Numerical Methods in Civil Engineering
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Lecture - 1
Introduction to Numerical Methods

First numerical method in civil **e**ngineering, in the first lecture I am going to talk about introduction to numerical methods.

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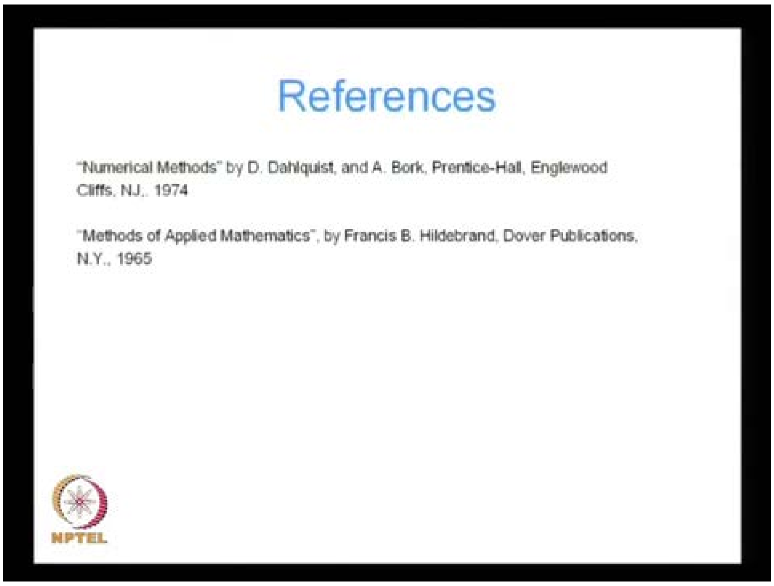

Before, going into further details, I would like to mention two references, which I am going to follow throughout this course, the first reference is numerical methods by Dahliquist and Bork, the second reference is methods of applied mathematics by Hildebrand.

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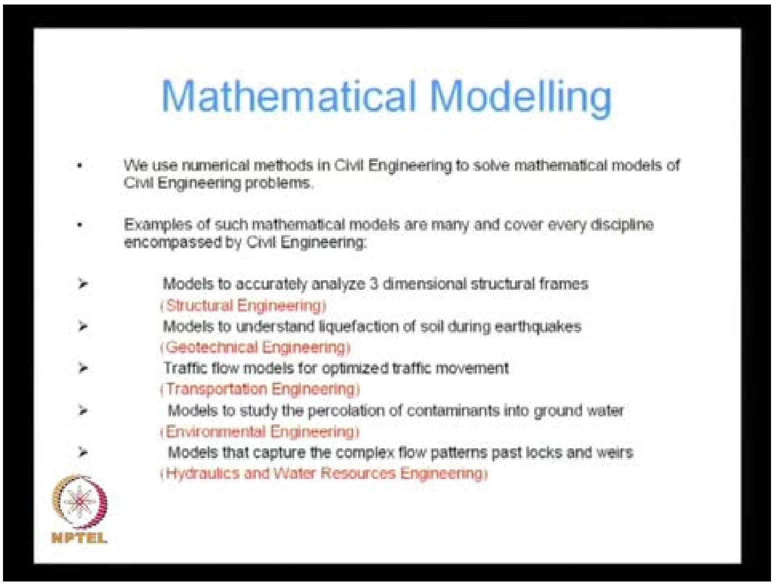

First I would like to talk about why it is necessary to do mathematical modeling, we use numerical methods in civil engineering to solve mathematical models of civil engineering problems. Examples of such mathematical models are many and cover every discipline encompassed by civil engineering, for instance in structural engineering, we need models to accurately, analyze dimensional structural frames. In geotechnical engineering, we need models to understand liquefaction of soil during earthquakes transportation station engineering for instance.

We used traffic flow models for optimized traffic movement an environmental engineering, we use models to study, the percolation of contaminants into ground water. And in hydraulics and water resources engineering, we use models that capture the complex flow patterns past locks and weirs.

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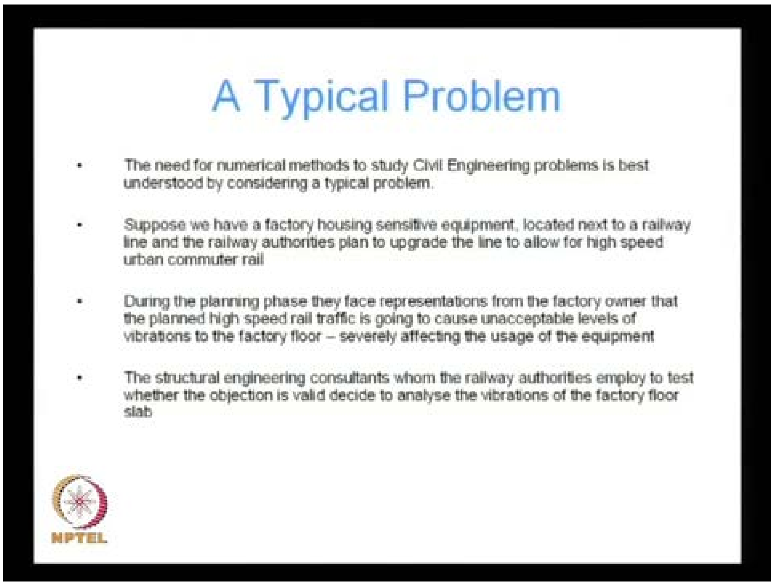

The need for numerical methods to study civil engineering problems is best understood by considering a typical problem, since I am a structural engineer I will choose a typical problem from the field of structural engineering. Suppose we have a factory housing sensitive equipment, located next to a railway line and the railway authorities plan to upgrade the line to allow for high speed urban commuter rail.

During the planning phase they face representations from the factory owner, that the planned high speed rail traffic is going to cause unacceptable levels of vibrations to the factory floor, severely affecting the usage of the equipment. The structural engineering consultants whom the railway authorities employ to test, whether the objection is valid decide to analyze the vibrations of the factory floor slab.

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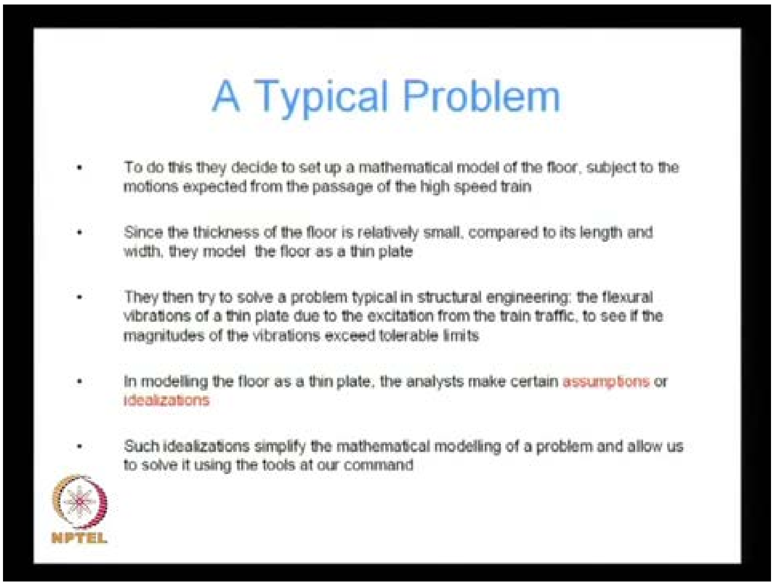

To do this they decide to set up a mathematical model of the floor subject to the motions, expected from the passage of the high speed train, Since the thickness of the floor is relatively small compared to its length. And width they model the floor as a thin plate, then they try to solve a problem, typical in structural engineering the flexural vibrations of a thin plate, due to the excitations from the train traffic, to see if the magnitudes of the vibrations exceed tolerable limits. In modeling the floor as a thin plate the analysts make certain assumptions or idealizations such idealizations, simplify the mathematical modeling of a problem and allow us to solve it using the tools at our command.

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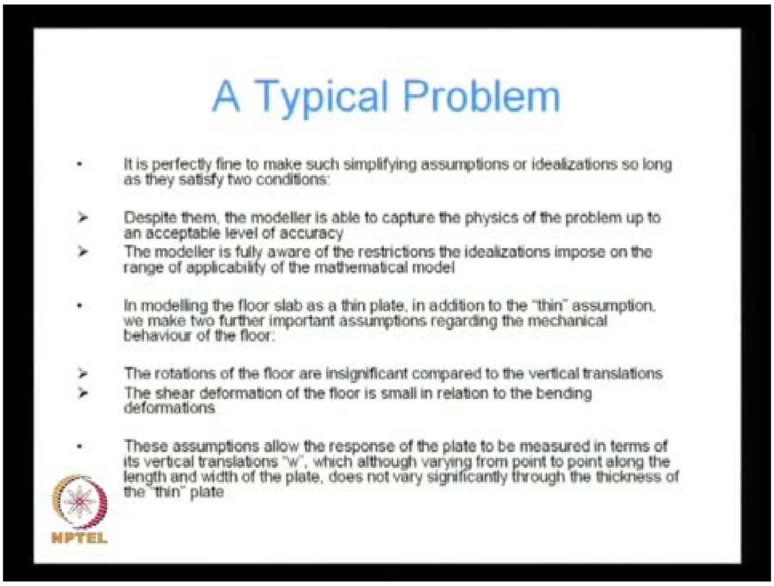

It is perfectly fine to make such simplifying assumptions or idealizations, so long as they satisfy two conditions, first despite them the modeler is able to capture, the physics of the problem up to an acceptable level of accuracy. Two, the modeler is fully aware of the restrictions, the idealizations impose on the range of applicability of the mathematical model.

In modeling the first floor slab as a thin plate, in addition to the thin assumption we make two further important assumptions regarding the mechanical behavior of the floor. First the rotations of the floor are insignificant compared to the vertical translations to the shear deformation of the floor is small in relation to the bending deformations. These assumptions allow the response of the plate to be measured in terms of its vertical translations w, which although varying from point to point along the length and width of the plate does not vary significantly, through the thickness of the thin plate.

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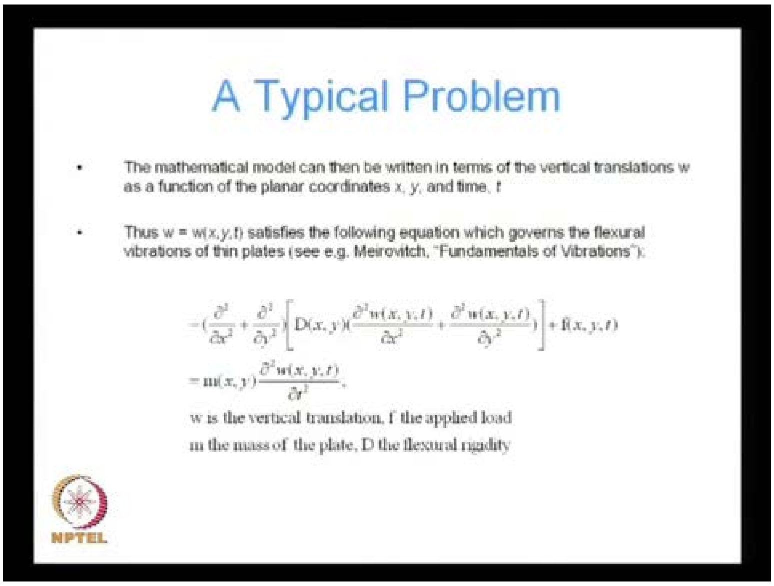

The mathematical model can then be written in terms of, the vertical translations w as a function of the planar coordinates x y and time t, thus w where w is the function of x y and t. Satisfies the following equation which governs the flexural vibrations of thin plates.

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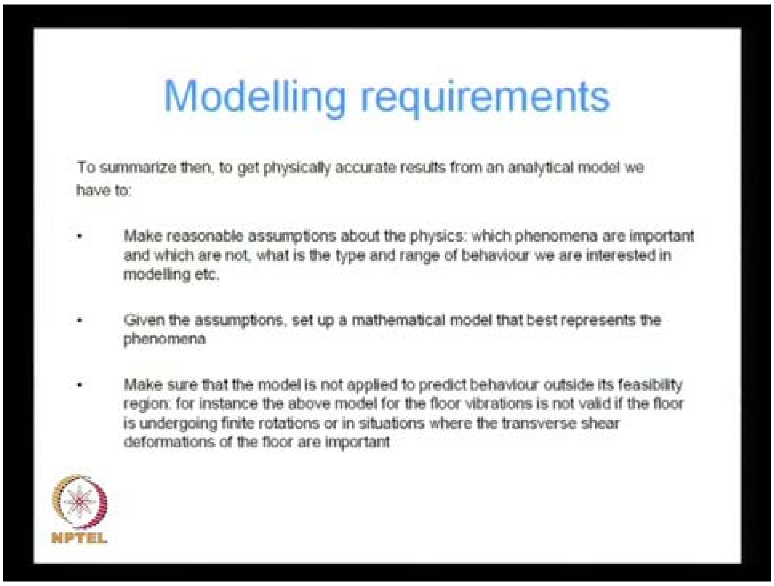

To summarize then to get physically accurate results from an analytical model, we have to one make reasonable assumptions about, the physics which phenomena are important. And which are not, what is the type and range of behavior, we are interested in modeling etcetera given the assumptions, set up a mathematical model that best represents the phenomena.

Finally, make sure that the model is not applied to predict behavior outside its feasibility region, for instance the above model for the floor vibrations is not valid, if the floor is undergoing finite rotations or in situations where the transverse shear deformations of the floor are important.

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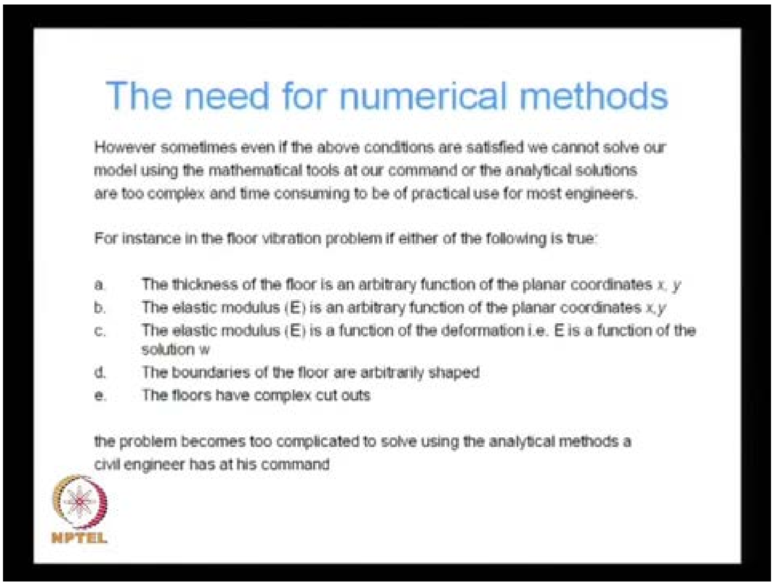

However sometimes even if the above conditions are satisfied, we cannot solve our model using the mathematical tools, at our command or the analytical solutions are too complex. And time consuming to be a practical use for most engineers, for instance in the floor vibration problem, if either of the following is true, either the thickness of the floor is an arbitrary function of the planar coordinates x and y.

The elastic modulus is an arbitrary function of the planar coordinates x and y, the elastic modulus is a function of the deformation, that is e is a function of the solution w the boundaries, of the floor are arbitrarily shaped or the floors have complex cut outs the problem, becomes too complicated to solve using the analytical methods a civil engineer has at his command.

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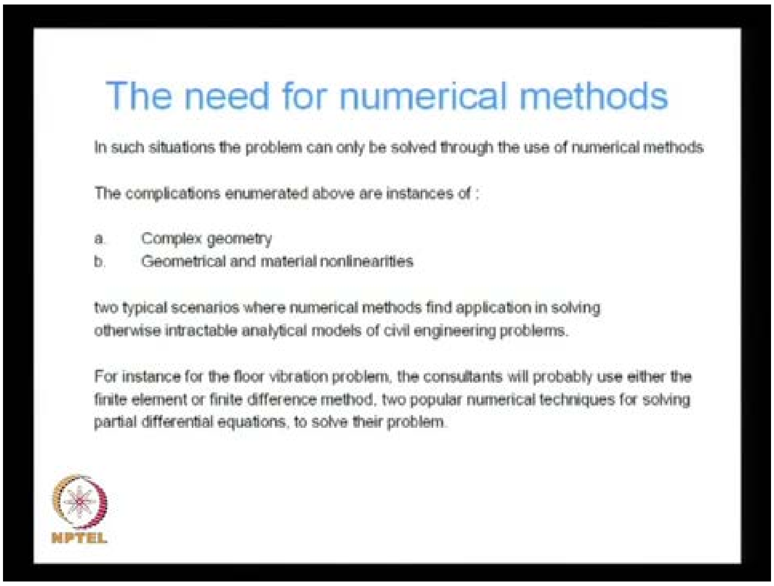

In such situations the problem can only be solved through the use of numerical methods, the complications enumerated above are instances of a, complex geometry geometrical and material nonlinearities, two typical scenarios where numerical methods. Find application in solving otherwise intractable analytical models of civil engineering problems, for instance for the floor vibration problem, the consultants will probably use either the finite element or finite difference method, two popular numerical techniques for solving partial differential equations to solve their problem.

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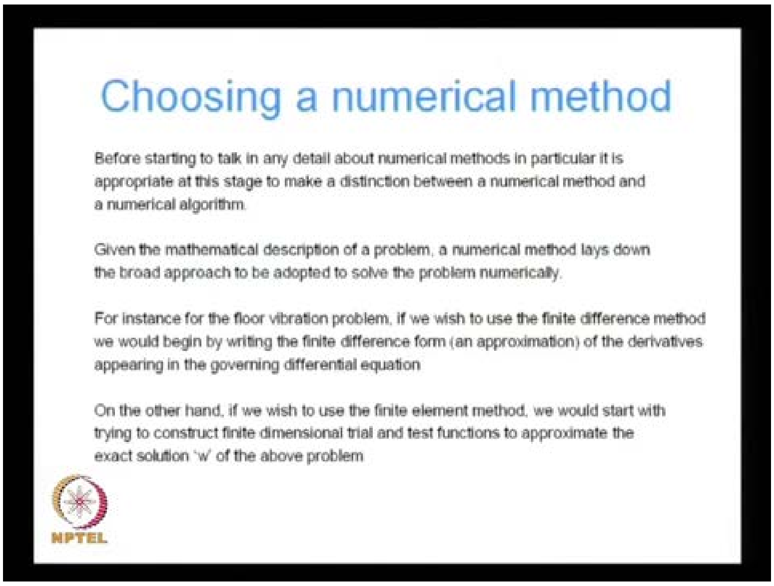

Before starting to talk in any detail about numerical methods in particular, it is appropriate at this, stage to make a distinction between a numerical method and a numerical algorithm. Given the mathematical description of a problem a numerical method lays down, the broad approach to be adopted to solve the problem numerically, for instance for the floor vibration problem.

If we wish to use the finite difference method, we would begin by writing the finite difference from which is an approximation of the derivatives appearing in the governing differential equation On the other hand, if we wish to use the finite element method, we would start with trying to construct finite dimensional trial and test functions to approximate, the exact solution w of the above problem.

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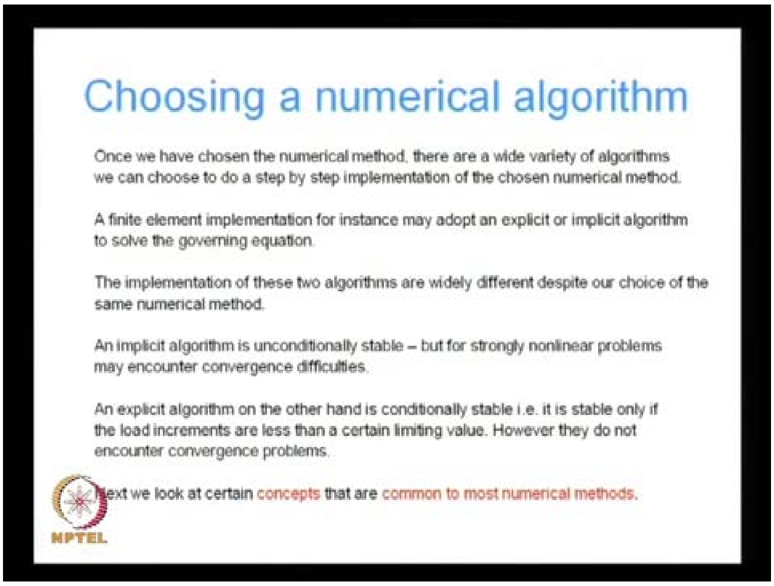

Once we have chosen a numerical method there are a wide variety of algorithms, we can choose to do a step by step implementation of the chosen numerical method. A finite element implementation for instance may adopt an explicit or implicit algorithm, to solve the governing equation. The implementation of these two algorithms are widely different, despite our choice of the same numerical method, that is the finite element method an implicit algorithm is unconditionally stable.

But, for strongly non-linear problems, may encounter convergence difficulties an explicit algorithm, on the other hand is conditionally stable, that is it is stable only if the load increments are less than a certain limiting value, however, they do not encounter convergence problems next we would look like, we would like to look at certain concepts that are common to most numerical methods.

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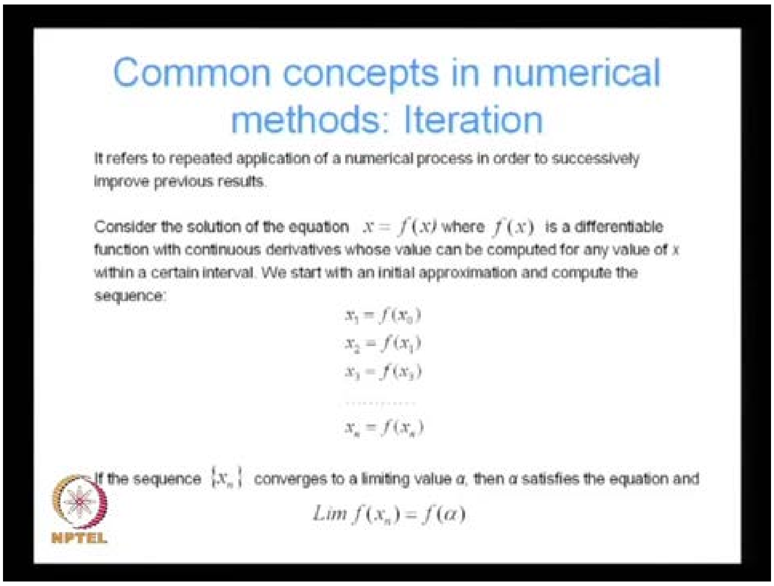

The first concept, that we wish to examine is the idea of iteration, it refers to repeated application of numerical process in order to successively improve previous results. Consider the solution of the equation x is equal to f of x, where f of x is a differentiable function with continuous derivatives, whose value can be computed for any value of x within a certain interval. We start with an initial approximation and compute the sequence, we start with the initial approximation x 0 and substituting x 0, in the and expression for f of x we compute the value x 1.

We use the new value x 1, again in the equation for f x to come up with the next iterate x 2 and so on, and so forth, until we get f of x n minus 1, which is equal to x n, if the sequence x n converges to a limiting value alpha then alpha satisfies, the equation and limit limiting value of f of x n is equal to f of alpha. So, alpha is the root of this equation.

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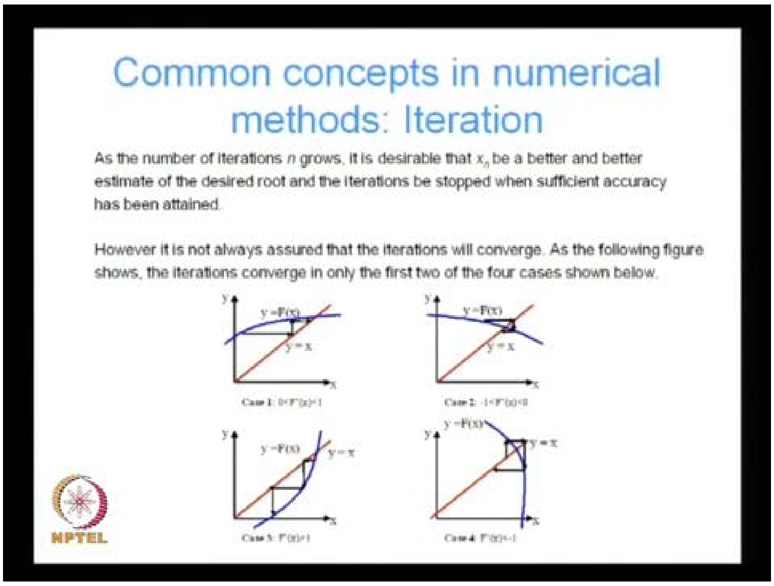

As the number of iterations n rows, it is desirable that x n be a better and better estimate of the desired root and the iterations be such stopped, when sufficient accuracy has been attained. However, it is not always assured that the iterations will converge as the following figure shows, the iterations converge in only the first two of the four cases shown below.

We start with an initial assumption and in case 1 and case 2, we converge to the true solution which is where the blue curve intersects, the red curve. In case 3 and 4, we can see that even, if we start near these two solutions are iteration process, takes us away from the root of the true solution.

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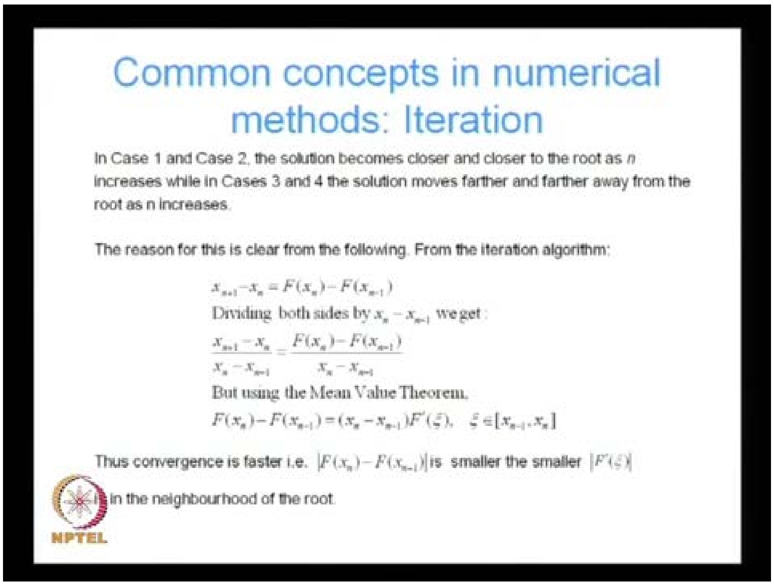

In case 1 and case 2, the solution becomes closer and closer to the root as n increases, while in cases 3 and 4 the solution moves farther and farther away from the root as n increases. The reason for this is clear from the following from the iteration algorithm, we can write x n plus 1 minus x n is equal to f of x n, minus f of x n minus 1, we called that according to algorithm x n plus 1 is equal to, f of x n and x n is equal to f of x n minus 1, dividing both sides by x n minus, x n minus 1, we get the following.

But, using the mean value theorem, we can write f x n minus f of x n minus 1 is equal to x n minus, x n minus 1 times the derivative of f evaluated at a value xi, where xi lies in the interval x n minus 1 and x n. Thus looking at the last equation it is clear, that convergence is faster that is the modulus of f of x n minus, f of x n minus 1 is smaller, the smaller f prime xi is in the neighborhood of the root, f of x n will be closer to f of x n minus 1, that is the iterates are going to convert faster the smaller is the value of the derivative f prime xi.

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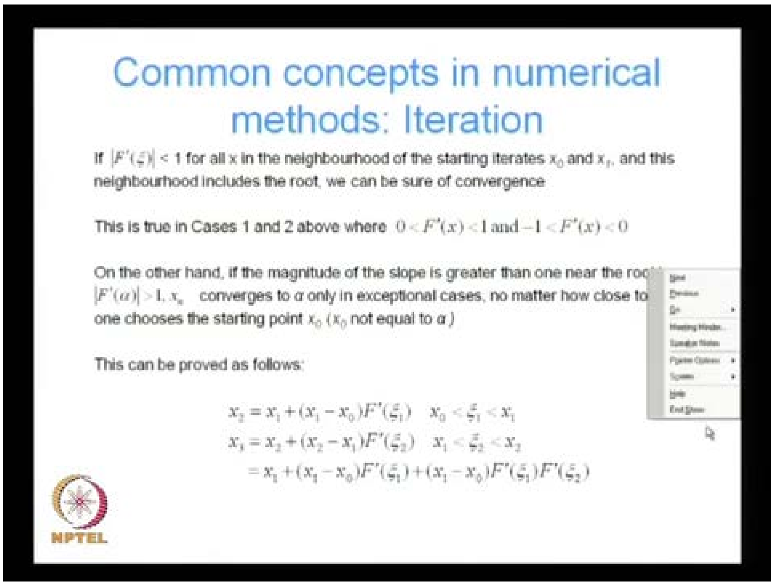

If mod f prime xi is less than 1 for all x in the neighborhood of the starting iterates x 0 and x 1 and this neighborhood includes the root, we can therefore, be sure of convergence. This is true in cases 1 and 2 above, where f prime of x is always greater than 0 and less than 1 in case of case 1 and f prime of x is greater than minus 1 and less than 0 in case of case 2, thus we can see that in case 1 and case 2 f prime of xi, the mod of f prime of xi will always be less than 1.

On the other hand, if the magnitude of the slope is greater than 1 near the root, that is mod of f prime alpha is greater than 1, x n converges to alpha only in exceptional cases no matter, how close to alpha one chooses the starting point x 0, for instance in case 3 and case 4 even though, we have actually started quite close to the root, you can see that we are moving away from the trial solution, this is because the slope at the root is greater than 1.

The absolute value of the slope at the root is greater than 1, thus as we iterate we move further and further away from the trial solution, while in case in case 1 and case 2, as we iterate we move closer and closer to the trial solution. We can come up with a quick mathematical proof for this can be proved as follows, we can write x 2 is equal to x 1 plus x 1 minus x 0 times f prime of xi 1 again according to the mean value theorem provided xi 1 lies between x 0 and x 1. Similarly, we can write x 3 is equal to x 2 plus x 2 minus x 1 times f prime of xi 2, where xi 2 lies between x 1 and x 2, using the first equation in the second, we can write x 3 in terms of x 1 x 0 and the derivatives f at xi 1 and xi 2.

Using the same idea by induction for any iterate, we can write the following for instance x n minus 1, we can write in terms of x 1 and x 0, according to this equation to the first equation. Similarly x n, we can write in terms of x 1 and x 0, according to the second equation subtracting, the second equate the first equation from the second equation, we get this equation x n minus x n minus 1 is equal to x 1 minus x 0 times f prime of xi n minus 1 times f prime of xi n minus 2 times up, which is the series extending up to f prime of xi 1.

Thus we can see that if the magnitude of f prime of xi I is less than 1 for all I modulus of x n minus x n minus 1, becomes smaller and smaller for large values of n and the iteration converges, f prime xi n minus 1 will be less than f prime xi n minus two times, f prime xi n minus 3 up to f prime xi n minus one and so on and so forth. As the as we add more and more terms the right hand side, becomes smaller and smaller and the left hand side converges.

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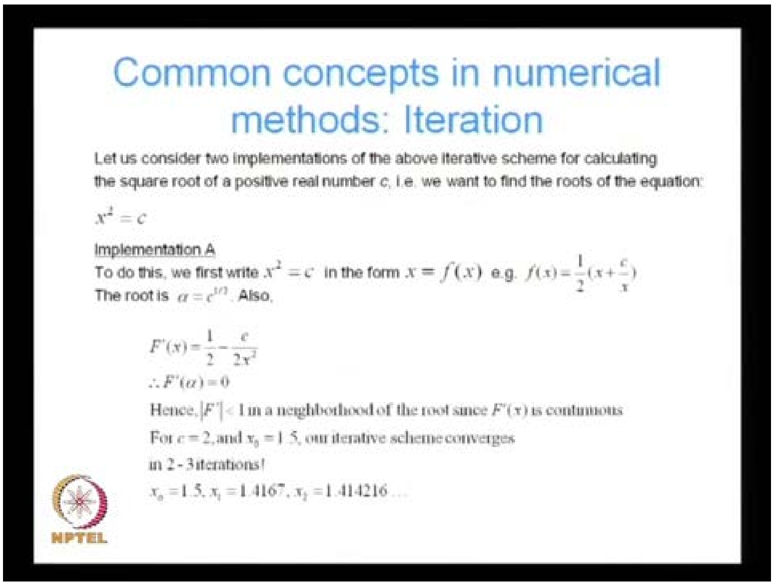

Next, let us consider two implementations of the above iterative scheme for calculating the square root of a positive real number c, that is we want to find the roots of the equation x square is equal to c. We look at the first implementation were we write f of x, as f of x is equal to half x plus c of x, the root obviously is alpha is equal to c of half, c to the power half, also it is clear that f prime of x the derivative of f of x is half minus c by 2 x square.

Therefore at the root alpha f prime of alpha is equal to 0, hence mod of f prime is less than 1 in a neighborhood of the root actually, it is 1 in a neighborhood of at the root and in at the neighborhood of the root, it has to be they must they must exist a neighborhood of the root, where mod of f prime is less than 1. Since, f prime of x is a continuous function for c is equal to 2 and with us starting case of x 0 is equal to 1 point 5 are iterative scheme converges in 2 to 3 iterations. We start with the x 0 is equal to 1 point 5 and by the time, we reach iteration number 2, we can see we are really close to the true solution which is root 2, 1 point 4 1 4.

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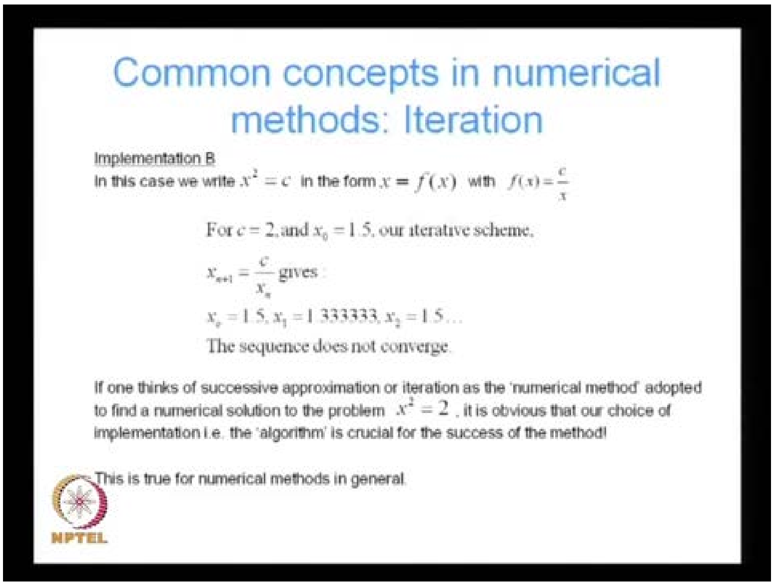

On the other hand, if we consider implementation b in which case we write f of x is equal to c by x for c is equal to 2 and x is equal to 1 point 5 are iterative scheme, which recall is x n minus 1 is equal to f of x n, which is equal to c by x n gives x 0 is equal to 1 point 5 x 1 is equal to 1 point 3 point 3 x 2 is equal to 1 point 5. So, it oscillates right the equate the sequence does not converge, If one thinks of successive approximation or iteration as the numerical method, adopted to find a numerical solution to the problem x square equal to 2.

It is obvious that our choice of implementation, that is the algorithm is crucial for the success of the method, the first implementation converged in two iterations by the second implementation is never going to converge, this is true for numerical methods in general.
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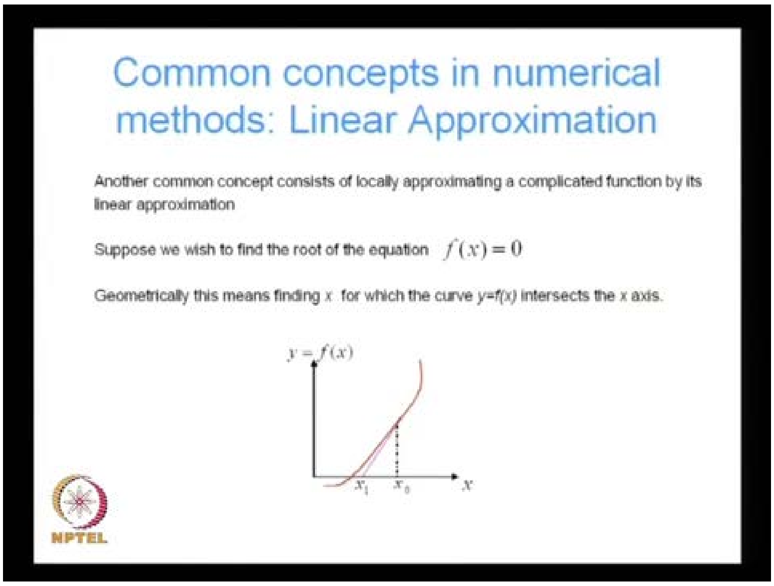

Another commonly used technique in numerical methods is linear approximation here, we locally approximate a complicated function by its linear approximation. Suppose, we wish to find the root of the equation f of x is equal to 0, geometrically this means finding x, for which the curve y is equal to f of x intersects the x axis. So, we start with the starting, case x 0 and then use a numerical method an algorithm to try to reach the solution, which is where the red curve intersects the x axis.

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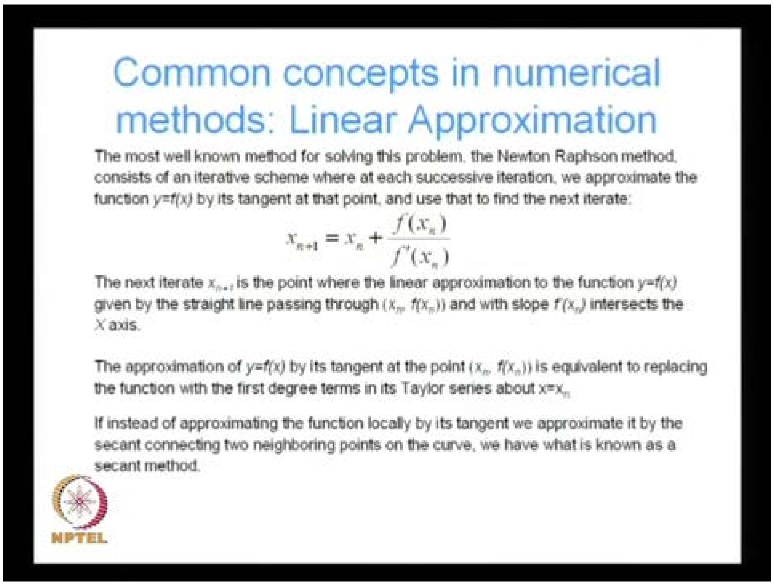

The most well-known method for solving this problem, the Newton rap son method consists of an iterative scheme, where at each successive iteration. We approximate the function y is equal to f of x by its tangent, at that point and use that to find the next iterate, for instance at x 0, we have approximated the true the curve y is equal to f of x by the blue line, the blue line being the tangent to the function y is equal to f of x I take 0, where the blue line meets the x axis that gives me by next iterate x 1.

So, x n plus 1 I can write as x n plus f of x n divided by f prime of x n, the next iterate x n by plus 1 being the point, where the linear approximation of the function y is equal to f of x given by the straight line, passing through x n and f of x n with slope f prime of x n intersects, the x axis. The approximation of y is equal to f of x by its tangent at the point x n f of x n is equivalent to replacing, the function with the first degree terms in its Taylor series about x is equal to x n.

If instead of approximating the function locally by its tangent, we approximate it by the secant connecting two neighboring points on the curve, we have what is known as a secant method, which is another commonly used method for linear approximation of a non-linear function.

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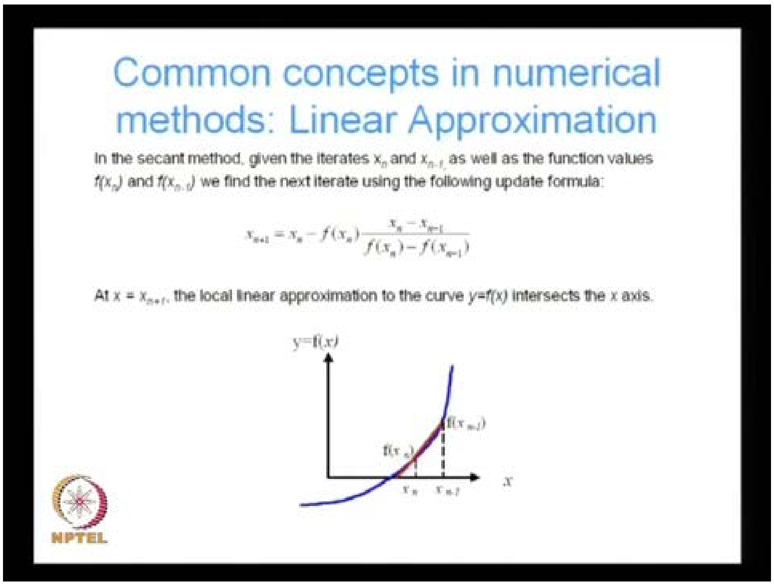

In the secant method given the iterates x n and x n minus 1, as well as the function values f of x n and f of x n minus 1, we find the next iterate using the following update formula at x is equal to x n plus 1 the local linear approximation to the curve y is equal to f of x intersects, the x axis. As you can see we have tried to fit a line through the values of the function at iterate x n minus 1, which is f of x n minus 1 and x n which is f of x n. And then where the line meets the x axis, that is going to give me by next iterate x n plus 1.

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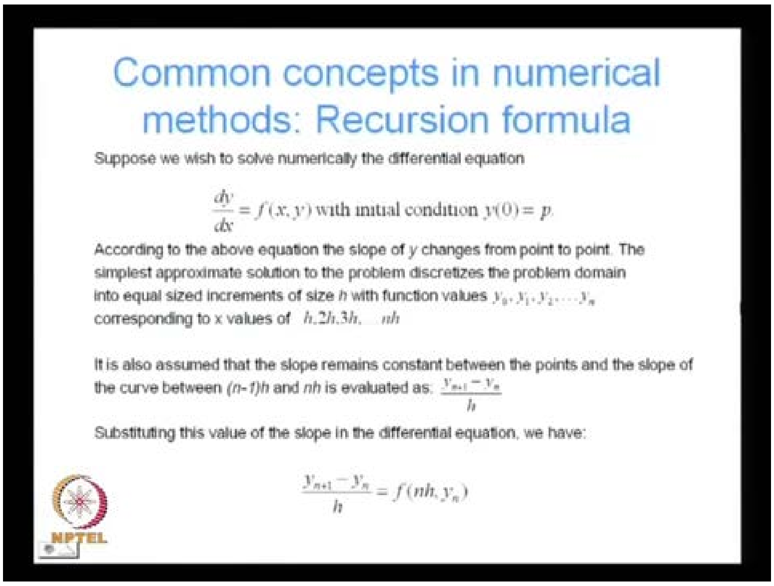

The third common numerical concept that, we wish to talk about is the idea of recursion, suppose we wish to solve numerically the differential equation d y d x is equal to f of x y with initial condition y 0 is equal to p. According to the above equation, the slope of y changes from point to point, the simplest approximate solution to the problem discreteness, the problem domain into equal sized increments of size h with function values y 0, y 1, y 2 and y n, corresponding to x values of h, 2h, 3h and n h.

So, at x is equal to 0, we have y is equal to y 0 at x is equal to h, we have y equal to y 1, at x is equal to 2 h, we have y equal to y 2 and so on and so forth. It is also assumed that the slope remains, constant between the points and the slope of the curve, between n minus 1h and n h for instance is evaluated as y n plus 1 minus yn divided by h. Substituting this value of the slope in the differential equation, we have y n plus 1 minus y n divided by h is equal to f n h, which is the value of x and y n which is the value of y.
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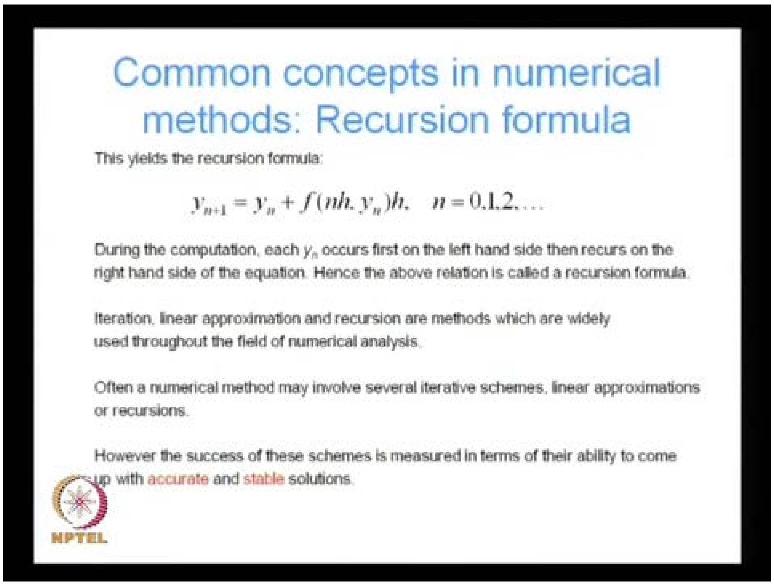

We can rewrite this equation to yield the recursion formula, we can rewrite the previous equation by keeping y n plus 1, on the left hand side and moving the other terms to the right hand side So, y n plus 1 is equal to y n plus f of n of h y n times h, this is going to be true for all values of n, n is equal to 0 1 2 and so and so forth, from this equation you can see that each y n occurs first on the left hand side, then recurs on the left hand on the right hand side of the equation, hence the above relation is called a recursion formula.

So, if we know a starting value for y, if we know y 0 for instance using this recursion formula, we can successively calculate y 1 y 2 y 3 up to y n. And we can continue the recursion process, until my recursion formula converges that is until my value y n minus

1, becomes equal to y n iteration linear approximation and recursion are methods, which are used widely throughout the field of numerical analysis, often a numerical method may involve several iterative schemes linear approximations or recursions.

However the success of these schemes is measured in terms of their ability to come up, with accurate and stable solutions accuracy and stability are two prerequisites of any successful, numerical algorithm in the following we are going to talk about each of these concepts in greater detail. We will start with talking about the notion of stability, following which we will talk in greater detail about accuracy.
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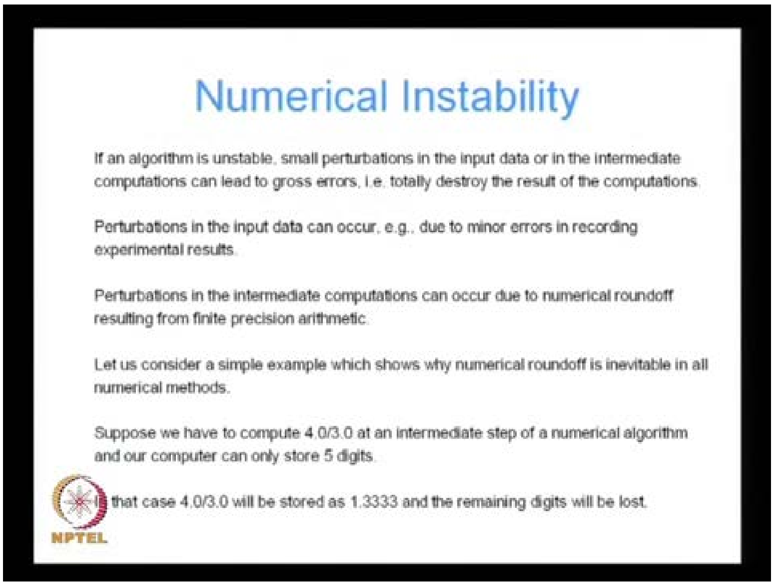

What do we mean by numerical in stability, if an algorithm if a numerical algorithm is unstable small perturbations in the input data or in the intermediate computations can lead to gross errors, that is the totally destroy the result of the computations. Perturbations on the input data in the input data can occur for instance, due to minor errors in recording experimental results.

If the input data to a numerical algorithm is experimental data and there are minor errors in the experiment which are always lightly, then if it is an unstable algorithm those minor perturbations can give me totally erroneous results. Perturbations in the intermediate computations, can occur due to numerical round of resulting from finite precision arithmetic.

Any computer; however, sophisticated only deals with that many numbers, so any number can be if a computer has accuracy up to t digits, it can only approximate a number using t digits for instance. Suppose we have to compute 4 by 3 at an intermediate, step of a numerical algorithm and our computer can only store 5 digits in that case 4 by 3 will be stored as 1 point 3333 and the remaining digits will be lost.

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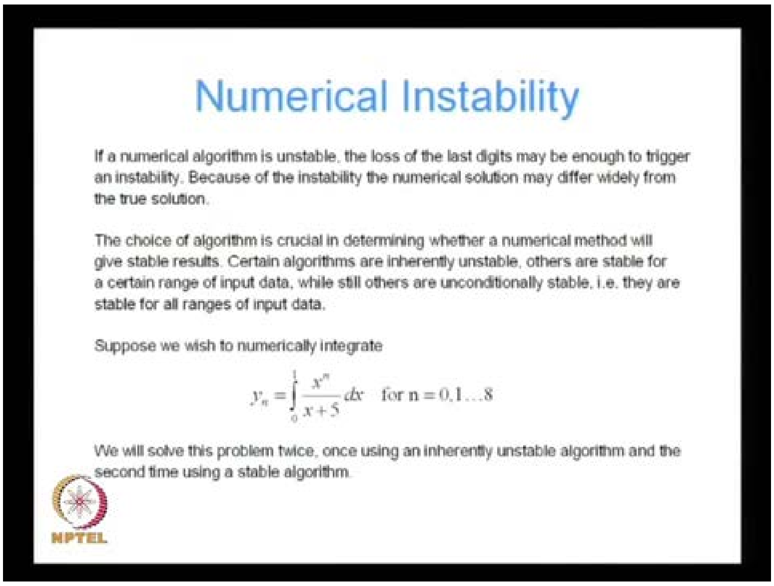

If the numerical algorithm is unstable the loss of the last digits, may be enough to trigger an instability, because of the instability the numerical solution may differ widely from the true solution. The choice of algorithm is crucial in determining, whether a numerical method will give stable results, certain algorithms are inherently unstable, others are stable for a certain range of input data, while still others are unconditionally stable, that is they are stable for all ranges of input data.

Suppose, we wish to numerically integrate, the following integral y n equal to integral of x to the power n, divided by x plus 5 and we wish to integrate within the limits 0 and 0 to 1 for values of n equal to 0 1 through 8. We will solve this problem twice, once using an inherently unstable algorithm and the second time using a stable algorithm.

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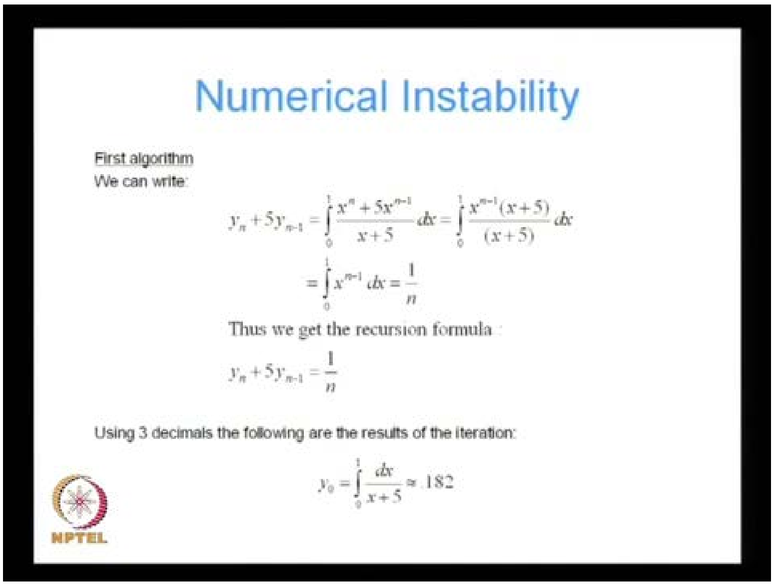

For the first algorithm, we write y n plus 5 times, y n minus 1 is equal to integral of x n plus 5 x to the power n minus 1 divided by x plus 5. Recall that this is our integrant, so y n is this y n minus 1 is equal to integral from 0 to 1, x to the power n minus 1 divided by x plus 5. So, we can write this and doing certain simplifications, we can reduce it to 1 by n, if we perform this integration it comes out as 1 by n, thus we get the recursion formula y n plus 5 times y n minus 1 is equal to 1 by n using three decimals the following are the results of a iteration.

We start with y 0, which we calculate accurately up to three decimals places from which we get y 0 is approximately equal to 1 point 8 2, then using y 0 in our recursion formula, we get y 1 is equal to 1 minus five times y 0.

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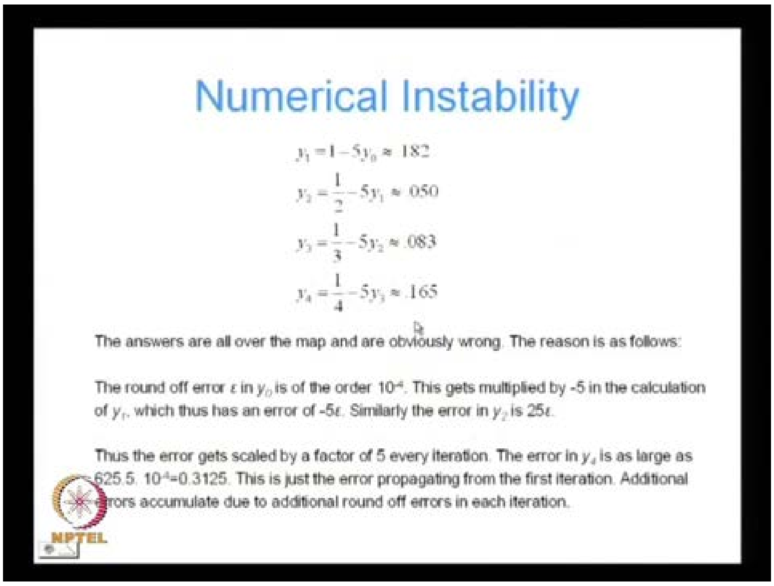

Which comes out as point 1 8 2 y 2 can be written as half 1 by 2 minus five times y 1, again according to our recursion formula, which gives me point 0 5 0, y 3 gives me 1 by 3 minus 5 times y 2 which is point 0 8 3 all being computed accurately up to three decimal places and y 4 gives me point 1 6 5. As you can see the answers are varying widely, they are all over the map and are obviously wrong, the reason is as follows.

Suppose the round off error in the computation of y 0 epsilon is of the order 10 to the power minus 4, which is the reasonable, because we are using three significant digits in our computations. So, the error in y zero is of the order of ten to the power minus 4 in our second recursion relationship, this gets multiplied by minus 5 in the calculation of y 1. Because, you can see there is 5 times y 0, so whatever error there is in y 0 gets multiplied by factor of 5, therefore y 1 has an error of minus 5 epsilon.

Similarly, y 2 will have an error of 25 epsilon, because again we are multiplied y 1 by a factor of 5, thus the error gets scaled by a factor of 5 in every iteration, the error in y 4 is as large as 625 times 5 times 10 to the power minus 4, which you recall was an initial error the error in y 4 is actually point 3125. So, you can see that the error has increased by at least three orders of magnitude, this is however note is just the error propagating from the first iteration.

Epsilon was the error in the first iteration, additional errors accumulate due to additional round off errors in each iteration, because in each iteration we are doing computations up to only three decimal places.

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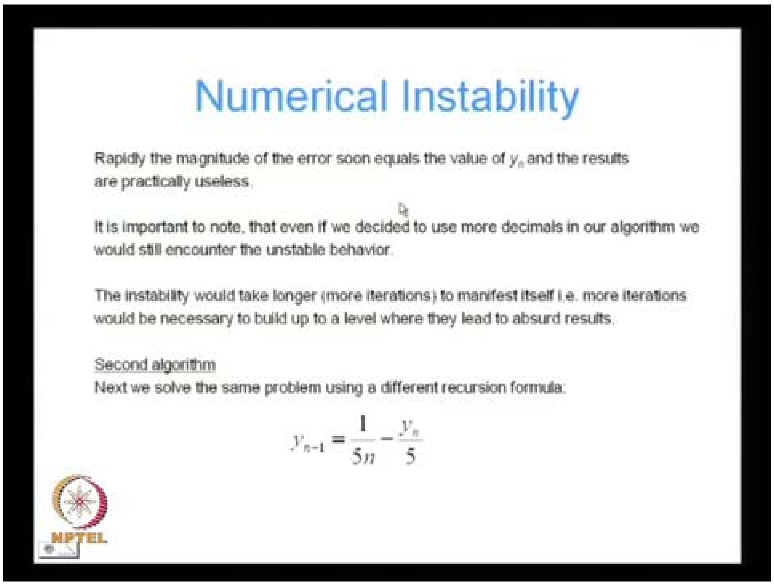

Rapidly the magnitude of the error soon equals the value of y n and the results are practically useless, it is important to note that even if, we decided to use more decimals in our algorithm for instance, instead of using numbers with three decimals. If we use numbers to 5 or 6 decimals, we would still encounter the unstable behavior the only difference would be that the instability would take longer more iterations to manifest itself, that is more iterations would be necessary to build up to a level where they lead to absurd results.

So, this is not a problem of performing computations with two few, two little precession, it is an the problem is the algorithm we have chosen is inherently unstable. The second algorithm we use is slightly different in this case, we use a variation of initial recursion formula, which you recall was this instead of using this recursion formula. We use the following recursion formula, where you can see the only difference is that we have divided both sides, by we have brought the five down to the denominator in the right hand side, but those two equations are identical it is just that a recursion formula is different.

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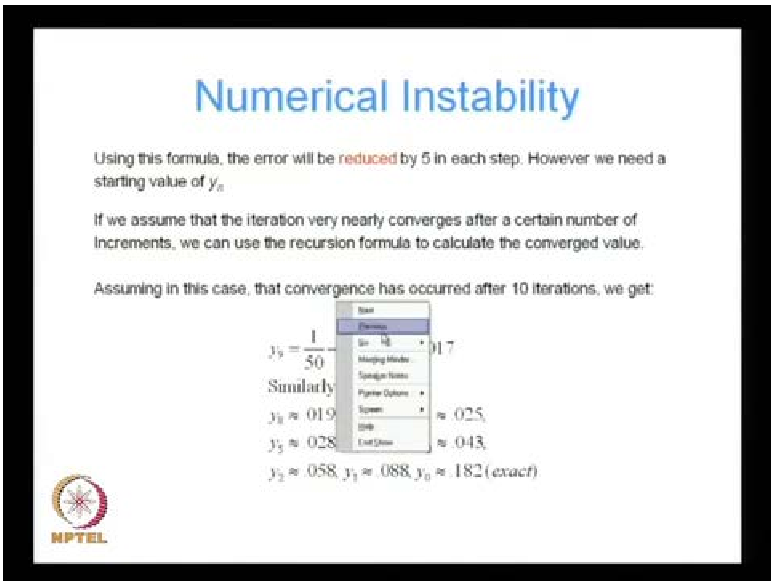

If we use this algorithm the error will be reduced by 5, in each step however, we need a starting value y n if we assume that the iteration very nearly converges after a certain number of increments. We can use the recursion formula to calculate the converged value, assuming in this case that convergence has occurred after 10 iterations, so let us go back to our previous slide.

So, we are assuming that the iteration has converged in 10 iterations, so y 10, y 9 is equal to 1 by 5 times, 10 minus y 10 by 5. Since, we have assumed that the iteration as converged in 10 iterations on the right hand side, we can replace by 10 by y 9, the assumption being that y 10 is almost exactly equal to y 9, since the iteration has converged in 10 iteration.

So, we can write y 9 is equal to 1 by 5 times 10, which is 1 by 50 minus y 9 by 5, which gives me y 9 is approximately equal to point 0 1 7, again notice, that we are do performing our computations only for up to three decimal places. We are we have not increased the accuracy of our computations, similarly doing the same using the same recursion formula, we get y 8 is approximately equal to point 019, y 7 is approximately equal to point 0 2 1, y 6 is point 0 2 5, eventually at y 0, we get point 1 8 2, which is the exact solution for this problem.

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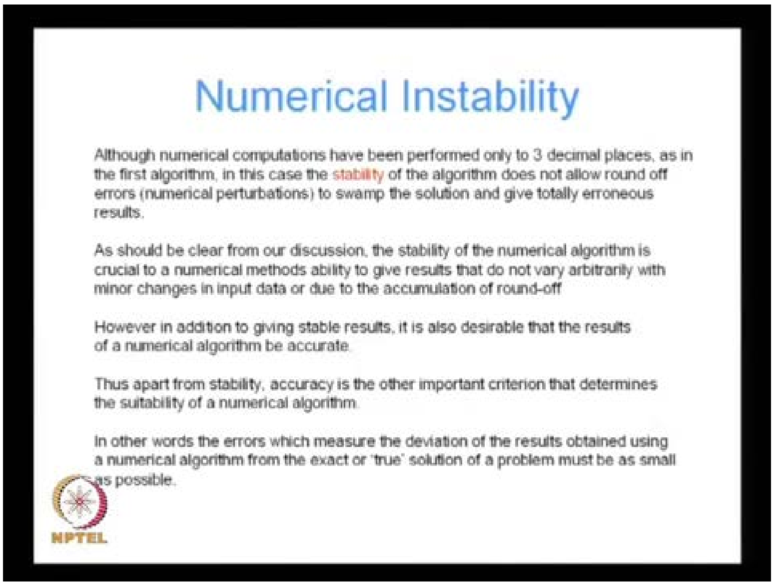

Although the numerical computations have been performed up to three decimal places as in the first algorithm, in this case the stability of the algorithm does not allow round off errors, that is numerical perturbations to swamp the solution. And give totally erroneous results. As should be clear from our discussion, the stability of numerical algorithm is crucial to a numerical methods, ability to give results that do not vary arbitrarily with minor changes in input data or due to the accumulation of round off.

However, in addition to giving stable results it is also desirable the results of a numerical algorithm be accurate, thus apart from stability accuracy is the other important criteria, that determines the suitability of a numerical algorithm. In other words the errors, which measure the deviation of the results obtained using a numerical algorithm from the exact or true solution of a problem must be as small as possible. This can be analyzed through what is known as error analysis, if the next lecture in the series, we are going to talk up in greater detail about error analysis. Since error and stability are the two criteria, which determine the effectiveness of a numerical algorithm.

Thank you.