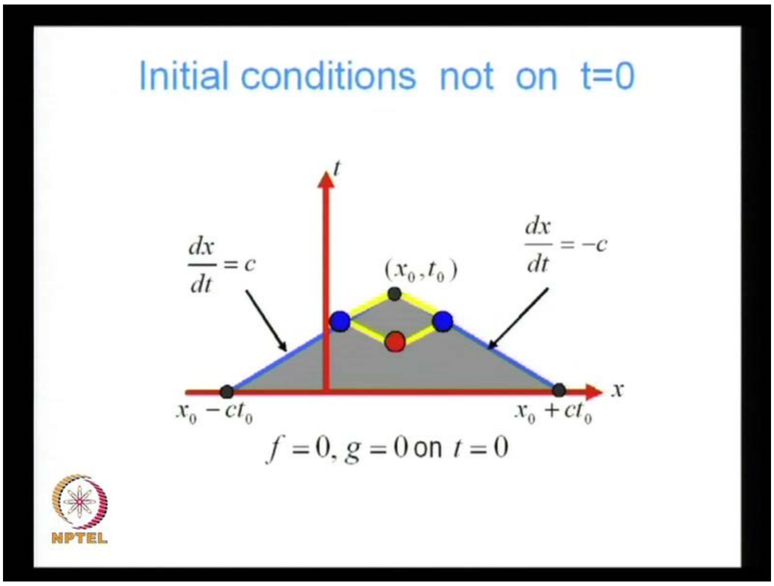
Numerical Methods in Civil Engineering
Prof. Dr. Arghya Deb
Department of Civil Engineering
Indian Institute of Technology Kharagpur

Lecture - 23
Analytical Methods for Hyperbolic and Parabolic PDE's

It is a twenty three of our series on numerical methods in civil engineering, we will continue with our discussion on analytical techniques for solving second order linear partial differential equations. Last time, we look at methods for solving hyperbolic partial differential equations and specifically, we looked at the method of Eigen functions. And, we started looking at the method of characteristics which yields a lot of geometric insight into the problem of solving the wave propagation problem in a space time domain.

(Refer Slide Time: 00:54)



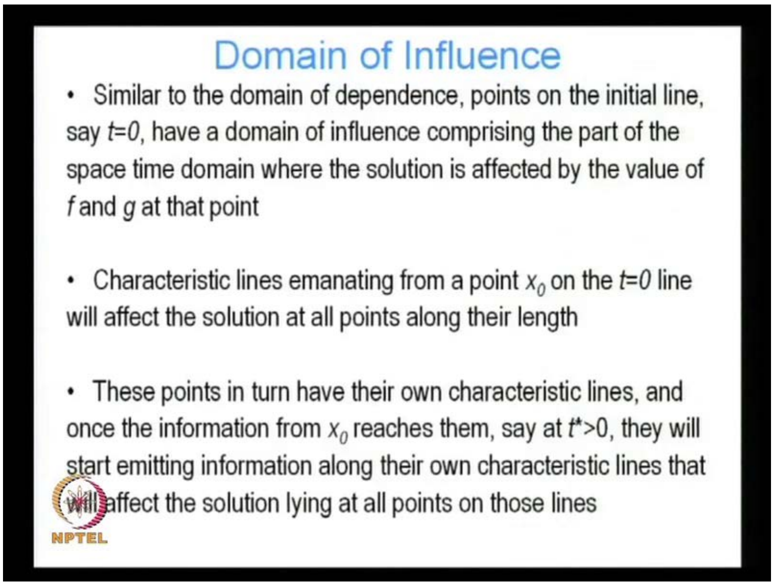
Specifically, we said that given any point in this space time domain, any point x 0 t 0 in the space time domain I can find this solution to the wave equation at that point based on the initial conditions initial conditions are prescribed at time t is equal to 0 then it depends on the part of the t is equal to 0 axis, which lies between these two lines x with slope plus c and minus c.So, all the information all the initial information in this part of the domain within these two black dots can affect the solution here right. In addition, we said that to effect the solution at this point right any source any source of disturbance it should be able to traverse the path between that source and this point right.

And, that path must lie along the characteristic lines that path must lie along the characteristic lines then that path must be must be it must be able to traverse that path within the time t 0 right. if it is not able to traverse that path within the time t 0; that means, that source will not affect the solution at the point x 0 t 0 right. So, for instance this if this if there is some disturbance at this red point for that to affect the solution at x 0 t 0 the it has to traverse along the characteristic lines passing through that red point that is these two lines these two yellow lines and once it reaches points on the characteristic lines passing through x 0 t 0.

This information can propagate from these blue dots to this line right. So, the disturbance here goes and affects the solution here; and the that and the solution here affects the solution here because the information from here can go to the can affects the solution here because it is passing through the same characteristic line. Right, but the sums total the net path from here to here; it must be able to traverse this net path from here to here within time t 0 right. If it is not able to do that then this disturbance here is not going to affect the solution at x 0 t 0 right.

So, the information travels only along the characteristic lines; number one and the information travels with wave with the speed equal to the wave speed right. So, that is the that those are the two things we can take away from here. So, we said that for this point x 0 t 0 this shaded area; this shaded area in gray is the domain of dependence right. The solution here depends on things happening within this domain right disturbances initial conditions anything happening within that domain is going to affect the solution at x 0 t 0 right.

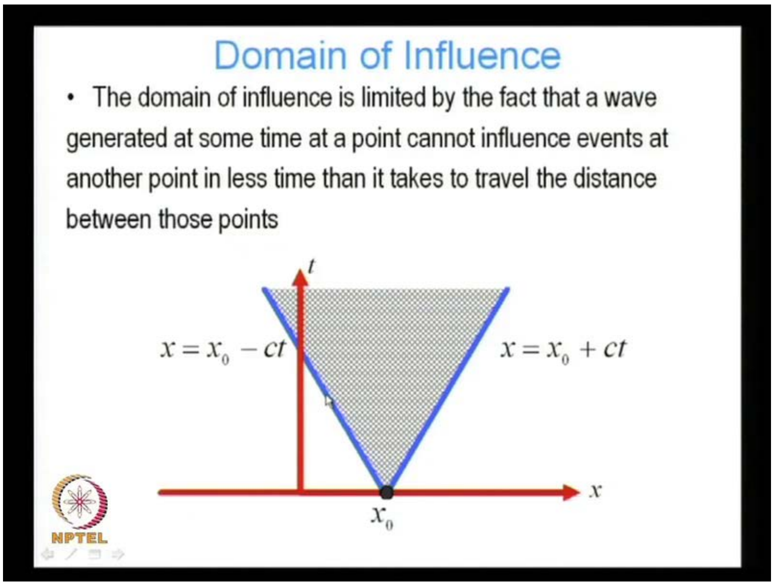
(Refer Slide Time: 04:06)



Similar to the domain of dependence there is something called the domain of influence, it is basically judge the reverse which is at points. This tells me if there is a point in the x t region in the x t domain what is the part of the domain, which affects that the affects the solution at x 0 t 0 right. So, the solution here depends on this region this is the domain of dependence. The other way of looking at it is to look for the domain of influence, suppose I have disturbance at a certain point what is the conceivable region in the space time domain which the disturbance at that point can affect right.

So, that is the domain of influence of that point right. So, similar to the domain of dependence points in the initial line say t equal to 0 have a domain of influence comprising the part of the space time domain, where the solution is affected by the value of f and g at that point; f and g being my initial conditions, f and g being my initial functions. So, characteristic lines emanating from a point x 0 on the t is equal to 0 line will affect the solution at all points along their length that we already know.

(Refer Slide Time: 05:25)

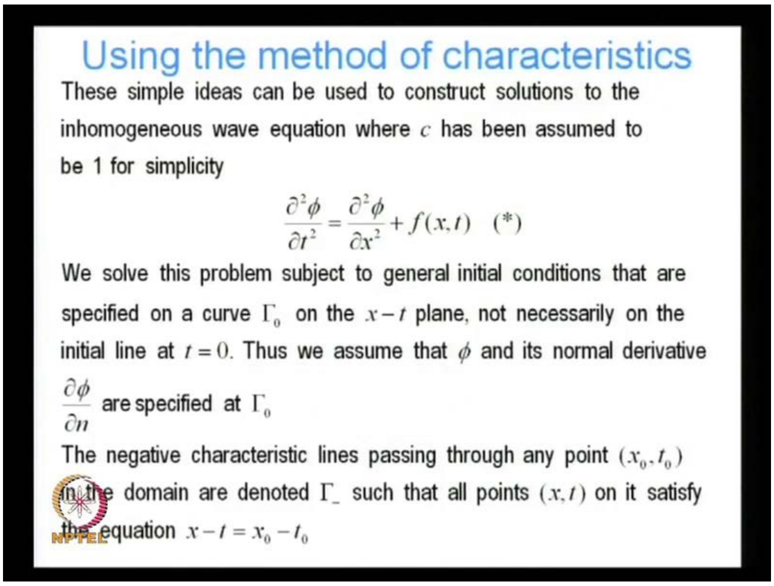


Suppose, I have the point on the t is equal to 0 line, So any point lying on these blue lines are going to be affected by the initial conditions here. But, what is also important is that any point in this shaded region is also going to be affected by their solution; by the initial condition here. Why is that well you look at a point here right; this point the information from this point is going to move along characteristic lines passing those through this point right. Characteristic lines, means lines with slope plus c n minus c so, for instance I do not have a point here, but if I bring my cursor here and focus on this region right. So, information from here is going to move along the characteristic lines passing through that point right. So, the characteristic lines passing through that point.

So, the information here is going to travel in this direction right. Similarly, the information here in the on the other characteristic line is going to travel in this direction. So, the information here can propagate to this entire region right. Because this entire region, is the region through which the this information along this and the along this line; and this line can get propagated into this region right. where two characteristic lines intersect right that in that region we can see the influence of this point right. So, characteristic lines emanating from point x 0 on the t equal to 0 line will affect the solution at all points along there that is a given.

But these points in turn have their own characteristic lines and once the information from x 0 reaches them say at time t star they will start emitting information along their own characteristic lines that will affect the solution at all points lying on those lines. So, the domain of influence is limited by the fact that a wave generated at some time at a point cannot influence events at another point in less time than it takes to travel the distance between those two points right. So, that is how we find out the domain of influence.

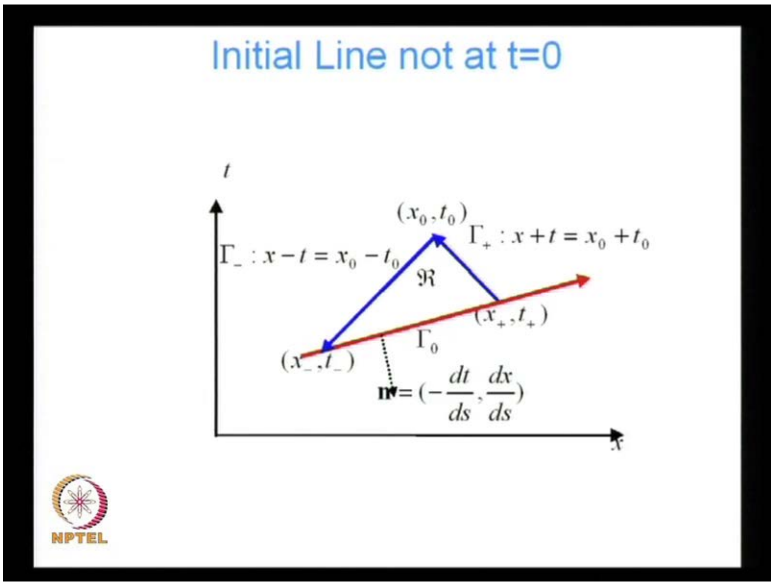
(Refer Slide Time: 07:52)



So using, so this is this was a general discussion of method of characteristics. Let us use the method of characteristics to find the solution to this inhomogeneous wave equation why is it inhomogeneous because on the. On this side, you have a forcing function f x t right the previous wave equations that we looked at they were all homogenous wave equations. So, this is an inhomogeneous wave equation with f x t source term; f x t and in this case for simplicity I have assumed that the wave speed c is equal to one right. So, we will sort this problem subject to general initial condition. Up till now, we have looked at the initial condition meaning conditions on the line t equal to 0 right. That always may not always be true right I can have initial conditions at any time right. suppose, I have a problem I have I gave I specify the conditions at time t equal to 5 right.

So, at time t is equal to 5, I gave some disturbance; I gave some start startup conditions and then for time t greater than 5 that is going to propagate right. So, the initial conditions need not be restricted at time t equal to 0 right. So, in this case I am going to subject it to general initial conditions that are prescribed on a curve gamma 0 on the x t plane not necessarily on the initial line at t equal to 0.

(Refer Slide Time: 09:18)



So, I have a curve gamma 0 in the x t plane right. you can see that it is not aligned along the line t is equal to 0. So, on this curve; on this curve I prescribed certain initial conditions and you can see these initial conditions are not all at the same time right. So, at any point on this these are all, these are points where I prescribe the initial conditions. So, at this point at the space; the space and time are given by x and t right at this point another point on this curve this space and time may be different; it may be at a different time right.

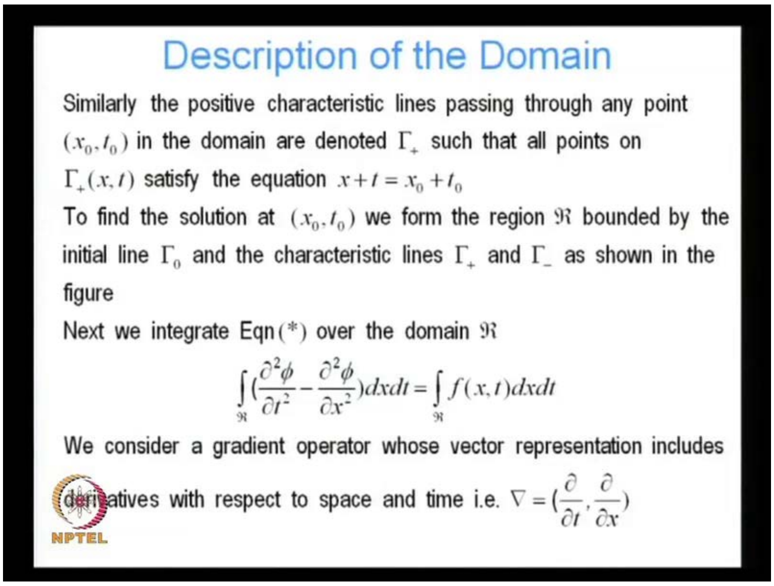
So, these are the points at which these are points in the space time domain where I give my initial condition right; that means, at different points they are setting of this at different times right. So, I have I have a location in space right. So, it is like I think of somebody has laid a charge in space right lead charges in space right and those charges are going off at different times right.

So, at different locations in space the initial condition is starting at different times. So, not necessarily the initial line at t is equal to 0. Thus, we assume that phi and its nominal we will find that we need not only to prescribe phi that is the actual primary variable suppose the displacement but we also need to specify its normal derivative at gamma 0 right. So, to fully define the problem not only do we need to specify phi along this red line the variable; the primary variable you also need to specify its derivative how phi is varying in the direction of the normal to this line right. How phi is is varying at the direction of normal to this line at different locations along that line. So, with these are the initial conditions that we need to specify along gamma 0.

So, the negative characteristic line passing through any point x 0 t 0 in the domain are denoted gamma minus. So, these are the negative characteristic lines this gamma minus right and these are the positive characteristic lines. Here is my point, x 0 t 0 through this point I have these two lines right. And, these two lines each having slope equal to one right one and minus one in this case because our wave speed is one right. So, slope of one and minus one passing through that point and this point; these characteristic lines intersect my gamma 0 my initial line right my initial condition line along which I prescribe my initial condition and the information from here is going to travel to this point along these characteristic lines with a speed of one.

So, all points on the negative characteristic line satisfy the equation x minus t equal to x 0 minus t 0 that we have seen last time right. So, that is basically the equation of the negative characteristic line x is equal to x 0 minus t 0 plus t or x minus t equal to x 0 minus t 0.

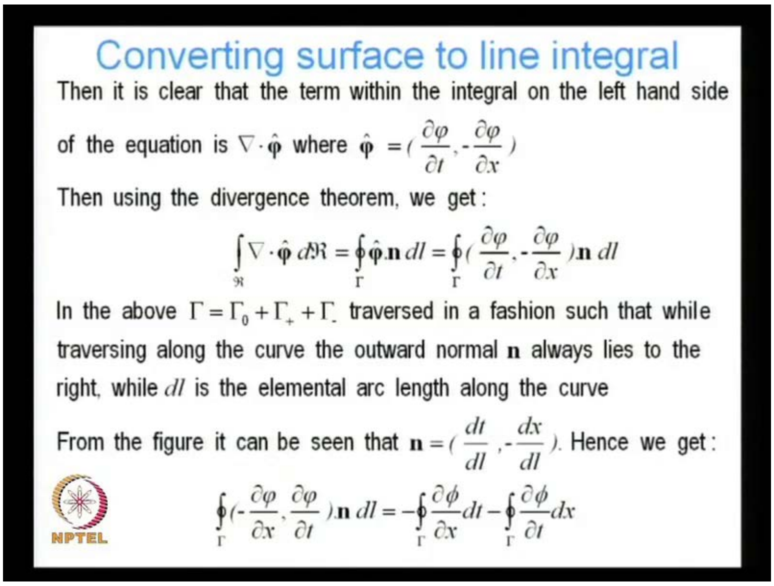
(Refer Slide Time: 12:49)



Similarly, we have the positive characteristic lines passing through any point x 0 t 0 in the domain denoted by gamma plus. Such that, all points on that line satisfy all points on gamma plus satisfy the equation x plus t equal to x 0 plus t 0, to find the solution at x 0 t 0 we form the region r bounded by the initial line gamma 0 and the characteristic lines gamma plus and gamma minus as shown in the figure.

So, we consider this region r right this region r, which is bounded by gamma minus, gamma 0 and gamma plus. Next, we integrate this equation which is my governing equation right my governing equation; which i integrate that over the domain r right straight forward integration of this over the domain r right. And then, we consider since the my primary my independent variables are x and t my gradient operator is nothing to comprises its components are partial derivatives with respect to t and x right. So, gradient is given by del del t del del x.

(Refer Slide Time: 14:10)



Then it is clear that the term within the integral on the left hand side of this equation is a divergence of phi hat, where phi hats if you look at this term this term del 2 phi del t square minus del 2 phi by del x square; this is equal to the divergence this dot of gradient operator acting on a vector and what is that vector. That vector is given by phi hat, where phi hat is equal to del phi del t minus del phi del x.So, if you take the gradient operator which was del del t del del x.

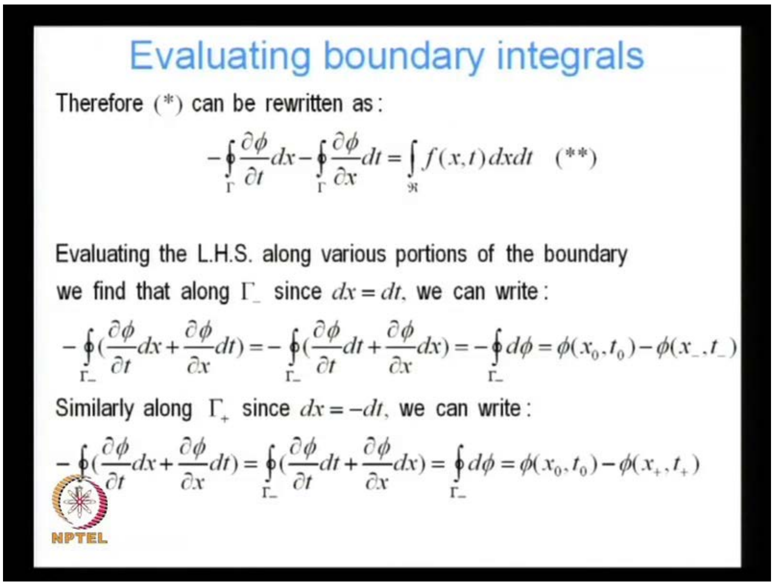
And, we dot it with this vector del phi del t del with whose components are del phi by del t minus del phi del x then we are going to get del 2 phi del t square minus del 2 phi del x square like here right. So, that is exactly what you are going to get. So, we can write. So, we can write this as this operator on the left hand side this operator on the left hand side I can write this as divergence of phi hat operating on the domain integrated over the domain r right integrated over the domain r and then I transfer I convert this volume integral into surface integral using the divergence theorem.

So, I get divergence of phi hat is equal to phi hat dotted with n and now the now the integral is over the surface right over the bound area of the domain which comprises of three parts gamma minus, gamma 0 and gamma plus right. So, phi hat dotted with n and then phi hat of course, is del phi del t minus del phi del x dotted with an d l right. Now, I have converted this integral over this region r to integral along these lines gamma minus; part of gamma, gamma 0 and gamma plus . So, in the above gamma is equal to gamma 0 plus gamma plus plus gamma minus is traversed in such a fashion that while traversing along the curve the outward normal n always lies to the right if you go back and take a look I am moving along this direction.

So, the outward normal is always lying to my right I move along this path I move along that path I move along. So, outward normal is always lying on the right hand on my right hand side as I move along this path. And, d l is the elemental arc length along that curve right d l is the elemental infinitesimal arc length along that curve. From the figure, it can be seen that the normal is actually given by d t d l minus d x d l if you take a look you can see here right the normal is minus d t dS it has a negative component right d t its pointing in this direction minus d t dS and d x dS right.

So, that is my normal and hence we get hence we get this integral as minus del phi del x del phi del t dotted with n for some reason I have just changed the order but that is not important right. So, minus del phi del x del phi del t dotted with n and if I do that I get minus del phi del x d t this d l d l cancels out right. So, I get minus del phi del x d t minus del phi d t d x right.

(Refer Slide Time: 18:24)



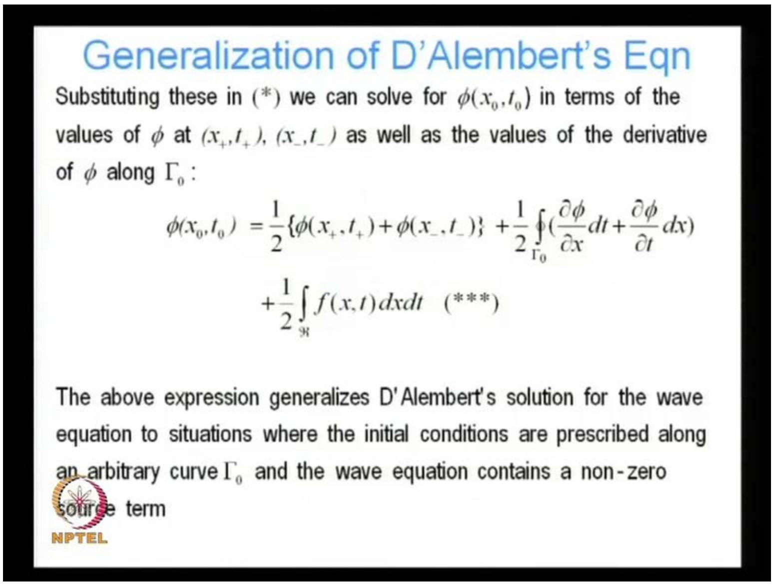
So, this integral that is that is what I get therefore, we can write my equation which was this which was this equation right. As this integral right and the integral of f of x t over the region r right. So, now, we are going to evaluate this integral along various portions of the boundary right. So, first we are going evaluate it along gamma minus and along gamma minus you will see that this line has got slope one right. So, along gamma minus as it moves along this direction it x decreases as it moves along this direction t decreases and the slope is one. So, d x is got be equal to d t right.

So, both have the same sign d x is negative d t is negative right. So, d x is equal to d t. So, along that line d x is equal to d t and we can write del phi del t d x plus del phi del x d t basically this term has is that now I can replace x by d t right along that line. So, I get del phi del t d t and del phi del x d t I can replace by d x right and if you look at this what is it is nothing, but the total derivative of phi right. So, that is nothing, but d phi.

So, that is the integral of this over gamma minus right and d phi and then if I evaluate that if I integrate that and evaluate it at the two end points of my path right that is phi of x 0 t 0 minus phi of x minus t minus why is that well my because my path is going from x 0 t 0 to x minus t minus normally it would be phi of x minus t minus minus phi of x 0 t zero, but there is negative sign in front. So, I get phi of x 0 t 0 minus phi of x minus t minus. Similarly, along gamma plus we can see that along this line we have d x d t d x is equal to negative of d t because as t increases, x decreases right and the slope is one.

So, d x is equal to minus d t and again we replace we replaced d x by minus d t and d t by minus d x right and eventually we end up with an integral and again a total derivative right of phi over along right it is pardon me its class right. So, then we integrate that and when we evaluate the function phi of x 0 t 0 minus phi of x plus, t plus. So, we have evaluated this integral gamma, over gamma minus and gamma plus but I have not evaluated it over gamma 0. So, that part is still remaining right.

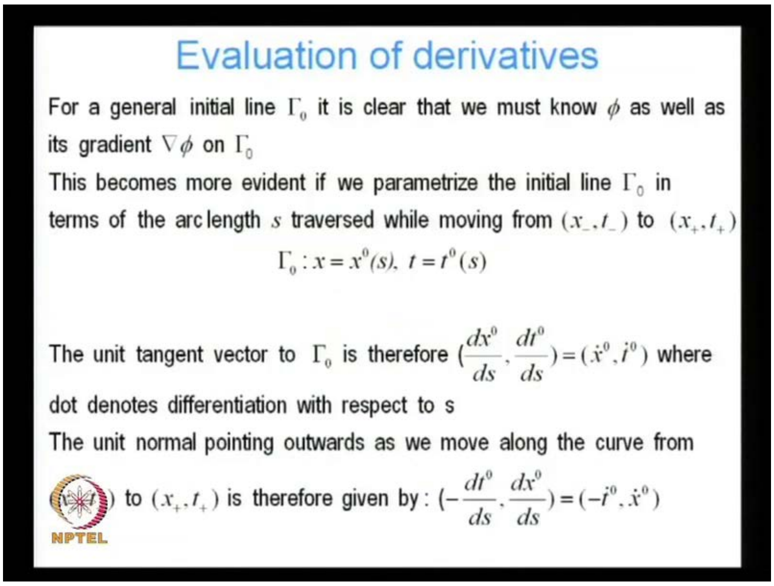
(Refer Slide Time: 21:34)



So, what I am going do is that I am going to substitute this in my this equation right and bring the integral over gamma 0 to the right hand side right. So, here this left hand side includes the integral over gamma 0. I am going to include that bring that to the right hand side, if I do that I am going to get something like this phi of x 0 t 0 is equal to half of phi of x plus t plus, plus phi of x minus t minus plus. you can see that I have brought this integral to the right hand side half of integral of del phi del x d t plus del phi del t del x over gamma 0 plus the source term right. So, this is my solution.

So, you can see that this expression actually generalizes the D'Alembert's solution for the wave equation to situations where I have the initial conditions not prescribed at t is equal to 0; not prescribed along the t is equal to 0 line and prescribed along any arbitrary curve gamma 0 number one. So, that is the one modification and the second modification is that I have a non-zero source term right. I have this additional term f of x t right function another function of f of x t it is like a source term right. So, for the non homogenous wave equation with non standard initial conditions; non standard many initial conditions not all give an t is equal to 0, this is my solution to the wave equation.

(Refer Slide Time: 23:18)

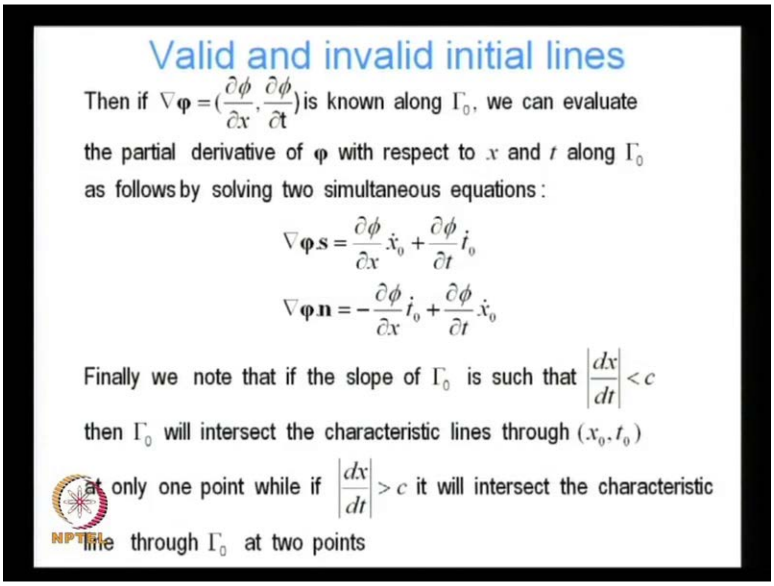


So, for a general line why. So, I said at the beginning not only must we know phi at gamma 0, we must also know the component; the dot product of the gradient of phi in the normal direction. Basically, I must know the normal derivative of phi along gamma 0 why is that well to understand that it is it make sense to parameterize my initial line gamma 0. So, I suppose that gamma 0, I can parameterize in terms of a parameter s right. And, while moving from x minus t minus to x plus t plus, I moving along gamma 0 right and along that path I have a parameter s right.

So, x is actually a function of s as is t a function of s along that line right. So, that one line can always be if I know the direction I can always parameterize it with one quantitative one scalar right and what is that scalar basically, its distance along that line right. So, as I move along that line my x and t changes. So, x is a function of distance along that line, t is a function of distance along that line right. So, x is a function of s and t is a function of s right and here I have said that function is x 0 and t 0 right x 0 a function of s gives me the x and t 0 another function of s gives me t right.

So, the unit tangent vector to gamma 0 is therefore, d x 0 dS d t 0 d dS it tells me how x 0 is varying with x s and t 0 is varying with s. So, now, you can see now I do not have partial derivatives any more I have total derivatives because x 0 and t 0 are functions of s only right. So, this is my unit tangent vector x dot 0 t dot 0 where dot denotes differentiation with respect to s and the normal pointing outwards as we move along the curve from x minus t minus to x plus t plus is therefore, given by minus d t dS d x 0 dS this we have already seen right and that is equal to minus t dot 0 x dot 0 .

(Refer Slide Time: 25:40)



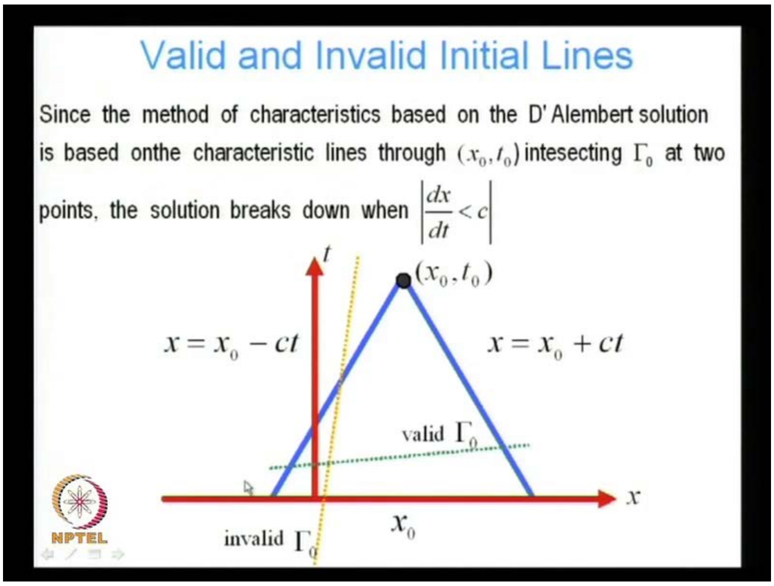
Then if I know grad phi which is del x del phi del t along gamma 0 we can evaluate the partial derivative of phi with respect to x and t along gamma 0 by solving two simultaneous equations. So, if I know the gradient right I take the projection of the gradient along the unit tangent vector s and that gives me del phi del s x dot 0 plus del phi del t t dot 0 right if you calculate that if you take this is my gradient vector if I take the dot of this with this vector right x dot 0 t dot 0 I've basically get this on the right on right hand side.

Similarly, if I take the gradient of this vector this vector and I take if I take the dot with this vector I get this term right. So, now So, we can solve for the partial derivates right if I know if I know these quantities right, if I know del phi del s and if I know del phi del n I can solve for del phi del x del phi del t del phi del x del phi del t these two simultaneous equations for my partial derivates, which I am going to solve right and I have of course, no x dot 0 t dot 0 because I know the path right I know the path as a function of s, the distance along travelled along that path.

So, that I know if I know my gradient if I know my gradient; and if I know the once I know the my gradient I can find out my del phi del n and my del phi del s by just taking the projection of my gradient vector along the unit vector; along the unit tangent vector as unit normal vector. So, those are those two quantities right. So, if I know the gradient, I can find out my partial derivates with respect to phi and t right.

So, that that explains why I need to know not only phi along my line gamma 0 I also need to know the gradient along gamma 0 right. Because in order to evaluate my partial derivatives, I need that information finally, one more point it is we note that if the slope of gamma 0 is such that d x d t is less than c then gamma 0 will intersect the characteristic lines to x 0 t 0 only at one point, while if d x d t is greater than c it will intersect the characteristic line through gamma 0 at through it should be x 0 t 0 at two points.

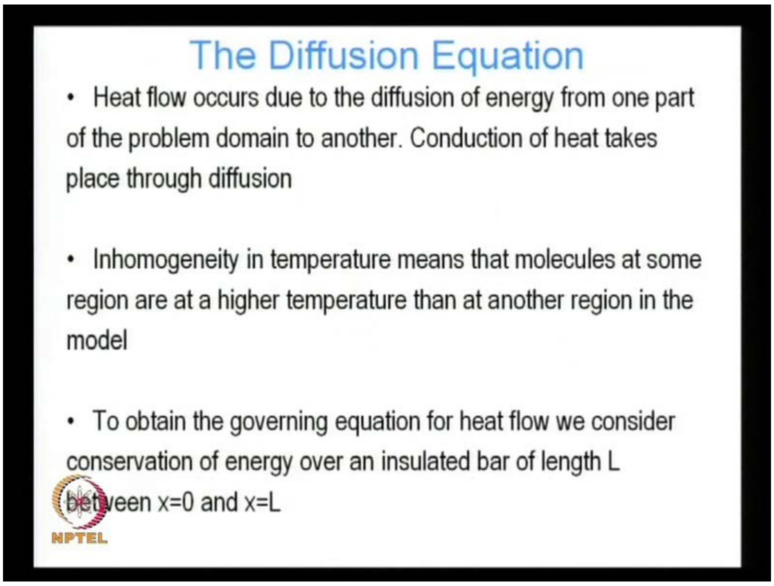
(Refer Slide Time: 28:41)



So, basically what I am trying to say I have tried to make it clear from this picture. So, if I have x 0 t 0 point right and this is my this either my brown dotted line or my green dotted line are my gamma 0 right. Suppose, the characteristic lines through x 0 t 0 are intersect this line only at one point. The slope of this line is such that the mod of d x d t is less than c. So, that there only characteristic line intersects this initial line right. In that case, my solution is going to breakdown because my prerequisite is that both of the characteristic lines passing through x 0 t 0 must pass through my initial condition line right. So, this green line is fine because both my characteristic lines passing through x 0 t 0 intersect this line intersect this line but look at the brown line this line x, x is equal to x 0 minus CT intersects that line, but this does not intersect that line except at sometime which is much beyond my time t 0 right. So, that is not going to work right. So, in this case the solution is going to break down. So, if I have the solution I initial line initial gamma 0, whose slope is less than c mod of d x d t is less than c.

Now, what I have did I write that well its basically this right. mod of d x d t is less than c in that case my solution is going to break down right. So, that is when the method of characteristics is not going to work anymore right is that clear. So, that is all I had to say about analytical methods for hyperbolic equations right. We looked at the method of Eigen functions, which I mentioned that it is a very general method right. It is not only valid for hyperbolic partial differential equations you can extend it to other types of partial differential equations as well as we are going to see very soon. But, then we are again we looked at another method which is method of characteristics, which is sort of specific to hyperbolic partial differential equations in particular, so very useful and very meaningful and very insightful of solving the wave equation.

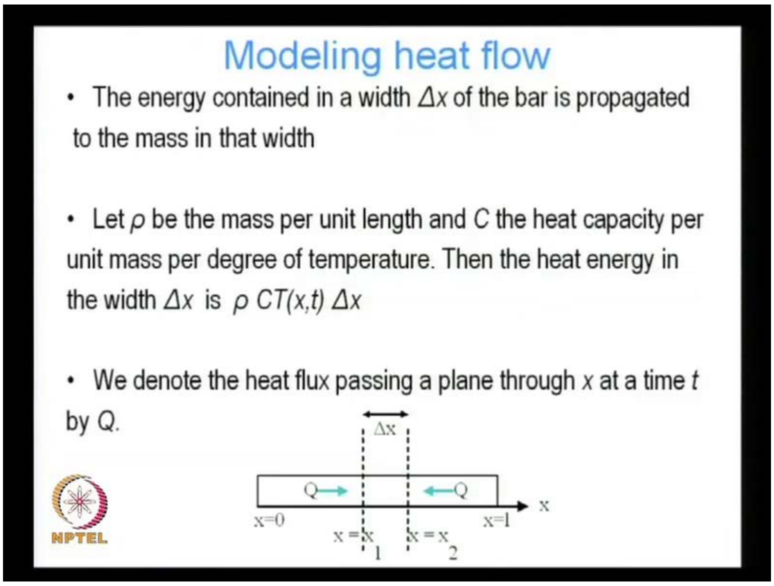
(Refer Slide Time: 31:22)



But let us move on to hyperbolic to parabolic equations and let us look at the diffusion equation. So, see main example while we talking about while we were talking about the canonical forms of the partial secondary order linear partial differential equation; I mentioned that the diffusion equation is a typical example of the parabolic form parabolic; canonical form of the second order equation right and the particular example of the diffusion equation is heat flow, is diffusion of heat through a metal right or through any substance which conducts heat right. So, heat flow occurs due to the diffusion of energy from one part of the problem domain to another conduction of heat takes place through diffusion. So, if you have a heat source somewhere the molecules their start moving more right they vibrate more they move more. So, they have more energy that energy gets transmitted over the domain heat propagates right.

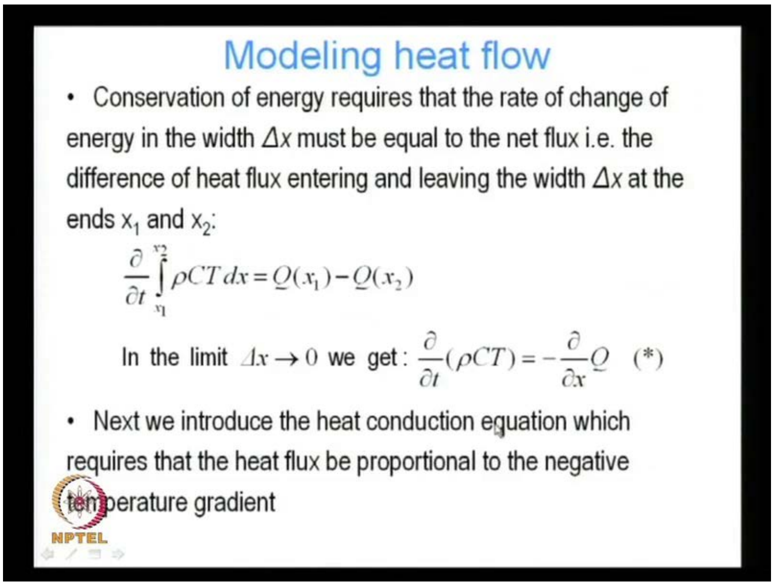
So, in homogeneity in temperature means that the molecules at some region are at a higher temperature then at another region in the model. And, basically what we want to do is to obtain the governing equation of heat flow right what is the equation, which tells me how the temperature is going to vary across a domain right. At the at the boundaries of, which suppose I have a certain heat flux right or prescribe a certain temperature right. So, in that case how is the temperature in that domain going to vary as a function of space as a function of time right that is what I want to model right to do that I use the principle of conservation of energy; I use the principle of conservation of energy and to make thing simple I look at a bar I look at the heat conduction problem, the diffusion problem in one d right I look at a bar which is of length L and I assume that it is lying between x is equal to 0 and x is equal to L.

(Refer Slide Time: 33:30)



So, the energy contained in its in its width delta x of the bar is propagated is proportional I am sorry this is proportional to the mass in that width right. It is proportional to the mass in that width. So, why well you know you know about specific heat capacity right if rho be the mass per unit length and see the heat capacity per unit mass per degree of temperature then the heat energy in that width delta s delta x is given by the density times delta x, which gives me the mass right times the heat capacity c times the temperature right. So, that is the total heat energy in that width delta x rho CT right, rho CT times the volume right which is delta x. And, we denote the heat flux passing a plane through x at a time t by Q right. So, this is my little width delta x it is bounded by those two planes and then I have heat flux through these planes right.

(Refer Slide Time: 34:42)

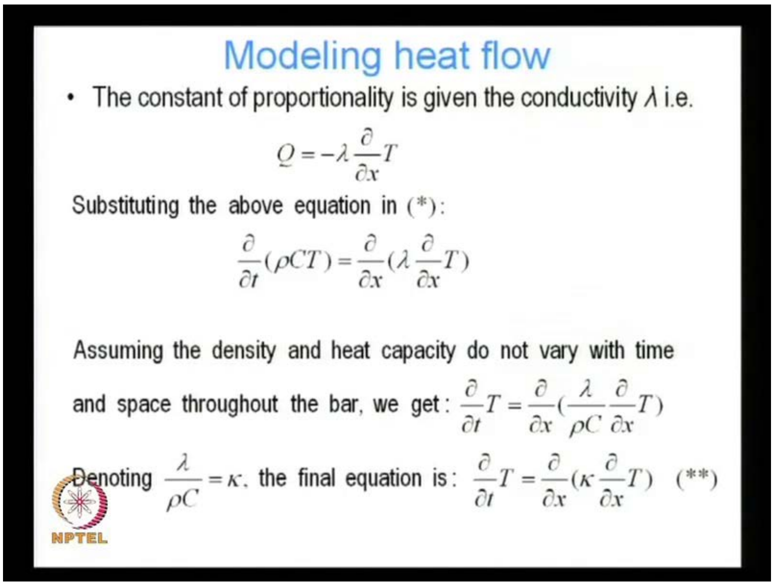


So, conservation of energy requires that the rate of change of energy in that width delta x must be equal to the net flux. The heat, amount of heat in that little volume it changes depending on the amount of heat that is flowing in through the boundaries and the amount of heat that is leaving through the boundaries right. So, the net the rate of change of energy in the width delta x must be equal to the net flux, that is the difference of heat flux entering and leaving the width delta x at the ends x 1 and x 2 right. So, we see that rate of changes so, this was my volume of heat energy in my little strip right rho CT d x integrated over x 1 and x 2 that was the total heat energy in that little strip and the rate of change of that heat energy is equal to the flux at x 1 minus the flux at x 2 right.

And, in the limit when delta x goes to 0 right what do we have we have del del t rho CT is equal to minus del Q del x. So, this Q x 1 minus Q x 2 I can now represent it as del Q right in the limit that x 1 and x 2 tend to 0 I mean x 1 minus x 2 tends to 0 and this integral becomes derivative right. So, I get that that right.

So, that is part of the problem, the other next I have to look at in look at the constitutive behavior. Basically, I have to see I have to introduce the heat conduction equation right which requires that heat flux be proportional to the negative of the temperature gradient right. So, it tells me how the flux is related to the temperature. So, heat is going to flow in the direction from higher temperature to lower temperature right. And, if the heat flow heat is going to be proportional to the difference in temperature divided by the in this case we can think of distance, but in multidimensions it is a basically a gradient right. So, if an one d which is the temperature at x 1 minus the temperature at t at x 2 divided by the difference between the distance between x 1 and x 2 right. So, that is the gradient and heat is flowing always in the negative gradient direction because it flows from higher temperature to lower temperature.

(Refer Slide Time: 37:19)

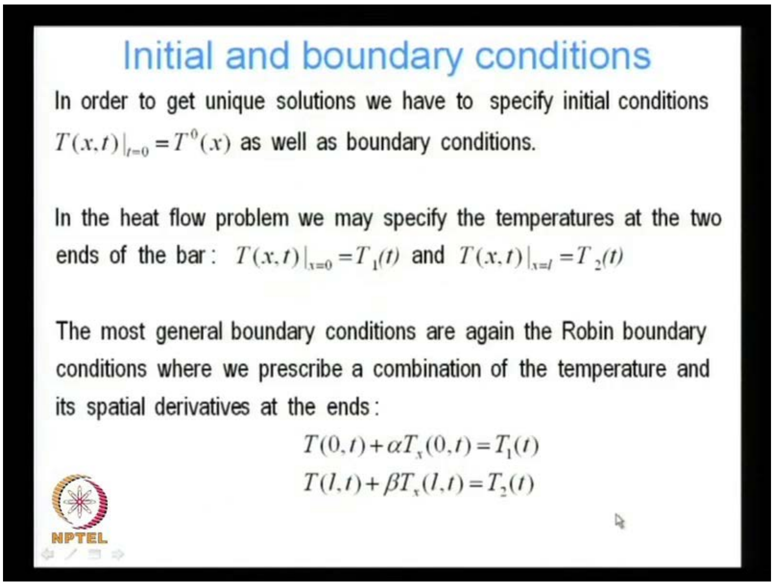


So, and then we say that that is, so we know that heat flux is proportional to the to the temperature gradient; negative temperature gradient and lambda is the is the constant of proportionality. So, I can write heat flux is equal to minus lambda times del del x T right.

And then I go ahead and substitute this in my equation here which I have got from energy conservation right this equation I got from conservation of energy.

So, I go ahead and substituent del Q del x in terms of lambda del T del x right and once I do that I get an equation like this right and then I assume that density and heat capacity do not vary not always true. But for the sake of simplicity, let me assume that the density and heat capacity do not vary with time and space throughout the path in that case I can pull this rho c down here right. So, I get del del T T is equal to del del x lambda by rho c del del x T. I denote lambda by rho c by kappa right and then the if I get a final form of this equation as del del T T is the temperature right small t is the time right. So, del T del T is equal to del del x kappa del T del x right.

(Refer Slide Time: 38:50)

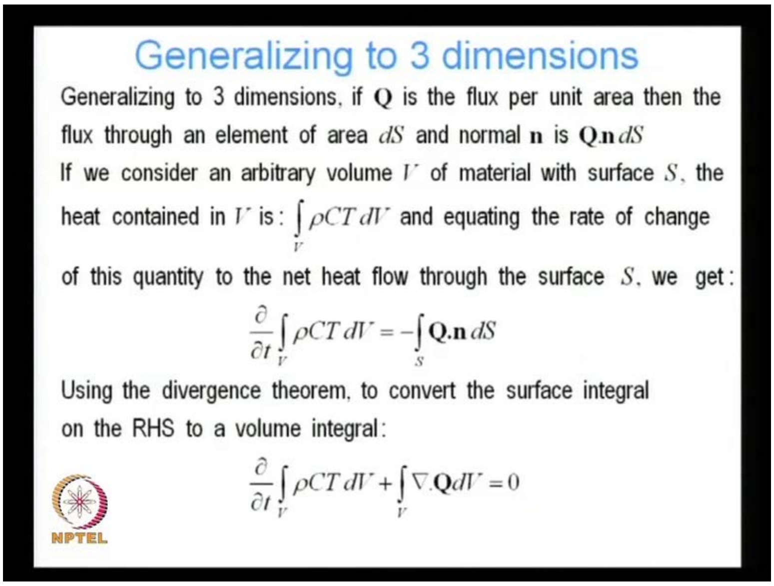


So, in order to So, this is my final equation this is my diffusion my equation right for heat; heat flow in order to get unique solutions again we have to specify initial conditions which tells me what is the temperature at time equal to 0 in my domain in my spatial domain right at time t equal to 0 how does my temperature vary in space. So, which is t 0 x which is a function of x right. So, that is my initial condition in addition I need boundary conditions and in this problem of bar one d bar I am going to specify my I have two boundaries at x is equal to 0 x is equal to l.

So, I am going to specify the temperature at x is equal to 0 and x is equal to l again as a function of time right. So, T 1 of t and T 2 of t. So, these are very simple boundary conditions I just wanted to mention in passing like I have also mentioned when I was talking about the wave equation that the boundary conditions need not be as simple as that right we can have we can have a mix of new Norman and Dirichlet boundary conditions. So, I can specify I can have a combination of the temperature and the gradient of the temperature specified at 0 similarly I can have a a linear combination of the temperature and the gradient of the temperature specified at l in this case it is really simple because I am just specifying the temperature at x is equal to 0.

So, this part alpha is equal to 0 and beta is equal to 0 for us right. So, it is a straight forward Dirichlet boundary condition, but you can have a combination of the temperature as well as its derivative specified at the boundary right most general form of the boundary condition, but even under these conditions it is possible to the problem is still well posed and it is still possible to get the solution for the heat equation.

(Refer Slide Time: 40:59)

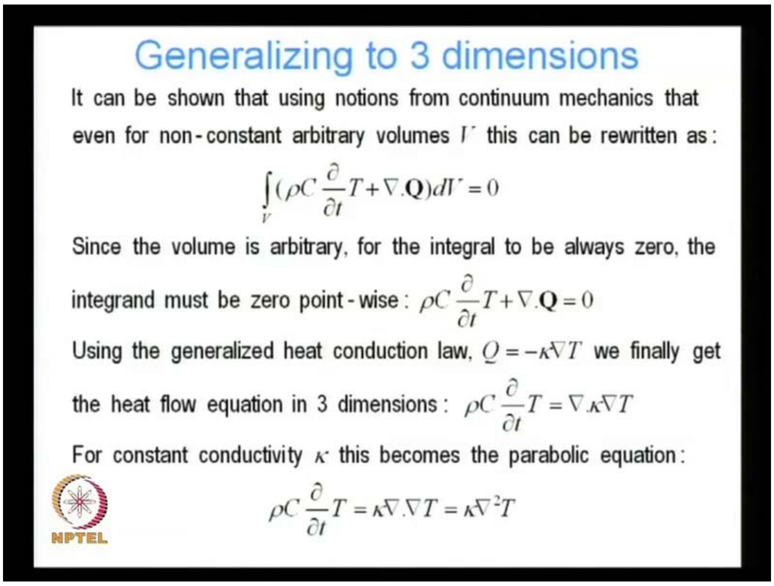


We can generalize this through three dimensions to 3D, I do not want to spend too much time on it but just to mention it in passing we say instead of the instead of saying that Q is the heat flux through a plane through a line at x is equal to x 1 or x is equal to x 2 we say it is the heat flux per unit area, where area denotes the boundary of a particular volume in 3 D space right. So, Q is the flux per unit area then the flux to an element of area dS is and the with normal n is given by Q dotted with n dS right, dS is the n dS is my area right area magnitude of the area given by dS infinitesimal area, n is the normal up to that area to that infinitesimal area and Q is the flux through that area.

So, the total flux through that area is given by Q dotted with n dS and next we consider an arbitrary volume v of material, which surface s and the heat contained in that heat volume v is again given by rho CT dV integral of v like we had earlier rho CT dx integral from x 1 to x 2. So, here we have rho CT dV integrated over my total volume right and then we equate the rate of change again we use the principle of conservation of energy and we say that any change in the energy contained in that little volume dV or any change in that volume V, which comprise infinitesimal volumes dV is given the rate of change of that energy has got to depend on the how heat is flowing in through the boundary.

So, right how heat is flowing in or how heat is leaving through the boundaries right. So, del del t integral of v rho CT dV got to be equal to minus Q dotted with n dS right. So, net flux through the boundary right and again we use the divergence theorem to convert the surface integral to volume integral. So, Q dotted with n dS I converted into divergence of Q dV right use the divergence theorem I get that and then I use.

(Refer Slide Time: 43:12)

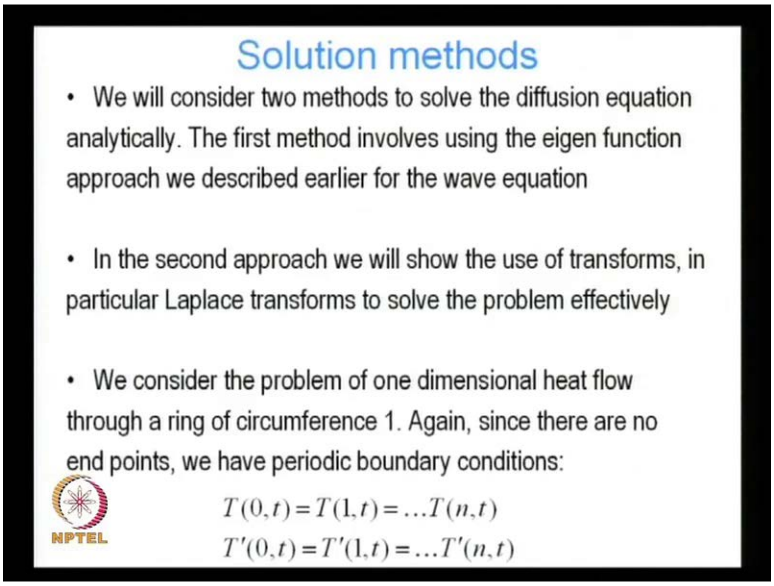


So, interesting thing is that I do not want to spend too much time on this, but it might be worth mentioning in passing is that I have this partial derivative with T outside the volume integral right well it well you can show using some arguments based on continuum mechanics right based on continuum mechanics that it is possible to bring that partial. Even if my volume v is not constant right it is varying with time right it is possible to bring that integral inside that partial derivative with respect to t inside that integral right.

And if we do that I get something like this rho c del del T, T plus divergence of Q integral over dV that is equal to 0 and since this volume is arbitrary for the integral to be always this is true for all arbitrary volumes right. So, you can make my volume go to a point right it is got to be true there so; that means, it must and I can do that at every for my every volume at around every point center around every point in my domain and if I do that I I in order to satisfy this question this must be 0 point wise right.

So, rho C del del T dT plus divergence of Q got to be 0 at every point in my domain and thus we get back an equation very similar to what we had for earlier one d right and then again we use my heat conduction law, which tells me that the flux is proportional to the negative of the temperature gradient and then I substitute that there I get divergence of k kappa times gradient of T right and in case kappa is a constant in case of constant conductivity kappa I can pull that kappa out. So, I get kappa times divergence of gradient of t which is nothing, but Laplacian of t. So, kappa times Laplacian of t. So, rho c del T del T is equal to kappa times Laplacian of T, which becomes my heat conduction equation in three dimensions right.

(Refer Slide Time: 45:39)

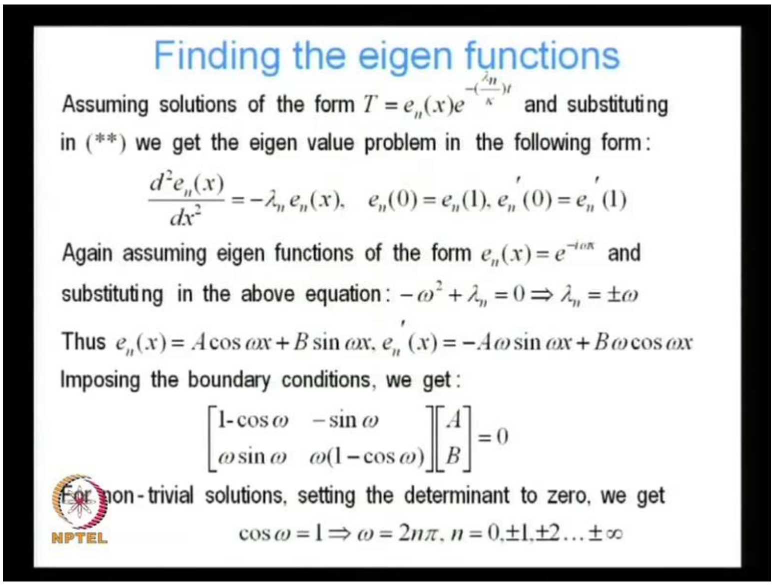


So, solution methods we will consider two methods to solve the diffusion equation analytically the first method involves using the Eigen function approach, which we described earlier for the wave equation; in the second approach we will use Laplace transform right we will use Laplace transform to solve the problem effectively right. So, that was one way is use to use the Eigen function approach. The other way is to use the transform based method in this case we are going to use the Laplace use Laplace transforms to solve that problem to consider the problem.

So, again we when we looked at the wave equation the first problem we looked at we considered wave propagation in a ring of unit circumference; of unit circumference and we looked at that condition and now again where when we looked at the heat flow problem we are again going to look at that geometry. So, we are going to consider a ring of unit circumference and we are going to consider heat flow along that ring and as we noticed in case of the wave equation for this particular geometry the boundary conditions are periodic right. So, if I have along if I my spatial dimension is my arc length along that ring once it completes one whole arc length once and since the whole circumference is one once it travels through one right I must recover I must my I must get back the same boundary conditions right.

So, it has got periodic boundary conditions, which tell me that the temperature at 0 must be equal to the temperature at one must be equal to equal to the temperature at n. Every time I am going around 1 2 3, I am reaching the same points at the same point the temperature are better be the same. So, T 0 t T 1 t must be equal to T n t and similarly the slope right here the slope when I say slope when I say prime I denote partial derivative with respect to the x right, where x is my distance along the circumference of the ring right. So, T prime 0 T must be equal to t prime 1 T and T prime n t. So, that is my that is I know what is my governing equation those are my boundary conditions right.

(Refer Slide Time: 48:04)



So, in this case let us do when I looked at the wave when I looked at the Eigen function approach for the wave equation I sort of assumed that you would know how to find the Eigen functions that I think that this particular problem I will go through it in detail right. How do you find the Eigen functions? let us go through that in detail right. So, to find the Eigen functions we assume solutions of the form T is equal to e n x e to the power minus lambda n kappa of t and substituting this in my governing equation, which was somewhere which we have left long back here if I substitute that particular form of my Tin this governing equation I get an equation like this d 2 e n x d x square equal to minus lambda n e n x right.

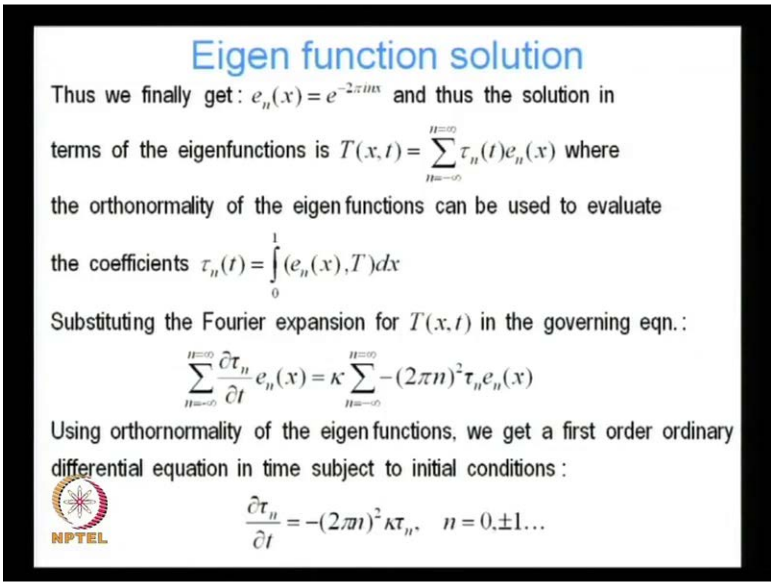
I get an equation like this, and it is it has these boundary conditions which are at e n 0 must be equal to e n 1 e n prime of 0 must be equal to e equal to n prime minus 1. So, this becomes my Eigen value problem how do I get my Eigen value problem well I do this substitution right I assume that t is of this form right with e n is the my Eigen function and this is my time dependence I assume my dependence of this of this nature if I substitute that I get this equation and then again. So, you can that this an equation in x right. So, I assume that my Eigen functions are of this form.

So, this equation now, I assume is of the form e to the power minus i omega x right and then once I substitute that here this differential equation becomes the algebraic equation in omega right I get an algebraic equation in omega and thus I get lambda n equal to plus minus omega right. So, my solution e n x is equal to e to the power minus i omega x. So, using de Moivre's theorem right I know I can write it as a cos omega x plus B sin omega x right. So, e n x is of this form right and omega is related to lambda the Eigen value like this right this is of the we have seen this is like an Eigen value problem right I have a linear operator and times some constants operating on that same function. So, this is of the form of an Eigen value problem lambda n is the Eigen value and I know the solution of this equation observed this form right. So, if a n is of this form then e n prime had better be this I have just taken derivative with respect to x and then I impose my boundary conditions what are my boundary conditions e n 0 is equal to e n 1.

So, evaluate e n at 0 equate that to e n at one. So, that gives me the top equation m one minus cos omega A minus sin omega B equal to 0 and then I evaluate e n prime i evaluate that at 0 and I also evaluate that at one and set it equal set them equal. So, I get the bottom equation omega sin omega omega 1 omega sin omega a plus omega 1 minus cos omega b equal to 0.

So, now, I have a linear system and a homogenous system to get non trivial solutions for A and B that is to get solutions when A and B are not both equal to 0 the determinant of that system must vanish right determinant, if I evaluate the determinant of that system and set it equal to 0, I find eventually I get this equation cos of omega equal to 1 and I know that cos of omega is going to be equal to 1 for omega equal to 2 n pi n is equal to 0 plus minus 1 plus minus 2 up to plus minus infinity. So, that gives my omega I know is equal to my Eigen value. So, I already know my Eigen values right I already know my Eigen values. So, I know my Eigen values I know now I know my Eigen vectors right I know my Eigen vectors up to those constants right up to those constants A and B right.

(Refer Slide Time: 53:01)



So, I know that. So, my Eigen vectors must be of the form e n x is equal to e to the power minus 2 pi i n x why is that I said that e n x is equal to minus i omega x right. So, omega is given by that. So, I know my Eigen vectors and thus I go back and remember that this is a self adjoint operator this is the self adjoint operator. So, its Eigen functions form a basis are complete they form a they are complete.

So, they form a basis for that function space. So, my solution t x t I can always express them as a linear combination of my Eigen functions. So, this t x t must be equal to the sum of this t tau n t which are some coefficients times my basis functions, which are my Eigen functions and I can evaluate these constants these, these coefficients which are functions of time by using the orthonomality of the Eigen functions right. So, may be this is a good place to stop and we will continue our discussion the Eigen functions solution for the heat equation. We will talk about the Laplace transform solution and then we will move on to last remaining type of second order linear partial differential equations those are elliptic equations.

Thank you.