Numerical Methods in Civil Engineering
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Lecture - 39
Integral Equations-II

In lecture 39 of our series on numerical methods in civil engineering, we will continue with our discussion on integral equations.

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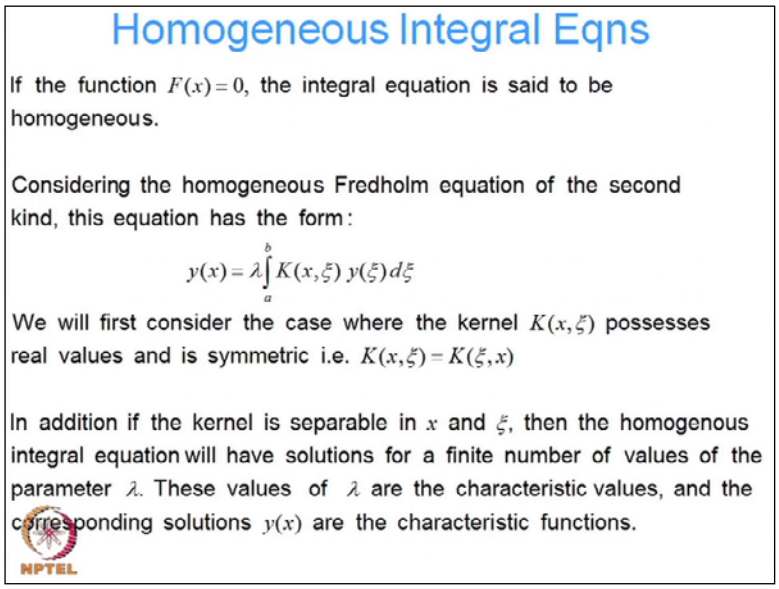

In the last lecture, we looked at the two main classes of integral equations. We looked at Fredholm equations and we looked at Volterra equations, and we also looked at Fredholm equations of the first kind and Fredholm equations of the second kind as well as Volterra equations of the first kind and Volterra equations of the second kind. Today, we will concentrate on homogenous Fredholm equations and the reason why we will spend some time discussing solutions to the homogenous Fredholm equations is because these solutions, particularly these Eigen values and Eigen functions of these Fredholm equations turns out to be very useful, and they are very useful in solving general integral equations. So, we will concentrate on the homogenous solution because the solution of the homogenous problem helps us in solving the general non-homogeneous problem.

So, what is a homogenous Fredholm equation? A homogenous Fredholm equation recall is of this form plus on the right hand side, there is a forcing function F of x. Now, if F of

x is equal to 0, the integral equation becomes a homogenous integral equation in this case homogenous Fredholm equation, and we will first consider the case when we have our kernel to this Fredholm equation K of x of xi. It possesses symmetry. It is symmetric and moreover, it gives real values for all values of x and xi. K of x of xi gives me a real value, right. So, it is a real kernel and it is symmetric. In addition, we will look at both the situations where the kernel is separable as well as when the situation arises, when the kernel is not separable, right. We will focus on separable kernels.

First the separable meaning that k of x of xi can be written as a function of x as the product of function of x and a function of xi, right. So, it is a separable kernel. So, dependence of x and xi of k x and xi on x and xi can be separated out in terms of dependence on a function of x, and a function of xi, right. Why we will focus on separable kernels first? Well, principally because these separable kernels, they give raise to finite number of Eigen values and Eigen functions. So, Fredholm equation with a separable kernel has a finite number of characteristic values and finite number of characteristic functions number one and number two that the insight we gain at looking at these separable kernels will help us solve the non-separable. It becomes much easier to understand the non-separable case.

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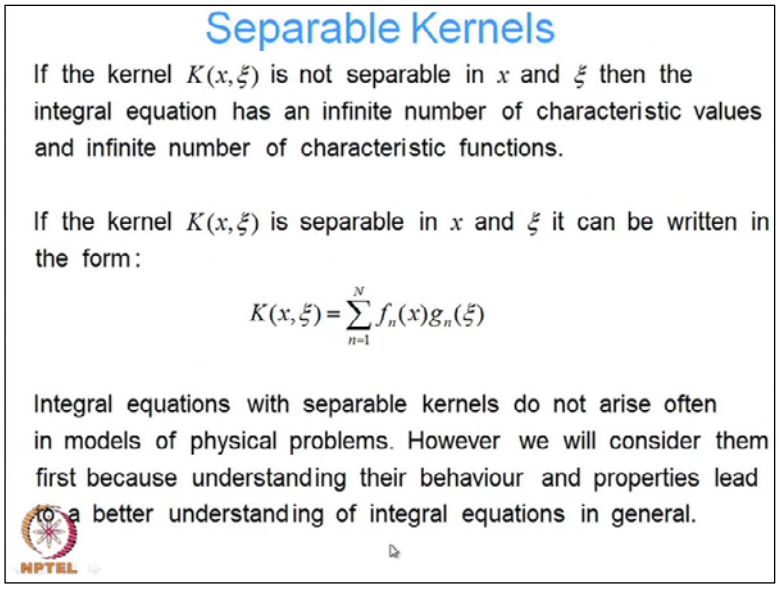

So, kernel is separable x and xi when we can split it up into the dependence on x and xi can be separated out, that is when K x of xi can be written as the sum of n number of

terms of this type, where this is a function of x and f n is a function of x and g n is the function of xi. So, this is a separable kernel in which case there numbers of Eigen, the integral equation possess a finite number of Eigen values and Eigen functions and if it is not separable in x and xi, it possesses an infinite number of characteristic values and infinite number of characteristic functions.

Now, as I said why do we concern ourselves with integral equations with separable kernels? Well, because they give us insight into the general problem, but we should remember it is important to point out that in most real world problems, when do we get integral equations as a result of some sort of modeling, right and that model when we do some modeling of a physical problem, we generally do not end up with an integral equation with a separable kernel in most of the times, right. Most of the times it is not possible to write out our kernel as sigma n is equal to 1 to n f n of x g n of xi to separate out the dependent on x and xi, but still we focus on this case because it is simple. First of all it simplifies things a lot. Number 2, it gives us some insight into the general problem.

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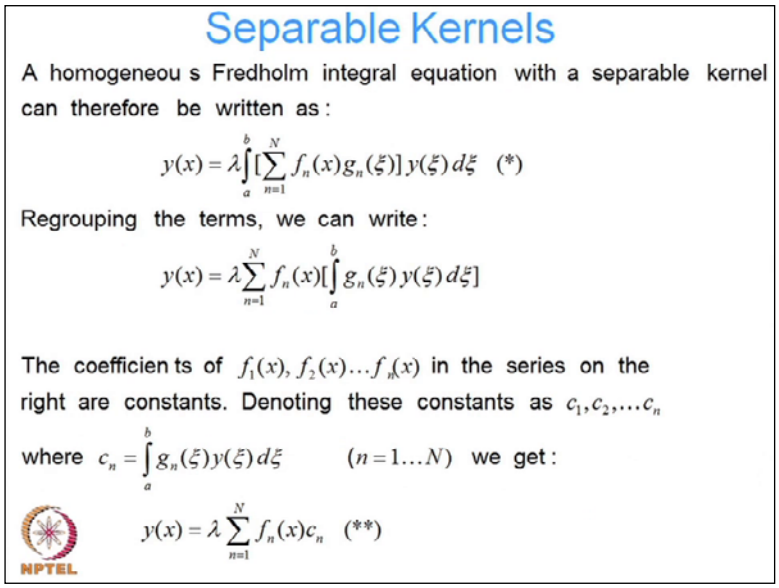

So, if the kernel is separable, you can see that we can write the homogenous Fredholm equation in this form where all I have done is, I have replaced the kernel k of x of xi in terms of this summation of these products of these functions of x and xi, right and then we regroup terms. I pull out the dependence on x outside the integral and leave the dependence of xi inside the integral. So, g n xi y xi, I pull it inside the integral and you

can see I have a series which is a function in x times, this thing which if I once integrates out, it becomes a constant, right and if I denote these constants as c 1 c 2 through c n, basically I denote each of these integrals a to b g n xi y xi d xi as a c n, then I can write this integral equation as y of x is equal to lambda times sigma n is equal to 1 to N f n a f n of x c n, where c n is the constant I get after evaluating this integral.

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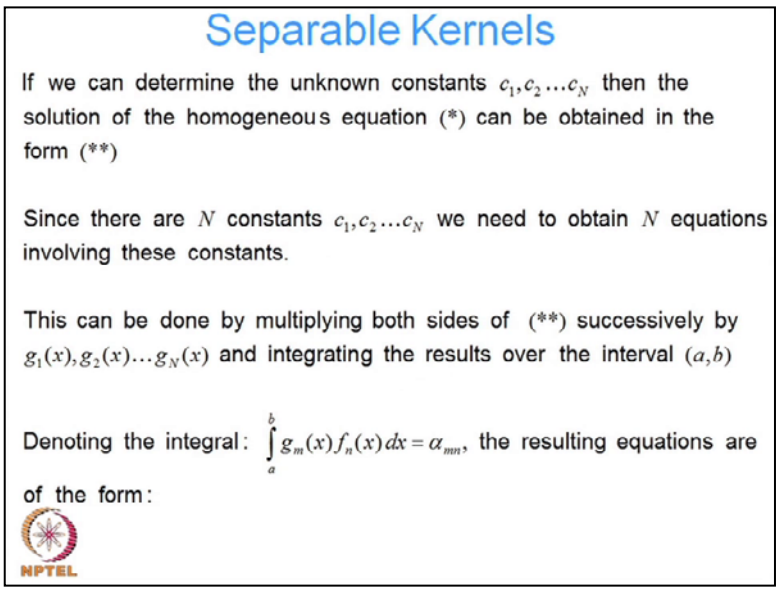

So, somehow we can determine these unknown constants c 1 through c n, you can see that I have a solution of my homogenous Fredholm equation because I know my f n of x. If I know my c n's I have solved, I have got my y in terms of f n fn of x and c n. So, that is the solution of my homogenous Fredholm equation. So, how do we find out these c n's? So, since there are n constants c n, we obviously need n equations to evaluate those n constants and we do that by multiplying both sides of this equation successively with each of the g's, right. So, we multiply both sides of this equation. First by g 1 of x, we get one equation and then I integrate over the limits a to b, right and then I multiplied by that equation by g 2 again. I integrate over the limits a to b and in this manner, I do this n times and I end up with n equations.

So, we will see that, but let us first look at some notation. We denote integral of a to b, g m x f n f n of x d x is alpha m n, right. We denote this as integral as alpha m n. So, basically we are looking at this equation and we are multiplying both sides, say by g 1 of x and we will integrate between a to b. We multiply the right left right hand side by g 1

of x. We are going to integrate between a to b, even integrate g 1 of x f n of x between a to b, right and that we are going to call alpha 1 g 1 of x integral of g 1 of x f n of x is going to give me alpha 1 n, right.

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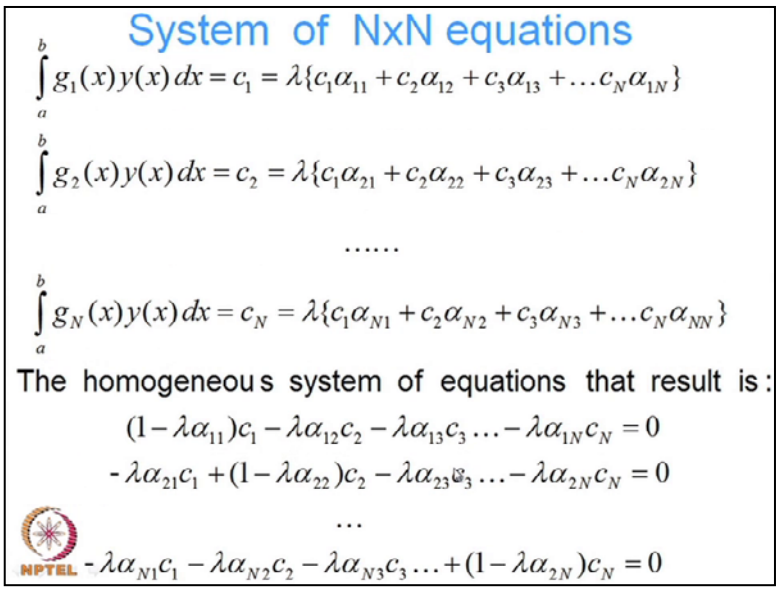

So, g integral of g m of x f n of x d x is going to give me alpha m n, right and if I do that, I end up with the system like this. First in equation integral of g 1 of x y of x d x is equal to c 1, right. That is c 1 because of the definitions of my c is just this, right. So, c 1 is equal to integral of g 1 xi y xi d xi and if I multiply both sides of this equation by g 1 of x and integrate, I am going to get c 1 on the left hand side. So, I am going to get c 1 and on the right hand side, I have lambda integral of g 1 of x f f, whatever it is f of x and then I get it if its f 1, then I get alpha 1, right. If it is say f 2, I get alpha 1 2. If it is f 3 integral of g 1 x f 3 of x, that is going to give me alpha 1 3 and so on and so forth until I get c N alpha 1 N.

Similarly, I do for all. I multiply it again with g 2 and integrate between a to b, I get this equation and finally, I multiply by g N of x integrate between a to b and I get this equation and grouping pulling all the terms to the left hand side, we look at the first equation. So, there is c 1 here and c 1 here. So, I get 1 minus lambda alpha 1 1 c 1. So, I get that term and when I bring this term to the left hand side, I have minus lambda alpha 1 2 c 2 minus lambda alpha 1 3 c 3 and finally, minus lambda alpha 1 and c N is equal to 0, and do it for all the equations. So, I get n equations in terms of my n unknown c 1

though c N.

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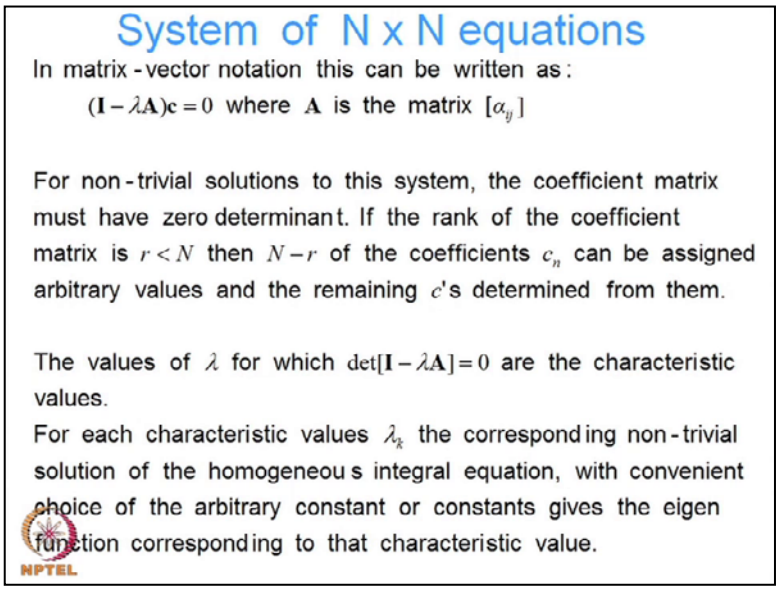

I also have a coefficient matrix which can be written in this form. So, basically my coefficient matrix is 1 minus lambda alpha 1 1 minus lambda alpha 1 2 minus lambda alpha 1 3 minus lambda alpha 1 n. So, I can write it as the identity matrix minus lambda times the alpha matrix, right. So, the identity matrix minus lambda times, the alpha matrix which I denoted by a here, right. So, I get in matrix vector notation, I get a system like this, I minus lambda A c is equal to 0, right and you know that for this system to have non-trivial solutions requirement is that the coefficient matrix must have 0 determinant, right. It cannot have full rank.

So, it has 0 determinants, and that means, its crank cannot be full. That means, if I have n, if I had if I had n equations, then that means, its rank must be less than N. So, the rank r must be less than N and if the rank of the matrix is r, then N in that case n minus. This should be familiar to all of us because we have looked at this. We have looked at linear systems in detail earlier on. So, if the rank is r, so N minus r of the coefficients can be assigned independently arbitrary values and the remaining coefficients can be determined from them. The remaining c's can be determined from them.

So, the values of lambda for which this determinant I minus lambda a, this coefficient matrix, its determinant become 0 are the characteristic values and for each of those characteristic values, the corresponding non-trivial c vector, right. The non-trivial

solution of this system, the corresponding c vector will give us the Eigen functions, right. How will they give us the Eigen functions? Well, they will give us the Eigen function from this equation, right. Those once we know these c's we substitute those c's for that lambda and we get y, right. That y is the Eigen function of the integral equation.

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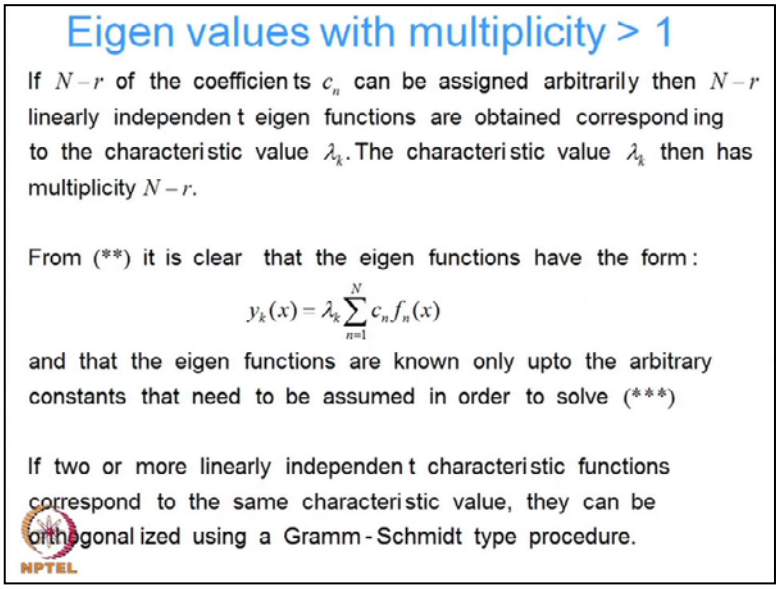

So, if n minus r of the coefficient c n can be assigned arbitrarily, then N minus r linearly independent Eigen functions are obtained corresponding to the Eigen value lambda a k and in that case, the characteristic value lambda k has multiplicity N minus r, right. So, corresponding to that single Eigen value lambda k, there are N minus r linearly independent Eigen functions, right. So, it has multiplicity N minus r and as I just showed the Eigen functions, then we will have the form y k of x is equal to lambda k sigma n equal to 1 to n c n f n of x. It is clear that the Eigen functions are known only up to the arbitrary constants that need to be assumed in order to solve that equation, basically this equation, right.

So, since the determinant is 0, we cannot have unique solutions, right. So, we have to assume some of the constants, right. The other constants c are evaluated in terms of those constants, right. If we have to assume the values of N minus r constants, then N minus r constants can be assigned arbitrary values, right. They can be assigned arbitrarily. The remaining r constants are evaluated in terms of those N minus r constants, right. So, that means, these Eigen functions are known only up to the arbitrary constants. That need to

be assumed in order to solve my characteristic equation, right and basically this equation, the characteristic equation would be determinant of that, but to solve this equation, that the number of arbitrary constants that need to be assumed would lead to the Eigen functions being known only up to certain number of arbitrary constants, right.

So, we have to assume values of those arbitrary constants in order to get those Eigen functions, but what is important is that even when I have these arbitrary constants, even when I have multiplicity say N minus r, it is possible. So, what did we see? We saw that for lambda k, if I have N minus r arbitrary constants, then that means, I have n minus r linearly independent Eigen vectors corresponding to lambda k, but even if they are linearly independent, but we can make them orthogonal to each other. How can we make them orthogonal to each other? Well, remember the Gram-Schmidt Orthogonalization procedure which we looked at when we were looking at systems of equations. When we said that if you have, say n minus r linearly independent vectors, it is always possible to come up with N. It is always possible to orthogonalize them, right.

How we did orthogonalize them? We took a first vector, right. When it came to finding the second vector, we projected out the component of the first vector from the second vector, and we make sure that the second vector was orthogonal to the first vector. That is called Gram-Schmidt Orthogonalization and we use that for vectors. It is possible to do the same thing for functions, right. So, we start with an Eigen function. We assume a certain Eigen, one of the linearly independent Eigen functions. The next Eigen function we project out the part which we have already assumed, right. So, the next Eigen function we make sure that it is orthogonal to the first Eigen function, and we proceed in this way. So, eventually we end up with n minus r, not only linearly independent, but actually orthogonal Eigen functions.

So, they can be orthogonalized using a Gram-Schmidt type procedure. So, in general it is possible finally to end up with for n orthogonal Eigen functions, right. It is possible to end up with n orthogonal Eigen functions, right. We know that if we have multiplicity more than 1, those Eigen functions are going to be linearly independent, but then we can orthogonalize them, right. We can orthogonalize them using the Gram-Schmidt procedure. So, eventually for n by n system, it is possible to end up with n orthogonal Eigen functions.

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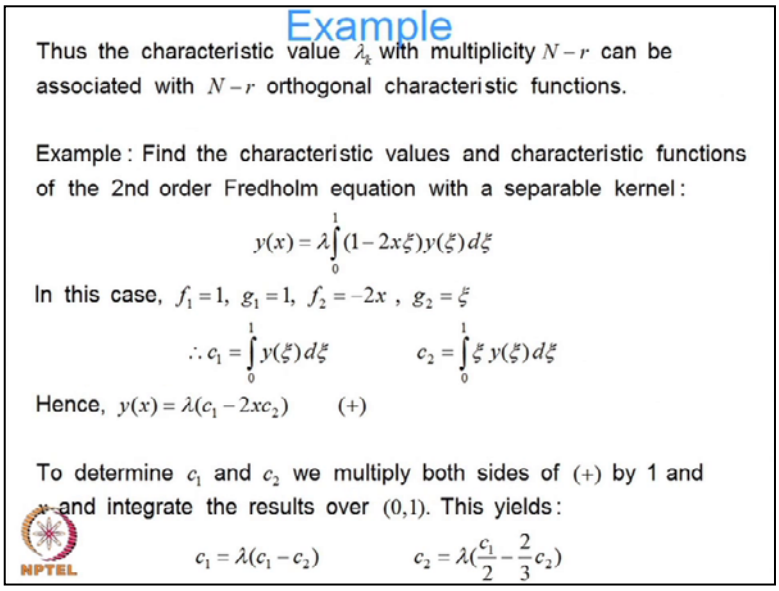

So, that is the characteristic value lambda k with multiplicity n minus r can be associated with n minus r orthogonal characteristic functions. So, up till now we have dealt with the theory, but I think some of the theory will become clear if we look at a specific example, right and the specific example we are going to look at is going to be a second homogenous Fredholm equation with a separable kernel, and it is going to be of type two, right. It is going to be of type two y x is equal to (( )).

So, you can see this kernel is y 1 minus 2 x xi. It can be written in as a separable kernel, right. It can be written as a term 1 minus 2 x xi. So, I can think of that as comprising a function f 1 and a g 1, f 1 times g 1 with both equal to 1. That is going to give me the first term. One and the second term is going if I consider the x dependence to be equal to minus 2 x and the xi dependents to be xi. So, I have minus 2 x xi. So, that is the second part of the kernel, right. So, that fits the form which we showed earlier on.

Basically, it is of this form, right. So, n is equal to 2. Now, f 1 is equal to 1, g 1 is equal to 1, f 2 is equal to minus 2 x and g 2 is equal to xi, right. So, that is a homogeneous Fredholm equation with a separable kernel and then we evaluate our c 1 and c 2. Recall that c 1 is nothing but integral of g 1 xi with y xi over that interval and c 2 is nothing but g 2 integral of g 2 xi with y xi. So, c 1 gives me integral of 1 with y xi, c 2 gives me integral of xi with y xi evaluated over the limits 0 to 1. Therefore, I get y x is equal to lambda c 1 minus 2 x c 2 because remember we can write our equation in this form, right

c 1 f 1 x lambda 1 c 1 lambda c 1. Basically, this is the form lambda c sigma lambda f 1 c 1 plus lambda f 2 c 2. So, we can write it like this, lambda c 1 minus twice x c 2 and again we want to determine c 1 and c 2.

So, what do we do? We multiply the left, take this equation, multiply first by g 1 integrated within the limit 0 to 1. Next, we multiply both sides by g 2, integrate the limits and integrate again over 0 to 1. If we do it with 1, we get this equation and if we do it with x and integrate with over 0 to 1, we get that equation, right. You can see this x square term is going to give me x cube.

So, that is going to after integration, it is going to give me x cube. So, there you have that here and c 1 x, if we integrate, that is going to give me a square term. If I integrate between 0 to 1, I will get a half here, right and if I multiply with 1, so that is going to give me x and 0. Once that gives me 1 here, if I multiply with 1 and integrate, that is going to give me x square by 2. 2 2 cancels out. So, I have minus c 2, right. So, this is what I get after multiplying first with g 1 of g 1 and then with g 2 and integrating between 0 to 1.

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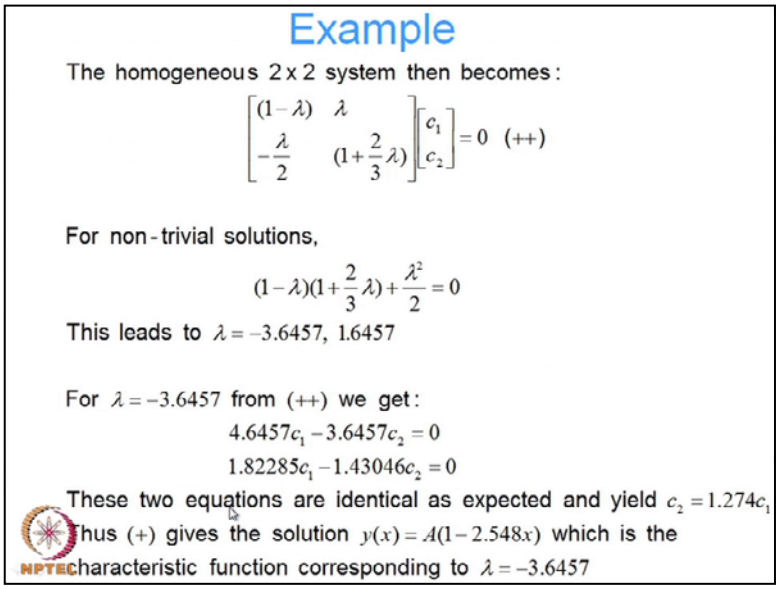

So, I finally end up with a system like this. It is a 2 by 2 system. Again for it to have non-trivial solutions, the determinant of that coefficient matrix must vanish. That gives me my characteristic equation for lambda, right. I solve for lambda, it is quadratic. So, I can easily solve it and I get lambda. In this case equal to minus 3.6457 and 1.6457. If I

substitute lambda is equal to minus 3.46457 in this equation, I get these two equations and it is clear that these two equations are actually identical because the determinant is 0.

So, they are linearly dependent. These two equations are linearly dependent. They are identical and all they tell me is that c 2 can be written in terms of c 1 c 2 is equal to 1.274 and therefore, it gives me if I put c 2 is equal to 1.274 c 1. In this equation, I get my Eigen function which is y of x is equal to some constant a times 1 minus 2.548 x. You can see there is a constant because I am writing c 2 in terms of c 1. C 1 still stays there, right.

So, there is one constant there, right. You can see that right here the rank is 1, right. So, n minus r is 1. So, I have to assume one constant value in order to know both my in order to evaluate both my c's, right. So, I have to assume one of them to be some arbitrary value. So, the Eigen function is now known up to a single arbitrary constant. It is known up to a single arbitrary constant because my rank is 1. So, N minus r is equal to 1 and I know that it is known up to N minus r arbitrary constants. In this case, it is known up to one arbitrary constant, right. So, this is the characteristic function corresponding to lambda is equal to minus 3.6457.

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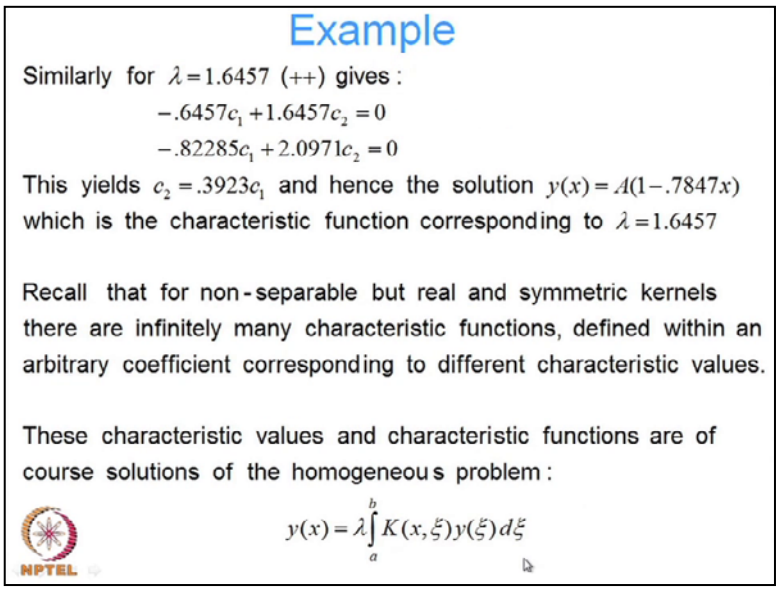

Similarly, I can evaluate for lambda is equal to 1.6457. This equation gives me this system. If I solve this, I can get c 2 in terms of c 1 like this and again, I get my Eigen function in terms of another arbitrary constant and this is my Eigen value. So, again I

know the Eigen function up to single arbitrary constant, and this is the Eigen function corresponding to lambda is equal to 1.6457.

So, this was for separable kernels, for non-separable kernels instead of having a finite. So, you can see that in this case since I could separate out my kernel into two products of x and xi f 1 x g 1 xi plus f 2 x g 2 xi, I got two Eigen functions corresponding to two Eigen values, right. In case, I cannot separate it out. I have infinitely many Eigen functions and infinitely many Eigen values, right.

So, in case for non-separable, but real and symmetric kernels, they are infinitely many Eigen functions, characteristic functions defined with an arbitrary coefficient corresponding to the different characteristic values, right. This need not be one arbitrary coefficient depending on the rank. They can be n. There can be n minus r arbitrary coefficients as since we are looking at. So, these characteristic values and characteristic functions of course are solutions of this system because that is by definition, right.

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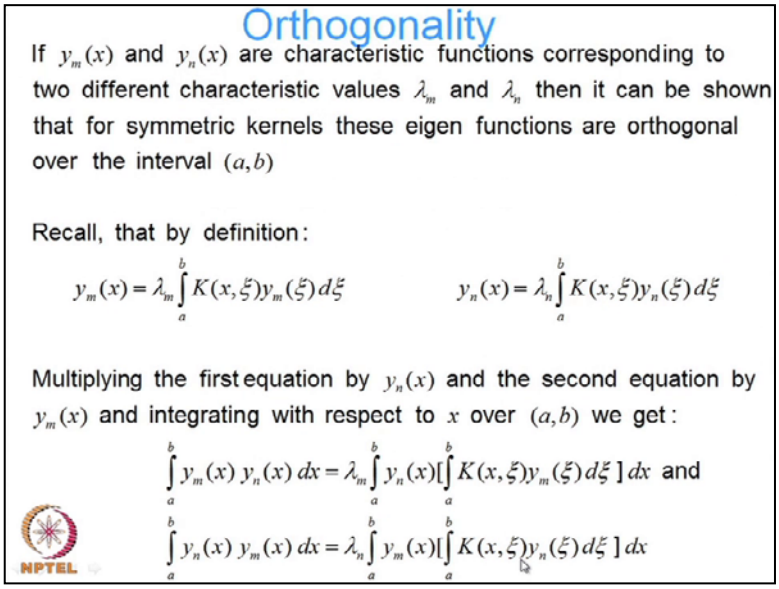

So, the characteristic values and the characteristic functions, they must satisfy this system and it turns out that if y m and y n are characteristic functions corresponding to two different characteristic values lambda m and lambda n, it turns out that if my kernel is symmetric. If my kernel is symmetric, then these Eigen functions are orthogonal to each other in the interval a to b. If you remember that for my system of equations, my Eigen function for symmetric coefficient matrix, my Eigen vectors were orthogonal to

each other, right. They were orthogonal to each other. Similarly, here even if I have infinite number of Eigen functions. So, long as my kernel is symmetric, my Eigen functions are going to be always orthogonal to each other and we can show that quite easily.

So, suppose we have two Eigen functions, lambda m and lambda n called, sorry two Eigen functions y m and y n corresponding to two Eigen values lambda m and lambda n. That means, they satisfy these two equations, y m and lambda m satisfy this equation while y n and lambda n satisfy this equation. So, you multiply the first equation y n of with y n of x, that is with this Eigen function and then again integrated over a to b and similarly, we multiply this second equation with y m of x and integrated over a to b. So, what do I get for this equation? I have on the left hand side integral of a to b, y m of x y n of x d x and then here I have lambda m y n of x this and then there is this integral a to b k x xi y m of xi d xi d x because I am integrating now with respect to x, right.

Similarly, the second equation I multiply with y m of x and integrate with respect to x within the elements a to b. Let us look at this first. The first equation, basically this one after we multiply with y n of x, I get this. Let me focus on this right hand side and if I focus on that right hand side, what do I get?

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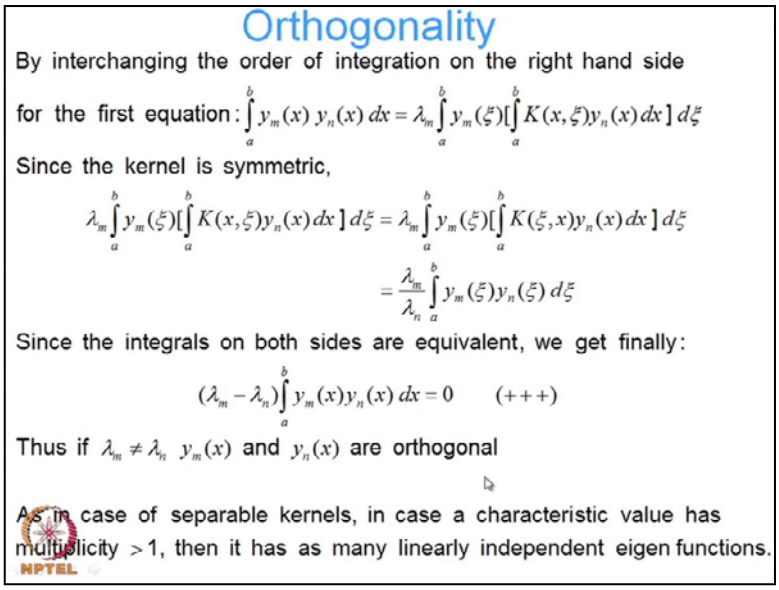

So, this is my right hand side and then what am I doing is, I am changing the order of the integration because earlier I was integrating first with respect to xi and then I was

integrating with respect to x. So, I change the order of integration. I integrate first with respect to x k x xi y n x d x. You can see I have pulled y n inside, right. Y n was outside; right and I have pulled that y n inside. So, f k x xi y n x d x integral over a to b, I have y m xi is integral over xi. I have pushed it outside and then I am first integrating with respect to x, then with respect to xi. So, I have this, but this is not really in the form of my integral equation because this is k x xi y n x d x, right.

If k x xi, but are actual integral equation was of the form where the second, the integrand what we were expecting with respect to that was the second variable in the kernel, right. That was the second variable in the kernel, and if the kernel is symmetric, then k of x xi is equal to k of xi of x. Otherwise, not right, but we have assumed that our kernel is symmetric. So, we can write k of x, xi is equal to k of xi of x because the function k is a symmetric function of xi and x. That is why I can interchange xi and x. If k were not a symmetric function of xi and x, I could not do that. If k of x and xi was x cube plus xi square x plus xi x square terms like that, if it were not symmetric in x and xi, I could not have done that because I am assuming that it is symmetric in x and xi. I can write k of x of xi is equal to k of xi of x, right.

I know now that I have done that, it is in the form of my homogenous integral equation, Fredholm integral equation of the second kind. So, this term within brackets is nothing but y n of x, right. It is nothing but y n where y n of xi is nothing but y n of xi. If we look at the term within second brackets, then that is nothing but y n of xi. So, I have in lambda m a to b y m of xi y n of xi and of course, I have to divide it by lambda n because this term within this bracket is equal to y n of xi divided by lambda n, right. It is y n of xi divided by lambda n from the definition of my homogenous problem, from the definition of my Eigen problem, right.

So, this is equal to integral of y n xi y n xi d xi lambda m divided by lambda n and this is equal to that. So, then this I can write it as I bring it I can write it as lambda m minus lambda n because this you can see these this integral is identical right this integral is identical. So, I can write it as lambda n minus lambda n integral a to b y m of x y n of x d x must be equal to 0 and since, my Eigen values are distinct that is lambda m is not equal to lambda n. After all I assumed that these are distinct Eigen values corresponding to distinct Eigen functions, right. So, if I do that, then lambda m cannot be equal to lambda n which means that in order to satisfy this integral in y of m must be orthogonal to y of n

over that interval a to b. Is that clear?

So, distinct Eigen values correspond to orthogonal Eigen functions, right. So, similarly as in case of separable kernels, in case Eigen values have multiplicity greater than 1, if I have multiplicity greater than 1, then I know that I have N minus r Eigen functions corresponding to a particular Eigen value. Each of those Eigen functions are going to be orthogonal to each other. They are going to be orthogonal to each other and similarly, I can again do a Gram-Schmidt type, sorry they will be linearly independent and I can again do a Gram-Schmidt type orthogonal orthogonalization procedure to end up with N minus r orthogonal Eigen functions. Is that clear?

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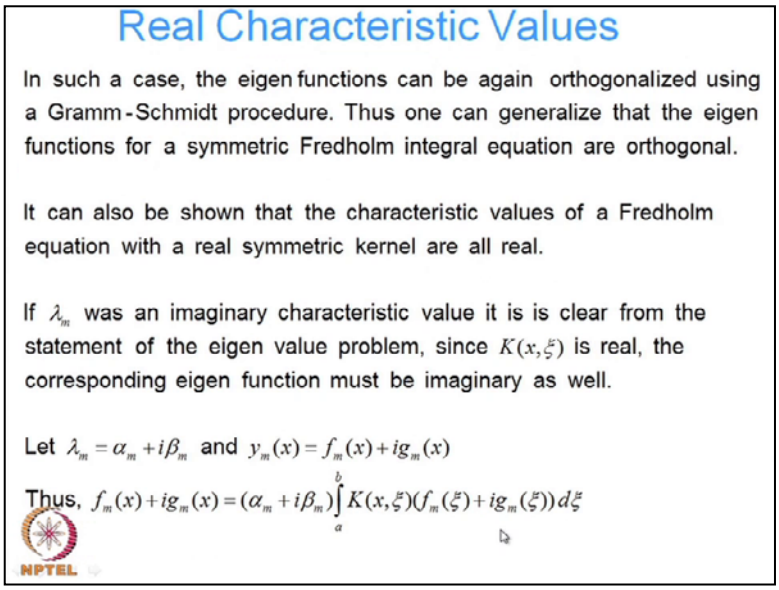

So, separable kernels in case of characteristic value have multiplicity greater than 1. Then it has as many linearly independent Eigen functions and in such a case, I can again orthogonalize using a Gram-Schmidt procedure. Thus, one can generalize that the Eigen functions for a symmetric Fredholm integral equation are orthogonal. Symmetric Fredholm equation does not matter if the kernel is separable or not even if it is non-separable. We always end up with Eigen functions which are orthogonal to each other, and it can in addition, it can be shown that the characteristic values of a Fredholm equation with a real symmetric kernel are all real, right. They cannot be imaginary. They are all real. It is very similar to our system of equations, right. For a symmetric matrix, we are assured that the Eigen values are all real, right. The Eigen values of a symmetric

matrix are real.

So, in this case if I have a kernel which is symmetric, in addition if the kernel by itself is real, the kernel by itself is not imaginary. So, if I have a real symmetric kernel, in that case my Eigen values are always going to be real, right. My Eigen value of my integral equation is always going to be real. That should be quite obvious because let us go back and look at that equation.

Suppose we look at something like this, right. So, if I have real symmetric kernel, then this is imaginary. If this is imaginary, it is certain that this has to be imaginary as well, right. Is that clear? If this is imaginary, let us go over it once more. If lambda n has an imaginary characteristic value, it is clear from the statement of the Eigen problem since k of x xi is real, then the corresponding function Eigen function must be imaginary as well, right.

Since, this is real, if this is imaginary, then the Eigen function has to be imaginary as well because the right hand side is imaginary, right. So, it has to be imaginary as well, right. Is that clear? So, let us look at a situation where I have an imaginary Eigen value, right. I have an imaginary value and I know in that case my Eigen functions got to be imaginary as well, right. My Eigen function has got to be imaginary as well. So, I can write it in this form, right. So, alpha m plus i b m, that is my imaginary Eigen value and this is my imaginary Eigen function, right.

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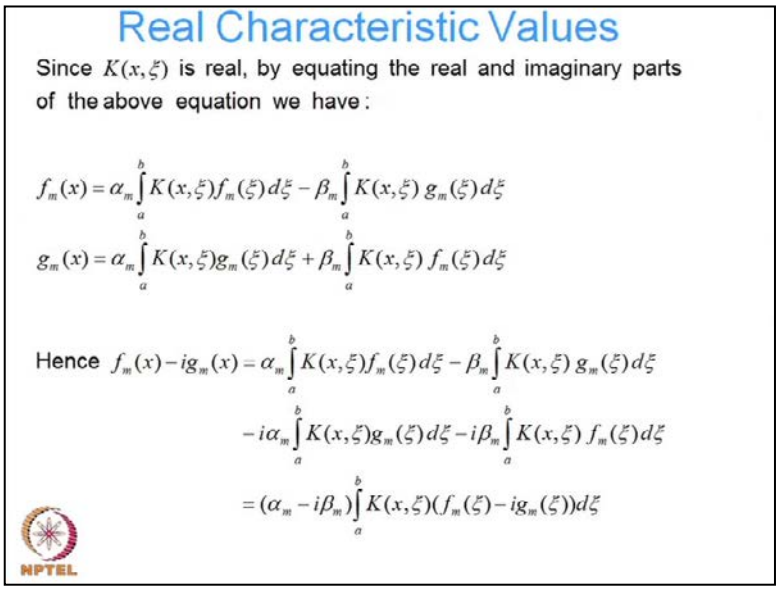

So, if I can write it like that and then I group the real and the imaginary terms together and I equate the real and imaginary parts and if do that, then I get f of f m of x is equal to this. I get g m of x is equal to that, right. By just grouping the real and imaginary parts I get that and then if I write f m minus i g m and I use these definitions of f m and g m in terms of alpha m beta m, then I am pulling those terms together. I finally end up with something like this and by just looking at this, what do you see that corresponding to? So, if f m plus i g m is an Eigen function corresponding to an Eigen value alpha m plus i beta m, it means that f m minus i g m is also an Eigen function corresponding to an Eigen value alpha m minus i beta m.

So, if I have a complex Eigen value, its complex conjugate is also an Eigen value and since for a complex Eigen value I have seen that I must have a complex Eigen function. That means, the complex conjugate of that Eigen function is also the Eigen function corresponding to that complex conjugate Eigen value, right.

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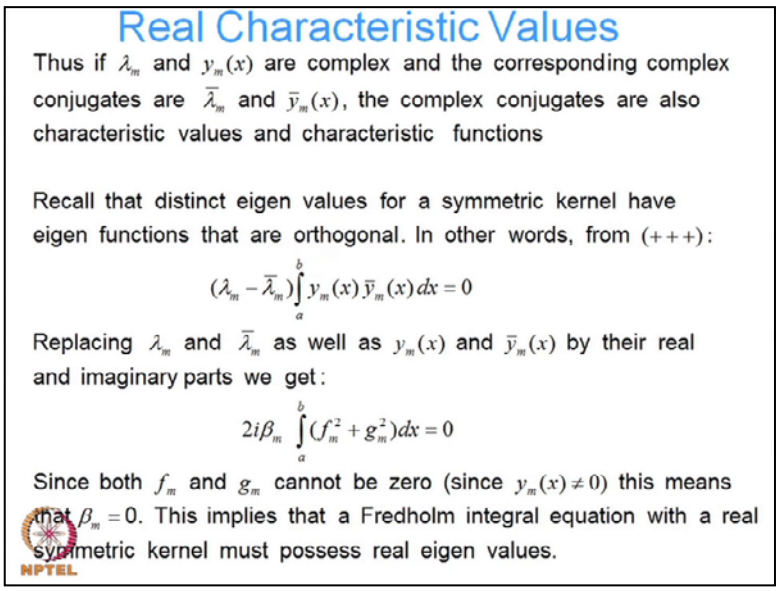

So, if lambda m and y m are complex Eigen values and complex Eigen functions and the corresponding complex conjugates are lambda bar m and y bar m, the complex conjugates are also Eigen values and Eigen functions. We have just shown that, right and recall that for distinct Eigen values, we have this condition, right. So, lambda m and lambda bar m are distinct Eigen values because they are not equal, right. One is the complex conjugate of the other. So, I can always write that equation because I know that

for any two distinct Eigen values, this condition has to be satisfied. So, this has to be satisfied for an Eigen value and its complex conjugate as well because I know that the complex conjugate of an Eigen value is an Eigen value also, right.

So, this condition has to be satisfied and then replacing lambda m and lambda bar m as well as y m and y bar m by their real and imaginary parts which are this, alpha m plus I beta m and f m plus i g m. In case of y m, we end up with an equation like this and I know that both f of m. So, this involves f of m square plus g of m square. The term within the first bracket, there can only be 0. If both f of m and g of m are 0, but if both f of m and g of m are 0; that means, my y m which is equal to f of m plus i g m has got to be 0. That is not true. Y of m is not equal to 0. That means this cannot be 0. So, the only way this can be 0 if beta m is equal to 0.

What does it mean? My Eigen value was alpha m plus i beta m. So, beta m is equal to 0 meaning that the imaginary part is always going to be equal to 0. So, my Eigen values are always going to be real. For real symmetric kernel, I am always going to have real Eigen values, right. This implies that a Fredholm integral equation with a real symmetric kernel must possess real Eigen values.

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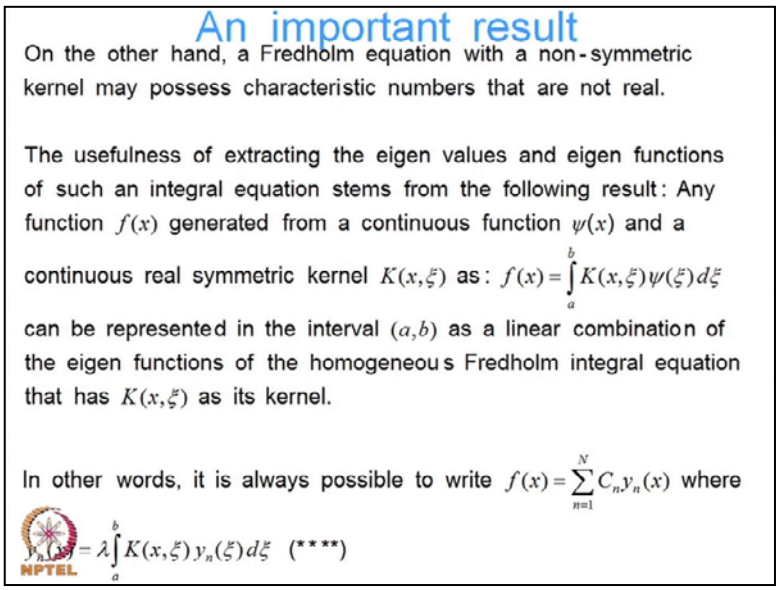

On the other hand, a Fredholm equation with a non-symmetric kernel may possess Eigen values that are not real. That is certain that can happen, right. Non-symmetric kernel can always have Eigen values which are not real. So, why do we spend so much time

discussing the homogenous problem and how to extract the Eigen values and the Eigen functions? Well, the reason is because it helps as these Eigen values and Eigen functions, they help us find the solutions of the non-homogenous problem as well, right. They help us find the solutions of the non-homogenous problem as well usefulness of extracting the Eigen values. Eigen functions of an Eigen results from this because this following result, right. Any function f of x generated from a continuous function psi of x and a continuous real symmetric kernel k of x xi.

So, suppose I have any arbitrary function psi of x, the only requirement is that psi of x be a continuous function and suppose I integrate, I take the kernel and I take the product with psi of x and I integrate with a and b within the limits, and I generate a function f of x, then I am assured that in that case I can always represent f of x as a linear combination of the Eigen functions of the homogenous Fredholm integral equation with k of x xi as this kernel, right. You know what is the homogenous Fredholm integral equation? With the kernel k of x xi, we have known how to find its Eigen values and Eigen functions.

It turns out that any function f of x which I can get by doing this integral, I can represent this f of x as a linear combination of my Eigen functions of my homogenous Fredholm equation, and we will see its use in a few slides. In other words, it is always possible to write f of x as sigma n equal to 1 to n c n y n of x, where y n of x is an Eigen function of this homogenous Fredholm equation. Remember this f of x, I generated completely arbitrarily, right. The only condition is that these functions psi is a continuous function. It can be any arbitrary continuous function. It may not have any relation with the Eigen functions, right.

So, if I generate any function, if I take any arbitrary function psi of x a integrated, I operate with the kernel k of x of xi, right. Basically when I operate, I mean integrated with over the limits a to b, take the product with k x and xi and integrate it and I generate a function, right. That function I can always represent in terms of as a linear combination of the Eigen functions of this homogenous Fredholm equation, right. This is an important and very powerful result.

Analogously think for linear system, what we know if I have a matrix residual symmetric-asymmetric matrix, right and I know that its Eigen vectors are going to be orthogonal. What do I know? If I have a dimension n, then I know that the Eigen vectors

of a are going to form a orthogonal basis for the n dimensional vector space. That means, any arbitrary vector of dimension n, I can represent as a linear combination of those Eigen vectors. So, similarly here any arbitrary vector f of x which I obtain like this, I can represent as a linear combination of my Eigen functions y n of x. It is very important that, very similar to what happens in case of linear system.

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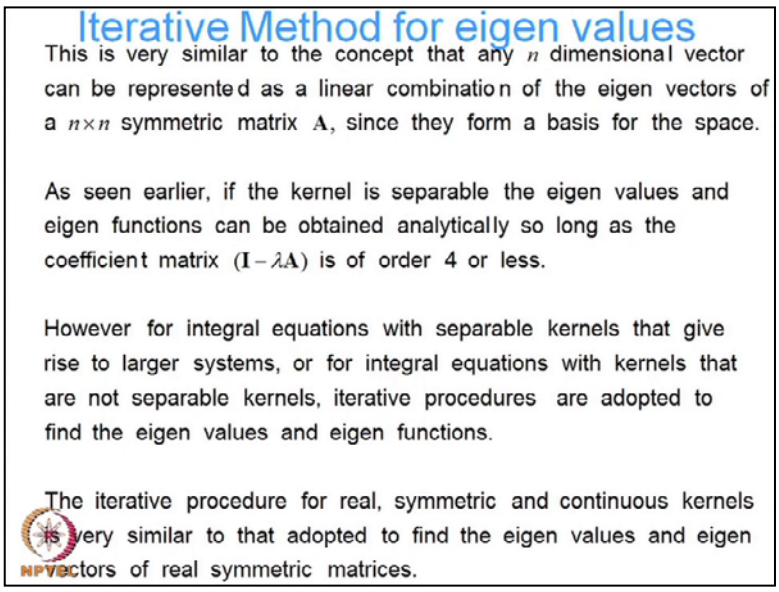

So, this is very similar to the concept that any n dimensional vector can be represented by a linear combination of the Eigen vectors of n by n symmetric matrix a, since these Eigen vectors form an orthogonal basis, right. So, that is very important result which we will use very soon, but at this point we digress and we mention that when we need to find. So, you can sort of motivate why it is important to find the Eigen values and Eigen functions because then we can use this very beautiful property which allows us to represent any arbitrary function f of x generated by operating on the kernel on any continuous function. I can split it up in terms of the Eigen functions, right.

That is why it is very important to find the Eigen functions, but we have seen that we can find the Eigen functions for separable kernels. Yes, we can find the Eigen functions relatively easily, but remember what do we have do is to find the Eigen functions, find the Eigen values. We have to solve this. We have to evaluate determinant of I minus lambda a is equal to 0, right and what is going to be the order of that equation is going to depend on the size of my matrix A.

Since, a was 2 by 2, but if a is large, then that characteristic function is going to have higher powers of lambda. Since, a was 2 in our example, problem we had a quadratic. We had a quadratic there. If a is greater than 4, then we know that we can only find the roots analytically only if it is fourth order or less, right. If it is fourth order characteristics is 4 or more, I cannot find analytic solutions, right. So, then what happens? Then I cannot it enclosed form, right. I have to do it iteratively, right. I have to do it iteratively.

So, I have to evaluate the Eigen values and the Eigen functions in an iterative fashion, right and we will look at how to do that again. The way to do it is very similar. If you remember when we were talking about linear systems, we also looked at iterative methods to evaluate the Eigen values and Eigen functions, and we found that when we do that, when we do these iterations, we first find the largest Eigen value and the corresponding Eigen function and then we can progressively find the next smallest Eigen value, the corresponding Eigen function, then next smallest Eigen values and so on and so forth.

So, doing a similar iterative procedure, it is possible to find the Eigen values and Eigen functions of Fredholm integral equation as well, and we are going to do that in the next lecture. Once we do that, we will talk about how to use our Eigen values and Eigen functions to solve the non-homogenous problem in whatever time is left because next lecture is probably our last lecture. We will look at some numerical methods for solving integral equations as well may be we cannot do it in much detail, but we will at least introduce some numerical methods for solving integral equations.

Thank you.