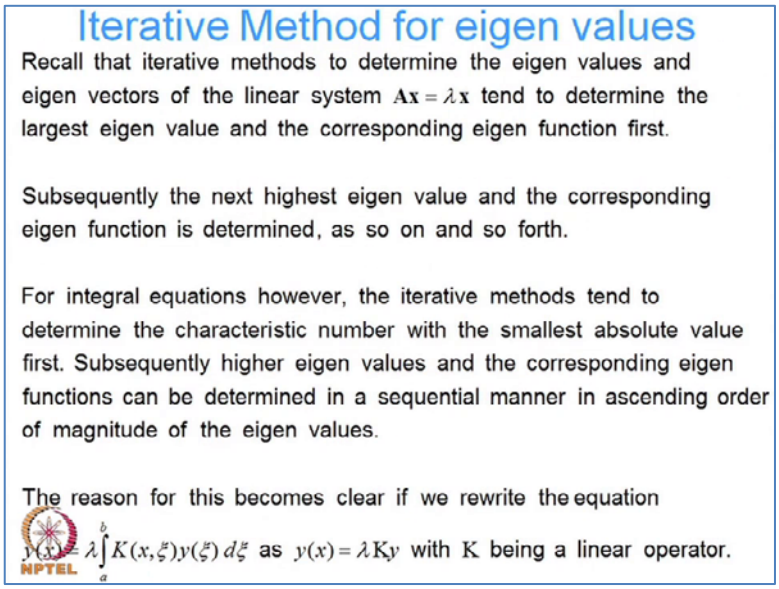

Recall again that if for the linear system, we encountered an iterative method for solving the Eigen values and Eigen functions and there we have got the first estimate was of the highest Eigen value, and the corresponding Eigen vector used in iterative scheme which

gave us the highest Eigen value and the corresponding Eigen vector. In case of integral equations, it is just the reverse, right. Our iterative scheme gives us the lowest Eigen value and the corresponding Eigen vector. Why is that? The reason for this becomes clear. If we rewrite this integral, this Eigen value problem as an op in terms of an operators instead of y x is equal to lambda. This if I write y x is equal to lambda K times y, where K is no longer the kernel, I am thinking of K as an operator, right.

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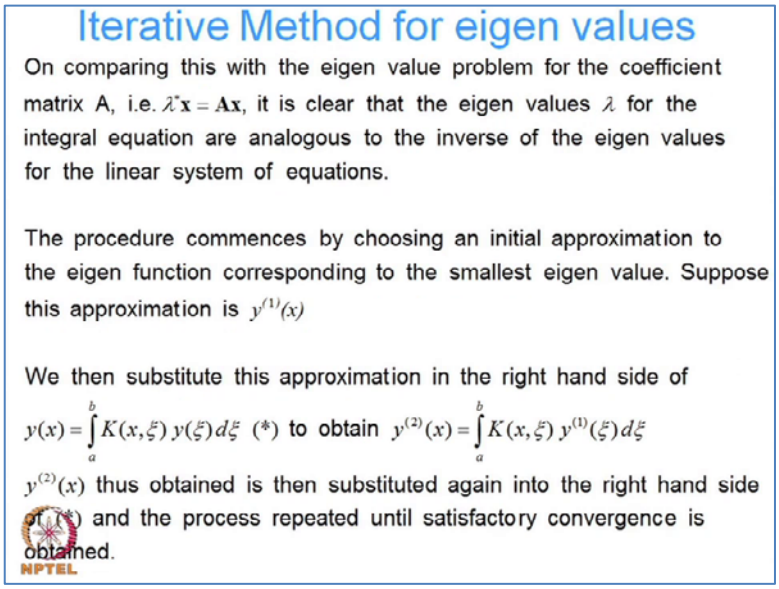

I compared this with my Eigen value problem for a linear system and for the Eigen value problem, for the linear system is lambda star x is equal to K a x. So, a x is equal to lambda x while here it is 1 by lambda y x is equal to K y, right. So, you can see that the Eigen value for the integral equation is analogous to the inverse of the Eigen value for the linear system of equations, right.

So, you can understand that when we are using an iterative procedure, we end up with the highest Eigen value for the linear system. When we are trying to use an iterative procedure for the integral equation, we will end up with the smallest Eigen value and the corresponding Eigen vector just because the Eigen value problem is a little different instead of a x is equal to lambda x. So, this is my coefficient x, this is my linear operating on x is equal to lambda x.

Here I have lambda times the linear operator is equal to y, right. So, it is just lambda is on the other side of the equation, right. So, because of that our iterative procedure is

going to give us an estimate of the smallest Eigen value rather than the largest Eigen value. How do we go about the iterative procedure? Well, we start with some function y 1 of x which is our estimate to the smallest Eigen value. We construct a function which we think may not be even close with, but it is an estimate, our first guess to the lowest Eigen value and then we operate with the kernel, and then we come up with something else, with another function, right.

So, this was my input. I put it on the right hand side, I came up with this on the left hand side and I do this repeatedly. So, y 2 of x again I put it on the right hand side, operate on it with the kernel, and come up with an improved estimate. Why is it an improved estimate? We will talk about that later on, but for the time being let us assume that we get another estimate.

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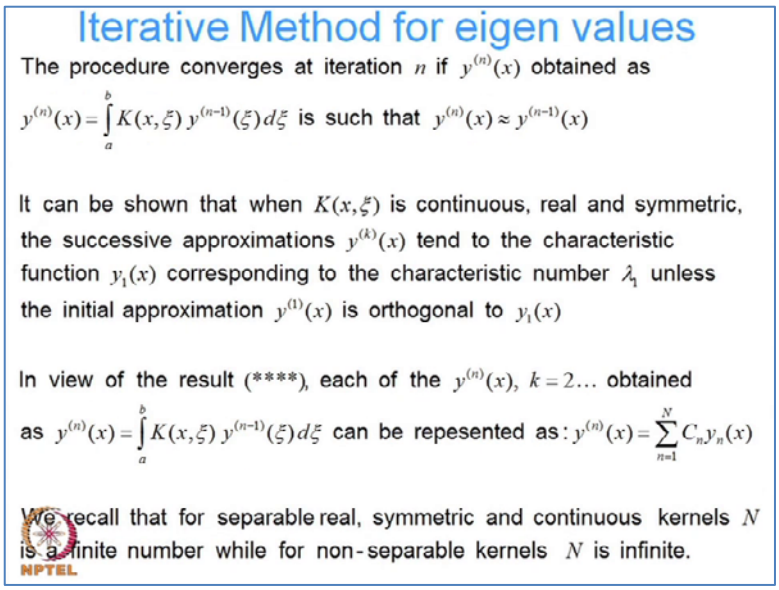

We keep on doing this until satisfactory convergence is obtained. What do we mean by satisfactory convergence? Well, two iterates are about the same right up to a certain tolerance to define some sort of tolerance, but these two iterates are about the same in some norm, right. Their difference is very acceptable and is beyond is below a certain cart of value, right. In that case, they are close. It can be shown that when K xi is a continuous real and symmetric, we can prove that these successive approximations 10 to the characteristic function corresponding to the smallest Eigen value. We can show that and unless there is a limitation, but for the time being let us leave the limitation aside, but

it can be shown that in most cases, whatever guess we take to y 1 of x if we perform this operation repeatedly, we are going to end up with the smallest Eigen value and the corresponding Eigen vector, right.

So, how is that? Well, we will use the result which we just encountered which says that if you have a function, if you have a continuous function and you operate on that with the kernel and you get another function, that function can be represented as a linear combination of the Eigen functions, right. That was the result we just encountered here, right. That was our result here, right. So, look at the form. It is exactly the same.

So, what we are saying is that the operation that we are repeatedly performing, we are operating, we are assuming that my y 1 xi is a continuous function, my y 2 xi my y 2 x that I get after operating on that with the kernel, I will be able to express that as a linear combination of the Eigen functions of the integral equation from my result before, right. We recall that for separable real symmetric and continuous kernels, we have a finite number of N. The number of Eigen functions is finite while for non-separable kernels N is infinite.

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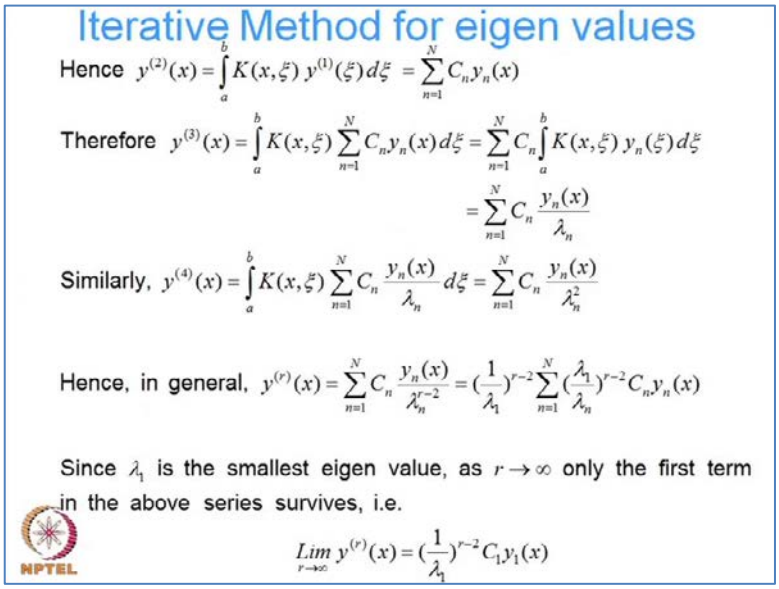

So, we do this and we get y 2 of x which is something like this which we know can be expressed like this because y 1 of xi are initial guess is a continuous function. So, now when I put y 2 of x in this equation back again on the right hand side to get my y 3 of x, look what we can do. We can pullout the summation outside and inside I have an integral

a to b K x xi y n xi d xi, but each of these y n xi's are the Eigen functions of my integral equation, right. So, that means, integral of a to b K x xi y n xi, it is going to be equal to 1 by lambda y n of x 1 by lambda n y n of x because this times lambda n I know is equal to y n of x because this is a Eigen function.

So, I can write this as sigma n is equal to 1 to capital N c n y n of x by lambda n, right. I can write it like that. So, again I substitute. When I do it again, when I substitute y 3 of x to the right hand side, I just substitute this right again. I pullout sigma c n and again I have integral a to b K x xi y n xi which, sorry this should be xi. I apologize, this should be xi.

Then I end up with y n of x by lambda n. And finally, I get sigma n equal to 1 to n c n y n of x by lambda n square, right and if I do it r times, I can write get in general that y r of x is equal to just analogous to this, right. Just by induction we can see that y r of x will be equal to sigma n is equal to 1 to N c n y n of x lambda n divided by lambda n to the power n minus 2. Look at this y 4, you have lambda n square.

So, y r, it will be lambda n to the power r minus 2, right. So, we can write this and then I do a little bit of manipulation. I multiply top and bottom by lambda 1 to the power r minus 2 and I bring 1 by lambda 1 to the power r minus 2 outside, and I have this sort of an equation. Now, look at this. Look at this term. The term within this summation sign, I know that lambda 1 is the smallest Eigen value.

So, when n is equal to 1, this term is going to become 1, but for n greater than 1, this term is always going to be less than 1. It is always going to be less than 1 and what does it means that if r goes to infinity as r becomes larger and larger, all those other terms except the term for n is equal to 1 are going to become smaller and smaller, right. Those are the terms that become even smaller, right.

So, that means, those in the limit that r goes to infinity, those terms are going to go to 0. Only the first term is going to survive and I will be left with this 1 by lambda 1 to the power r minus 2 c 1 y 1 of x. So, you can see that as I increase the number of iterates, my iterate is going to 10 to the first Eigen function and whatever I get here, I can get my Eigen value from there, right lambda 1 from there.

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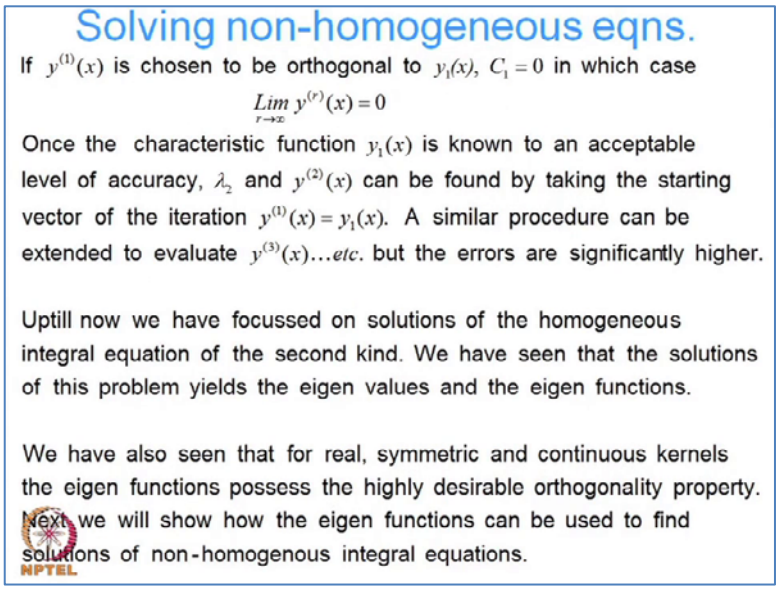

However, if our first iterate by some chance I chose my first iterate to be orthogonal to the Eigen function corresponding to the lowest Eigen value, then my r goes to infinity. My y r is going to go to 0 because this c 1 has got to be 0 because c 1 is the coefficient of the first Eigen function in my expansion, right. There c 1 is the coefficient of the first Eigen function.

So, if by some chance I choose my y 1 such that it is orthogonal to my first Eigen function, when I say first Eigen function I mean the Eigen function corresponding to the smallest Eigen value. In that case, I mean this thing will break down, right. I will not be able to find, but that is relatively rare situation, right. So, this iterative method is guaranteed to yield the Eigen function and the smallest Eigen value. Similarly, once I have found this smallest Eigen value and the corresponding Eigen function, I can use that as the starting iterate, right.

In that case, I will end up with y 2 because I know that what I get at the end, it will not contain y 1, right. When I say y 1, I mean the first Eigen function because if I choose, so I will end up with y 2 and if I do this and after I find y 2, I can find y 3 and so on, but this is actually one must qualify this and say that it does not really work. That way it works very well usually to find the first Eigen value and the first Eigen function, but beyond that up to second third, it breaks down. Why? Because it is not the errors are so high. The errors are sufficient. So, whatever you get does not really it can be shown. I have not

done that because it can be shown that the higher order Eigen functions, it is not possible to get them accurately using this method. It is only the first Eigen function, maybe the second, but not much beyond that, right.

So, until now we have focussed on solutions of the homogenous integral equation of the second kind, and we have found ways of finding the Eigen values and Eigen functions, and we will see that for real symmetric and continuous kernels, these Eigen functions possess highly desirable orthogonality property which will allow us to use this Eigen functions to solve non-homogeneous integral equations.

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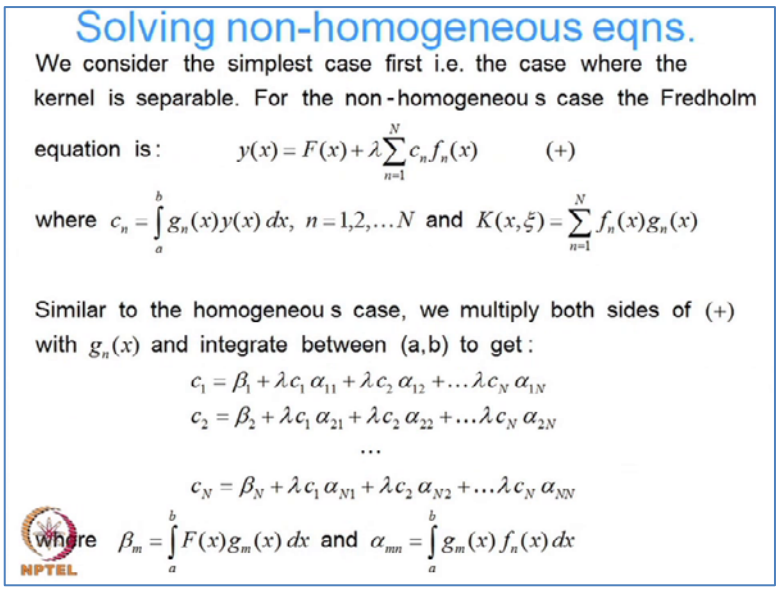

How are we going to do that? Well, let us look at this simple case where we have a separable kernel, but earlier we looked at the situation where it was homogenous, but now we are considering a non-homogenous situation, where f of x is not equal to 0 and we are interested in finding the solution of that equation. Again these we have defined already c n, I defined earlier and K x xi is equal to f n x g n x because it is separable, right. Again, I have a mistake here. It is g n of xi, right f n of x g n of xi K of x xi is equal to that and similar to the homogenous case if we multiply both sides of this with the g 1 g 2 g 3 and so on and integrate within the limits a to b, I end up with a system of equations. This system of equations is identical to the previous system except for these terms beta 1, beta 2 through beta n which now appear because I have this integral also on the right

hand side f of x g m of x d x. I have this integral also on the right hand side. That is why these additional terms beta 1 through beta n.

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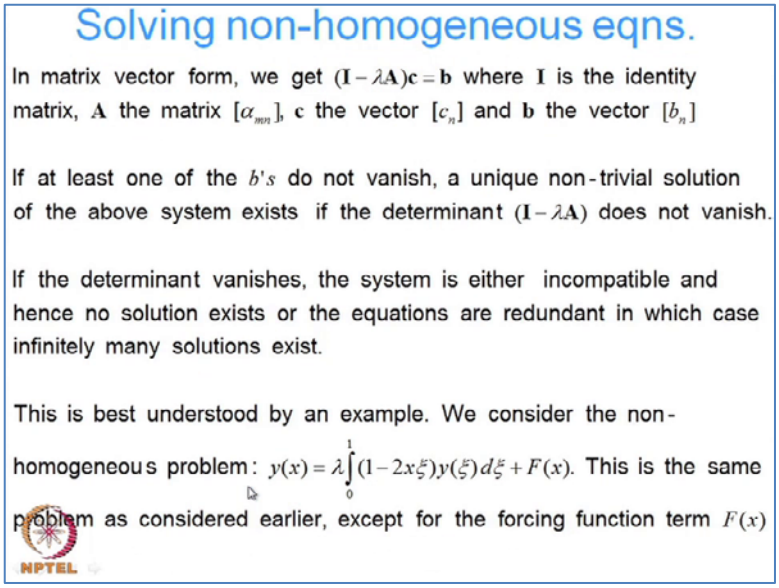

Then finally, we end up with a matrix system like this. I minus lambda a c is equal to b. Recall for the homogeneous system, I had I minus lambda a c equal to 0, right. So, I is the identity matrix, a is the matrix of alpha m n, where alpha m n is defined like this. These are all familiar. So, I am not spending too much time on this because we have seen this before. So, now, we have this system like this if at least one of the b's do not vanish, and this coefficient matrix is invertible, then we can find a solution, right. A unique non-trivial solution to the above system exists if the determinant I minus lambda A does not vanish.

What happens if the determinant vanishes? Well, if the determinant vanishes, the system becomes either it is incompatible. That means, there are no solutions possible or they are linearly dependent in which case there are infinite number of solutions possible, right. We will look at this situation by considering an example where we are considering and this was an example we considered earlier except that now we have non-homogenous, right. There is a forcing function f of x.

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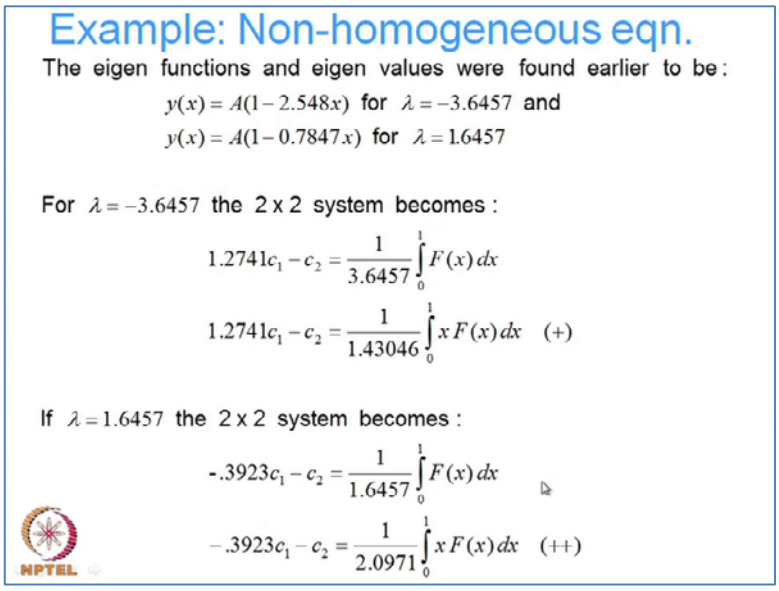

So, let us look at this and we found the Eigen values and the Eigen functions for this problem. Earlier, the Eigen values were lambda is equal to minus 3.6457 and lambda is equal to 1.6457, and these were the corresponding Eigen functions. Now, suppose lambda is equal to one of the Eigen values, then this 2 by 2 system becomes the following, right. It becomes the following and if lambda is equal to 1.6457, this becomes the following, right.

You can see that the left hand side is identical, right. The left hand side of these two equations are the same, but the right hand sides are different. The right hand side of this equation is different from this as is the right hand side of this from this, right. So, what does this mean? This means that this system is going to be incompatible unless this is equal to this. Similarly, this system is going to be incompatible unless this is equal to that, right.

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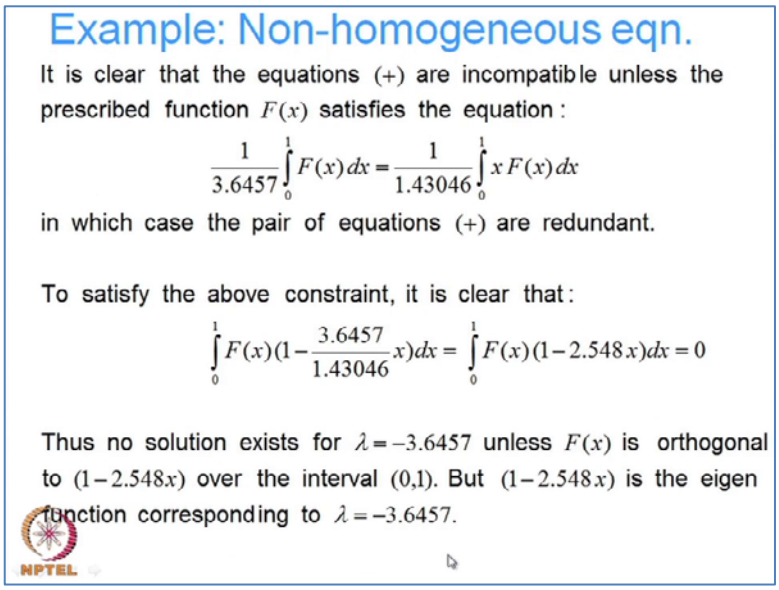

So, it is incompatible unless this condition is satisfied, this is equal to this. So, to satisfy this constraint, we can see that this condition has to hold, right. If this is equal to this bringing the terms, to bringing this to the left hand side and pulling out F of x, I get this equation, right. What does it mean? That means, my f, but look at this. What is this? This is actually the Eigen function corresponding to that Eigen value, right. So, it turns out that in this case, in order to have a solution, my F of x must be orthogonal to the corresponding Eigen function, right. 1.25, this is the Eigen function corresponding to that. So, unless F of x is called orthogonal to that Eigen function, I am not going to be able to get the solution, right.

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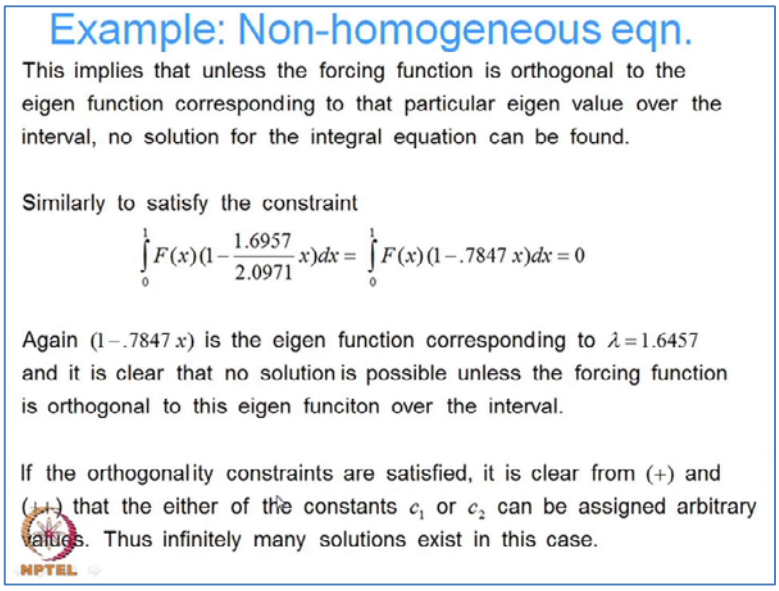

Similarly, for the other case in order to make the equations compatible, they must satisfy this equation and if I rearrange the terms, you can see that f of x in 1 minus this integral over 8, 0 to 1 must be equal to 0. So, that means, my forcing function must be orthogonal to my Eigen function, right. So, the forcing function must be orthogonal to the Eigen function. If the orthogonality constraints are satisfied, then it is possible to come up with a solution, but go back to this system. This means that these two equations, when this is equal to this means that these two equations are identical, right. So, I can find the solution, but only up to an unknown constant, right. If I assume one of the constants c 1, I can find c 2 or vice versa, right. So, I know the solution up to an unknown constant. That means I have infinitely many solution because that unknown constant, any arbitrary value can be assigned arbitrary values. Thus, infinitely many solutions exist in this case.

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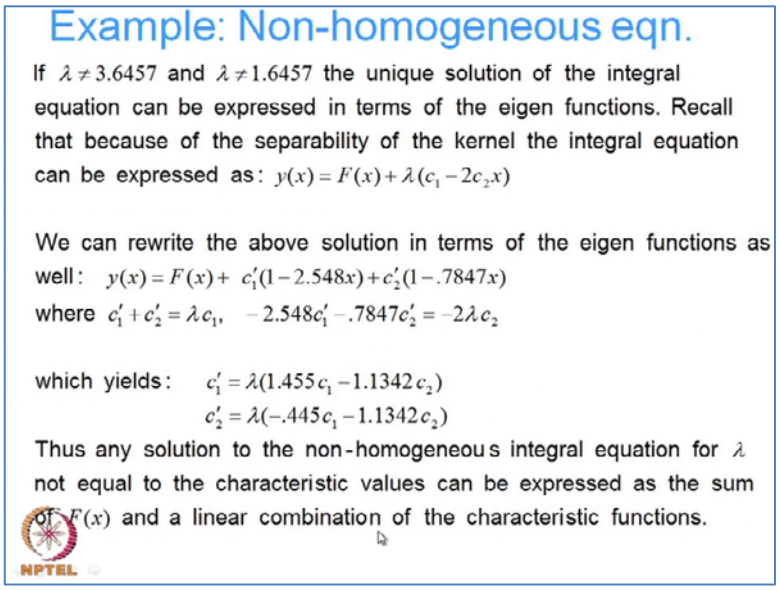

So, if lambda is not equal to this, so the unique solution can be expressed in terms of the Eigen functions. Let us see how. So, recall that this is because separability of kernel, the integral equation can be represented by this. We have seen before because of this separability of the kernels, we can write the solution in these terms and then we can rewrite the above solutions in terms of the Eigen functions?

So, if I write this as I know that the solution can be expressed in this form. We have seen that before, right. Now, suppose I say that I am going to write it in terms of the Eigen functions. This is one Eigen function; I know this is another Eigen function. That means, my c 1 prime and c 2 prime that gives two equations for c 1 prime and c 2 prime, right. So, c 1 prime plus c 2 prime must be equal to lambda c 1 because this is the constant term, c 1 prime plus c 2 prime is the constant term here and here, the constant term is lambda c 1. So, equating the constant term, I have this equation and the term involving x is minus lambda 2 c 2 x here.

So, here this must be equal to minus 2 lambda c 2, right. So, that gives me two equations for c 1 prime and c 2 prime. I solve them. So, I can get c 1 prime and in terms of c 2 and c 1 prime and c 2 prime in terms of lambda and c 1 and c 2. So, what does it mean? That means, the solution to the non-homogeneous integral equation is equal to the forcing function plus the forcing function plus linear combinations of my Eigen functions, right.

So, that means the solution of the non-homogeneous integral equation for lambda not equal to the characteristic value can be represented as the sum of f of x, and a linear combination of the characteristic function, you can now see by knowing the characteristic functions is so very important because if I know the characteristic function, then I can write down the solution straight away, right.

I can write down the solution straight away like this, right and then it is just a matter of evaluating these constants c 1 prime and c 2 prime, right. It is just a question of evaluating the constants c 1 prime and c 2 prime which I can do if I know the boundary conditions, right. So, that is why it is very important to that Eigen analysis. It is so very important to finding the solutions of integral equations.

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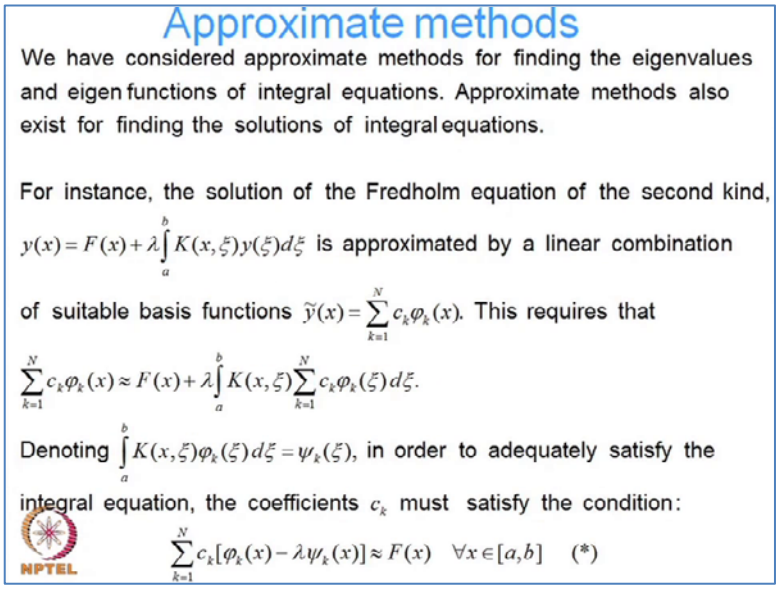

So, we have talked about ways of solving integral equations, we have talked about the Eigen function approach, but after all, this is supposed to be a course on numerical analysis. So, we would like to talk a little bit in whatever time remains about approximate methods for solving integral equations, right. So, how can we do that? Well, we will talk about that in the context of the Fredholm equation of the second kind which I know is given by this, right and the solution to this is usually approximated by a linear combination of suitable basis functions. I say that if I am going to construct an approximate solution to this integral equation as a linear combination of some independent basis functions, right.

If that is a good approximate solution, this requires that if I substitute this here, right. So, this becomes c k phi K x is approximately equal to f of x plus lambda K x xi and then I substitute that here this must be approximately equal to that, right. If this is a good estimate to the solution, then I denote this thing integral, a to b K x xi phi K xi d xi. Basically, all the terms involving xi as another function psi K of xi, then in that case just for convenience of notation, just for easier writing, so in that case this condition becomes this sigma c k phi K minus lambda psi K approximately equal to f of x. This condition has to be satisfied if this is a good approximate solution, right and this has to be satisfied for all x belonging to a and heading interval a b.

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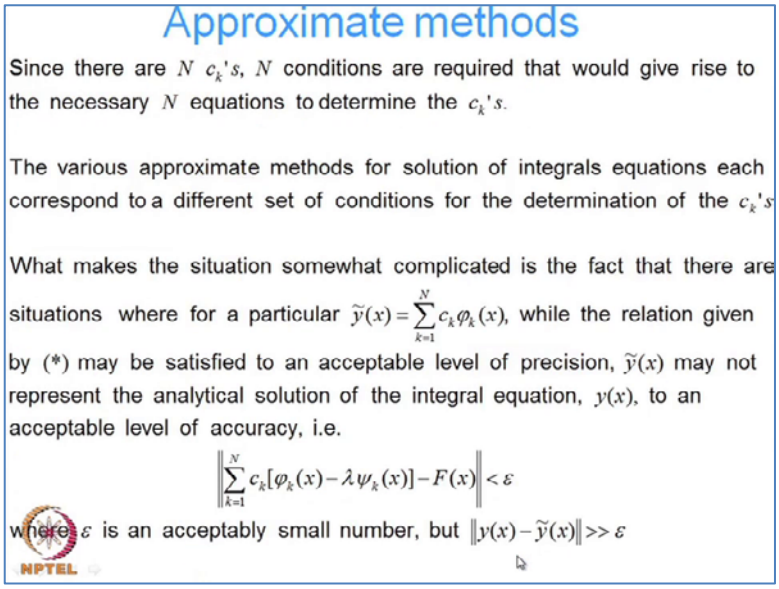

So, now, you can see that. So, basically it is a question of determining these constants c k, right. So, we are assuming that we know our basic functions. When we know that, we know the independent where we know the x. The way the basic functions in which I am going to expand my approximate solution, right and I have to find out what are the values c k, such that what I construct the approximate solution I construct is a good estimate to my true solution.

So, then c k must satisfy this relationship, right and you can see that there are n c k's. So, I need conditions, I need n conditions to evaluate each of these n c k's. What are those n conditions? What are those? So, there are different methods for approximate methods for solving integral equations. Each of those methods, they are a different way of

determining or satisfy of supplying the conditions to determine the c k, right. So, let us go back to that equation I have to determine the c k's, there are n c k's.

So, I need n conditions. How am I going to come up with those n conditions? There are for instance, there is a collocation method. The collocation method will tell me this is the way of coming up with the n conditions for determining the n c k's. Then there are ways for instance the Galerkin method. For the Galerkin method, it will give a different set of n conditions to come up with these c k's, right. So, the different ways of the conditions to determine the c k's that determines the approximate method for solving that problem.

So, the various approximate methods for solution of the integral equations correspond to a different set of condition for the determination of the c k's, right. However, there is a little K v hat I want to introduce here. Suppose I come up I satisfy that condition, right. Suppose I satisfy that condition sufficiently meaning that this minus this is lesser than in some norm, it is less than some small number.

So, that means, that I am satisfying my criterion right here up to an acceptable level. Does that mean the solution that I get why tilde K is equal to sigma K equal to 1 to n c k phi K is that going to be close to the exact solution by a very small amount is not necessary. So, it is very important and some way it is complicated. In fact that for a particular y tilde x via the relation, this relation may be satisfied that relation may not be satisfied, norm of y x minus y tilde x may still be large, right. So, in that case it satisfies the integral equation. It satisfies represent the analytical solution to y, it may not satisfy the analytical solution closed enough. That means, my y tilde x may not be close enough to the analytical solutions, that is a problem.

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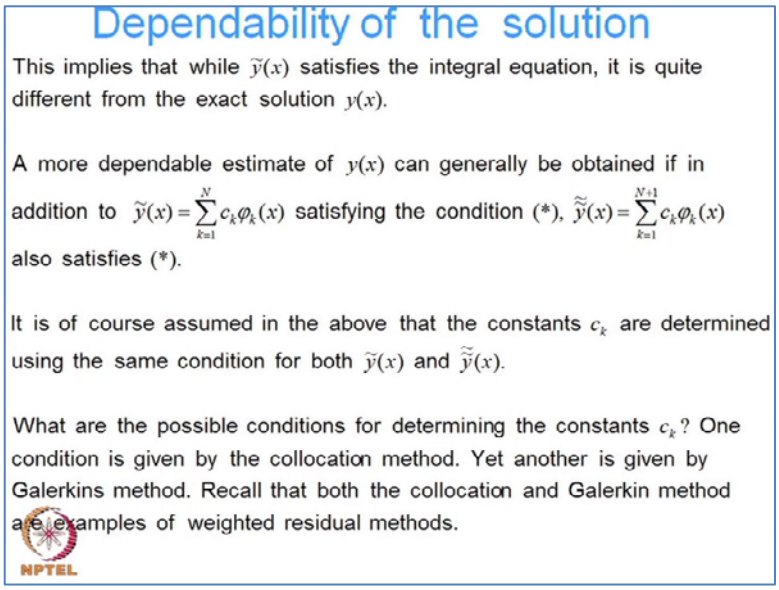

So, typically how people get around, it is that they find out y tilde x for say n basis functions for n constants and n basis functions and then they find out y tilde, they find out another approximate solutions for one more basis function, and one more constant. If these two are close enough, then it is quite likely that my approximate solution is actually close to my true solution. So, my y tilde x is close to y of x usually. So, what are the possible conditions for determining the constants? We will talk about two methods if time permits. Very briefly we will talk about the collocation method and we will also talk about the Galerkin method.

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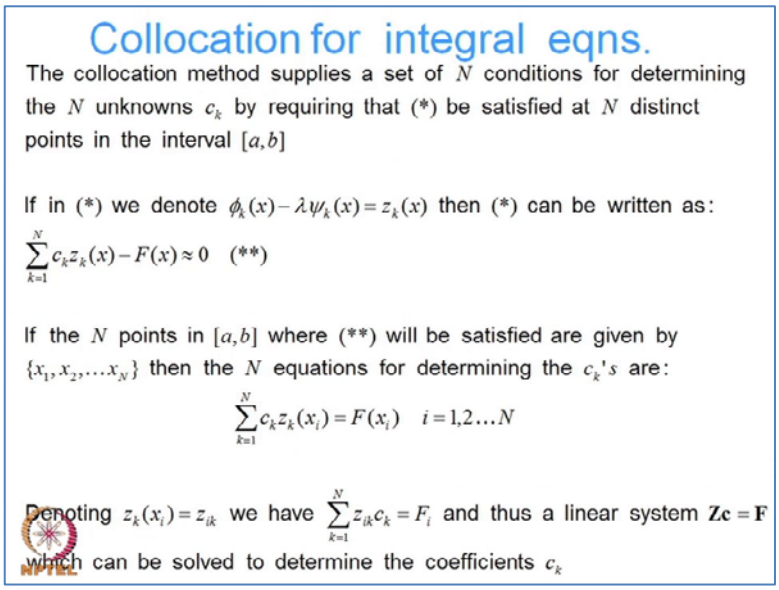

In the collocation method, what we do is that we say we have to satisfy that condition, right. The collocation we have to satisfy this condition and we are going to satisfy that condition at n discrete points, right. So, the collocation method supplies a set of n conditions for determining the n unknown c k by requiring that this equation be satisfied at n discrete points, right.

It says that at x is equal to 1, x is equal to 2, x is equal to n, this equation is satisfied, then what do I have? I have n equations because I write down this equation for x is equal to x 1. I have one equation. I write down this equation for x is equal to x 2. I have another equation I write down that equation for x is equal to x n, I have one as the n-th equation.

So, I have n equations, I have n unknown constants c n, I solve that n that system of n equations and I find out my coefficient c, right. So, that is the way to find the collocation method to find out the constants. So, in stars this is again some notation I am using. If I denote this term phi k minus lambda psi k as an intermediate variable z k, then I have sigma.

My criterion becomes sigma K equal to 1 to n c k z k x minus f of x is approximately equal to 0 and then I satisfy that at n points, I try to satisfy that at n points and then I have a linear system and I solve that linear system I find out the coefficients c k, right. So, that is the essence of the collocation method which we have encountered earlier also in different context, but this is how we can use it to solve integral equation.

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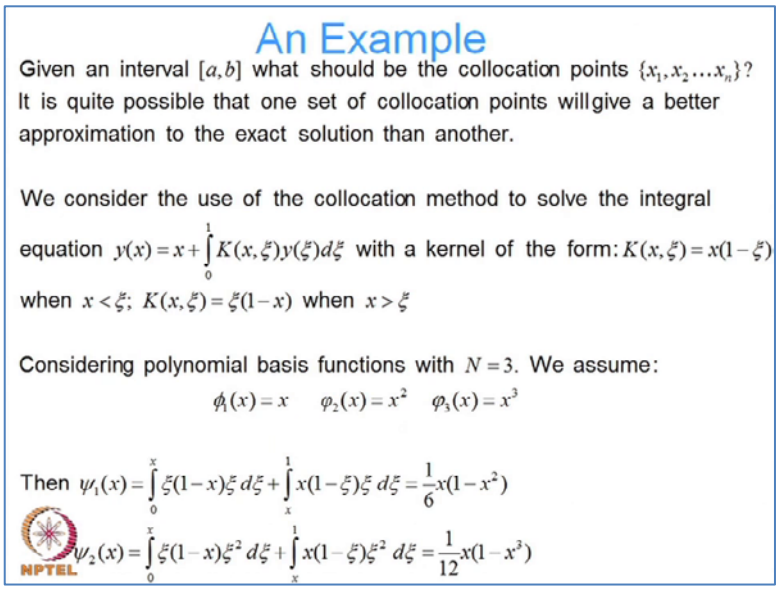

So, here is a little example where I have used the collocation method to solve this problem. So, here is my integral equation. This is the forcing function x, right and the kernel is of this form. So, kernel is K x xi is equal to x 1 minus xi when x is less than xi, and K x xi xi 1 minus x when x is greater than xi and now, suppose I choose my polynomials, my basis functions to be polynomials up to order 3.

So, I choose the first basis function to be x, the second to be x square and the third to be x cube. So, my approximate solution is c 1 phi 1 of x c 1 x plus c 2 x square plus c 3 x cube. I am interested in finding those c 1, c 2, c 3 using the collocation method. So, I find out my intermediate variable psi 1 of x is equal to integral of the kernel times, this phi 1 which we denoted earlier.

So, if we do that, here we are integrating between 0 to x and between x to 1. When I integrate between 0 to x, x is always greater than xi. So, this part of the kernel is going to contribute xi 1 minus x. When I integrate between x to 1, xi is always greater than x. So, this part of the kernel, this branch of the kernel is going to contribute. So, I have x 1 minus xi x 1 minus xi d xi and then I integrate between these limits I get psi 1 of x is equal to that here I get psi 2 of x is equal to this, where I use phi 2 of x here, right. Same thing except that instead of using phi 1 of xi, I have used phi 2 of xi. You can see I have replaced xi by xi square here. That is the only difference, right and then I evaluated psi 2 of x.

Similarly, I evaluate psi 3 of x, right and then if I go back to the solution, I have c k z k c k is nothing but phi k of x minus lambda of psi k of x. I just devaluated psi k of x, right.

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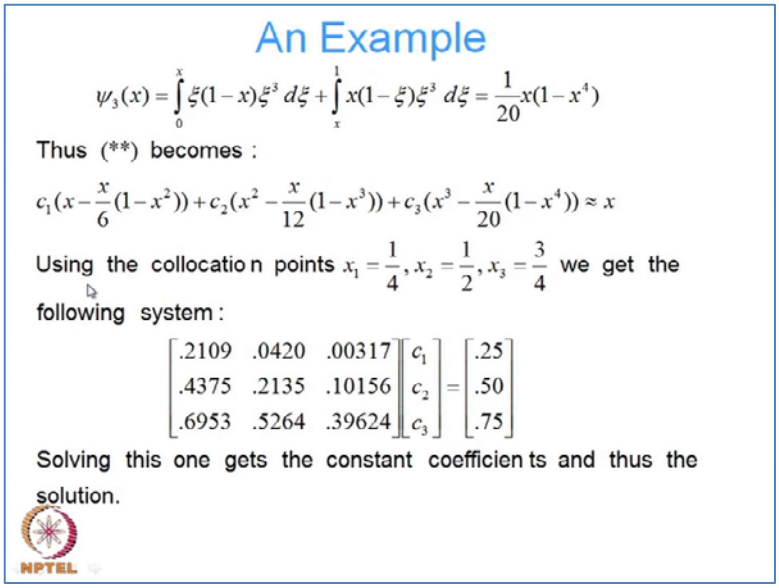

So, now, I can write my solution as c 1 x minus, this is of course, my psi 1, right. That is my psi 2, this is my psi 3 and this is nothing but my phi K of x, right. So, this is my phi 1, this is my phi 2, phi 3 and this must be equal to that. So, this condition has to be satisfied for the solution to be good, right. This condition has to be satisfied for my approximate solution to be good. Then how am I going to use that? I am going to use the collocation method. So, I am going to satisfy, I have got three constants.

So, I will say that I am going to satisfy that equation at three different points, right and if I satisfy that equation at three different points, I get a system of equation in 3 by 3 systems for my c 1, c 2 and c 3. I invert this coefficient matrix, I solve for my c 1, c 2, c 3. I know my solution, right.

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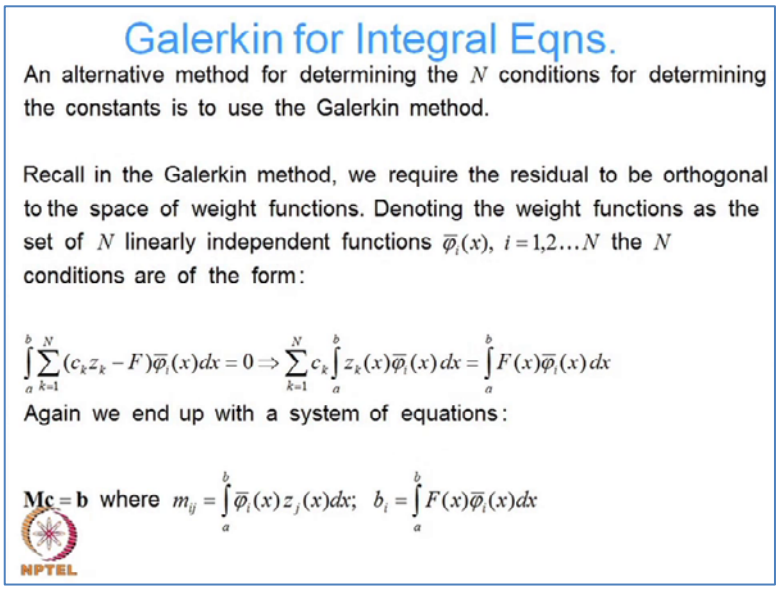

So, that is an example of using the collocation method for solving an integral equation like we have an alternative method for determining the n conditions, for determining an alternative method which gives the n conditions for determining the constants is known as Galerkin's method. So, in the Galerkin's method, recall the requirement is that whatever is my error, whatever is my residual, it must be orthogonal to a set of weighting functions, right and these weighing functions are again linearly independent set of functions.

So, what is the residual here? The residual here is nothing but this term because this has got to be equal to 0. If this is approximately equal to 0, then I know that my approximate solution is good, right. It is smaller, better is my approximate solution. So, I am going to take this residual and integrate it with respect to these basis functions with these weighing functions.

So, here you can see that. So, denoting the weighting functions as a set of n linearly independent functions phi i bar x i is equal to 1 through n, then conditions are of this form sigma k equal to 1 to n c k z k minus f phi bar i x d x and then I pullout this summation outside sigma k equal to 1 to n c k integral a to b z k x phi bar i of x and then that must be equal to f of x phi bar i of x, right.

So, this is my condition again. I can evaluate these integrals, I can evaluate the z k if the z K phi phi bar i x I know, right. These are my known basis functions z k's. I have found

out from earlier that was my z k y phi k of x minus lambda psi k of x. Remember, now there are two sets of independent, linearly independent functions that is the psi k which I used to construct my approximation, right. My approximation y tilde is equal to sigma c k psi k. So, the psi k's are the basis functions for my approximation and in addition, I have another set of basis function which I am denoting as phi bar of i of x which are my waiting functions, and these waiting functions are also linearly independent.

So, the requirement is that this condition be satisfied. Again, I end up with a linear system, m c is equal to b where m i j is equal to that condition and b is equal to that condition, right. I invert this equation, I solve for c I, get my constant. I can find my solution and I know that this solution satisfies the orthogonality requirement with respect to the weighting functions, right.

So, this is a way to find. So, this is my Galerkin equation. So, in case I choose my phi bar, phi bar to be of the same form as my psi's. In case I chose my phi bar to be of the same form as my psi's, my basis function for the approximation, then I have the standard Galerkin formulation also known as the Bubnov Galerkin formulation, right. However, if I choose my phi bars to be different from my psi's, then I have what is called the Petrov Galerkin formulation where my waiting functions are different from my basis functions which I use to construct my approximate solution which are in certain context. They are known as the trial functions, right. So, that brings us to the end of our lecture.

So, thank you very much.