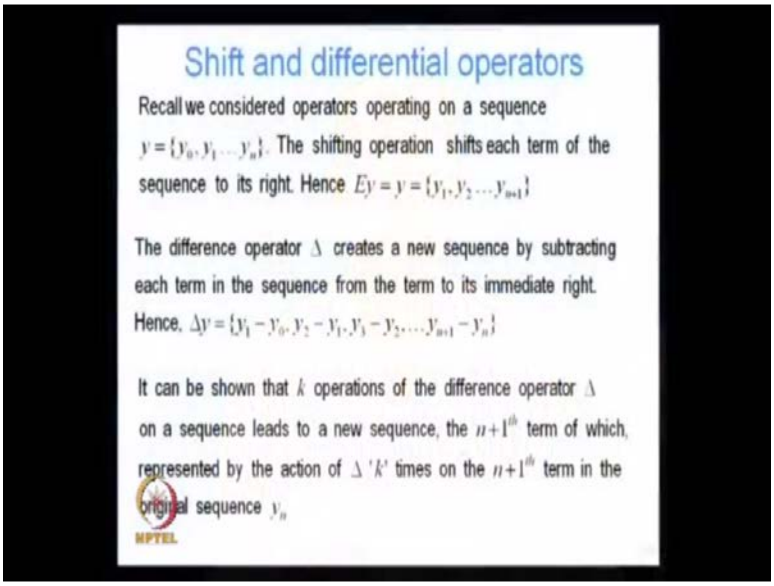
Numerical Methods in Civil Engineering
Prof. Arghya Deb
Department of Civil Engineering
Indian Institute of Technology, Kharagpur

Lecture - 27
Differential Operators

In lecture 27 of our series on numerical methods in civil engineering, we will continue with our discussion on differential operators, recall we considered operators operating on a sequence y is equal to y 0 y 1 through y n.

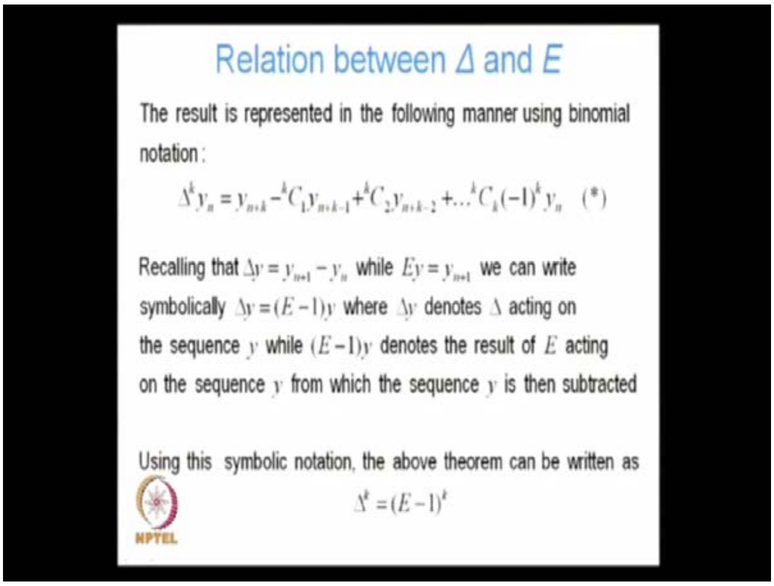
(Refer Slide Time: 00:37)



and we define two operators, we define the shifting operator which we denoted as E and which operates on the each term of the sequence and shifts It to the shifts each term to the and we also define the difference operator delta which creates a new sequence by taking the difference between each term In the sequence and Its next term. So, for y for in the 0th position, we had y 0 in my original sequence in the new sequence it will be y 1 minus y 0. So, the term on the minus the original term similarly, for other terms in the sequence.

It can be shown that k operations of the differential operator delta, lead to a new sequence the n plus 1th term of which is represented by the action of delta. K times on the n plus 1th term in the original sequence y n. So, the n plus 1th term in the original sequence is y n because, it starts from y 0 and what I am just saying Is that. If you operate k times on that operator that is the resultant sequence is the action of delta, k times on the n plus 1th term in my original sequence which was y n.

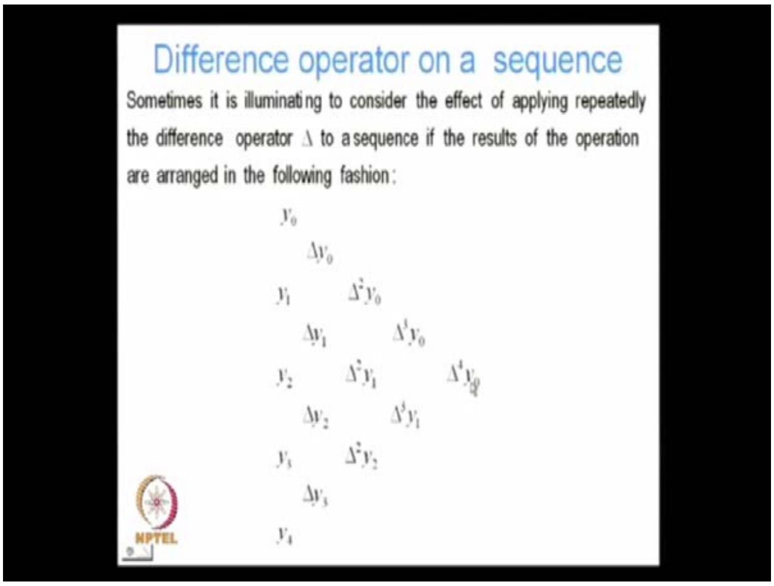
(Refer Slide Time: 01:53)



So, the result Is represent as you can understand If you operate k times on the y n you are going to get more and more terms. This first time when you operate delta on y n you are going to get y n plus 1 minus y n If you operate again on that as you operate on y n plus 1 as well, as y n similarly, you the terms become larger and eventually, the result of operating k times on y n with the delta operator can be represented by something like this. We saw this expression last time and we also tried to give a proof on that using Induction. So, here I am just mentioning the final result. So, that was where we stopped last time.

Now, let us take a step back and recall that delta y n delta y Is equal to y n plus 1 minus y n and since E y Is equal to y n plus 1. We can write symbolically delta y Is equal to E minus 1 y because, E operating on y Is going to be y n plus 1 minus y n; So that, Is the equivalent to the operating on y with y n with delta, So, we can write this delta Is equal to E minus 1. We can write delta Is equal to E minus 1 and using this symbolic notation the above theorem, the above theorem meaning this result this result can be written as delta k Is equal to E minus 1 k, because they are equivalent, So, we can write this as well as E minus 1 k operating on y n.

(Refer Slide Time: 03:43)

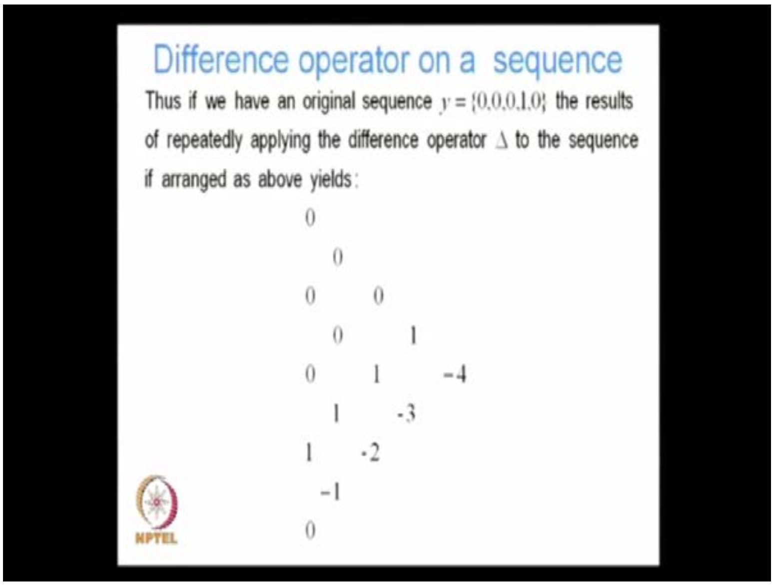


So, sometimes it makes Its yields a lot of Insight, If we look at a sequence y 0 y 1 y 2 y 3 y 4 and then see the results of operating on the sequence repeatedly with the delta operator and arrange the terms In this fashion. So, here what I have done I had originally y 0 y 1 through y 4.

First I operate on this. So, delta y 0 that is equal to y 1 minus y 0 delta y 1 which Is equal to y 2 minus y 1 delta y 2 y 3 minus y 2 and delta y 3 y 4 minus y 3. So that, Is operate first on with once the delta operator with the difference operator once then If I operate again, So, let us see what del 2 y 0 would be that would be del y 1 minus del y 0 del 2 y 1 would be del 2 y 2 minus del y 1 del 2 y 2, would be del y 3 minus del y 2 similarly, If you operate 3 times del 3 y 0 Is going to be del 2 y 1 minus del 2 y 0 del 3 y 1 del 2 y 2 minus del 2 y 1 and del 4 y 0 Is del 3 y 1 minus del 3 y 0.

So, If you arrange this sequence, this difference the action of the difference operator on the sequence the repeated action of the difference operator on the sequence In this fashion Sometimes It yields a lot to It gives a lot of Insights for Instance In the next problem.

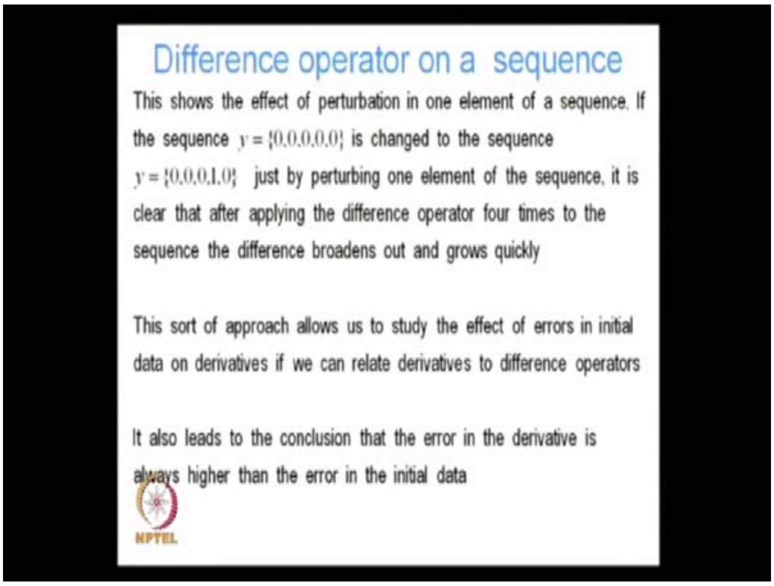
(Refer Slide Time: 05:22)



suppose, my original sequence was 0,0,0,1,0 and I operated on that repeatedly with my difference operator and you can see that after I operate first time I get 0,0,1 minus 1 second time I get 0, 1 minus 2, third time I get 1 minus 3, fourth time I get ((Refer Time, 05:48)) minus 4. So, what does this tell me this tells me that, If my original sequence was actually 0,0,0,0,0 and then I Introduced a minor perturbation to that sequence In the fourth term In the sequence I made It 1.

So, If my original sequence was 0,0,0,0 all the terms were 0 In my sequence, If I operated on that sequence with the difference operator 4 times what would I get? I would still get 0 Del 4 y 0 would still be 0. But, then If I Introduce some minor perturbation one term In the sequence undergoes a slight change, It changes from 0 to 1 In this test. The fourth term In the sequence changes from 0 to 1 then If I look at the difference way down the road after I have taken repeated differences you can see that the difference Is much larger It has become amplifier that minor change that change In the fourth member of the sequence that change In 1 has now become minus 4.

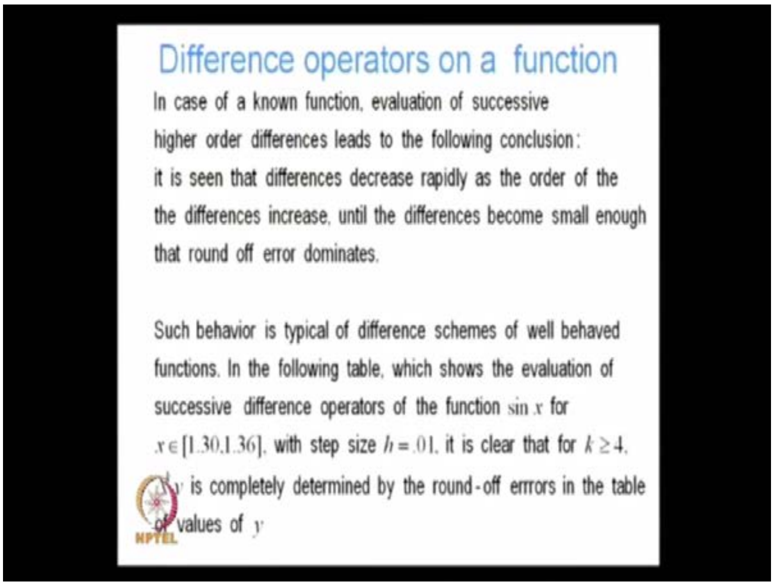
(Refer Slide Time: 06:57)



So, this shows the effect of perturbation In 1 element on the sequence, which changes by just 1 element then after difference operator it, has grown 4 times. So, this tells us why In any numerical method you will see that the If you look at the derivative, the derivative the error In the derivative Is always higher than the error In my original variable for Instance, If I know many of you are familiar with the finite element method where suppose, we have the primary variable as the displacement, So, and then we are Interested If you look at the results of the solution, we are look at the displacements we are look at this strange, we look at this stresses; So, suppose there Is an error In the displacement and then we look at the error In the stresses the error In the stress will always be higher than the error In the displacement. Whether the error In the displacement Is due to discretization error or It Is just a truncation error that error In the derivative will always be higher this exactly shows why because, repeated operations of the difference operator, It always multiplies It always scales magnifies the error.

So, this sort of approach allows us to study the effect of errors In Initial data on derivatives. If there Is a ((Refer Time-08:27)) here, If we can relate the derivatives to the difference operators I have shown this for the difference operators but, when If you have to claim If I claim this for the derivatives I have to relate the derivatives to the difference operator, In the terms of that you can do that; So, the same thing carries over for derivatives and error In the Initial data It gets multiplied, It gets scaled several times then more times you take the derivatives more Is the error scaled by. It also leads to the conclusion that the error in the derivative Is always higher than the error In the Initial data.

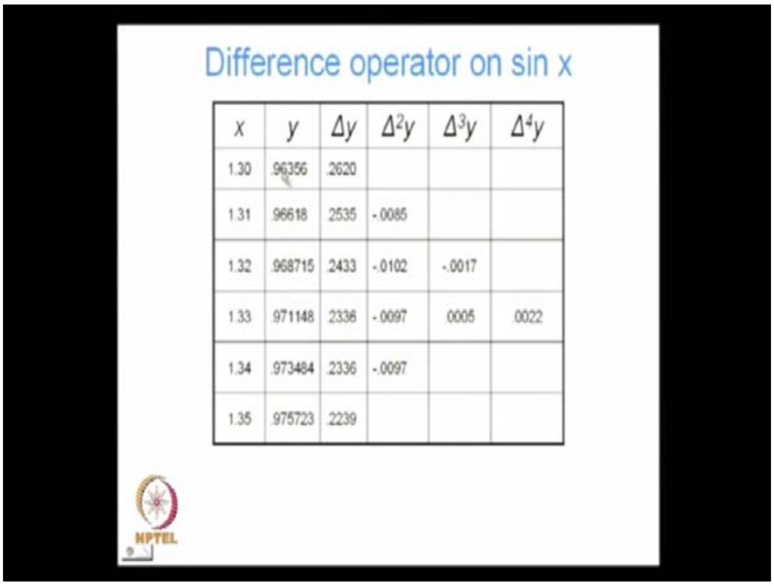
(Refer Slide Time: 09:07)



up till now we have looked at difference operators on a sequence, we can as well operate we can as well apply these different operator, In a function So, In case of a known function evaluation of success of higher order differences leads to the following conclusion. We can show that also, it is seen that differences decrease rapidly as the order of the differences Increase until the differences becomes small enough that round of operators, the round of error dominates. So, It Is now that If you If Instead of a sequence I look at a function, If I see, If I take repeated differences If I take repeated differences on a function, It Is found that the differences decrease rapidly and then until finally, the differences are only different by the round of error. So, If In my original function values I had certain round of error then after repeated differences what is going to show up is just the round of error.

Such behaviors typical of difference schemes of well behaved functions, In the following table which shows the evaluation of successive difference operators of the function sin x So, If we looking at the function sin x over the range x belongs to 1 point 3, 0 and 1 point 3, 6 and we are evaluating It at step size of point of 0, 1; we can see that after we take the fourth difference the It Is totally dominated by the round of error In the values of y let us look at this little table .

(Refer Slide Time: 10:54)



x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.30	.96356	.2620			
1.31	.96618	.2535	-.0085		
1.32	.968715	.2433	-.0102	-.0017	
1.33	.971148	.2336	-.0097	.0005	.0022
1.34	.973484	.2336	-.0097		
1.35	.975723	.2239			

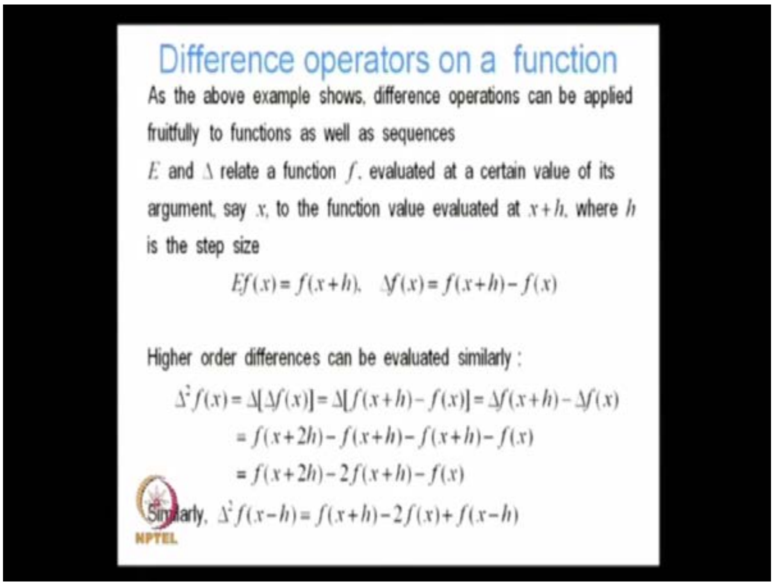
So, here now we are looking at difference operators on a sequence on a function rather than a sequence and the function that we are considering Is sin x and I am evaluating that sin x over an Interval from 1 point 3 0 to 1 point 3, 6 and I am evaluating It at steps of point 0,1; So, 1 point 3, 0 1 point 3, 1 these are my values at which I am evaluating the function sin x and these are the values, the results of sin x , these are the results of sin x and here I have taken the difference between this and this and this and this and so on and so forth.

Then I have taken the difference again, So, In this case this minus, this becomes that, this minus this, becomes this minus this, becomes that and then I have taken the difference of third time you can see that these are becoming smaller and smaller these magnitudes are becoming smaller and smaller the magnitudes of these differences are becoming smaller and smaller until they are so small that this may be as small as the round of error Involved In the calculation of this.

If, we have calculated this with a round of error up to the fourth decimal place then this difference Is the same as the round of error which I used round of which I used In my calculation of this sin x. So, this is this may become as small as the round of error is that clear. So, Let I hope It Is clear. The first thing that I talked about was the fact that when you have a when you have a perturbation, In the Initial data when you take repeated difference that perturbation gets amplified. So, that means that when you have an error In the Initial data that error becomes amplified but, here I am saying that when you have a certain function that you are evaluating at certain Intervals, at a certain fixed Interval If I take repeated differences those things become smaller, this differences become smaller does not say anything about the error the error the error might If there was an error here If there was an error In the calculation here, when I take repeated differences that Is also going to get amplified, that Is for sure that holds every time all I am saying here Is that this differences this magnitude of this differences are becoming smaller but, the error can become larger.

If there was an error there that Is going to get magnified this Is the actual difference the actual difference Is becoming smaller but, the error If there was an error In my calculation of sin x somewhere then when I do this repeatedly that error Is going to become larger and larger that Is why when I say that this becomes this Is equivalent to the round of that means that Is actually reflected by that. So, If there was a round of error In the calculation of sin y have I take repeated differences that round of error Is going to get multiplied that Is a It Is going to become larger It Is going to get magnified but, now when I take the repeated differences the function value the difference value Is also becoming smaller the difference value Is also becoming smaller the actual. So that, so that round of error, If It Is Increasing It may be become actually larger than the true difference it become larger than the true difference Is that clear I hope I did not confuse you.

(Refer Slide Time: 14:46)

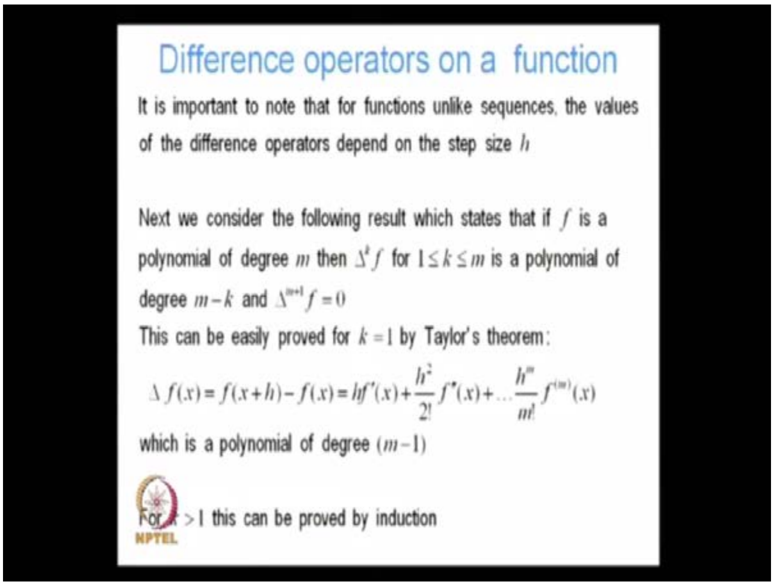


So, as the above example shows difference operations can be applied fruitfully to functions as well as sequence. So, E and delta relate a function f evaluated at a certain so It Is very similar, So, I do not want to elaborate this but, this Is very similar to how we operate on sequence so E f of x Is nothing but, f of x plus h x evaluated at the after a after a move one step forward, So, If I am If I have uniform grad If I have a uniform grad If E f of x Is f evaluated at x plus next at x.

At the next point on the grad so f evaluated at the next point on the grad which Is x plus h similarly, delta f x f x Is equal to f evaluated at the next point on the grad minus f evaluated at x similarly, we can evaluate we can evaluate higher order differences, So, del 2 f x Is equal to del f x I operate on del f x I operate with on f x with the difference operator and then I operate again.

So, operating on f x with a difference operator I get f x plus h minus f x and then I operate again with the difference operator, So, I have write first on f x plus h, So, I am going to get f x plus 2 h minus f x plus h minus del operating on f x will give me f x plus h minus f x, So that, Is what I am going to get; If I pull the terms together I get f x plus h minus 2 f x plus h minus f x similarly, del square f x minus h Is given by this same logic x minus h del f x minus h Is equal to f x minus f x minus h and operate on that with del again I get that value.

(Refer Slide Time: 16:50)



So, It Is Important to know that for functions unlike sequences, the values of the difference operator depends on the step size It depends on h remember for sequences, It only evaluated, It only depended on the function value on the various terms; In the sequence on the various depended on y n y n plus 1 y 0 y 1, now It depends not only It depends on the step size It depends on the step size.

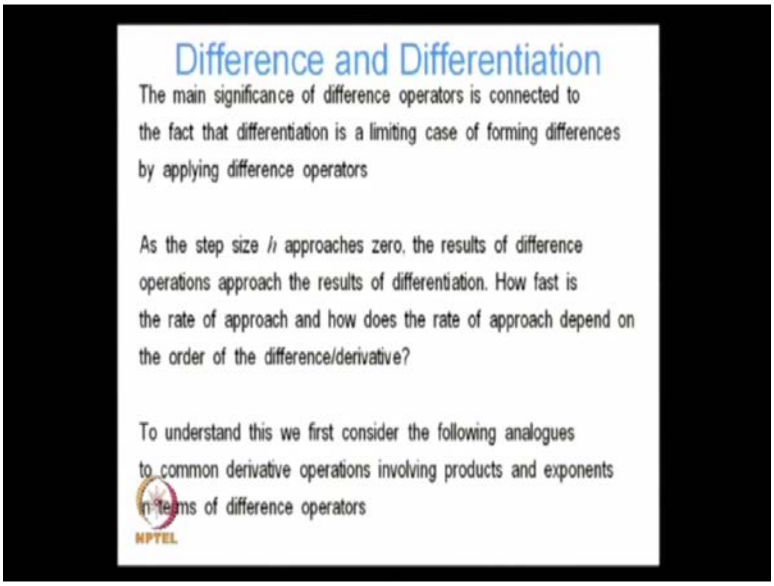
Next, we look at the following results which states that If f is a polynomial of degree n then Del k f for k lesser than or equal to m greater than or equal to 1. Is a polynomial of degree m minus k and delta m plus 1 f Is equal to 0, If you think of derivative this Is very, this should be very familiar to you, If I have a function which Is a polynomial of degree f and I take the derivative k times I take the derivative k times then the order Is reduced to m minus k think of a polynomial x to the power n.

If I take the derivative k times then I have the term will become x n minus k, So, the same thing holds for the difference operator what It Is saying that, If f Is a polynomial of degree m then I If I operate on that function k times with the difference operator then I am going to get a polynomial of degree m minus k similar to what I have would get If I operated with f k times with the different, differential operator Instead of the difference operator.

And then If I operate n plus 1 times on f I am going to get 0 think of a polynomial of degree x to the power m If I take m plus 1 derivative of x to the power m I am going to get 0 similarly, exactly the same thing happens for the difference operator to show that well If we can show that for k equal to 1 by using Taylor’s theorem; So, delta f x Is equal to f x plus h minus f x by definition and then I expand f x plus h about x, In Taylor series. So, what do I get f x plus h f prime of x f x f x cancels and I have h f prime of x plus h square factorial 2 f double prime of x through h m factorial m f mth, mth derivative of f evaluated f at x which It Is clear, Is a polynomial of degree m minus 1 It Is a polynomial of degree m minus 1, where Is the leading order term Is given by this h f prime of x and that Is the first derivative of the polynomial of degree m. So, that is a polynomial of degree m minus 1.

So, this term will be m minus 1 that term will be m minus 2 and so on and so forth so what does this tell me that if I and this Is actually equal to that, this Is exactly equal to my difference the result of my difference operation. So, this tells me that If I take the first difference of f x and If f is a polynomial of degree m then delta f x Is going to be a polynomial of degree m minus 1 and for higher orders of k we can prove It similarly, by Induction.

(Refer Slide Time: 20:32)

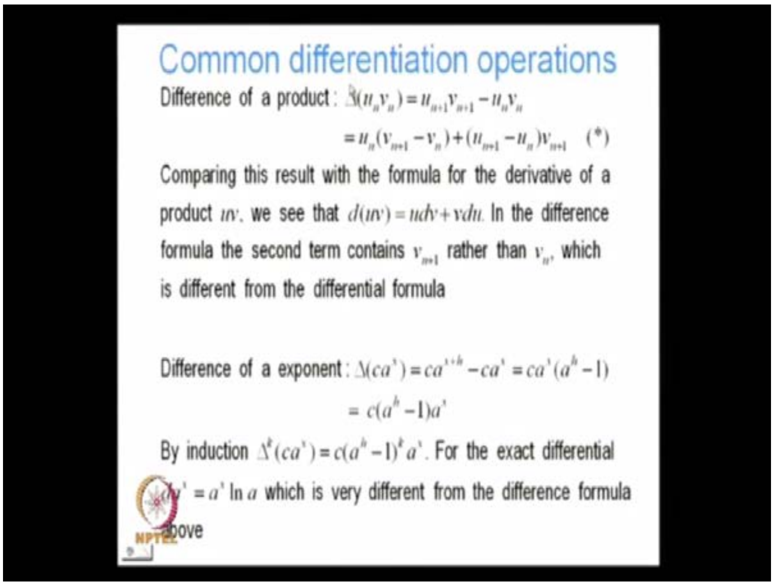


the main significance of difference operators Is connected to the fact that differentiation Is a limiting case of forming differences by applying difference operators. So, the difference operators as I make this step size as If I take, If I consider, difference operators operating on a function and I reduce my step size I reduce my h In the limit that h goes to 0 It can be shown that my difference and derivative operators become very similar So that, Is why they are useful that Is why these difference operators when we solve a differential equation numerically. We can use these difference operators to approximate my differentiation operations.

So, as the step size approaches 0 the results of difference operations approach the results of differentiation the question Is how fast Is the rate of approach number 1; how fast does the difference operator approach the derivative as h goes to 0 number 1 and number 2 how does the rate of approach change with the order of the difference order of the derivative Is the rate of approach the same for the first order difference operator and the first order different derivative and the second order difference operator and the second order derivative, the third order difference operator and the third order derivative before we look at this we first consider the following analogous operations, which are very similar to for.

When you first look at derivatives you look at things like derivative of a product and things like that derivative of an exponential or derivative of something raise to the power of something. So, you looked at those derivatives first when we looked at derivatives like In high school or somewhere now I want to look at very similar operations using the difference operators I want to apply the difference operator on a product, I want to apply difference operator and exponent and see the result look at the result and compare to what I would get, If I apply the derivative on those same things, So, If I apply the difference operator on a product how does the result compare to the derivative of the same of that same product let us look at that.

(Refer Slide Time: 23:04)



So, If I apply If I look at the difference of a product, So, I have a product u, n, v, n I have a product u n v n and I am operating on that with a difference operator. So, by definition this Is going to be u n this thing evaluated at n plus 1 minus this thing evaluated at n, So, that Is going to be u n plus 1 v n plus 1 minus u n v n.

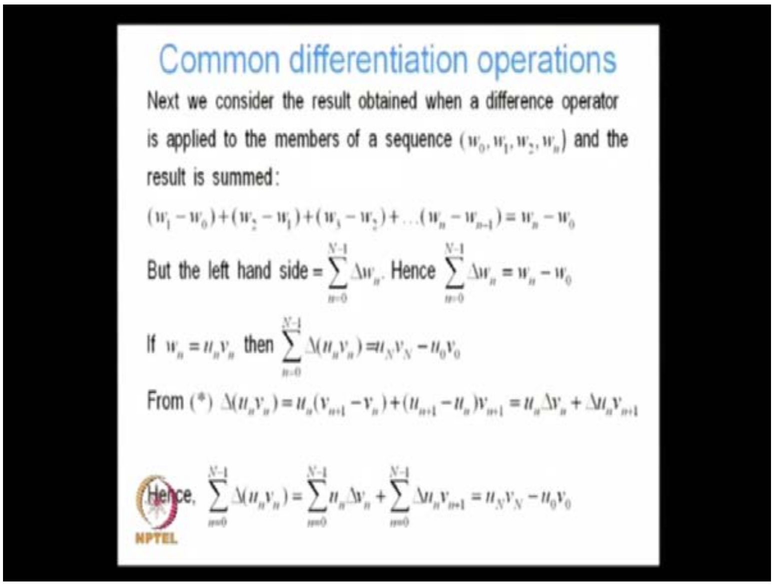
So, I can write this as u n v n plus 1 minus v n plus u n plus 1 minus u n v n plus 1 you can see that the only terms that are going to survive are going to be u n plus 1, v n plus 1 and u n v n this term u n v n plus 1 and u n and u n v n plus 1 they are going to cancel out because, they are going to negate each other. So, I can write It like that so let us compare this result with the formula for the derivative of a product u v derivative of a product u v d u v Is equal to u d v plus v d u.

Now, look at this If we think of this as del v as del v, So, I have u del v plus, If I think of this as del u del operating on u n del u n and this has del v n, So, I get del u n v n plus 1 which Is very different we have d u v Is equal to u d v, So, this term Is similar then you have u n plus 1 u n you can think of that Is d u analogous to d u that then here It should have been v n but, It Is actually v n plus 1. So that, Is that is the difference.

So, In the difference operator formula the second term contains v n plus one rather than v n which Is different from the differential formula this tells us that this operation Is not exactly equal to the derivative this difference operation Is not exactly equal to the derivative It will become close to the derivative when my n plus one Is very close to n when my v n plus 1 is very close to v n when my step size is small but, It is not exactly equal to the derivative, So that, is the point I want to emphasis friends next let next look at let us look at the difference of an exponent.

So, del c a to the power x, If I evaluate that So, by definition that Is equal to the function evaluated at x plus h minus the function evaluated at x So, that is equal to c a x to the power x plus h minus c a to the power x, I pull out c a x I get a to the power h minus 1 I can write this as c a h minus 1 Into a to the power x, by Induction If I operate on this k times with the difference operator I can show that this becomes c a h minus 1 to the power k times a x. Now, see what you would get, If I used exact differentiation d a to the power x would have given me a x l n a which Is very different from the difference formula, which we calculated here which is very different from you can see that difference operators and differential operators are not really Identical, even for very simple operations most elementary derivatives here also the difference operator Is giving something which Is not really the same as the as the differential operator.

(Refer Slide Time: 27:05)

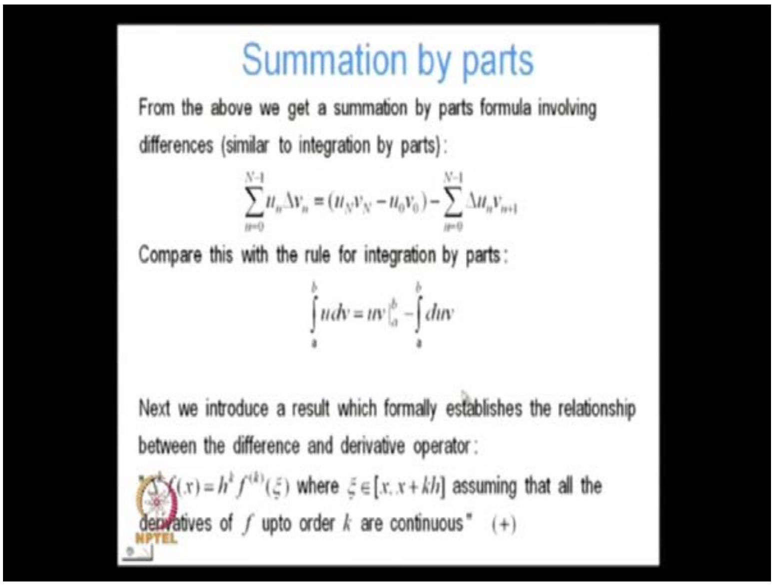


Next we consider the result obtained when a difference operator is applied to the members of a sequence w 0 w 1 w 2 through w n and there should be, there it is, it means w 3 w 4 w n sorry I missed the ellipsis and the result is sum, So, If I operate or with a difference operate on this sequence. If I operate on the first term I am going to get delta w 0 which is going to give me w 1 minus w 0 plus and then I am adding the result and the result is sum then I operate with a difference operator on w 1, I get w 2 minus w 1 operate with a difference operate on w 2 I get w 3 minus w 2 and so on and so forth w n minus w n minus 1 I get this and you can see that If I add them together I am left with w n minus w 0 but, the left hand side is nothing but, sigma n Is equal to 0 n minus 1 del w n why because, this del w 0 which is this term plus del w 1 which is that term and then del w n minus 1 which Is this term which Is this actually should be small n this should be small n which Is that term.

So, the left hand side s actually this and the hand side Is w n minus w 0. Now, we assume that w n is actually given by the product of 2 terms u n v n so each term in the sequence w 0 is given by the product of u n v. So, w 0 is equal to u v w 1 equal to u 1 v 1 v 1 and so on and so forth. So, w n so let us assume that w n Is equal to u n v n then what do I get the left hand side become sigma n equal to n minus 1 delta u n v n and that must be that by definition that by no so that Is equal to w n w n Is equal to u n v n minus w that Is equal to u v but, by definition delta u n v n Is equal to not by definition but, just from my previous result which used the definition.

Let us look at the previous result. That is this delta u n v n is equal to u n delta v n plus delta u n v n plus 1. So, that Is that u n delta v n plus delta u n v n plus 1 So, we can write n equal to sum n equal to n minus 1 delta u n v n Is equal to sigma so I am just taking the summation and then I am replacing this by that u n delta v n plus sigma n equal to n minus 1 delta u n v n plus 1 and I know that this is must be equal to this u n v n minus u, v. Now, I am going to collect terms from the above we get sigma u n delta v n this term is equal to u n v n minus u v and then I bring this term to the minus sigma n equal to n minus 1 delta u n v n plus 1.

(Refer Slide Time: 30:48)



So, I just brought 1 term to the and I get this thing, So, sigma n equal to u n delta v n is equal to u n v n minus u v minus sigma n equal to n minus 1 delta u n v n plus 1 compare this with the Integration by parts. So, as I Integrate u d v within the limits a b then the Integration by parts rule tells me that this is u v evaluated at v n a difference between b and a minus Integral a to b d u v look at this is very similar, So, this is u d v If you think of delta as d, So, u n delta v n u d v is equal to u n v n, So, I have the evaluated at the other end of the Integral minus u v evaluated at the first at the first point In the Interval minus Integral of a to b g u v del u n v n plus 1.

Very similar except for that difference that is v n plus 1 and here I have v, So, If It had to be exactly analogous this should have been v n, this should have been v n but, It Is similar, So, next we Introduce a result which formally establishes the relationship between the difference and the differential operator and the derivative operator, so what is that result. That result say tells me that if I operate on the function f of x k times with the difference operator then I get h to the power k h being the size of my step size times the function f the kth derivative of the function f but, that kth derivative is not evaluated at x It Is evaluated at xi which Is very Important.

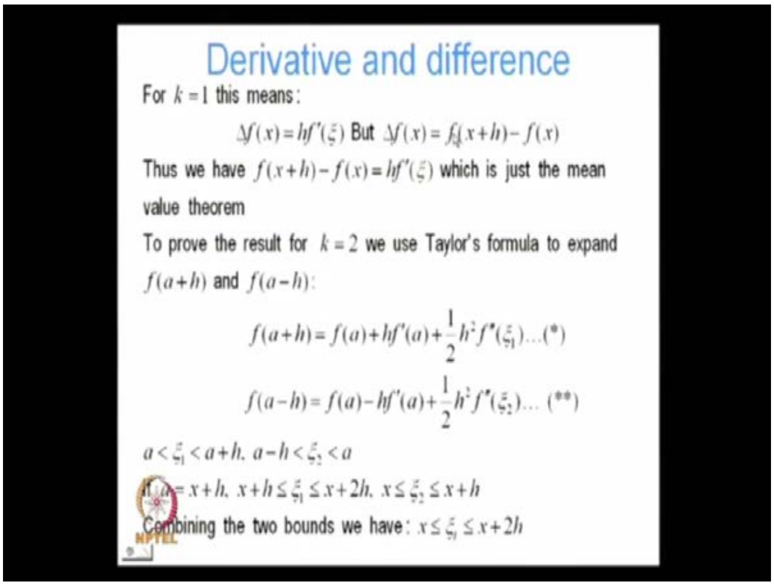
It tells me that If I operate on the function f k times I am going to get something very similar to that kth derivative of f I am going to get something very similar to the kth derivative of f but, the kth derivative of f not evaluated at x the point at which I am taking the difference but, at some other point xi and how what Is the range of xi what Is xi where xi can be can be any point which belongs to the Interval x to x plus k h xi can be any point which belongs to the Interval x plus k h. Of course, this assumes that the kth derivative exists. So, that the function is k derivatives up to order continuous derivatives up to order k.

So, this Is very Important It tells me that well, If you take k differences you are going to get something like the kth derivative but, not at the same point It is evaluated at a point which is shifted which may be shifted from the original point and the extend of the shift is of course, going to depend on the your step size, It is going to depend first on the step size and also In the order of the derivative the higher the order.

If I take the same step size and I look at higher and higher differences so the shift the range of the shift becomes larger because, It Is getting multiplied by k so for the first derivative the point the exact that the derivative Is will be evaluated at will belong to any Interval between x and x plus h, for the second derivative It is can belong to any Interval between x and x plus 2 h. So, as you take more and more derivatives the range becomes larger.

But, If you reduce the step size again the range becomes smaller, So, as you Increase the derivative the range becomes larger as you reduce the step size the range becomes smaller, So, for k equal to 1 this means that delta f x is equal to h of f prime of xi. Let us go back and take a look so delta f x delta f x is equal to h times f prime where the first derivative of f evaluated xi. So, It is not equal to h f x that is very Important.

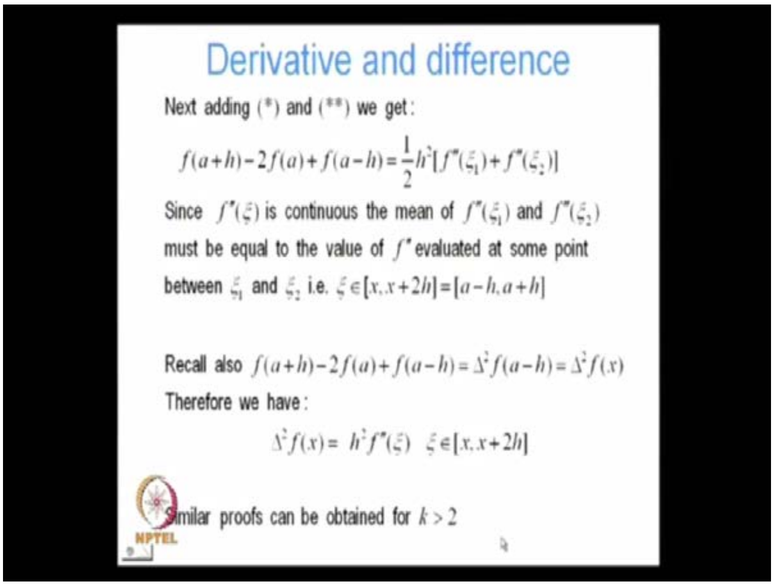
(Refer Slide Time: 35:51)



But, recall the delta f x Is equal to f of x plus h minus f of x therefore, we have f of x plus h minus f of x is equal to h times f prime of xi and I know that xi must belong to the Interval x to x plus h which Is nothing but, my mean value theorem which Is nothing but, my mean value theorem. So, to prove the result for k equal to q k equal to 2 we use Taylor’s formula to expand f of a plus h and f of a minus h and we will define a In terms of x later on but, for the time being let us look at f of a plus h and f of a minus h If I use Taylor’s formula on f of a plus h I get f evaluated at a plus h f prime a plus this term this remained term whether xi 1 belongs to the Interval a and a plus h because, I am evaluating It about xi 1 belongs to the Interval a to a plus h and let us look at f of a minus h It Is equal to this term plus, this term and xi 2 must belong to the Interval a minus h to k a minus h to a.

So, we have xi 1 belonging to a and a plus h and xi 2 belonging to a minus h n a. Now, if I define a Is equal to x plus h then I get xi one Is lesser than or equal to a which Is equal to x plus h lesser than or equal to a plus h which Is equal to x plus two h similarly, xi 2 Is greater than a minus h a minus h Is nothing but, x and xi 2 is less than a which is equal to x plus h. So, this tells me that xi one must lie In this range xi two lies In this range If I combine these 2 bounds what do I get for xi 1 as well as xi 2 so if I so this makes a statement about xi 1 about the bounds In xi 1 this makes a statement about the bounds In xi 2 If I want to make a combined statement on bound; So, I want to calculate bounds on both xi 1 and xi 2, So, I take the lowest value which Is x and the highest value which Is x plus 2 h, So, both xi 1 and xi 2 must lie between x and x plus 2 h.

(Refer Slide Time: 38:39)



So, then that what do we do? We add these two. So, I add these two I add these 2 so I get f a plus h plus f a minus h then I get f a plus f a that gives me 2 f a this term cancels out. So, I have f a plus h plus f a minus h minus 2 f as if I bring that to the left hand side. That is this and that is equal to half h square f double prime xi 1 plus f double xi 2. So, that is equal to this plus this half h square f double prime xi 1 plus half h square f double prime xi 2.

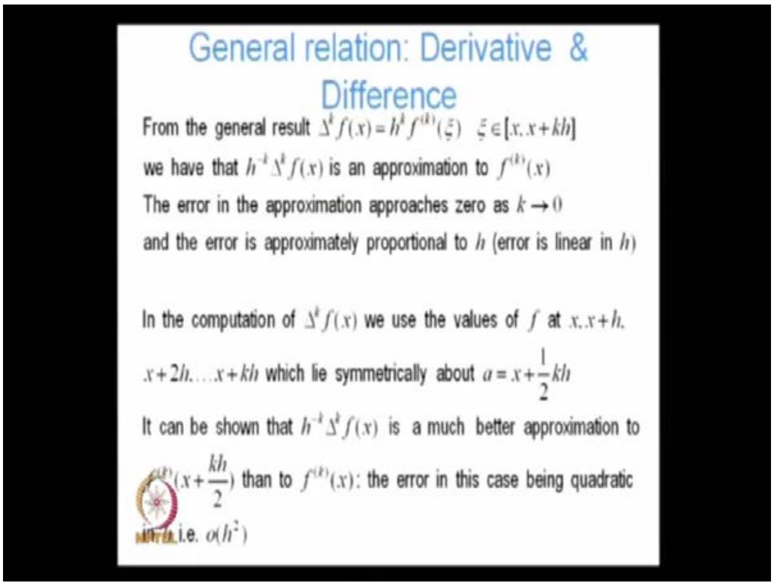
So, I have that term. Now, since f double prime xi is continuous f double prime is a continuous function. So, the mean of f double prime xi 1 and f double prime xi 2 must be equal to the value of f double prime evaluated at some point between xi 1 and xi 2; so f double prime evaluated at some point between xi 1and xi 2 and therefore, xi must belong to the Interval. This as we know that both xi 1 and xi 2 belongs to this interval so the xi So, If the function xi which Is equal to the mean of these 2 the function I mean f double prime xi is the mean of this function that xi must lie also In this interval is that clear that xi must also lie In that Interval.

So, and this Interval again I am expressing In terms of a and h taking keeping in mind that x is equal to a plus h, a is equal to x plus h. So that, gives me a minus h a plus h so xi must belong to a minus h and a plus h recall also that f a plus h minus twice f a plus f a minus h is equal to del square f a minus h. We saw that before and where we saw it that we saw it sometime here, Del square f x minus h is equal to f x plus h minus.

So, f a plus h this Is equal to del square f a minus h and a minus h Is equal to x, So, that Is equal to del square f x, So, what do I get finally, I get del square f x Is equal to h square times f double prime of xi where f double prime of xi is the mean of f double prime xi 1 and f double prime xi 2 and I know that xi belongs to the Interval x to x plus 2 h So, again we are getting a proof for k equal to 2. Recall our original general expression.

Del k f x Is equal to h k f k xi, So, for k equal to 2 also we have shown that this Is true. So, Del square f x is equal to x square f double prime xi where xi again belongs to the Interval x to x plus 2 h remember the Interval, we said was x 2 x plus k h. Now it is exactly x plus 2 h because, we are looking at the second difference. Similar proofs can be obtained for k greater than 2, so k this is the relation between the difference operator and the differential operator. So, It is equal to the difference of the result of the difference operation but, evaluated at a point which Is not exactly at the same point.

(Refer Slide Time: 42:57)



So, from the general result Del k f x is equal to h k f k xi. Xi belongs to x to x plus k h we have that h k h to the power minus k times Del k f x is an approximation to f k of x why is It an approximation; because, it is not evaluated at x. It is evaluated at xi that is if it is not exact because, of this. So, It is just an approximation to f k x and the approximation will be as good as the Interval as the Interval becomes smaller the approximation is going to become better and better, So, If x is very close to x plus k h. Well, your derivative and differential operator are going to be close, If not they are going to be wrong or there will be errors and the error you can see is approximately proportional to the step size because, the error is like k h. So, it is proportional to the step size, So, It Is linear In h it is linear In h.

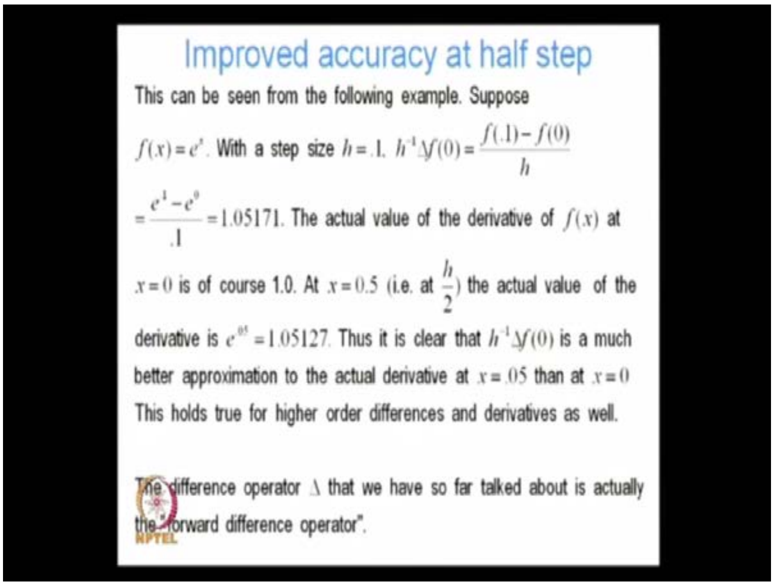
One more thing In the computation of delta k f of x, we use the values of f at x plus h x plus 2 h up to x plus k h and you will note that these values this x plus h x plus 2 h x plus k h lie symmetrically about x plus half k h, So, all the, all the function values that I am evaluating at these different points they are symmetrical about x plus half k h, the points at which I am evaluating the function are symmetrical about x plus half k h; Now, it can be shown that h to the power minus k times delta k f x is actually a much better approximation to f to the derivative to kth derivative evaluated at x plus k h plus 2 that is at the midpoint of my Interval in the midpoint of my Interval rather than at 1 end of the Interval at x, So, this operation this operation which I got after operating on f k times with the difference operator and If I divide that by the step size raise to the power k.

I know that is an approximation to the derivative evaluated at x and the error we know depends on this step size but, actually It is much closer that it is less of an approximation If I evaluate the If I look at the point x plus k h by 2, It Is less of an approximation If I look at the point x plus k h by 2 rather than looking at the point x.

So, I have the center of the Interval, If at the center of the Interval the difference the kth difference is much closer to the derivative than at the end of the Interval the derivative being evaluated at the center point, derivative being evaluated at the end. So, again let me repeat so If I look at the kth difference and I look at the derivative evaluated at the end of the Interval and I look at the derivative evaluated at the midpoint of the Interval the kth difference is much closer to the exact derivative of the function evaluated at the midpoint of the Interval rather than at 1 end of the Interval.

So, that is because, It turns out that the error in this case Instead of being linear with the step size is actually quadratic with the step size If I am evaluating it at the midpoint. So, the midpoint gives the difference is much closer to the derivative at the midpoint than at 1 end of the Interval.

(Refer Slide Time: 47:11)



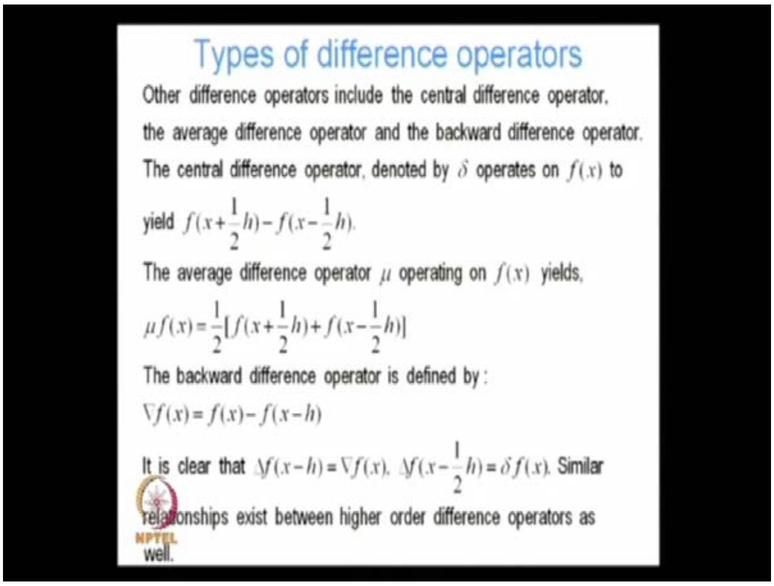
So, we let us look at little example and show what whether this is true. So, In the example, we look at is the exponential function f of x is equal to E to the power x and suppose, we are going to evaluate differences with the step size of h Is equal to point 1 then if I look at delta f, delta f is going to be f evaluated at plus h minus f evaluated at and h Is point 1. So, f evaluated at point 1 minus f evaluated at divided by my step size h which is point 1; So, that is going to be E to the power point 1 minus E to the power divided by point 1 which is going to be 1 point, 5, 7,1. How, closes then we want to look at how close this value is to the derivative of f evaluated at well, the derivative of f evaluated at is 1 derivative of E to power x is x.

So, the derivative of f evaluated at is 1, So, it is quite different I mean, It is not quite its 5 percent all; So, It is 1 point 5, 1, 7, 1 and here It is 1 but, Instead of evaluating It at if I evaluate the derivative at this should point 5, that is at h by 2 my step size is h; So, evaluated at midpoint of the step size midpoint of the step size, So, remember for the linear for the first derivative the Interval size is x plus x 2 x plus h, So, I am evaluating it at the midpoint of the Interval my first point is my end point is point 1 I am evaluating it at half, that half, that at the midpoint of that Interval, So, I want evaluate It at point 5.

So, If I evaluate E to the power point 5, If I evaluate the derivative of E to the power x which is the same as E to the power x, if I have if I evaluated It at point 5, I get 1 point 1 5, 1, 2, 7 which is actually much closer to the difference than the derivative at x is equal to thus. It is clear that h minus h the power minus 1 delta f Is a much better approximation to the actual derivative at x is equal to point 5 than at x is equal to it is much better approximation the midpoint of the Interval rather than at the end point.

This holds true for we just showed it for the first difference on the first derivative It holds true for higher order, differences and higher order derivatives as well. Let us take a step back now and the difference operator that we have talked about the delta operator that we have talked about is actually 1 particular difference operator, there are many other difference operators this delta operator. This delta difference operator is known as the forward difference operator. Why is it called the forward difference operator because, delta operating on f at x is equal to f x plus h minus f of x. So, I have to look forward by 1 step h and then I take the difference at f of x to calculate that difference operator but, there are other difference operators as well, for Instance we have things like the backward difference operator, the central difference operator the average difference operator and things like and all turns out that all these operators are related to each other so let us take a quick look.

(Refer Slide Time: 51:14)



So, other difference operators include the central difference operator the average difference operator the backward difference this for Instance it we are going to denote by delta. The central the y the central difference operator, So, that is f x plus half h, So, delta operating on f of x is equal to f x plus half h minus f x minus half h then there is a average difference operator mu which operating on f of x yields half f of x plus h plus half f of x minus h, So. at x I look at I take half a step forward I take half a step back evaluate the functions at those locations, I take the average of that and I said that is the result of my average difference operator.

The backward difference operator on the other hand is f x minus f of x minus h so which I denoted by this nabla sign f of x is equal to f of x minus h. So, I am looking back now Instead of looking forward I am looking at x minus h Instead of looking at x plus h I am looking at x minus h and even by just cursory examination you can see that these difference operators are related to each other for Instance the forward difference operator operating on f evaluated at x minus h, is equal to the backward difference operator operating on x. And similarly, you can relate the central difference operator to the similar relationships exist between higher order difference operators as well, So, we will continue with our discussion on difference operators in the next lecture.

Thank you.