Numerical Methods in Civil Engineering
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Lecture - 26
Series Solutions for Elliptic PDE's and Introduction to Differential Operators

In lecture 26 if our series on Numerical Methods in Civil Engineering, we will wind up our discussion of analytical solutions for partial differential equations by talking about series solutions for elliptic to partial differential equations, and then move on to numerical methods for solving them we will talk about differential operators.

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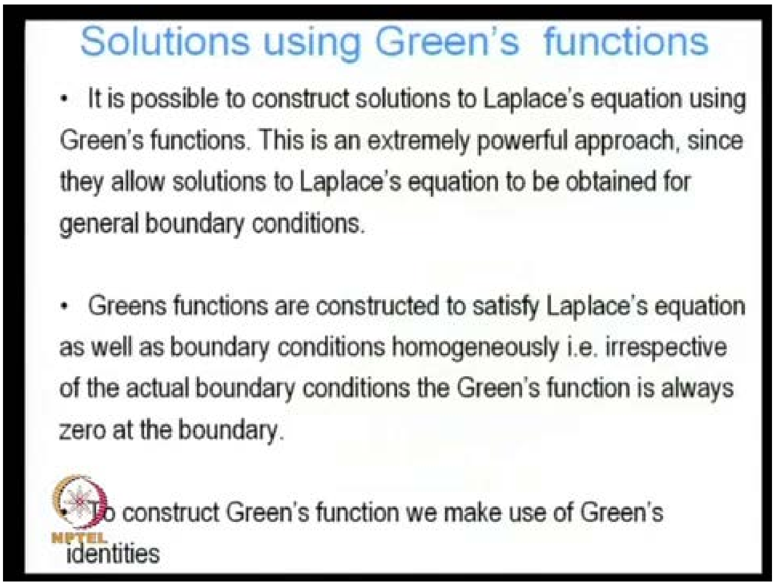

Before moving on to see before actually talking about series solutions I want to recapitulate little bit, I just want to talk briefly about something we discuss last time that is the method of green's function for solving partial differential equations, why do I want to do that. Because, this method is crucial to our development of series solutions, we saw that it is possible to construct Laplace's equation, solutions to Laplace's equation using green functions. And what are these green functions, these green functions they satisfy Laplace's equation, and they satisfy the boundary conditions homogeneously.

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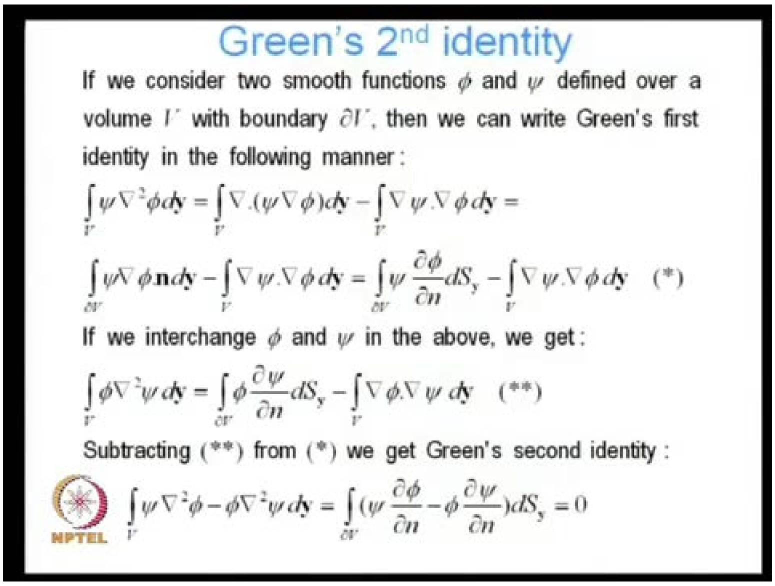

And to construct green's functions we used green's identities, we talked about the first second and the third identity of green. We define smooth functions phi and psi defined over a volume V with boundary del V in which case we get the first identity psi laplacian of phi is equal to psi del phi del n over del v and minus grad psi dotted with grad phi over v. So, that was our first identity then this was our second identity which we got basically from the first identity by interchanging phi and psi and then subtracting that from this first identity. And this was our second identity which was psi laplacian of phi minus phi laplacian of psi is equal to psi del phi del n minus phi del psi del n.

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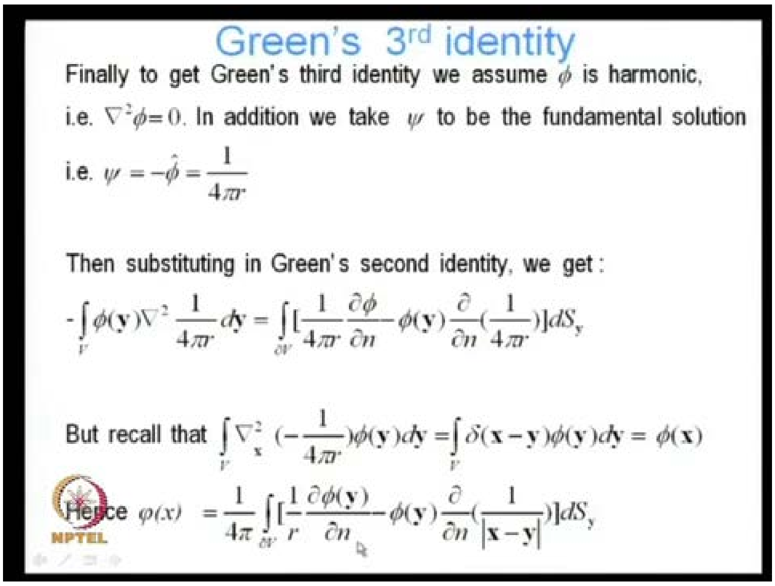

And then we looked at green’s 3'rd identity in order to get green’s 3'rd identity we assume that the function phi is harmonic that is laplacian of phi is equal to 0. In addition we assume that psi is the fundamental solution to Laplace's equation, and we recall that their fundamental solution is given by psi is equal to 1 by 4 pi r in spherical coordinates.

Where r is the position from the reference from the origin, r is the distance from the origin not the position actually here, r is the distance from the origin. And we saw that this solution this phi 1 by 4 pi r, satisfies Laplace's equation everywhere in the domain except at the origin, where it becomes undefined it becomes and actually becomes and that lead to our discussion of the Dirac delta function that lead to the Dirac delta function.

So, we substituted that in green’s second identity and use the fact that laplacian of minus 1 by 4 pi r is actually equal to the Dirac delta function delta x minus y operating on phi y equal to phi x hence we got phi x is equal to this. So, then we said that this tells me that if I know the value if phi satisfies Laplace's equation, and I know both the value of the function as well as it is gradient at the boundary, then I can use this equation to find out my solution to Laplace's equation. So, using these boundary conditions see this integral is over del v which is the boundary, so provided I know this and I know that at the boundary I can solve for phi x.

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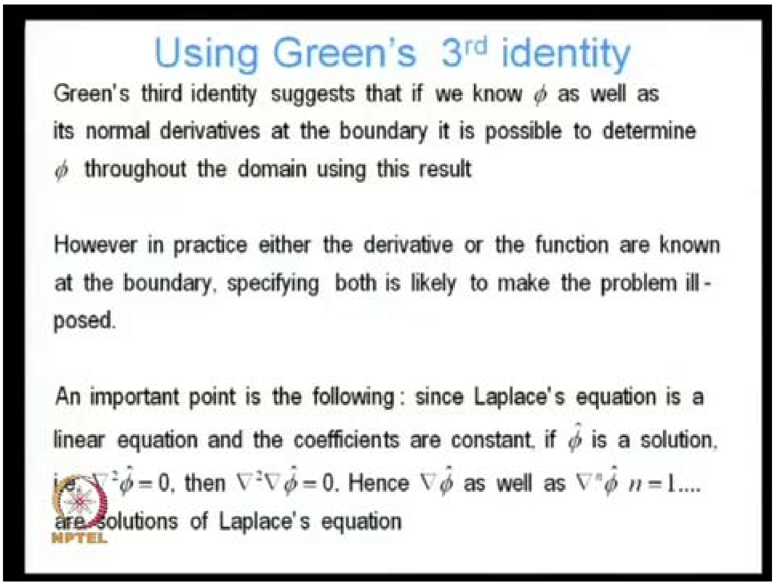

Then I went on to say that it is almost never possible to know both these values at the boundary because, otherwise the problem becomes ill-posed, and the solution to that was to look at green’s functions solution.

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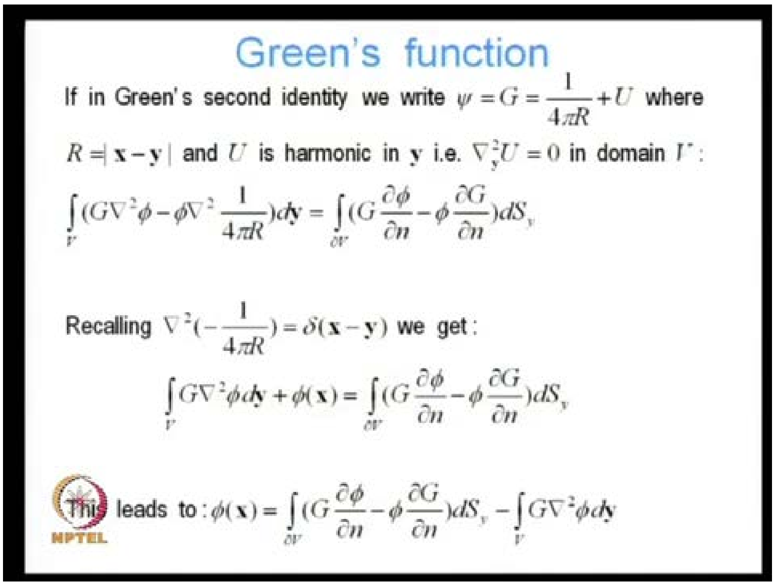

And say that psi is not actually equal to 1 by 4 pi r, but it is equal to 1 by 4 pi r plus another function U, where U is the is also harmonic laplacian of U is equal to 0 in domain V. In which case we got the following phi x is equal to G del phi del n minus phi del G del n minus d laplacian of phi d y, and if phi satisfies Laplace's equation this part becomes 0 by definition, and we have phi of x is equal to G del phi del n minus phi del G del n. And now if we construct the green’s function to be G is if we construct it in such a way the G is equal to 0 everywhere in the boundary.

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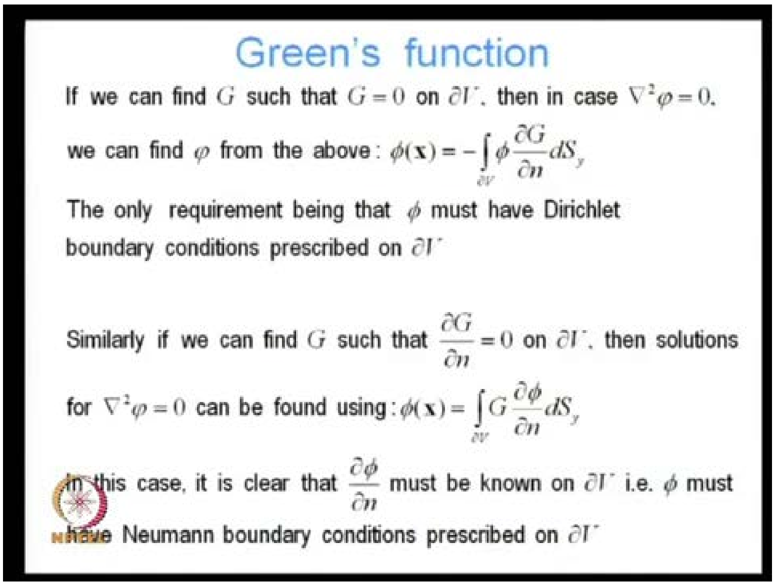

Then we get phi x is equal to minus integral del v phi del G del n d S y, so the only requirement that only thing that we need to know is that we must know phi all over the boundary del G del n is known green’s g is a known function. So, del G del n is known if we can find out how phi is distributed on the boundary, then we can always find phi, so this gives us a way for finding phi when we have dirichlet boundary conditions.

On the other hand if G satisfies the condition del G del n equal to 0 on del V then solutions can be found using this expression phi x is equal to G del phi del n. So, in this case we have to know del phi del n on the boundary that is we need Neumann boundary conditions on phi to be prescribed on the boundary, I think that is where we stopped last time we talked about green’s function and that is where we stopped.

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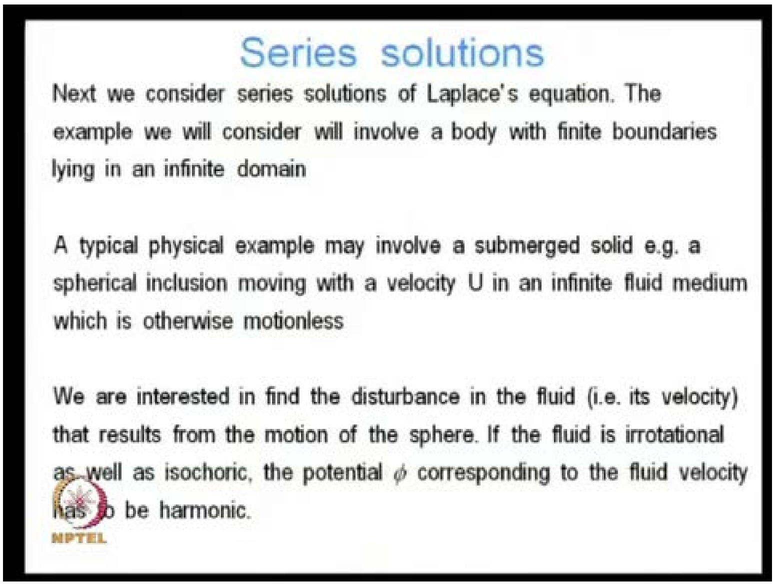

Next we want to consider series solutions of Laplace's equation, the example we will consider will involve our body with finite boundaries lying in an infinite domain. So, you can think of a body or solid lying with in an infinite fluid for instance a spherical inclusion, a spherical solid moving with a certain velocity U in an infinite fluid medium.

Which is otherwise motionless think of simple example would be a submarine, a submarine flowing through the ocean the ocean is still, the submarine moving through or any just the simplest example is a sphere moving through a very large water body a water tank, suppose where the boundaries are, so far away that we can assume that the water tank is infinite.

And what we are interested in knowing is that this sphere is moving with a velocity of U, but what is the effect of that motion of this sphere on the fluid. So, we are interested in finding the disturbance in the fluid that is it is velocity that results from the motion of this sphere. And if the fluid is irrotational as well as isochoric, irrotational meaning curl of the velocity is equal to 0, and isochoric means divergence of the velocity is equal to 0, then there is a potential and by taking the gradient of the potential we can find the fluid velocity, and the potential satisfies Laplace's equation, so the potential is harmonic. So, again in this case we have we have back to finding the solution of Laplace's equation because, once we find phi we can find the velocity, where v is the gradient the velocity of the fluid is the gradient of phi.

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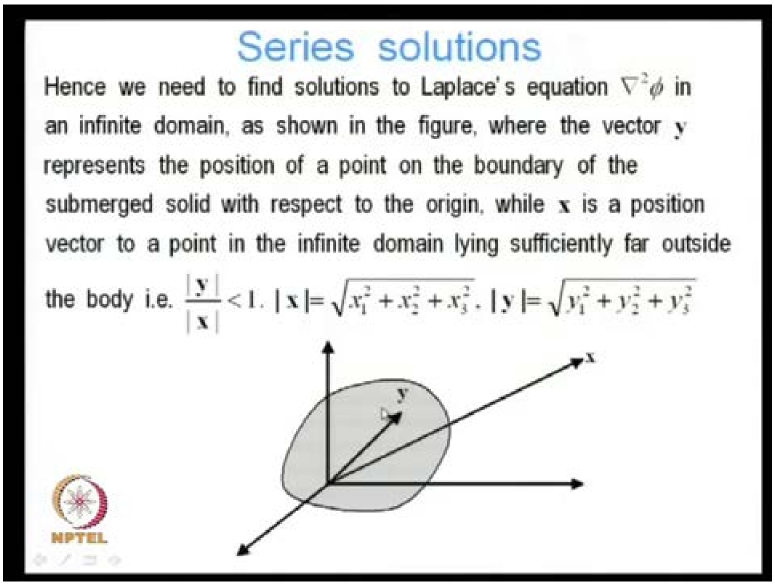

So, hence we need to find solutions to Laplace's equation, laplacian of phi in an infinite domain. So, this is my infinite domain here is my little solid body, and here is my origin of my spherical coordinate system, and y is the vector from the origin to any point on the surface of my solid body. And x is a point far out in the infinite domain, where I want to find my solution, where I want to find my velocities.

So, the vector y represents the position of a point on the boundary of the submerge solid with respect to the origin, while x is a position vector to a point in the infinite domain lying sufficiently far outside the body. So, this mod of y by mod of x is less than 1 that is a requirement, and mod of x is of course, equal to the length of this vector and mod of y is the length of that vector.

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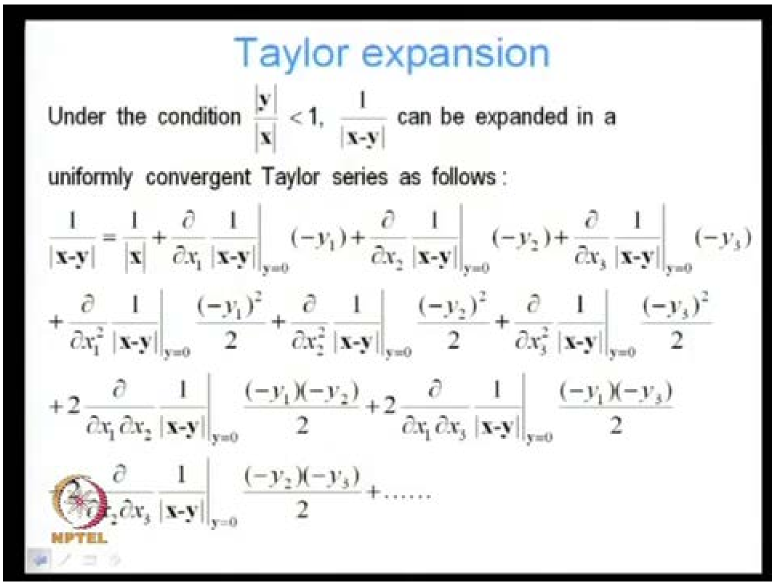

So, under the condition mod of y by mod of x is less than 1, we can expand we call our fundamental solution is given in terms of 1 by r. So, 1 by if I can expand 1 by mod of x minus y using a Taylor series, and if this condition is satisfied the Taylor series is uniformly convergent. So, mod of 1 by mod of x minus y that is I am expanding about x and my displacement from x is given by y, so I am expanding about x. So, 1 by mod of x minus y is equal to the value of the function, when y is equal to 0 plus the value of the derivatives at y equal to 0 times the perturbation times of change in y.

So, partial of this with respect to x 1 partial of 1 by mod of x minus y with respect to x 1 times the change in the first coordinate of y first minus y 1. Similarly, partial with respect to x 2 change in the second coordinate of y that is y 2 partial with respect to x 3 change in the third coordinate of y 3. So, that is the first term in my Taylor series, then I have the subsequent terms which are which involve the second derivatives del del x 1 squared del del x 2 squared del del x 3 squared. And then the mixed derivatives del del x 1 del x 2 del del x 1 del x 3 del del x 2 del x 3 and so on and so forth.

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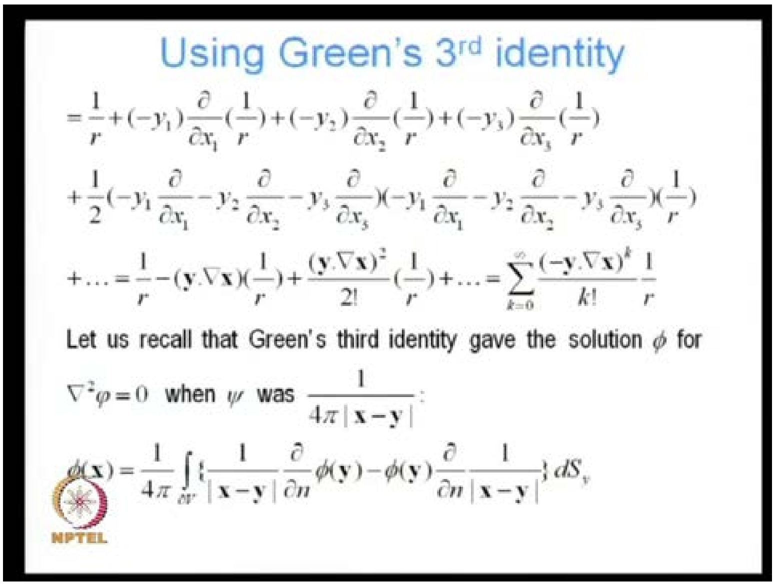

And then, so this actually becomes if I denote 1 by mod x by r, so this becomes 1 by r plus minus 1 by x 1 del del x 1 1 by r, you can see why that would be 1 by r because, y is equal to 0. So, 1 by mod of x minus y becomes 1 by mod x which is equal to 1 by r, so that we get something like this 1 by r minus y 1 del del x 1 1 by r plus minus y 2 del del x 2 1 by r minus y 3 this thing, and then the second derivative.

Second derivatives you can see I can write it in a convenient fashion by taking the product of this the coefficient of the operator as it acted in the first derivate, which was minus y 1 del del x 1 operating on 1 by r minus y 2 del del x 2 operating on 1 by r minus y 3 del del x 3 1 by r. So, it is as if I pull out 1 by r and I am looking at the operator minus y 1 del del x 1 minus y 2 del del x 2 minus y 3 del del x 3 operating on 1 by r.

And you can see that these second derivatives are actually represented by this operator operating on this operator, and then the result operating on 1 by r it can be written like that. So, if I represent this operator as y dotted with grad x, you can see this is y dotted with grad x because, it is y 1 actually it is minus y dotted with grad x because, it is minus y 1 del del x 1 minus y 2. So, grad x is del del x 1 i plus del del x 2 j plus del del x 3 k. So, this is y dotted with grad x with a negative sign.

So, this equation can be written compactly as 1 by r minus y dotted with grad x 1 by r minus y dotted with grad x operating 2 times. So, y dotted with grad x operating on y dotted with grad x, we are writing that as y dotted with grad x squared by factorial 2 operating on 1 by r plus higher order terms. So, I can write it in index notation as a summation as minus y dotted with grad x to the power k divided by factorial k operating on 1 by r, so that is my Taylor series expansion of 1 by mod of x minus y.

Now, let us go back to green’s 3'rd identity and remember that green’s 3'rd identity give us the solution for phi, when laplacian of phi was equal to 0 and when psi was 1 by 4 pi mod of x minus y. So, phi of x in that case we found was this, this we have already seen the green’s 3'rd identity from green’s 3'rd identity, what we are going to do, we are going to replace that series expansion for 1 by mod of x minus y in green’s 3'rd identity, and then we are going to integrate with respect to y.

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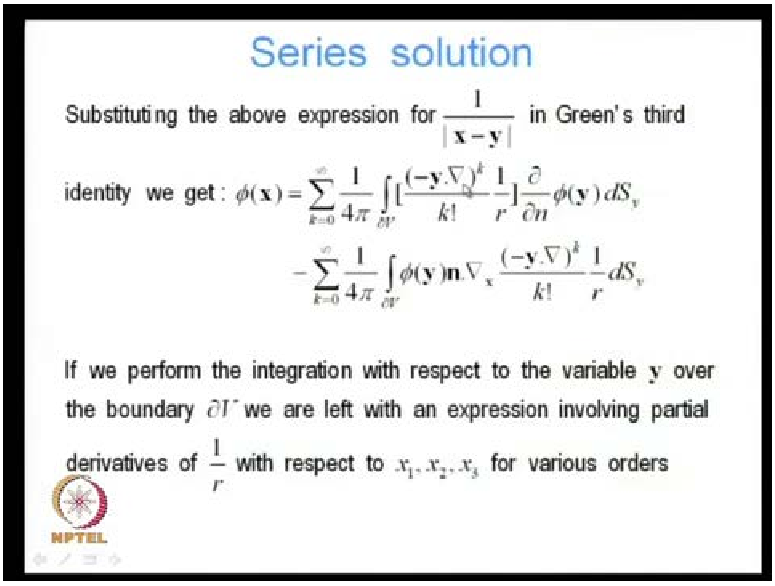

So, substituting the above expression for 1 by mod of x minus y in green’s 3'rd identity we get the following phi x is equal to sigma k equal to 0 to infinity 1 by 4 pi integral over the boundary minus y grad I dropped the x because, that sort of understood to the power k operating on factorial k 1 by r I could have put an x here. So, this x is actually this is not really x which is the gradient with respect to x.

So, this is gradient with subscript x is that clear I hope I should have probably being more careful with that, but I meant that. So, gradient subscript x and then finally, I have dropped the x, and denoted that just by the gradient, so y dotted with, so I get this sort of an expression. And then I integrate this is a function of y that is a function of y, so I integrate this I perform the integration over y, if we perform the integration with respect to the variable y over the boundary del V we are left with an expression involving partial derivatives of 1 by r with respect to x this grad is with respect to x. So, partial derivative of 1 by r with respect to x 1, x 2, x 3 for various orders is that clear I think that should be fine.

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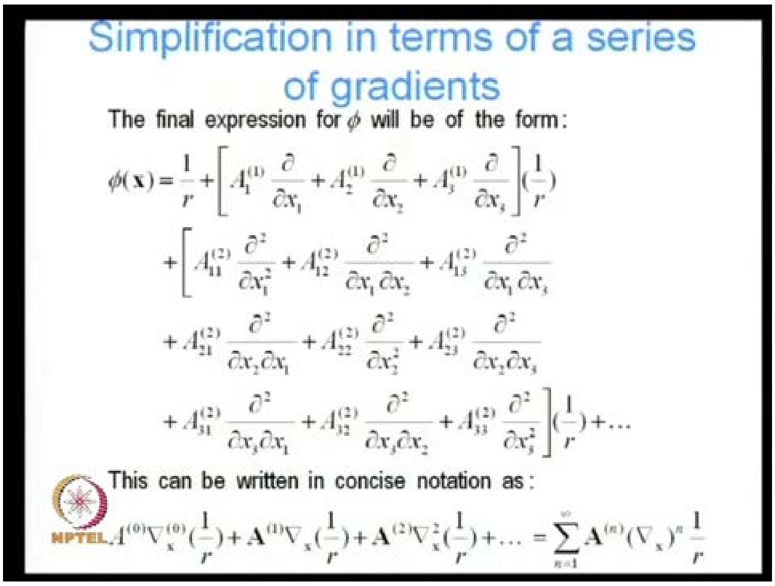

So, the final expression for phi would be of this form, so after we integrate out the y dependence. So, integrate out all the terms which depend on y in this expression, we get an expression for phi like this, so the we get a first term which is 1 by r plus this derivative terms del del x 1 del del x 2 del del x 3, and these coefficient terms. The coefficient terms involve the integral with respect to y, whatever I get after integrating with respect to y those are my coefficient terms.

So, those coefficient terms appeared here, and I have 1 by r here similarly I have these coefficient terms here, and I have operating on 1 by r. So, writing this in concise notation we have A 0 grad x 0 1 by r plus A 1 grad x 1 by r plus A 2 grad x squared 1 by r and so on and so forth. Where these this is my A 1, this is an A 1 vector, A 1 vector whose components are A 1 1 A 2 1 A 3 1, A 2 is no longer a vector it is a tensor and it is components are A 1 1 2 A 1 2 2 A 1 3 2 A 2 1 2 A 2 2 2 A 2 3 2. So, this becomes a tensor, this is a vector that is a scalar.

And all of them act on these gradients of various orders of 1 by r, so this I can together I can submit up as sigma n is equal to 1 to the power infinity A n grad x to the grad x n 1 by r. So, that is my final series solution, and this series solution is in terms of what are known as spherical harmonics they are known as spherical harmonics.

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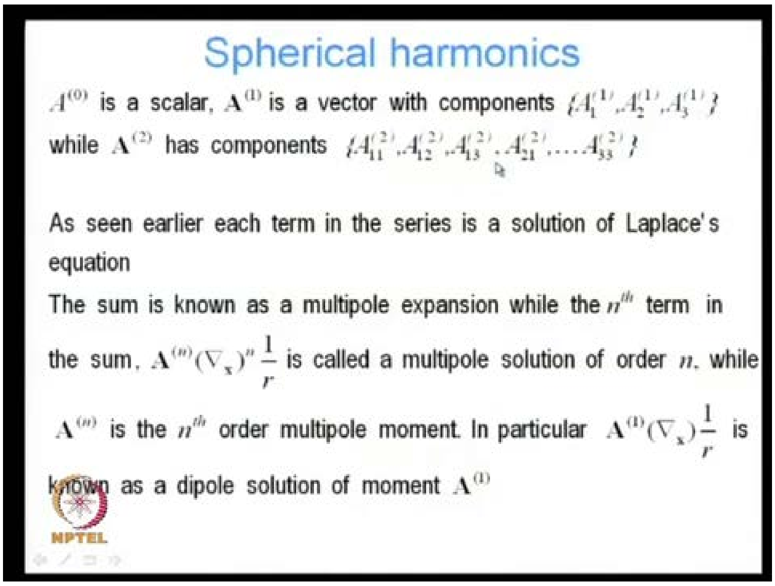

As I said A 0 is a scalar, A 1 is a vector with components like this A 2 has components like this, but let us go back to this term. So, each of these coefficients whether they be scalars, vectors or tensors they are operating on this grad x with raised to some power operating 1 by r. Now, each of these terms we saw last time each of these terms whether it be grad x 1 by r or grad x squared 1 by r or grad x 3 1 by r each of them are solutions of Laplace's equation we saw that last time.

Because, laplacian of x has got constant coefficients, so laplacian of grad of something is also going to satisfy Laplace's equation. And since grad of something satisfies Laplace's equation, second gradient is also going to satisfy, so each of these terms grad x and 1 by r they are solutions of Laplace's equation. And these are known as spherical harmonics of various order x to the power 0, x to the power grad x 1 by r grad x squared 1 by r grad x cubed 1 by r these are known as spherical harmonics.

So, we can think of that as these are the basis for my solution for Laplace's equation phi in terms of some coefficient, some coefficient times some basis. And these basis these are the spherical harmonics are like my basis for the solution of Laplace's equation, like we looked at the Eigen function approach, we looked at the Eigen functions were the basis. So, these in my series solution, these spherical harmonics are my basis for the solution of Laplace's equation.

So, as seen earlier each term in the series is a solution of Laplace's equation, and this sum is known as the multipole expansion, this sum is known as the multipole expansion. And n'th term in the sum A n grad x raised to the power n 1 by r is called a multipole solution of order n, A n is known as the n'th order multipole movement. In particular A 1 grad x 1 by r when n is equal to 1 is known as the dipole solution, dipole solution with moment A 1. So, we got a solution for spherical harmonics as a solution for Laplace's equation, series solution in terms of spherical harmonics. And we saw that each of those spherical harmonics is actually a solution of Laplace's equation by itself.

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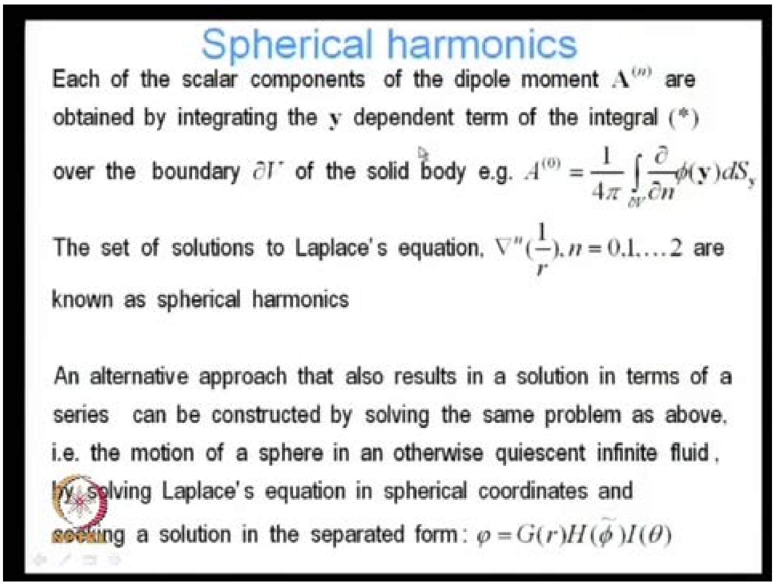

So, each of the scalar components this is of course, I have said before, but just to reiterate each of the scalar components of the dipole moment A n are obtained by integrating the y dependent terms of the integral of that integral of this integral integrating the y dependent terms, we get those coefficients. So, get those coefficients over the boundary del V of the solid body for example, l A 0 can be obtained like this.

And the set of solution of Laplace's equation given by that are known as spherical harmonics. So, that is one approach to the series solution of Laplace's equation, and alternative approach uses the method of separation of variables, it uses the method of separation of variables with which I am sure all of you are familiar.

So, it assumes that the solution of Laplace's equation in spherical coordinates, it is going to depend on all 3 spherical coordinates, it is going to depend on r it is going to depend on phi, and it is going to depend I have replaced phi with phi tilde this to distinguish from the solution which is also phi. So, it is going to depend on r it is going to depend on phi tilde it is going to depend on theta, and if I am assuming that if the solution can be written out in separable form.

So, the functional dependence on r is can be written separately from the functional dependence on phi tilde and as well as the functional dependence on theta. So, it can be written out in separable form, and then we are going to solve the same problem which is the same problem meaning the motion of a sphere in an infinite fluid. So, if we do that again we have to solve Laplace's equation, and then I am going to make a little assumption here, I am going to assume that the solution does not have theta dependence.

So, I am going to assume you can solve most general way assuming that the solution depends on r t tilde as well as theta. But, for the time being I am going to show I am going to solve the problem assuming that there is no theta dependence, so the solution phi depends only on the radial distance r and on phi tilde. And we will see that in that case we have in order to obtain the solution, we have to solve the Legendre's differential equation.

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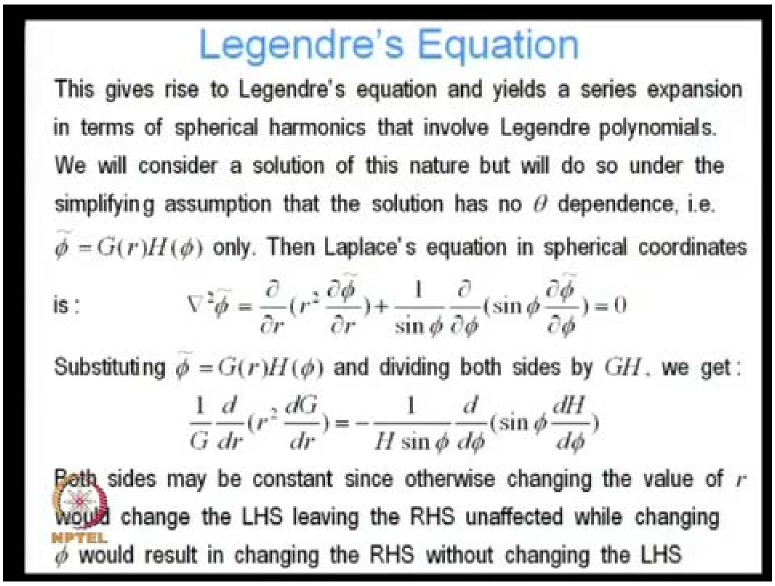

This gives rise to Legendre's equation, and yields a series expansion in terms of that involves Legendre's polynomials. Now, we are going to get a series expansion in terms of Legendre's polynomial, so this part you just ignore in terms of Legendre's polynomials. So, we will consider a solution of this nature, but we will do, so under the simplifying assumption that the solution has no theta dependence that is phi tilde is equal to now I have mess things up wide badly the notation.

Because, my solution now is actually I am calling phi tilde, and the angular dependence is phi and the radial dependence is r. So, I want to phi. So, laplacian of phi tilde is equal to 0 sorry for the confusion in the rotation, so then Laplace's equation in spherical coordinates is given by that which we have seen before, only difference is that now I have removed the theta dependence I only have the dependence on phi.

And then I substitute phi tilde is equal to G of r H of phi if I substitute that there and I divide both sides by G H G H I get this equation. And you can see this becomes an ordinary differential equation because, this involves only derivative with this r has not no phi dependence, while on this side there is no r dependence. So, this becomes an ordinary differential equations, and in this case both sides must be constant because, if we change r, we will change the left hand side while the hand side would not change that is impossible. So, what does that mean; that mean both sides must be constant, so basically now we have to solve these two differential equations equate them to a certain constant and solve them.

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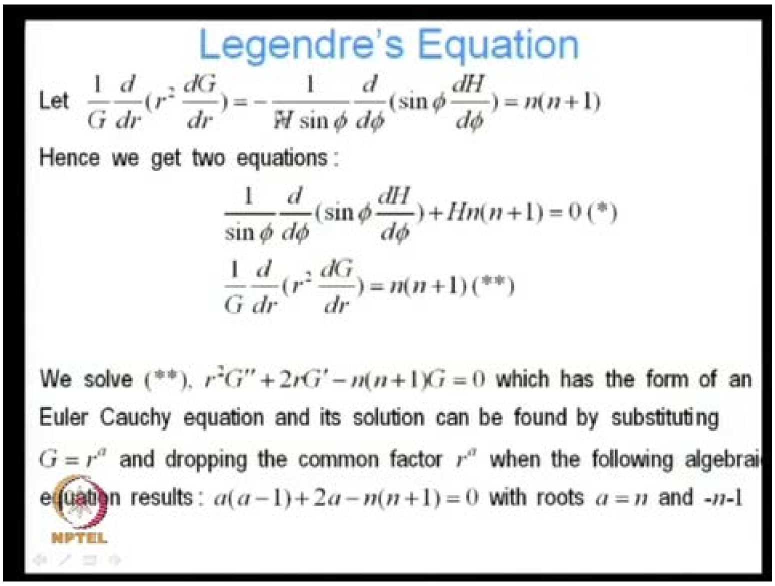

So, we did not let us suppose that 1 by G d d r r squared d G d r is equal to n n plus 1 and 1 minus 1 by H sin phi d d phi sin phi d H d phi, which is basically these two equations both of them are equal to n times n plus 1 why did I use n into n plus 1. Because, that is the standard form of Legendre's equation, we will see this very soon, so we get two equations 1 by sin phi d d phi sin phi d H d phi plus H n n plus 1 is equal to 0, one equation.

And the other equation is 1 by g d d r r squared d G d r is equal to n n plus 1, so we solve the second equation first, the second equation has a very nice form you can see why because, you can see the first term involves r squared and it has a derivative with second derivative with respect to G the second term involves r and has involves first derivative with respect to G, while the third term does not involve any r it just has G.

So, the solution this is known as an Euler Cauchy differential equation, so standard form and the solution to this equation can be found is in terms of r to some powered a, the solution if the differential equation is of this form it is a standard form. And what is that standard form, that standard form is known as the Euler Cauchy equation, and for the Euler Cauchy equation the solution is given by r raised to some power.

You can see if you raise if that is; obviously, going to satisfy that equation r a if I take the second derivative of that, that is going to give me r a minus 2 r a minus 2 r squared that is going to give me r a. So, it will be some coefficient times r a second term again is going to give me r a minus 1 times r, so that is going to be r a time some coefficient in terms of a, the third term is just going to be r a. So, every term is going to have r a I pull out r a and then I have a algebraic equation in terms of the coefficient a.

In this case since a second order it is a quadratic, it is a quadratic equation in a I solve that quadratic equation I get the roots of a in terms of n , but in this case the roots come on to be n and minus n minus 1.

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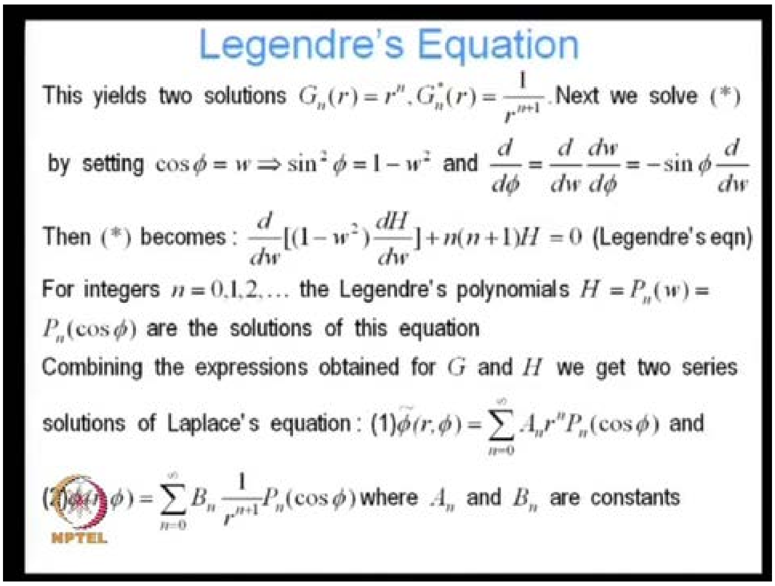

So, what does that tell me that tells me that in this case the solution G there are 2 solutions 1 G involves r to the power n, the other solution involves r to the power n plus 1 r to the power a is the solution. So, a was n and minus n minus 1, so my G is there are 2 solutions 1 solution is G is G n equal to r to the power n and the other part which I denote by G star which is equal to 1 by r n plus 1.

So, there we have solved the first the second equation, and we have solved that equation and found that the solution involves powers of n positive as well as negative powers of n. And now we want to solve this second equation you can see the terms like sin phi d phi so; obviously, this is the candidate for transformation of variables, this is the candidate for transformation of variables.

So, what is the transformation we do well we set cos phi is equal to w, then sin squared phi is equal to 1 minus w squared and d d phi is can be written as d d w d w d phi which is equal to d w d phi. So, minus sin phi d d w because, cos phi is equal to w, so d w d phi is equal to minus sin phi, so minus sin phi d d w, so in that then if I substitute that in this equation, if I substitute that in this equation I finally, get an equation like this in terms of whether independent variable with the derivative is with respect to w.

So, d d w 1 minus y squared d h d w plus n times n plus 1 H is equal to 0 which is the famous Legendre's equation. And for integers n is equal to 0 1 2 this Legendre's equation has solution in terms of the Legendre's polynomials P n w, P n w is again cos phi, so P n cos phi are the solution of this equation.

So, if you remember these Legendre's polynomials they also are orthogonal to each other and they form a basis. So, they are orthogonal to each other, so combining, so they are lot the Legendre's polynomials are the probably one of most useful expansion, expansion in terms of Legendre's polynomials. In the very interesting properties, but at this point we do not want to go into that I just want to tell you that eventually we are going to solve Legendre's equation, solution of Legendre's equation is in terms of Legendre's polynomials.

And using those Legendre's polynomials we are going to construct our solution of Laplace's equation, construct our series solution for Laplace's equation like we did with using spherical harmonics. But, now we are instead of using spherical harmonics as our basis for our series solution, we are using the Legendre's polynomials, we are going to use the we are going to construct the series solution with the Legendre's polynomials as our basis.

So, combining these expressions for G and H we get two series solutions for Legendre's equation, the first series solution involves the positive powers of r, we saw that G n has got two solutions, the first solution was the positive powers of r Legendre's polynomials, the second solution involves the negative powers of r Legendre's polynomials.

And A n and B n are constants you can ((Refer Time: 34:12)) by looking at this equation, you can see that this gives the solution in two entirely different domains, two entirely different domains. Why because, this gives me the solution outside my rigid body, outside my sphere as r goes to infinity, this is going to go to 0 this cannot be the solution inside this sphere.

Because, when r goes to 0 this is going to blow up, so this is going to give me the solution outside the sphere that is going to give me the solution A n r n P n cos phi. This part is going to give me solution inside the sphere, even when r goes to 0 this part remains well behaved. So, that is going to give me the solution inside this sphere inside my rigid sphere are my inclusion whatever it be, and this part this B n 1 by r n plus 1 P n cos phi is going to give me the solution outside this sphere.

Because, this solution is well behaved when it when we go to infinity, when r goes to infinity this part of the solution is well behaved, when r goes to 0 this part of the solution is well behaved, so this gives the solution in basically two different parts of the domain.

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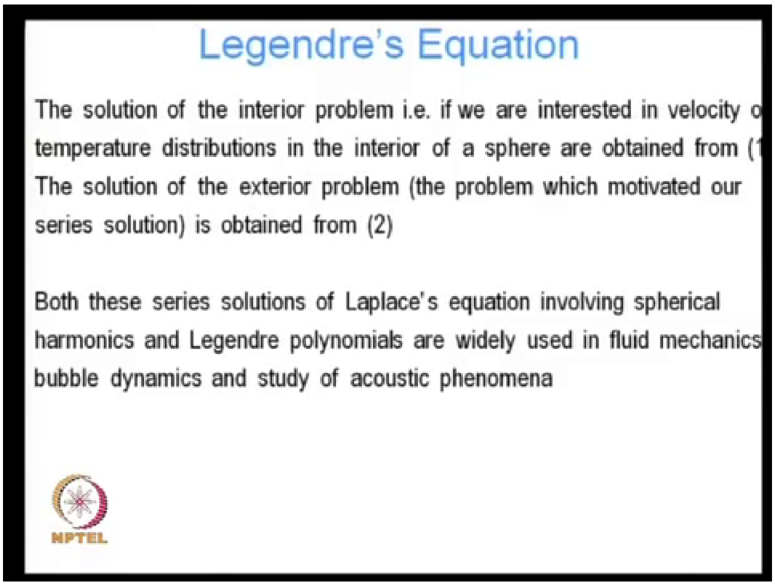

The solution of the interior problem that is if we are interested in velocity or temperature, whatever if we are solving a different problem we were interested in temperature. In the interior of this sphere are obtained from 1 from the solution 1, the solution of the exterior problem, the problem which motivated our series solution. Because, we are interested in the velocity in the fluid due to the motion of that body in that infinite domain is obtained from the second part second solution.

So, both this series solutions of Laplace's equation involving spherical harmonics as well as Legendre's polynomials are widely used in fluid mechanics, bubble dynamics and study of acoustic phenomena, there may be many other applications, but I am familiar with these applications have seen this when we use that. So, I am talking about this, but there are I am sure there are many, many other applications of these series solution of Legendre's of equations is that clear.

So, that brings us to an end of our discussion of analytical methods for partial differential equations, just to recap we looked at the 3 canonical forms of partial second order partial differential equations, with constants. And then we looked at the solution elliptic we looked at hyperbolic solutions, we looked at d'alembert's solution, we looked at the solution in terms of Eigen functions, the Eigen function solutions. Then we looked at the parabolic solution, the parabolic equation the solution for the parabolic equation in terms of again Eigen functions.

And we also looked at the solution for the parabolic equation using transforms using Laplace's transforms. And finally, we looked at elliptic equations, Laplace's equation, we looked at the fundamental solution, we looked at how we can solve that equation using green's function,which is a very powerful technique. Because, it can used for very, very different boundary conditions, various boundary conditions and then finally, we looked at series solutions for Laplace's equation, using series both in terms of spherical harmonics as well as Legendre's polynomials. So, these are some powerful analytical techniques for solving second order partial differential equations.

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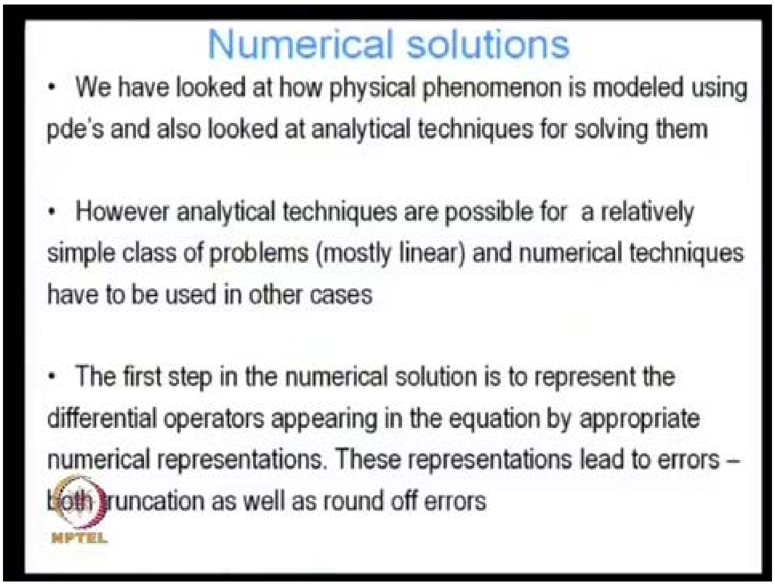

Linear partial differential equations, very important to remember that, so we have looked at how physical phenomenal is modeled, using partial differential equations. And we have also looked at analytical techniques for solving them; however, analytical techniques are possible for relatively limited class of problems, for instance we have seen mostly linear problems, and numerical techniques have to be used in other cases.

The first step in the numerical operation is to numerical solution is to represent all those differential operators, which we saw in our equations del del x squared del del t all those things we have to represent them in terms of difference operators. This we have to represent them in terms of appropriate numerical representations, and inevitably those representations lead to errors both truncation errors as well as round off errors. So, we have to look at how we represent these operators, and most importantly we have to understand what is the error that gives rise to any particular representation.

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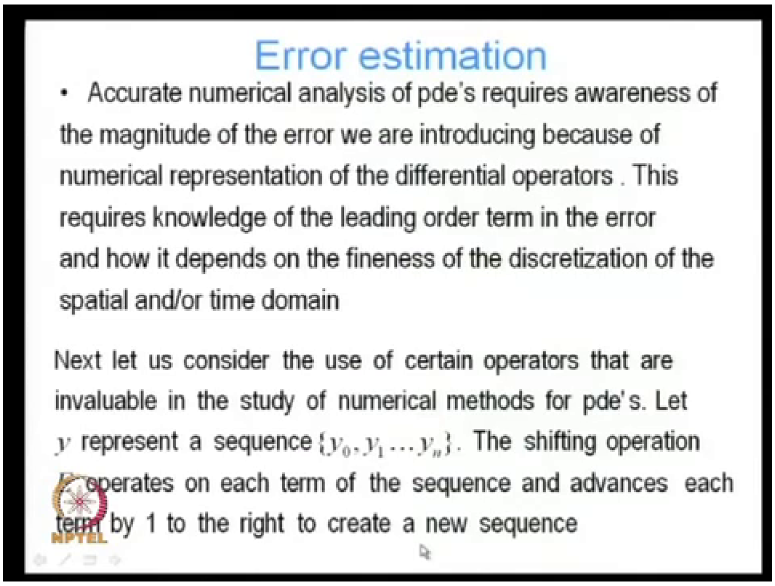

So, accurate numerical analysis of partial differential equations require awareness of the magnitude of the error we are introducing. Because, of numerical representation of the differential operators, this requires knowledge of the leading term in the error expression we want to know the leading term. So, we know the order of the error due to either discretization in this space are the time domain, so we will consider the use of certain operators that are invaluable in the study of numerical methods for partial differential equations.

And these two operators we have I am going to talk about first, first is the shift operator and then I am going to talk about the difference operator. What is the shift operator well

I have if I have a sequence y 0, y 1, y 2, y n, the shift operator basically shifts that sequence. So, it shifts each term to one term on the right, so y 0 becomes y 1, y 1 becomes y 2, y n becomes y n plus 1, the shifting operation E operates on each term of the sequence, and advances each term by 1 to the right to create a new sequence.

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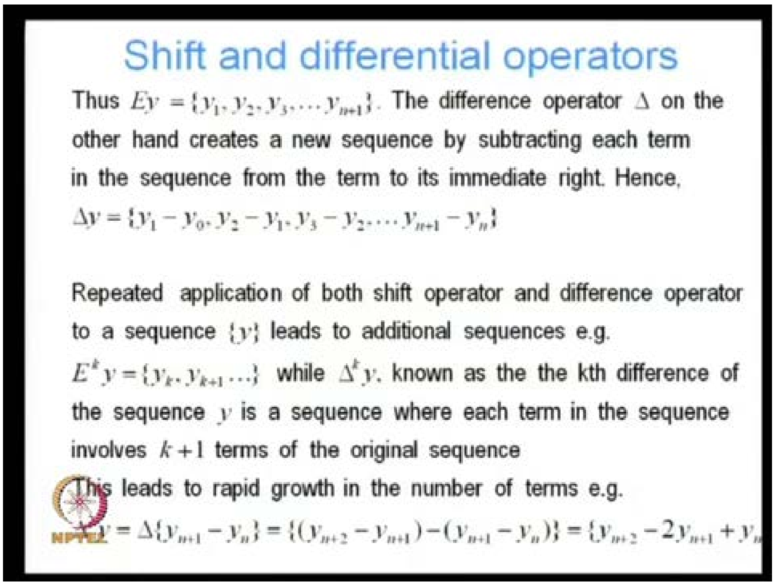

Thus E y is equal to y 1, y 2, so y 0, y 1, y 2, y 3, y n now becomes after I have operated with that on E it becomes y 1, y 2, y 3 through y n plus 1. The differential operator delta on the other hand creates a new sequence by subtracting each term in the sequence from the term to it is immediate. So, my original sequence was y 0, y 1, y 2, y 3 through y n, so after I have operate on that sequence with the differential operator, my first term in the new sequence will be my second term in my original sequence minus the first term in my original sequence.

My second term in the new sequence will be my third term in my original sequence minus my second term in the original sequence and so on and so forth. So, my first term in my new sequence is y 1 minus y 0, my second term is y 2 minus y 1, my third term is y 3 minus y 2 and so on and so forth. So, this is the difference operator operating on the sequence y, so repeated application of both shift operator and difference, difference operator lead to additional sequences, you can see as you operate again and again I am going to get additional sequence.

For instance for the difference operator if I instead of taking let me call that as delta y I refer to the difference operator as delta. So, instead of delta y if I have delta squared y that is going to not delta y of the first term of that sequence were involved y 1 minus y 0, if delta squared y it is going to involve additional terms in the sequence functions E k y. So, if now we are talking about the shift operator E k y basically shifts each term in the sequence k places.

So, y 0 becomes y k, y 1 becomes y k plus 1 and so on and so forth, while delta k y known as the case difference of the sequence y is the sequence, where each term in the sequence involves k plus 1 terms in the original sequence, this leads to rapid growth in the number of terms. For example, del squared y you can see that is equal to del operate on typical term if I consider typical term in my sequence to be y n, it is del squared y that is equal to del operating on del y, del y in a typical term y n is y n plus 1 minus y n.

So, del y n plus 1 minus y n is equal to again this term has got to be shifted and subtracted from itself. So, y n plus 2 minus y n plus 1 minus del operating on this term that is going to shift that term to y n plus 1 and I am going to subtract y n from that term, if I combine those to together I have y n plus 2 minus 2 y n plus 1 plus y n. So, this operating on that gives me shift gives me y n plus 2 minus y n plus 1 this operating on y n gives me y n plus 1 minus y n. So, eventually I have get this, so this is the if I apply the difference 2 times if I apply the difference k times, you can see that there will be many more terms in my difference expression.

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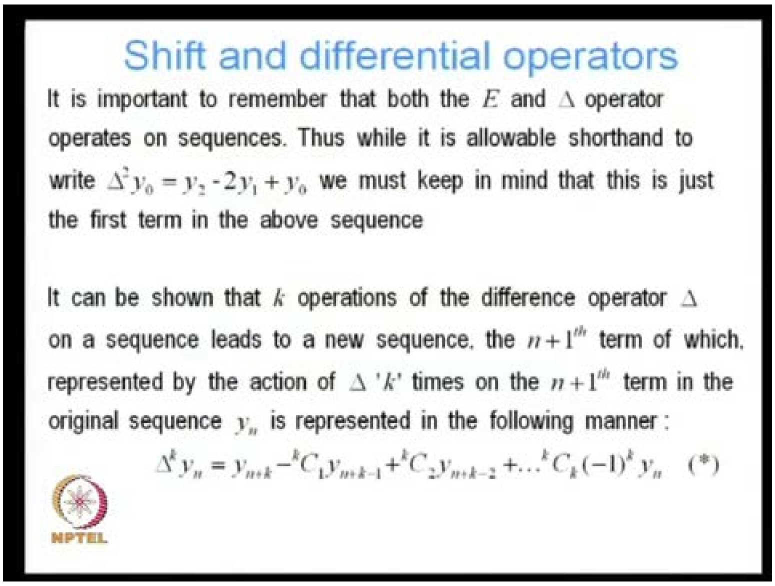

Now, it is important to remember that both the E and the del delta operator operates on sequences, thus while it is allowable shorthand to write del squared y 0 is equal to y 2 minus 2 y 1 plus y 0, as we just saw. As we just saw, we must remember that this is just the first term of the sequence del squared operating on y, so it is the first term in that sequence. The first term on that sequence meaning that term that results from del squared operating on the first term in my original sequence.

Now, it can be shown that k operations of the difference operator delta on a sequence leads to a new sequence. Then plus 1'th term of which is represented by the action of delta k times on the n plus 1'th term in the original sequence y n, suppose in my original sequence by n plus 1'th term was y n because, I started from 0. So, my n plus 1'th term was y n, and if I operate on that in y n term with n times with my difference operator then I am going to get this thing.

If I operate on this y n n plus 1'th term in my sequence in my original sequence k times I take the difference k times, then I will get something like this, where C is the combinatorial expression. So, k C 1 is factorial k divided by factorial 1S times factorial k minus 1, so this is my operation, so this can be this is going to be this well I am going to show it, but I am going to show that by induction.

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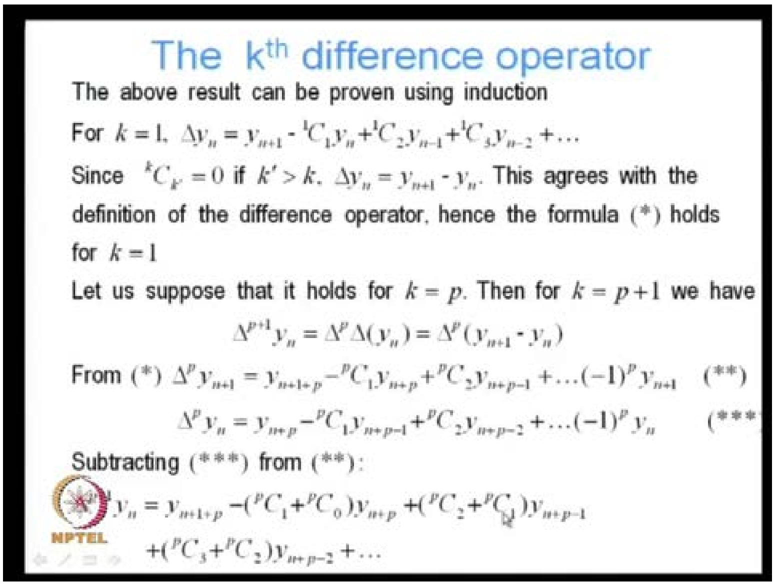

So, how am I going to do that well for k equal to 1 this expression, this expression is going to give me del 1 y n that is going to give me del 1 y n. So, I have just denote it by del y n, and that this expression is going to give me y n plus 1 minus 1 C 1 y n plus 1 minus 1 that is y n plus 1 C 2 plus y n plus n minus 1 and so on and so forth. But, you know that when this is 1 and that is 2, and this is greater than that all these things become 0.

So, only the first term is going to survive, the first term and second term are going to survive, so in that case what are we going to get, we are going to get y n plus 1 minus 1 C 1 y n plus 1 C 2 y n minus 1 plus 1 C 3, so on and so forth. And since k C k prime is equal to 0, if k prime is greater than k, so this is going to give me del operating on y n is equal to y n plus 1 minus y n, which agrees with the definition. So, we know that this definition that we just gave holds for k equal to 1, now let us assume that it holds also for k equal to p.

So, it is the same thing if p is equal to 1 if p is equal to 1 then we can show that it is since it holds for k is equal to 1, we can show that it holds for k equal to 2. And suppose it is holds for p k equal to p, then if I can show that it is holds for k equal to p plus 1 then I am all set, I have proved it by induction I have shown that it holds for 1, and if I also show that if it holds for k equal to p it automatically holds for k equal to p plus 1 then I can then I have shown that it holds for all k.

So, if it holds for k let us suppose that it holds for k equal to p then for k equal to p plus 1 we have delta p plus 1 y n is nothing, but delta p delta y n. So, I move out 1 I move out 1 of the operators out. So, delta p delta y n that is equal to delta p y n plus 1 minus y n delta y n is equal to y n plus 1 minus y n, and from my expression here for k equal to p I get that expression delta p y n plus 1 is equal to y n plus 1 plus p minus p c 1 and so on and so forth and delta p y n.

So, delta p y n plus 1 was that delta p y n is just if I replace n plus 1 by n I get this and then I subtract this from that, if I subtract this from that what do I have on the left hand side I have del p y n plus 1 minus del p y n, which is exactly this del p y n plus 1 minus del p y n. And that I know is equal to del p plus 1 y n, so the left hand side becomes del p plus 1 y n on the right hand side, this term is this term remains y n plus 1 plus p.

And then if I look at coefficients of y n plus p what do I have a minus p c 1, and then I have a 1 this 1 I can represent as p C 0 because, p C 0 is nothing, but factorial p by factorial p, so that is 1. So, that I can represent as p C 0, then if I look at the coefficients of y plus n y n plus p minus 1 what do I have I have p C 2 minus I might have made a mistake here or I might have made a mistake here, so that term is that term.

So, that involves p C 2 and p C 1 that term involves p C 2 and p C 1 the next term is going to involve p C 3 and p C 2 and so on and so forth, and now this p C 1 plus p C 0 or this p C 2 plus p C 1 I can combine together.

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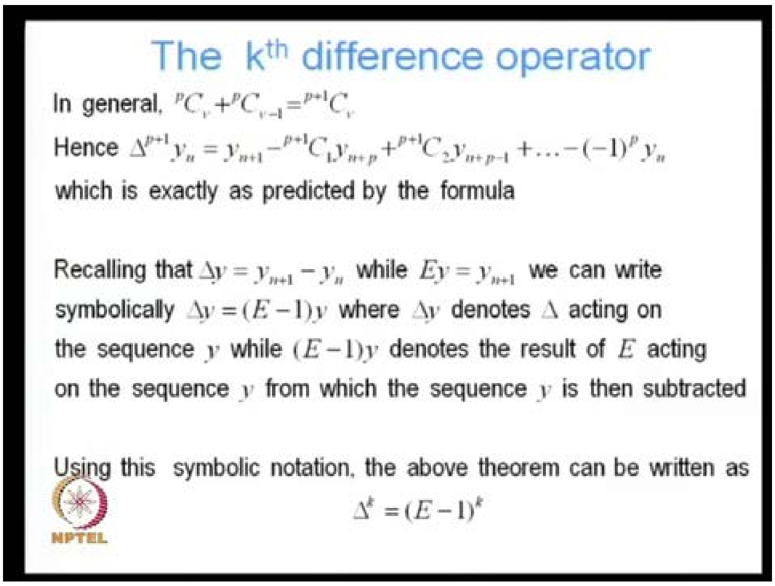

Why because, in general p C nu plus p C nu minus 1 is equal to p plus 1 C nu that is always true. So, this involves p C nu plus p C nu minus 1 terms like that, so I can represent that as p plus 1 C nu, and hence I can write del p plus 1 y n is equal to y n plus 1 I have missed some terms here y n plus 1 plus p minus p plus 1 C 1 y n plus p and so on and so forth, which is exactly as predicted by the formula.

So, we have seen that we can write the k'th difference operator in terms of this binomial this expansion involving this binomial coefficients. And it depending on the order of the difference operator we get more and more terms in those expressions, so next time we are going to continue with this. And then we are going to use, we are going to try to establish relationships between these difference operators, between these various orders of difference operators and various orders of derivatives.

So, if I have d n d x n I want to be able to if I have the n'th derivative of a function with respect to n a variable x, I want to relate that to my difference operator operating n times on that same function. And then I want to find out that what is going to be the difference between those two, between the derivative applied n times, and the difference operator applied n times, what will be the difference if I instead of using the derivative I use the difference operator what will be the error.

And how on what is going to be specifically, what is going to be the leading term in that error. And that will allow us to write our differential equations our partial differential equations in terms of differences, while at the same time being fully aware of what are the errors that we are introducing, when we introduce those differences in terms of instead of the derivatives.

Thank you.