Numerical Methods in Civil Engineering
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Lecture - 35
Spline Functions

In lecture 35 of our series on numerical methods in civil engineering, we will wrap up our discussion on orthogonal polynomials that we had for the last 3 lectures and then talk about Spline functions.

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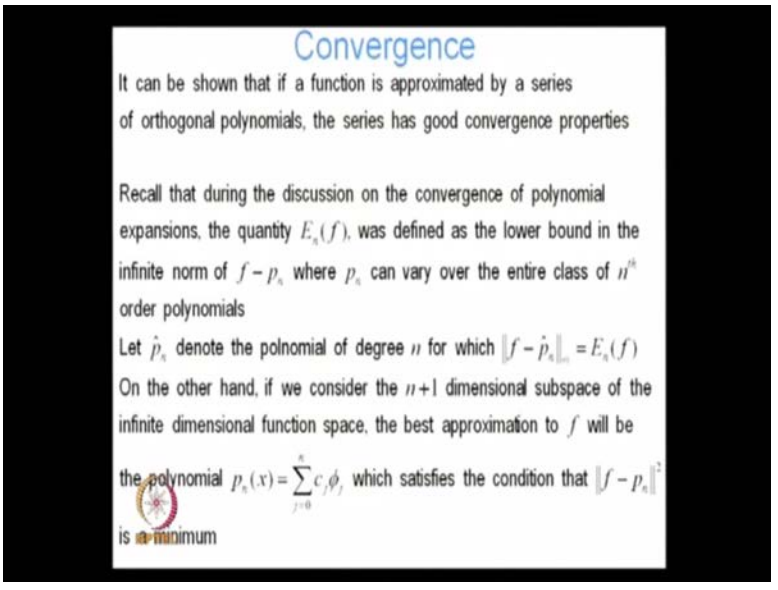

So, last time in the last lecture that we had on orthogonal polynomials, we were talking about convergence of orthogonal polynomials, and we said that if a function is approximated by a series of orthogonal polynomials, the series has good convergence properties. And we showed this by bringing up the E n f error, the error in the maximum norm for a polynomial in an interval and then we found the E n f recall what the E n f norm the error in E n f norm which is defined as the error in the maximum norm over the interval, and then we find the polynomial which gives the minimum error in the maximum norm, the minimum error in the maximum norm over the entire interval.

So, we want to relate that error, E n f error to the error which we get using the best approximation property. So, regarding during our on the discussion on the convergence of polynomial expansions, we talked about E n f which was defined is the lower bound in the infinite norm or the maximum norm of f minus p n, where p n can vary over the entire class of n or n th order polynomials.

So, we calculate f minus p n point wise over that interval and then find out the maximum in the error over that interval, the error being f minus p n. We find out the maximum error and then find out the polynomial p n belonging to the entire class of n th order polynomial which minimizes that error, which gives the smallest maximum error, which gives the smallest maximum error over the interval.

So, you look over all the points in the interval, find out the maximum error then search through all the n th order polynomials and find the n th order polynomial which gives the smallest maximum error over the entire interval so that, is E n f and we want to relate the error in the L 2 norm which we obtained for the best approximation using the least squares minimization to the E n f error.

So, we say that let p hat n denote t he polynomial of degree n, for which we get this E n f error. Let that be the polynomial which gives me the smallest error in the maximum norm over that interval and that polynomial I am going to denote by p n hat, that polynomial is p n hat. On the other hand, we consider the n plus 1 dimensional functions space, the best approximation to f we have seen earlier will be this polynomial, p n x is equal to sigma j equal to 0 to n c j phi because that satisfies this condition, that this ever norm in the L 2 in the L 2 sense the L 2 norm that is a minimum, f minus p n squares.

So, these 2 polynomials p n and p n hat are not identical, p n hat gives me the E n f. It is the polynomial corresponding to the smallest error in the maximum norm in that interval while p n gives me the best fit in the L 2 norm. It is gives me the smallest error in the L 2 norm and however these 2 error are related.

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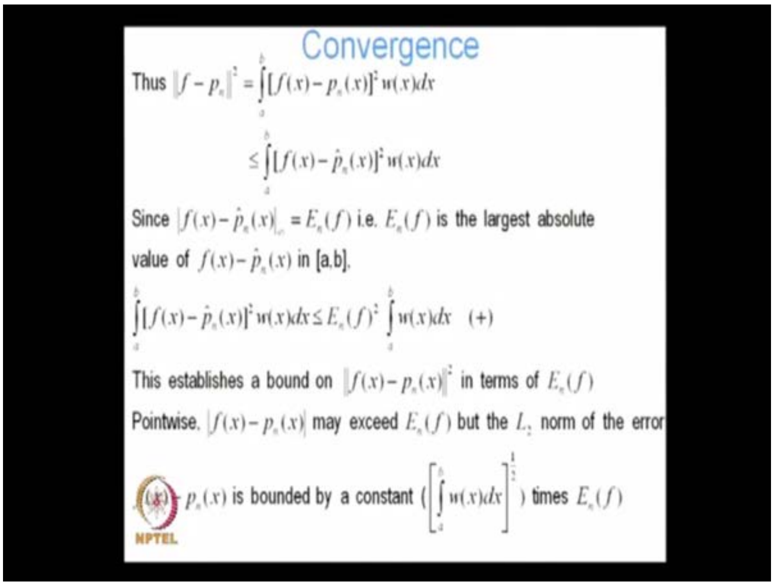

So, to do that we write out the error in the L 2 norms f minus p n norms square is equal to f minus p n x square w x d x because the inner product is defined in terms of a weight factor w x that w x, may be 1 if the weight factor is 1 but in general there is a weight factor w x. But, this must be lesser than or equal to f x minus p n hat x square w x, this is because f x minus p n hat x. f x. Since, this gives me the smallest error in the L 2 norm. So, if I replace p n by p n hat x. this has got to be greater than this because p n x gives me the smallest error in the L 2 norm. But, f x minus p n hat x in the infinite norm is equal to E n f, where E n f is the large, again this I should have fix this, this gives me the smallest error in the infinite norm. So, it is not really large, it is the smallest error in the infinite norm.

Now, so integral a b f x minus p n x square w x d x is lesser than or equal to E n f square w x d x, that must be, that has to be true because E n f gives me see it is in the maximum norm, it is error in the maximum norm so for that p n hat polynomial, the p n hat polynomial gives me the smallest error in the maximum norm for all the polynomials. But, it is the smallest error in the maximum norm so that, means that is the maximum error in that interval for p n p n hat x, is that clear, I do not know if it is clear. Let me try to explain again.

So, p n hat x is the polynomial which gives me the smallest error over all the polynomials in the maximum norm but the error in the maximum norm is the largest difference between f x and the polynomial in that interval. So, if I am considering the polynomial p n hat x, then f x minus p n hat x infinity is the largest difference in between f x and p n hat x within a b, with in the interval a b. So, instead of writing f x minus p n x square by taking the point wise difference, here I am taking the point wise difference between f x and p n hat x. But, E n f is the largest difference in a b, E p n hat, E n f is the largest difference in a b, It turns out that it is the smallest for all the, if I look at the all the polynomials in the class it is the smallest but for p n hat f, that is the largest maximum difference, largest difference in the absolute value.

So, if I replace f x minus p n hat x in this interval by E n f. So, it is obvious at this, it has to be lesser than or equal to E n f square integral a to b w x d x. I hope that is clear. So, now we already know that this is less than that, norm square in the L 2 norm is less than this.

But, now we are found that this is less than this. So, that means that this must be less than that, the norm of the error in the L 2 norm, the error in the L 2 norm square must be lesser than or equal to E n f square integral a to b w x d x. So, this establishes a bound on f of x minus p n x square interms of E n f. So, point wise f x minus p n x might exceed E n f but the L 2 norm of the error is bounded by this, the L 2 norm of the error is bounded by this times E n f. Is that clear?

So, point wise f x minus p n x is going to exceed E n f, that has to be true because p n x is not equal to p hat n x. So, f x minus p n x is going to exceed E n f at least at some points in that interval a b but in the L 2 norm of minus p n square is bounded by this quantity times E n f.

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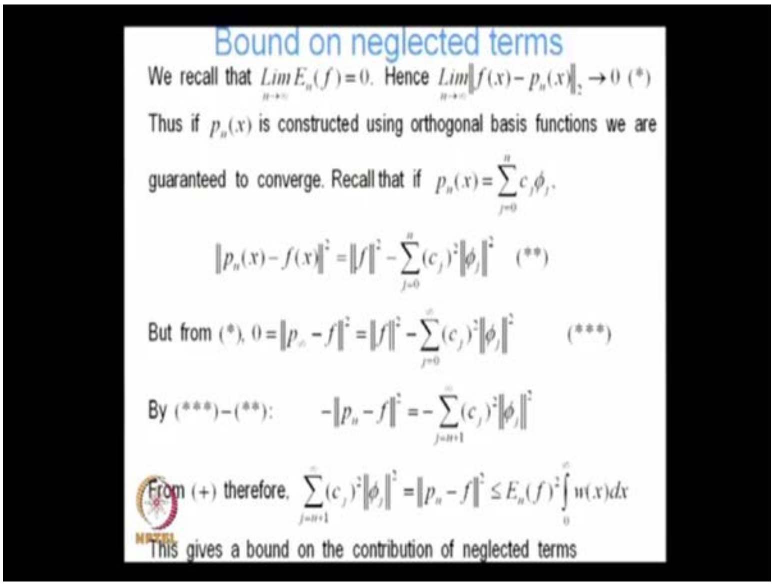

So, now we recall that E n f, when n goes to infinity is equal to 0, when my order of the polynomial becomes infinity, then we saw when we looked at convergence of polynomials that E n f becomes 0. So, in the limit when n goes to infinity when the order of my polynomial goes to infinity. That means that this since, this is bounded by that, this is bounded by that and that goes to 0 when n goes to infinity. So, this also has to go to 0 as n goes to infinity.

So, limit f x minus p n x in the L 2 norm must go to 0 as n goes to infinity. Thus, if p n x is constructed using orthogonal basis functions, we are guaranteed to converge. We are guaranteed to converge as we increase n, as we as n becomes larger and larger that error in the L 2 norm is going to go to 0 and we are going to converge.

So, if p n x is if it is not, we are not taking a infinite number of basic functions, if we have a finite number of basic functions, then we write p n x is equal to sigma j equal to 0 to n c j phi j, in that case this becomes equal to that. This we have seen before, p n x minus f of x are norm, that is norms square is equal to norm of f square minus sigma j equal to 0 to n c j square norm of phi j square.

But, from this expression p infinity minus f of x when n goes to infinity p infinity minus f norm square is equal to norm of f square minus sigma j equal to 0 c j square norm of infinity square and we know that this goes to 0. When n goes to infinity f x minus p infinity norm goes to 0 and this I can write exactly like that norm of f square minus sigma j equal to infinity, now it is no longer n its infinity c j square norm of phi j square and then if i subtract this minus this, what do I get? Well, I get minus p n minus f because this is equal to 0 this is equal to 0 minus p n minus f that is minus p n minus f norm of square minus f norm of f square, f square cancels out. So, minus sigma j equal to n plus 1 to infinity, c j square norm of phi j square because this is j is equal to infinity 0 to infinity, that is equal to j equal to 0 to n, So, the terms from 0 to n cancel out and I am left with the terms from n plus 1 to infinity.

Therefore, what do we have? Well, that means that from this expression then I go back to that expression. Sigma c j square norm of phi j square is equal to norm of p n minus f whole square form here ,write n plus 1 to infinity that must be lesser than or equal to this, because we have seen from here p n minus f square is lesser than or equal to that, p n minus f minus p n the same thing, f minus p n norms square is lesser than or equal to this times that.

So, what do I have this is equal to lesser than or equal to that; So, what does this give me this gives me, that if I do a finite dimensional approximation, If I consider a finite dimensional function space, instead of an infinite dimensional function space, I am going to get some error and what is that error, that error is given by this term, the terms which have neglected, j equal to n plus 1 to infinity and what this tells me is that the terms that I have neglected well they are bounded.

They are bounded and how are they bounded, they are bounded by this, they are bounded by this. So, the neglected terms have to be lesser than or equal to E n f square 0 to infinity w x d x, given the order of the polynomial given the number of terms in that, in my orthogonal series, in my series of orthogonal polynomials, I can calculate if I can calculate the corresponding E n f if I can correspond calculate the correspond E n f the error is founded by that.

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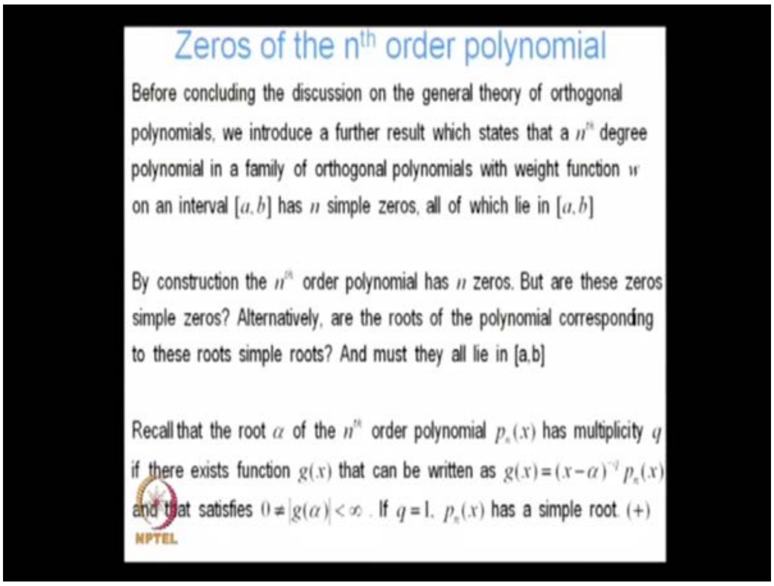

So, before moving on to splines, there is just 1 more topic I want to consider it is a little complicated but I think we go through it slowly it should be, it should not be too hard, So, we know that an n th degree polynomial has got n 0 and we are considering, we are trying to approximate a function f of x in the interval a to b in the interval a to any interval a to b and within that interval we have defined all these things, my inner product which is between the intervals a to b and things like that.

But, that function that my polynomials may exist outside a to b also, so how do I do, where will the 0s of that polynomial lie of this orthogonal polynomials, where will they lie, will they lie within the interval a to b or will they lie outside the interval number 1. That is a question the second, question I want to answer is that whether those 0 are they simple 0 or are they complex 0 I mean are they simple roots, are they complex roots I do not meant imaginary roots but roots with power more than1.

So, we want to talk about we want to in talk about our results which states that an n-ith degree polynomial in a family of orthogonal polynomials, with weight function w on an interval a b has n simple 0, which all lie in a b, that polynomial may exist you can also instance, we are going to talk about Legendre polynomials and Legendre polynomials they are typically the inner product is defined over the interval minus 1 to 1. It is possible that, can that polynomial can exist outside of 1 but that the 0 of the polynomial all lie between minus 1 and 1 .

So, by construction the n th order polynomial has n 0 but is these 0 simple alternatively that is are the roots of the polynomial corresponding to these roots, simple roots and do they all lie in a b in the interval a b. So, recall we talked about this earlier we defined what is a simple root, I said that a root alpha of the n th order polynomial p n has multiplicity q, if there is a function which can be written as g x is equal to x minus alpha to the power minus q p n x and this function is at alpha, If I evaluate this function at alpha it is always bounded and non-zero. It is non-zero and it is bounded, So, what does it says that if I divide p n by x minus alpha q times, I will get some quantity, So, x minus alpha to the power minus q, So, basically I am I have p n x in the denominator I have x minus alpha to the power q.

So, I am dividing it by x minus alpha q times and whatever, I have left is not 0, nor is it infinity at alpha, that means I have taken out all the roots when I divided it by x minus alpha to the power q times, I have managed to take out all the roots, If I up to do it if I can do it by just dividing it by x minus alpha only once then I have got just a simple root, if I have to divide it by two times x minus alpha square then i have what is called the root with multiplicity 2.

If I have to divide it by x minus alpha q times, before that function g x is no longer 0 or 0 at alpha that means, that alpha has a the function p n x has a root of multiplicity q at alpha and q is equal to 1 p n x has a simple root.

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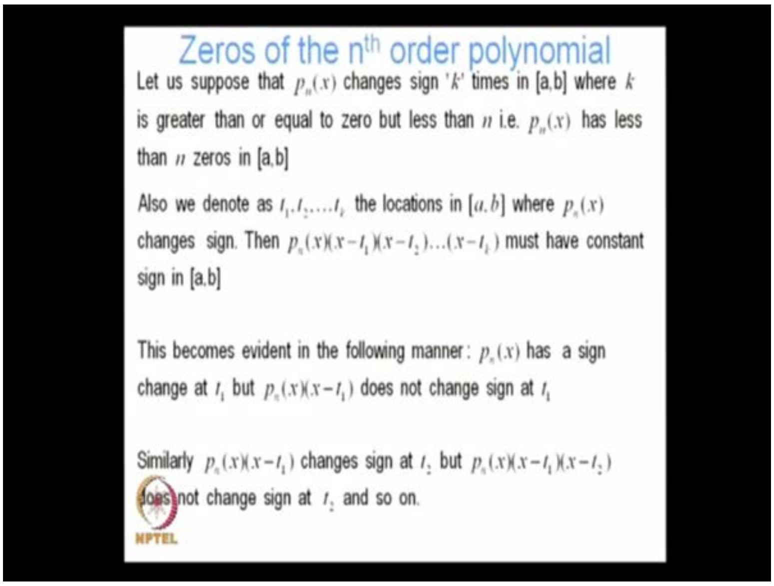

Now, let us suppose that p n x has only k roots in a b where k is less than n, So, we will show that, that is impossible we will assume that the number of roots in a b is less than n, I know p n x has n roots but I must going to assume that suppose, that p n x in a b has less than n roots has k roots, where k is less than n and I will see that what follows from that assumption is impossible, that cannot happen. So, that is going to show that is going to tell me that p n x must have n roots in a b.

So, let us suppose there are k roots and we denote as T 1 T 2 through t k, the locations in a b where p n x changes sign, changes sign means it has a root, it is crossing the x axis. So, p n x times x minus T 1 x minus T 2 x minus t k must have a constant sign in a b. why is that? We will let us look at it simply.

So, if p n x has a sign change at T 1 p n x times x minus T 1 does not change sign at T 1, we can understand that p n x is changing sign at T 1 then I multiply p n x by x minus t 1, suppose on 1 side p n x is negative. So, if I multiply it by x minus T 1, x minus T 1 is also going to be negative on that side, So, negative is going to be positive; similarly, where p n x was positive, So, it would be positive in that case, if I multiply by x minus T 1, it is going to make sure that p n x minus T 1 p n x times x minus T 1 does not change sign at T 1.

Similarly, if I multiply p n x minus T 1 into x minus T 2, I know that p n x changes sign at x at T 2 but if I multiply that thing by x minus T 2, I can also make sure that quantity does not change sign because whenever it changes sign x minus T 2 will compensate for that and it will prevent the sign from changing, is that clear. Similarly, if I do it for k points T 1 T 2, through t k I can make sure that p n x times x minus T 1 into x minus T 2 into x minus t k must have a constant sign in a b but what does this mean?

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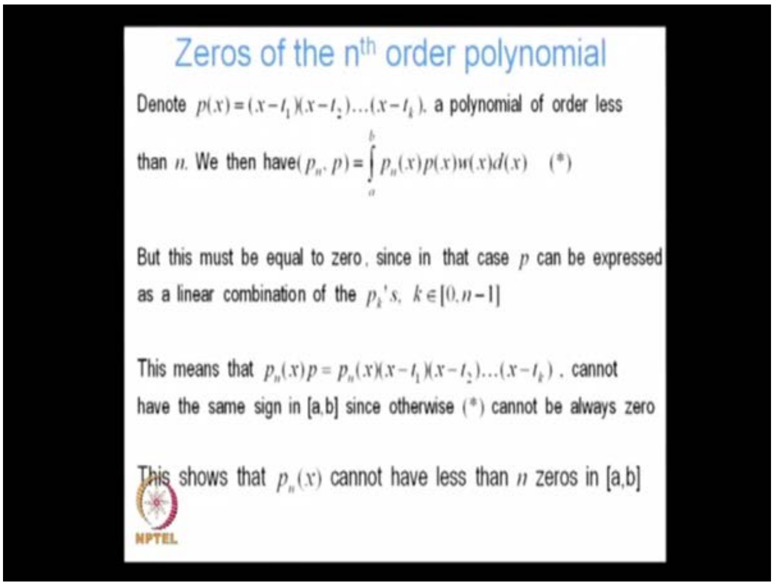

Well what does it mean is that this thing. So, if I take the inner product of p n, let me denote this x minus T1 x minus T 2 through x minus T k by p and I know p has to be a polynomial of order less than n, why because k is less than n, So, it is x minus T 1 x minus T 2 x minus T k So, p x is a polynomial of order less than n less than n. And now, I take the inner product between p n and p.

So, p n x p x w x integral between a to b, now this thing have must be equal to 0, why because p x is a polynomial of order less than n, now, p n, so it must be orthogonal so it must be orthogonal to p n, p n is that clear because p can be expressed as a linear combination of the p k's, k belonging to 0 to n minus 1.So, p is a p is polynomial of order k, k is less than n, So, maximum k can be is n minus 1,So since, this is a polynomial of order let us suppose, it is of order n minus 1 it is an order of polynomial of order n minus 1, so I can always it in terms of the basic functions the orthogonal basis functions up to n minus 1, because any polynomial of order n minus 1 I can write it into as a linear combination of my basis functions for that function space.

So, I can write it in terms of those basis functions and that is going to result in this thing, becoming 0 is that clear, because p n x has a higher order basis function is I do not know if it is clear p n x is of order n, p is of order n minus 1, So, p must contain some additional basis function, p n must contain more basic functions than p n minus 1 and that additional basis function, is going to annihilate all the basic functions in.

Is that clear, this means that since this is going to become 0, thus means this quantity cannot have a constant sign in a b since, otherwise star cannot always be equal to 0, is that clear; So, since this quantity cannot have the same sign in a b cannot have the same sign because otherwise this thing cannot be equal to 0 this shows that p n x cannot have less than n 0 in a b.

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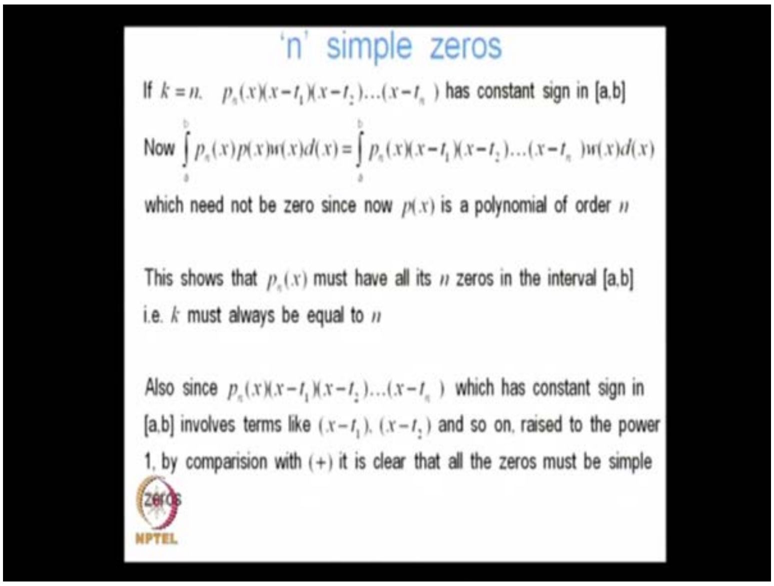

If k is equal to n on the other hand if k is equal to n p n x minus t 1 x minus t 2 x minus to t n has got constant sign in a b, because I have this thing, I multiplying by all these points, if it has a that is perfectly consistent because now, I have p n x p x w x this is which need not be 0. Since now, p x is a polynomial of order n this is a polynomial of order n. Now so that, if this p x by p x I mean x minus t 1 x minus t 2 through x minus t n. So, p n x p x that is also a polynomial of order n, is that clear? So, this shows that if the if this condition is only going to work, if p n, if k is equal to n k is equal to n and that means that p n x must have all it is 0's in the interval a to b.

Also since, this thing we have just seen as constant sign in a b. So, terms like x minus t 1 x minus t 2 and so on and it is raised to the power 1. So, you have, you can see this thing has got constant sign in a b that means it is not going to be 0 anywhere in a b. It has constant sign in a b, it is not going to be 0 anywhere in a b and you can see, it is this thing is p n x is multiplied by terms like x minus t 1 where t 1 is a root x minus t 2, where t 2 is a root and all of them have simple power of 1.

So, that means all these roots are simple roots, if it was not a simple root then x minus t 1 has to be raised to a greater power but this thing not to be equal to 0 in the interval is that clear? So that means that all these roots must be simple roots. That was a little involved but I hope you got the idea, may be if you took over it again it will become clear.

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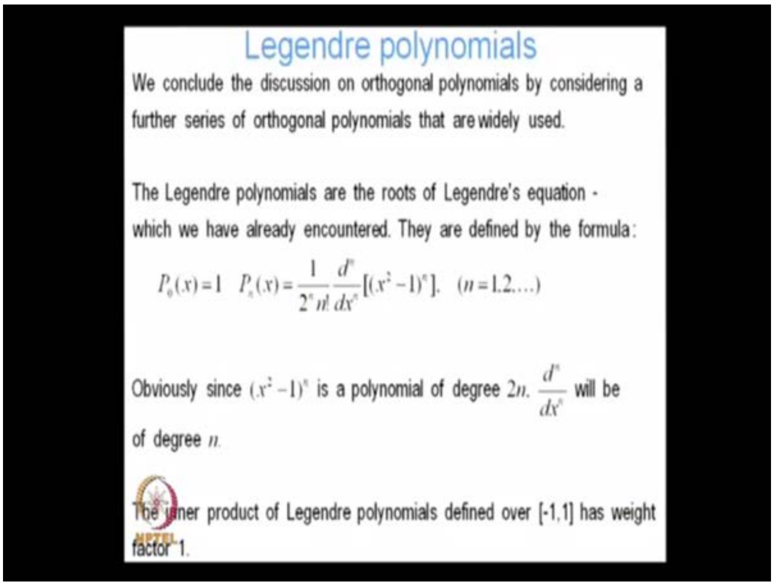

So, before winding up I just want to talk about a few common of just 1 in that actually 1very common type of series of orthogonal polynomials, which we have also encountered before, when we are talk about solutions for partial differential equations. So, I am talking about Legendre's polynomials and Legendre's polynomials I know are the roots they are defined by this differential equation, they are defined by this differential equation p n x is equal to that and p 0 x is equal to 1 and since x square minus n is a polynomial of degree 2 n if I take n derivatives of that, this will be of degree n and the inner product of Legendre polynomial's is defined over the interval minus 1 to 1 as a weight factor of 1.

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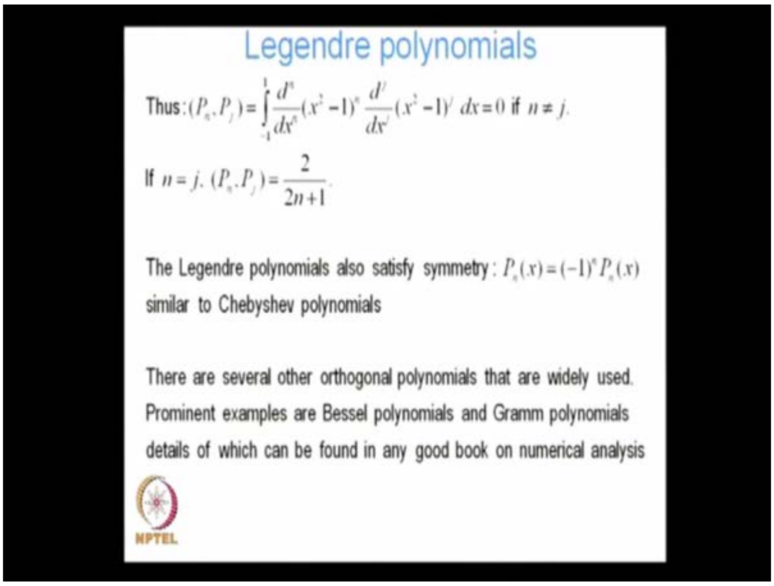

And because the inner product has a weight factor of 1, what does it mean? That means, the inner product is just of p n and p j is just defined by this, and since they are orthogonal this has got to be 0, if n is equal to not equal to j and if n is equal to j they are given by this. They also satisfy symmetry p n x is equal to minus 1 to the power n p n x similar to Chebyshev polynomial. So, there are many other orthogonal polynomials series of orthogonal polynomials for instance, there is the Gramm polynomials then there are the Bessel polynomials which can be obtained from Bessel functions, which are again solutions of Bessel differential equation or Bessel's equation and you can find them in any good book on numerical analysis.

So, with that we end our discussion of orthogonal polynomials. So, from this the main basic idea which we should take away is that, there are series of orthogonal polynomials and they allow us to expand, write a function in terms of a liner combination of the basic functions, which are the members of the series of orthogonal polynomials. So, an n th order polynomial I can write it in terms of basic functions, the basic functions n basis functions orthogonal to each other has a linear combination of those basis functions.

And there are results about the best approximation property and things like that which tell me that it is possible to find a polynomial which is possible to find coefficients basically, if given a particular set of basic functions. It is possible to come up with coefficients which minimize the error between the approximation and the function itself.

So, suppose have I a somebody else given me Legendre's polynomials and somebody has given me a function, then I can always find a constants c 1 through c n, constants such that if I take the corresponding Legendre polynomials, if I take p 1 I multiplied by c 1 p 2, I multiplied c 2 and them together and then I subtract it from my function f of x. the error is going to be the minimum possible it is going to the best fit. So, it is possible to come up with such constants, we have seen that.

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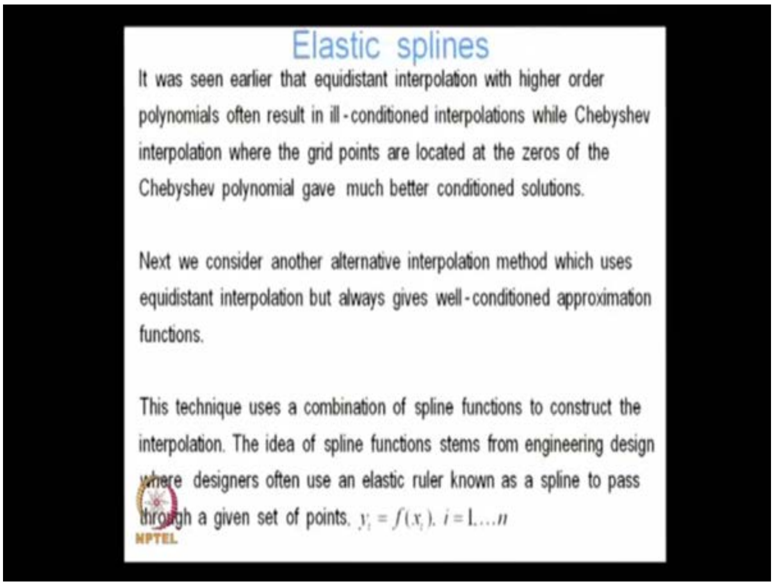

One last polynomial h that I am going to talk about before moving on to gale kin methods and similar such methods which rest on the idea of linearly independent basis functions is elastic splines, because I just want to mention this briefly because these are very useful elastic splines can also be used for heating. They can also be used as basic functions, they are very powerful, basis functions because of the fact that they allow us to approximate a function by suppose, I want to approximate a function over an interval a to b. So, what I can do is that, if I divide that interval into small sub intervals splines allow me to use a different polynomial over each of those sub intervals.

If I have a function f of x, over a sub over an interval a b and then I am going to use So, the all the previous approximations, that all the previous best fits and the orthogonal use of orthogonal polynomials, to find the best fit that we have talked about in that case those polynomials are defined over the entire domain a b. So, each phi 0 phi 1 through phi n are defined over the entire domain a b and I just take linear combinations I take those polynomials which are defined over a b I multiply them with a constant and then add them together and if I choose the constants in a certain way I am assure that is a best fit.

But, in this case splines are somewhat, different because in splines I am using a different you can think of it like I am using a different basis function in an each of those sub intervals. So, have my interval a b, I have divided it into small parts over each of those small parts I am using a different polynomials, I am it is like I am using a different basis function over each of those subintervals and then my resultant approximation is over a b, is just by adding all those approximations together. So, within x suppose I have divide it into x i x x i minus 1, x i x i x i plus 1 things like that so within x i minus 1 x i, I have a certain polynomial within x i x i plus 1 I have a certain other polynomial.

So, if my polynomial approximation is going to work, I have to ensure that my function values are continuous at the intervals and splines have this wonderful property, that those polynomials although they are different over each of these intervals, they are continuous at the interval boundaries, not only are they continuous but it turns out that the derivatives are also continuous

So, suppose I am using cubic splines, I know that not only is the function continuous, it is first derivative is continuous, it is second derivative is continuous but it is third derivative may not be continuous, So, if I use a spline of order j so all the derivatives up to order j minus 1 are continuous at the boundaries; So, it is like I m using separate basis functions for each of those intervals each of those intervals. So, for my basis functions having mathematical terminology my basis functions have local support, they are defined only over certain domains, they are locally defined but then they are continuous.

They ensure continuity at the interval boundary so this, is just a little background. So, we saw that equidistant interpolation earlier that with higher order polynomials. We often get ill-conditi1d interpolations and that is why we had to talk about Chebyshev interpolation, that the grid points are located are not equidistant, it turns out there is an alternative to using Chebyshev interpolation and that is using spline functions, we can always get a well-conditi1d approximation, even for equidistant grid points

So, this technique uses a combination of spline functions to construct the interpolation the idea of spline functions strengths engineering design, where designers often use an elastic ruler to pass a curve through a set of points.

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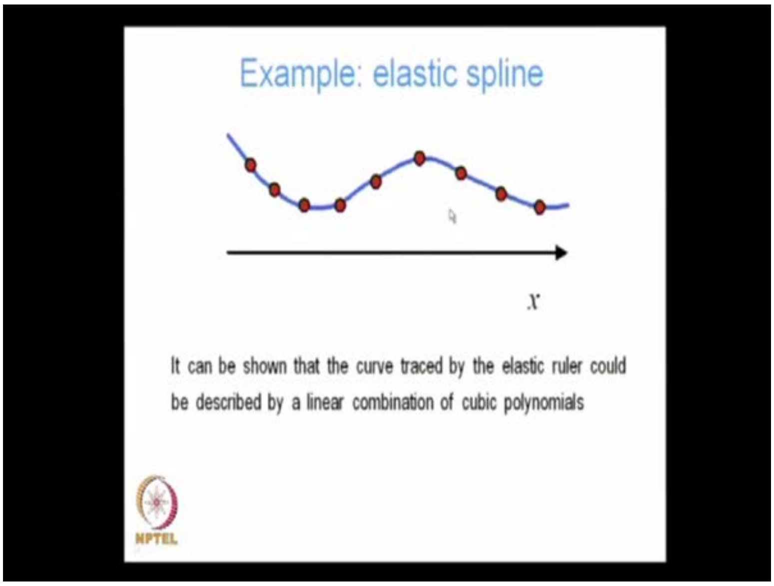

Suppose, I have a set of points and they have an elastic rule they used to have in the olden days, they used to pass, they used to find out what is the best fit through that point by passing that elastic ruler through those points and what they were using were basically spline functions. So, it can be used, it can be shown, that the actually cubic splines, it can be shown that the curve traced by the elastic ruler could be described by a combination of by a linear combination of cubic polynomials.

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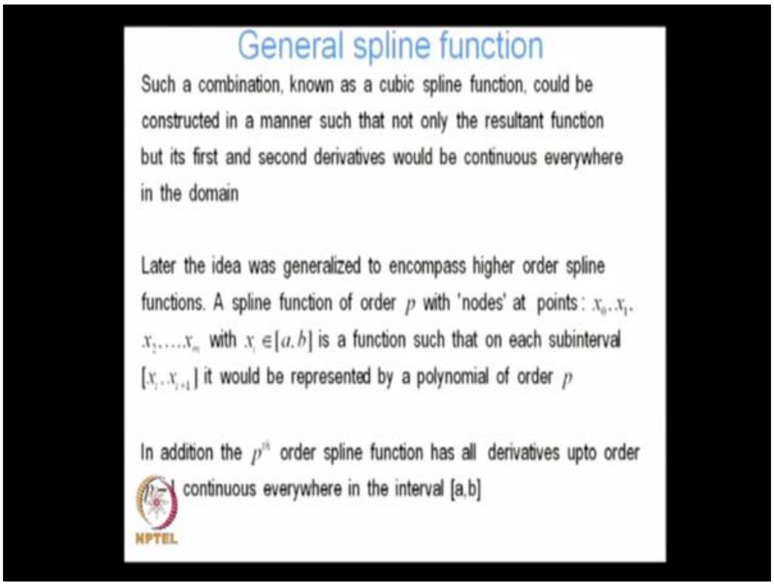

So, such a combination known as a cubic spline function to be constructed in a manner such that not only the resultant function, but its first and second derivatives would be continuous everywhere in the domain. So, originally it started with cubic spline because the curve that is the curve that the engineers used to use that could be described by a linear combination of cubic functions, cubic polynomials, but then mathematicians got to that idea they started generalizing it, they generalize it and they said that you could have splines of any higher order.

So, a spline function of order p with known's at points x 0 x 1 through x m with x i belonging to a b, see x i means all these x 0 x belonging to a b is a function such that, on each subinterval x i to x i plus 1. It would be represented by a polynomial of order p in each sub interval it would become a polynomial overall it is a linear combination of those polynomials, the overall function is a linear; So, you can think it is like earlier we were looking at linear combination of orthogonal basis functions, those orthogonal basis functions were defined over the entire interval a b, now I am looking at a linear combination of spline functions and the spline basis functions are each defined on each of those subintervals x i minus 1 x i x i plus 1 x i x i to x i plus 1 and so on, and so forth. So, in addition the p th order spline function has all derivatives up to order p minus 1 continuous everywhere, in the interval a b generalizing the idea from cubic splines.

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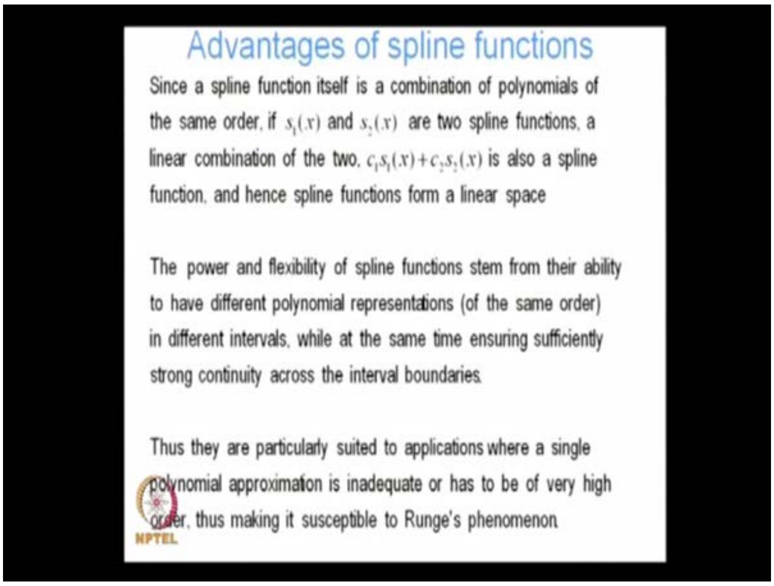

Since, the spline function itself is a combination of polynomials of the same order all of them are cubic. If you think of cubic splines all of them are cubic polynomials, if s 1 x and s 2 x are 2 spline functions a linear combination of the 2 c 1 s 1 x plus c 2 s 2 x 1 x is also a spline function and hence spline functions form a linear space they form a linear space because if I combine two basic functions 2 members of that space, I am still in that space, that is the I am not outside that space, whatever I get I am still in that space. So, it is a linear space.

The power and flexibility of spline functions stem from the ability to have different polynomial representations of the same order, in different intervals while at the same time ensuring sufficiently strong continuity across the interval boundaries, thus they are particularly suited to applications, where a single polynomial approximation is inadequate or if we have use the single polynomial my function has got lots of peaks and valleys and got lots of maxima and minima. So, I cannot I have to use a higher order polynomial. I have to use a higher order polynomial but I know that if I use a very high order polynomial I am going to run into problems, what is that problem? Those problems we have talked about run this phenomenon.

So, at the ends I am going to get very large errors, while at the center I might get a good fit to my function value but at the ends my error is going to be large and we found that if you use Chebyshev interpolation that can be problem, can be solved but this is another way of doing it using spline functions. And spline functions are good thing, they can use equidistant grid points also they do not have to use like the Chebyshev interpolation Chebyshev are 0’s of the Chebyshev. So, we will focus on cubic splines only because they are the types of splines which are most widely used.

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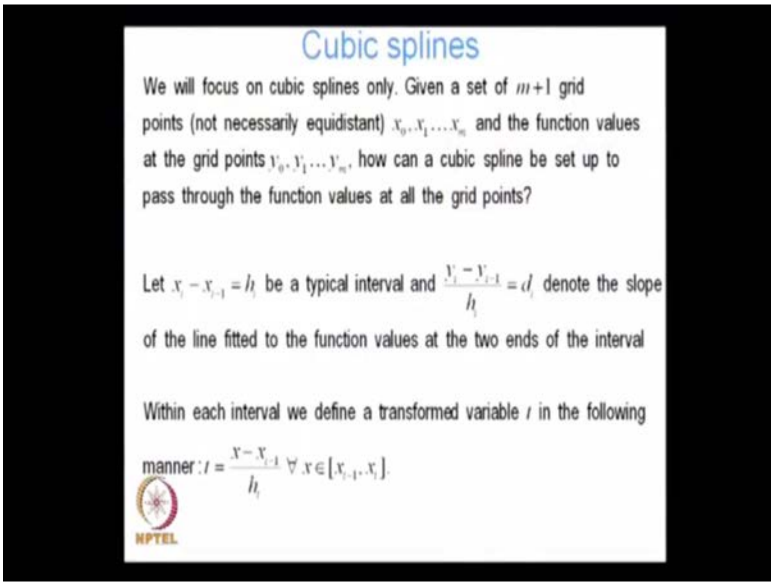

So, given a set of m plus 1 grid points which may be equidistant may not be equidistant x 0 x 1 through x m and the function values at the grid points f of x 0 equal to y 0 f of x 1 y 1 and y f of x m equal to y m, how can we set up a cubic spline to pass through the function values at all the grid points. Well let us try to find out how so let x i minus x i minus 1 is equal to h i be a typical interval and y i minus y i minus 1 by h i equal to d i denote the slope of the line fitted to the function values at the two ends of the interval.

And within each interval, we defined a transformed variable, t in the following manner t is equal to x minus x i minus 1 h i. So, at x is equal to x i minus 1 t is equal to 0; So, it is like I am parameter zing that interval t is at x is equal to x is equal to x i minus 1 t is equal to 0 at x is equal to x, i t is equal to 1 because x i minus x i minus 1 going to be h i. So, I am going to get t is equal to 1.

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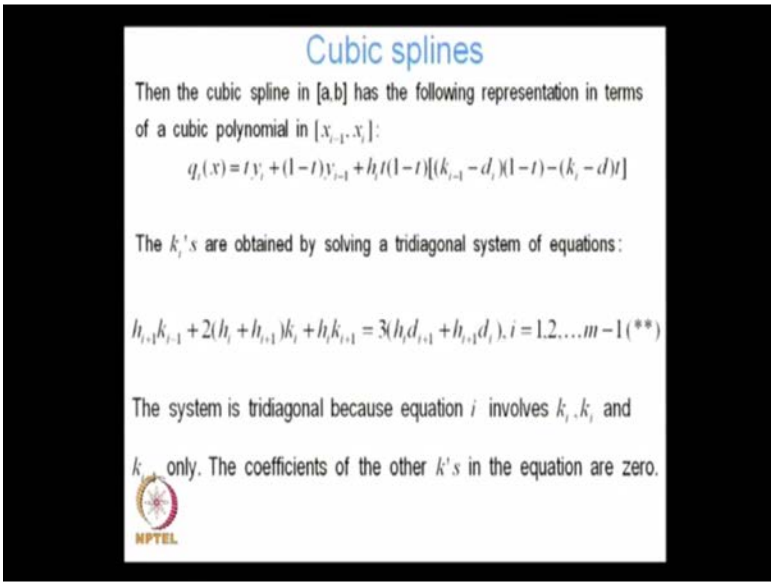

Then, the cubic spline in a b has the following representation in terms of a cubic polynomial, in x i minus 1 x i, So, within that interval x i minus 1 to x i, I said that my cubic spline has got different representations in each interval within the interval x i minus 1 to x i my cubic spline becomes this. Q i x now x goes only from x i minus 1 to x i is can be written like this t we have just defined y i is the function value at i y i minus 1 is the function value at i, i minus 1 x i minus1 h i is the size of my interval k i, I am going to talk about it later on d i we have just defined is the slope of the line fitted between y i minus 1 and y i and the k i's are obtained by solving a tridiagonal system of equations.

So, this is my cubic spline in that interval x i minus 1 to x i. So, I know everything here except the k's and will talk about the physical meaning for k i next slide. But, for the time being you assume that k's to get the values of the k i, I have to solve this system of equations why this system called tridiagonal, because you can see it involves terms like k i minus 1 k i k i plus 1.

So if i and this goes from i equal to 1, 2 through m minus 1, it goes through all the intervals, because I had m grid points. So, it goes through these i equal to 1 2 through m minus 1 and you can see that, if I write this is a matrix each row in that matrix is going to have only 3 non zero contributions from k i from k i minus 1 k i plus 1. So, it is the diagonal plus 2 terms plus row on either side of the diagonal column on either side of the diagonal. So, because equation I involves k i k i minus I am that is a mistake k i minus 1 k i and k i plus 1 only the coefficients of other k's in the equation are 0.

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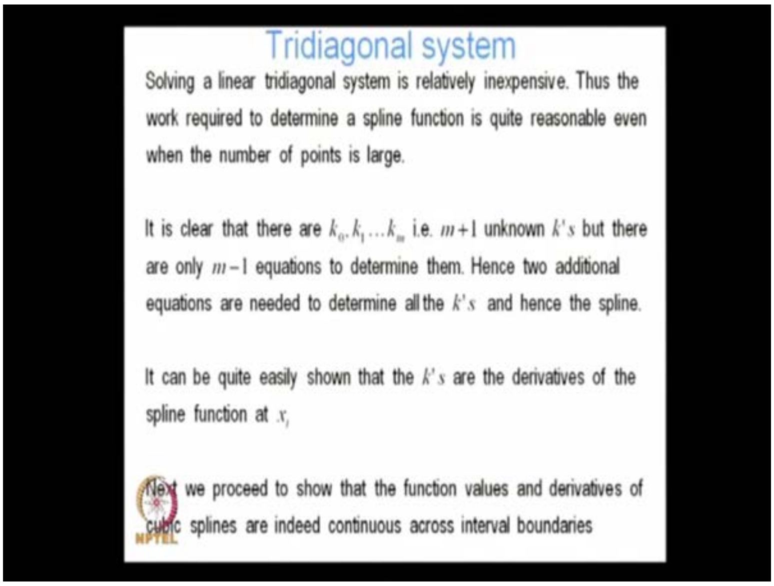

So, solving this system is relatively inexpensive because it is inexpensive the work required to determines spline functions is reasonable, it is not that hard. So, the only things we know everything if you fit this spline function, we know everything. We know x i y, we know since I know x i minus 1 x i, I know t i know I know h i, i know d this is d i, So, i d i i know all of those things the only thing that i need to find out by solving this system are these k i's and once i have found my k i's i can fit my i can get my cubic my cubic splines.

So, it is clear that there are k 0 k 1 through k m that is m plus 1 unknown k's. But, there are only m minus 1 equations. Is that clear? So, I have i equal to 1 two through m minus 1. So, I have m minus equations but there are m plus 1 unknown k's. So, how could I i find more unknowns than I have equations well I have to assume impose certain constraints. Hence, 2 additional equations are needed to determine all the case and hence the spline now it can be shown quite easy so that I did not define the case earlier. But, the cases are actually the derivatives of the spline function at x i at the grid points. So, the k's are actually the derivatives of the spline function at the grid points.

So, we want to show and I said that the spline function is continuous and it is derivatives are continuous at the interval boundaries; well I am going to show that.

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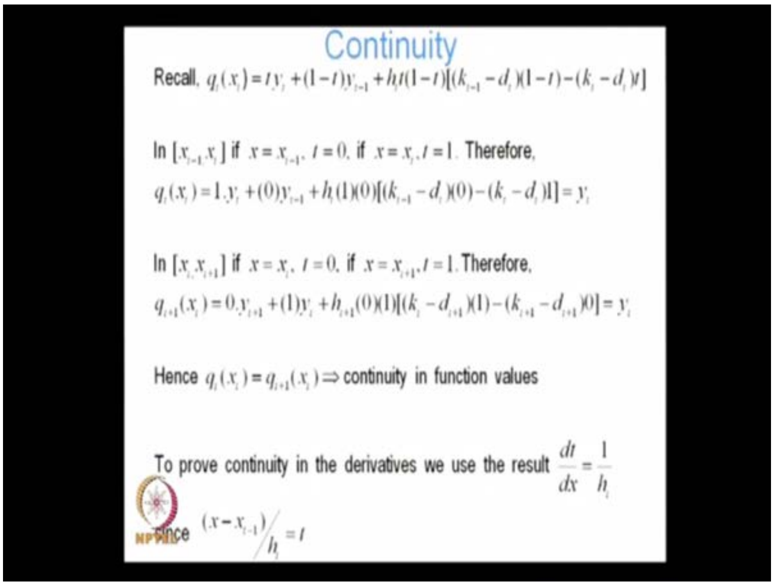

So, we said that this is the form of the spline function within a particular interval x i minus 1 to x i, So, in x i minus 1 x i if x is equal to x i minus 1 t is equal to 0 if x is equal to x i t is equal to 1; therefore, if I evaluate q i x i in now that is a mistake that is q i x, that is q i x this is any point in that in that interval. So, if I want to evaluate q i x i then t is going to become 1, So, I have 1 times y i this becomes 0 times y i minus 1 plus h i t is 1, but 1 minus t is 0. So, everything here becomes 0 and that is equal to y i which is has to be So, it just shows that this gives me at q at x i my cubic spline gives me, the matches the function value at x i; So, this we found by looking at the interval x i minus 1 x i.

Now, let us look at the interval x i x i plus 1. So, if x is equal to x i in that interval at the left hand of the interval if x is equal to x i then t is equal to 0 in that interval and if x is equal to x i plus 1 t is equal to 1, So, let us evaluate q i plus 1 that is the cubic spline at the i plus 1th interval I want to took the cubic spline for the i'th interval and i at the by i'th interval i mean the interval between x i minus 1 in x i and i evaluated it at x i . Now I am taking the cubic spline for the i plus 1th interval which is basically, the interval from x i to x i plus 1 and I am evaluating it at x i and I want to show that this formula is going to give me the same function value, if I can show that I have shown that my cubic splines are continuous at the interval boundary.

So, q i plus 1 x i so in this case if x is equal to x i t is equal to 0, So, I have 0 times y i plus 1 is its i plus 1, So, this becomes i plus 1, So, y i plus 1 times y i since again t is equal to 0, So, 1 times y i h i t is equal to 0, So, this whole thing goes away so I am left with y i, So that, shows me that q i plus 1 evaluated at x i is equal to y i q i evaluated at x i is also equal to y i So that, means that my cubic spline is continuous at the interval boundaries and this means that the function is continuous at the interval boundaries to prove continuity in the derivatives, we use the result that d t d x is equal to 1 by h i from definition since x i x minus x i by h i is equal to t d t d x must be equal to 1 by h i.

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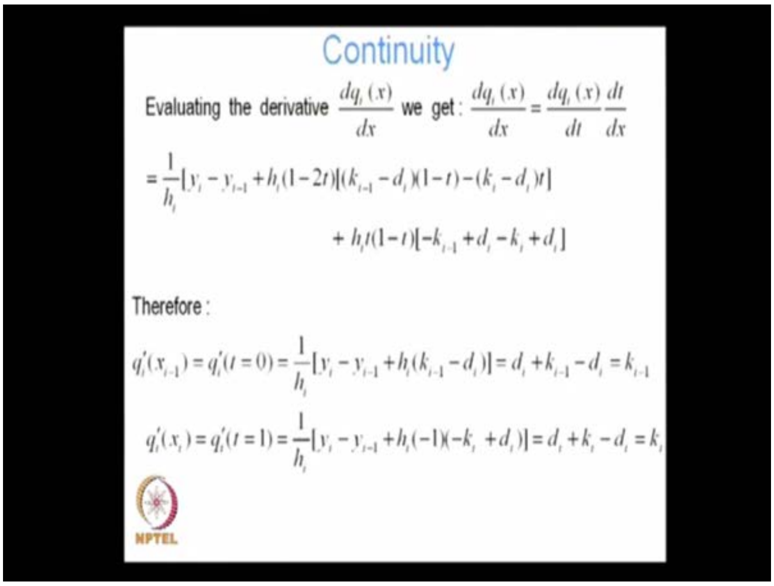

So, then we evaluate the derivative. So, this was my q i x. So, I evaluate q i x d q i x d x that is nothing but d q i x d t times d t d x because this is a function of t. So, I take d q i x d t times d t d x. And if I do that I get this expression and then if I look at this expression and I evaluate q i prime at x i minus 1 i get that t i minus 1, So that, from here you can see that this k i's are nothing but my the derivatives of my spline function. So, if I evaluate q i plus y prime at x i minus 1, I get k i minus 1, if I evaluate q i prime at x i, I get k i this by substituting these values, substituting the appropriate values of t I can show that is true.

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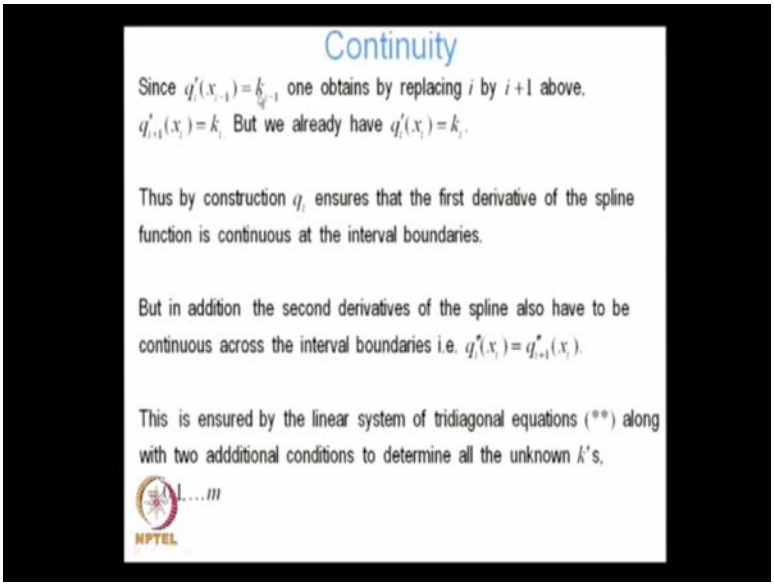

And since q i prime x i minus 1 is equal to k i prime minus 1. 1 obtains by replacing i by i plus 1 above q i prime q prime at i plus 1 x i is equal to q x i is equal to k i. So, we have seen q i prime at x i minus 1 is equal to k i minus 1, that we have seen from here. Then, instead of replacing i by i plus 1, I have q prime i plus 1, this x i minus 1 becomes x i k i minus 1 becomes k i. But, we have already shown that q i prime x i is equal to k i. That means that q i, i plus 1 prime x i is equal to q i prime x i that means that by construction the first derivative of the spline function is continuous across the boundary. So, in addition we said not only must the first derivatives be continuous the second derivatives of the spline functions must also be continuous.

That condition the fact that the second derivative of that is actually satisfied by this is that constraint which I meant i1 d earlier, that if I when I satisfied that I make sure that my second derivatives are also continuous at the interval boundaries. So, by construction the spline function is continuous at the interval boundary, the derivative is continuous at the interval boundary if in addition I satisfied that equation that makes sure that the second derivative is also continuous at the element, at the interval boundary. But, I said that I cannot solve that system by itself because there are 2 more unknowns than the equations. So, I have to use additional constraint equations and what are the constraint equations?

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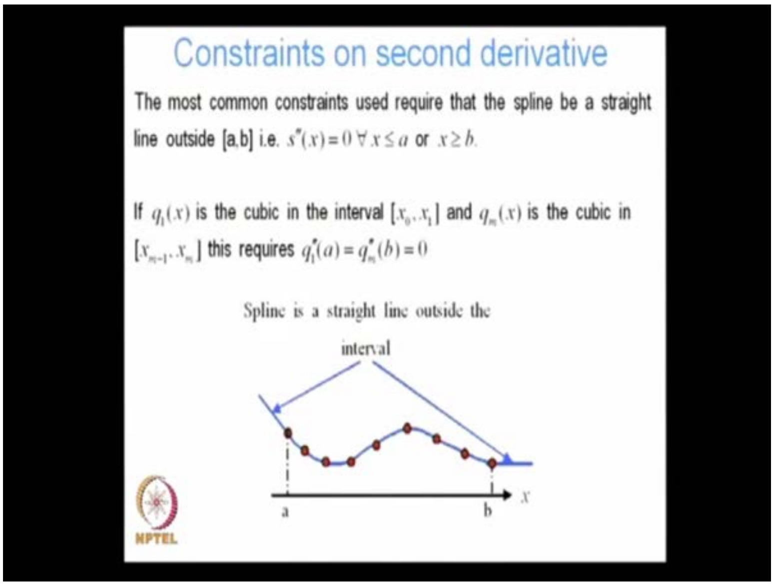

Typical constraint equations require that outside my interval the slope is a constant. So, I have 2 constraints equations. So, whatever is the slope here it remains the same then when x is less than whatever is the slope here it remains the same when x is greater than b. So, this imposes certain restrictions on the slope here. So, those are the additional restrictions that the spline we have straight line outside a and b that it is slope does not change outside that interval, So, this basically requires if q i x is the cubic in the interval x 0 x 1 and q m x is the cubic in this interval, this requires that this is equal to 0, that is equal to 0.

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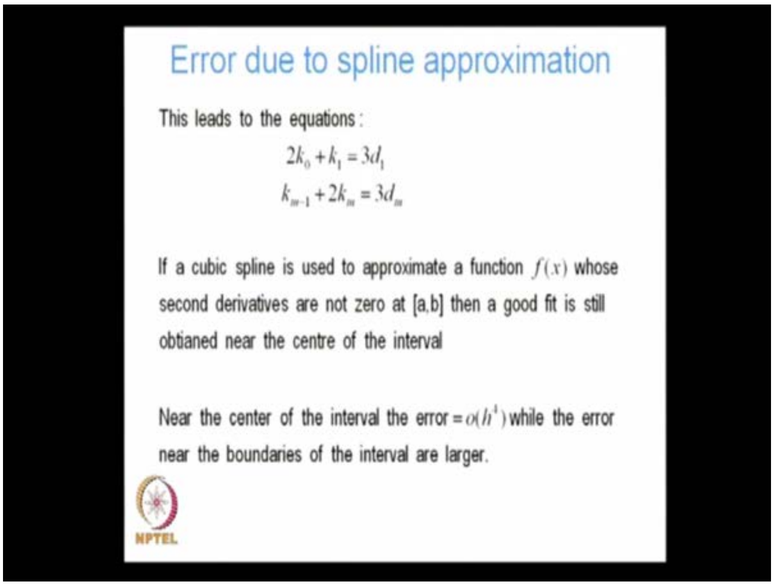

And this leads to these 2 equations. So, it may be we will just go over this next slide because we are running out of time. So, next class I wrap up spline functions and then we will go over to gale kin method where, we are going to use these concepts of orthogonal polynomial expansions orthogonal functions to allow us to solve partial differential equations, which are basically, will be prepared as for advanced numerical techniques like the finite element method.

Thank you.