Advanced Foundation Engineering

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Lecture - 35
Soil - Foundation Interaction (Contd.)

In last class I have discussed about various foundation model. Basically a two parameter models and how the limitations of Winkler model can be removed by using two parameter model, and then using non-linear soil parameters. Now, and then I have also discussed about the elastic continuum approach and elastic continuum model, and then isotopic elastic continuum, isotopic elastic half space and then an isotopic elastic continuum thing. Now, today I will discuss about the beams on elastic foundation. Now, if we place a beam generally footing is can be idealized as a beam for the analysis purpose, and then with the application of different types of load, different types of beams some which is finite beam or infinite beam.

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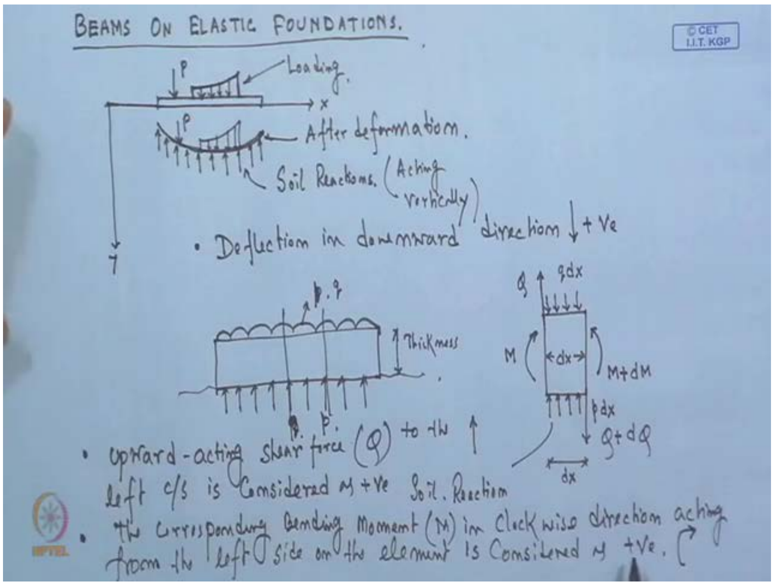

Now, if we place beam on a soil medium then what will be the response of that beam? How we can solve this that beam equation, so that thing we will explain today. Now, if I plot that beams on elastic foundation. In the beams of elastic foundation suppose, if this is the z axis or y axis. This is x axis and if we place a beam here and then we can apply different types of loading. Suppose, this is a u d l or we can play, put a concentrated load

P. We can put a varying u d l here and then this is the position of the beam before deflection and then after that the beam will deflect. And this will be the again, this is the concentration load concentrated load P and then the loading which is acting here, and this is the deflection line and here base of the beam there will be the reaction of the soil.

So, this would be the soil reaction where, we can analyze the soil reaction in this form. So, this will be the soil reaction and these are the applied load and this will be the deformation set. So, this is this form is before deformation and this is the after deformation. Now, reaction force to be assume to be acting vertically and opposite opposing the deflection of the beam. So, that means this reaction force that is assumed to be acting, this is acting vertically and it is opposing the deflection of the beam. Means we have the deflection in directed downward direction. So, that means here the deflection in downward direction is taken as positive and deflection in the upward direction is taken at negative.

So, although it is assumed that soil cannot take the tension, but here it is for the present analysis it assumed that, it can take for the analysis purpose. But that is as treated as hibbing of the foundation soil. So, that means basically the direction or the deflection in the downward direction is taken as the positive deflection. Now, if we and this is the reaction. Now, if we take one beam. This is the beam and which is subjected to a u d l q or p and reaction q. Or we can say this is q reaction is p. So, that means this beam is placed on the ground and this is the thickness of the beam, so now if we take one small component of this beam. Suppose if I take this small element and then if we draw the various forces and moment on the beam. So, this is acting in the this thickness is taken as this small element is a width of d x and whose q is acting here.

So, that means force this will be q into d x. Now, these are the reaction p is the reaction force that is p into d x. So, this is reaction force or soil reaction then that q where shear force acting in the left side is q and one moment is acting in the clockwise direction is M in the left side. Similarly, the moment it will act in the anti-clock, clockwise direction in the right side. That is M plus d M and Q will act another shear force that is Q plus d Q that will act in the downward. So, these are the forces and moments for the small segment of d x. So, that is q is the applied load, p is the reaction, soil reaction, M is the moment. Capital Q is the shear force in the website.

Now, the sin conversation that is used that upward acting shear force. So, here Q to the left cross section is considered as positive. So, the upward acting shear force in the left side is considered as positive and the corresponding moment and the corresponding bending moment M in clockwise direction acting from left side on the element is taken as as positive bending moment. So, we have the sign can being change for the one for the deflection, one for the shear force and one for the moment, so for the, and in this analysis deflection the downward direction in the downward direction is positive. The shear force acting upward direction in the left side is positive corresponding bending moment acting clockwise direction for on the left side of on the element is considered as positive.

So, these are the sign convention where the clockwise moment in the right side left side is positive, shear force upward direction left side positive and deflection downward direction is positive, so next one that how to solve this beam expression.

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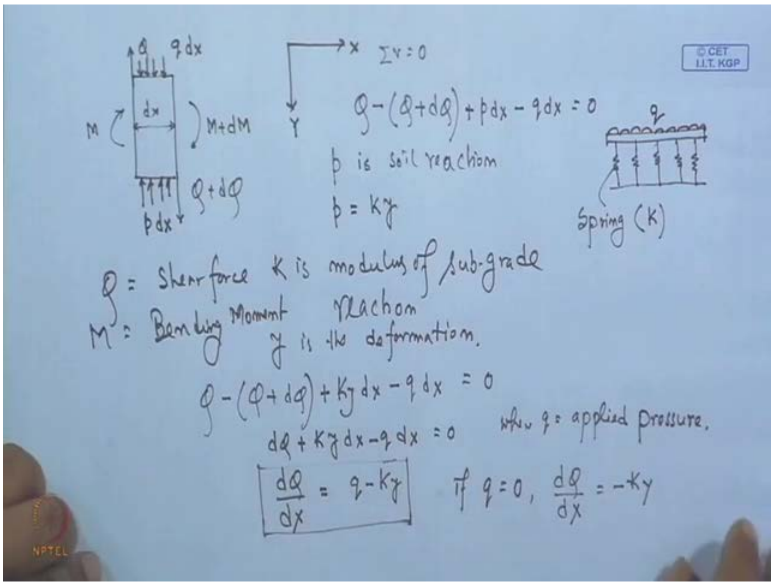

Suppose, again if we draw the same element of d x this is the Q and here, if I draw the, this is x axis coordinate system this is y axis and then shear force then downward direction right side Q plus d Q and the moment M, moment this side M plus d M. The force that will act this q into d x and the reaction that will act p into dx, so this is the force acting on the beam, this is that is soil reaction which is given on the beam and this is shear force left side in this moment. Now, if I write the expression that is you can write Q minus Q plus d Q, all the vertical forces so summation of vertical force that is 0.

So, Q minus Q plus d Q plus so upward direction that is plus p into d x minus Q into d x that is equal to 0.

Now, here the soil p is soil reaction. So, p small p is is soil reaction. In early case now, if this beam suppose this beam is placed on a soil, where this soil are idealized as spring. This is spring with stiffness K or K is the modulus of sub grade reaction. So, on this beam q is acting. It is the u d l. So, these soils are giving the reaction and the soil soil this soil, is giving the reaction and soil is idealized by spring. So, you can write that q p will be equal to K into y, where K is the spring constant or modulus of sub grade reaction. K is modulus of sub grade reaction, y is the deformation. So, that is the principle of Winkler model that is any force which is proportional to the deflection and the k is the spring constant.

So, finally, if I put this p in the final expression, that will be plus K y into d x minus q into d x that is equal to 0. So, finally... So, q will be cancelled out so d Q plus k y d x minus q d x equal to 0 where q is the applied pressure and here it is analysis so, that means we can write the d Q by d x that will be equal to q minus K y. So, this is the reaction. Now , if we do not have any applied pressure or soil applied load. That means if q is equal to 0 that means there is no force that means this q is applied 0 then the expression will be d Q by d x minus K into y. y y is the deformation or deflection and K is the modulus of sub grade reaction.

So, we have that expression final expression of the beam that d q by d x equal to so, that means here Q this is the shear force, M is the bending moment. This is the shear force M is the bending moment y is the deflection. Now, finally next step then how to calculate the other part of the, once we will get the final expression of the beam. Then how we will calculate the other reaction or other forces that means the form that beam how we will calculate.

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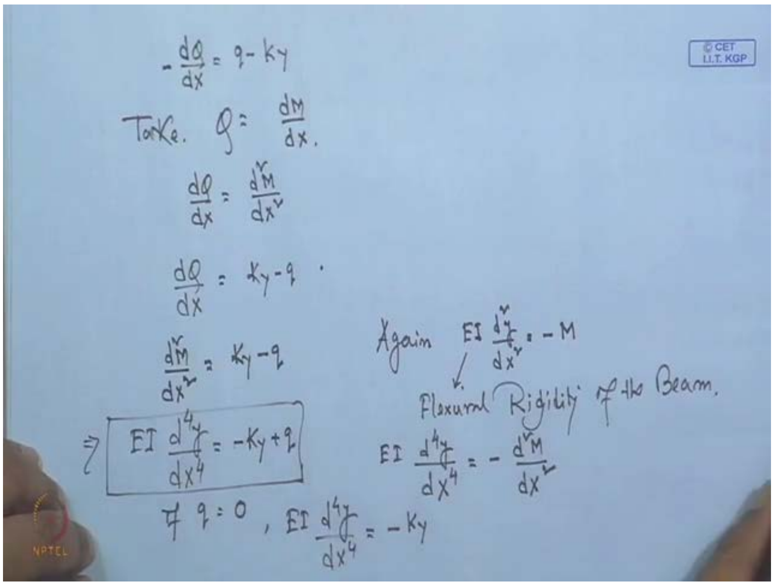

So, Q is the rate of moment that means d M by d x. So, we can write d Q by d x. That is equal to d square M by d x square. So, we will put this expression in the final equation that we will get that, d Q by d x that is equal to K y minus q or we can take that is because here this negative term is there. So, we will get d K Q finally. So, that means finally, from here we will get d Q by d x, K y into minus q So, we will replace this expression by d square M by d x square equal to K y minus q. So, in the previous expression, if we get that expressions.

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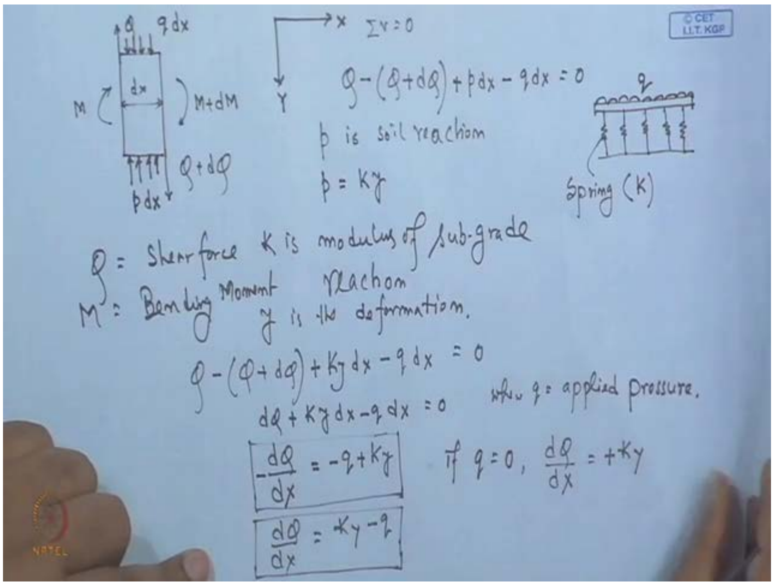

So, here. There is a negative term. So, we can write this is minus q plus K 1. So, this will be plus K 1. So, d Q by d x is K y minus q. This will be the final expression of the, because there is negative term is there so, we can write in this form. So, if there is no q so, there will be K into y. So, finally, from this K into this y minus q we put d square n d x square and again, we know that E I d square y d x square is equal to minus M. where E I is the flexural rigidity rigidity of the beam. So, again we know this expression. So, finally, for d square M by d x square, we can if we differentiate these equation twice. Then we will get E I d to the power 4 y d x to the power 4. That is equal to minus d square M d to the power d x square.

Now, you put this expression into here, then we will get that E I d to the power 4 d x to the power 4 equal to minus K y plus q. So, again if q is equal to 0 then there will be E I d to the power 4 d x to the power 4 will be minus K into y. So, now we have that expression that E I d to the power 4 y d x to the power 4 minus K y plus q. Now, you have to solve these expressions for the different condition.

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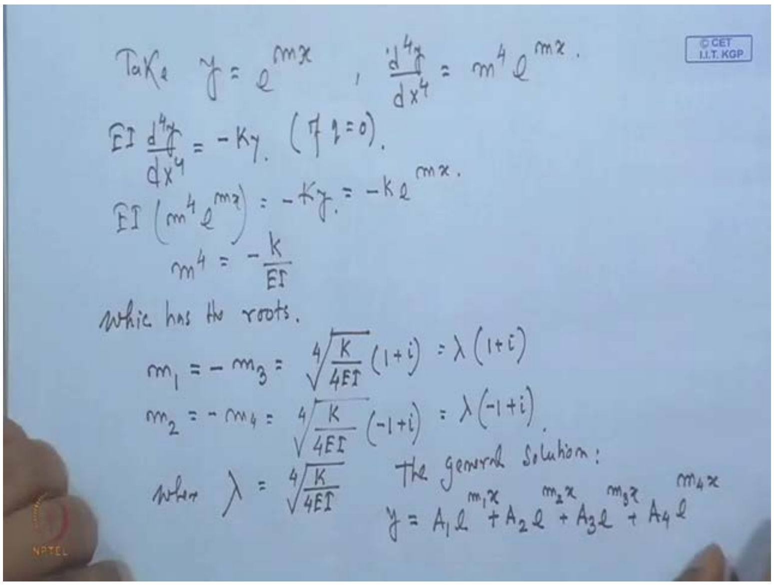

Now, once we if we put take y is equal to e to the power m x, then we have the expression E I d to the power 4 y d x to the power 4 equal to minus K into y if q is equal to 0. So, now if we differentiate this y 4 times then we will get this d to the power 4 y d x 4. That gives you m to the power 4 e m x. So, now, if we put this expression here E I into m to the power 4, e to the power m x that is equal to k into y and here y is also e to the power m x. So, this e to the power power m x will be cancelled out. So, m to the power 4 is equal to minus k by E I.

Now, this m which has two roots. Basically, this m which has this 4 root that is one is m 1 it is equal to minus m 3 that is equal to K 4 E I 1 plus and this is lambda 1 plus 5. Another one m 2 will minus m 4 that is equal to four K four E I minus 1 plus. So, that is equal to lambda minus 1 plus I, where lambda is 4 K 4 to the E I. 4 root y into K by four E I, so in general solution of this equation. Solution will be y equal to a 1 e to the power m 1 x plus a 2 e to the power into x, plus a 3 e to the power m 3 x plus a 4 e to the power m 4 x. So, why at a 1 e to the power m m 1 x, a 2 e to the power m 2 x, a 3 e to the power m 3 x and a 4 e to the power m four x. Now finally, so we will get this expression.

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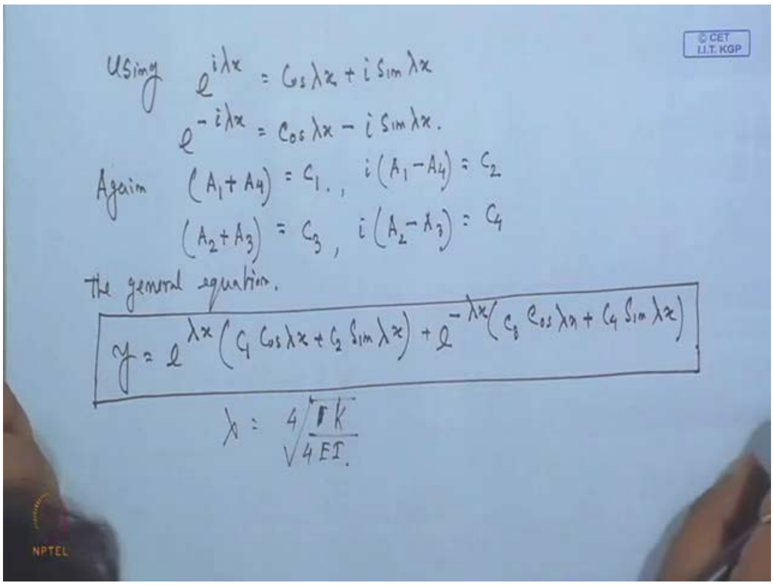

Now, using e to the power i lambda x is equal to cos lambda x plus i sin lambda x and e to the power minus i lambda x this is equal to cos lambda x minus i sin lambda x. So, again if we take that A 1 plus A 4 is equal to C 1 and i A 1 minus A 4 is equal to C 2. Similarly, a 2 plus A 3 that is equal to C 3 and i A 2 minus A 3 is equal to C 4. Then the final expression general equation will be, y is equal to e to the power lambda x C 1 cos lambda x plus, C 2 sin lambda x, plus e to the power minus lambda x C 3 cos lambda x plus C 4 sin lambda x. So, this will give us the final expression of this beam where lambda is given 4 root k four E I.

So, this lambda is basically the, non dimensional value of flexural rigidity of the b. Now, here we have four constant C 1 C 2 C 3 and C 4. Now, we have to determine next step is that, we have to determine this four constant then how we will determine this four constant? This is the final expression of the beam and then how we will determine this four constant of this expression. That will depend the, to use the boundary condition to solve this four constant. Now, before we solve this four constant, then we have another value that how to determine the bending moment slope from this expression because we have this expression final expression.

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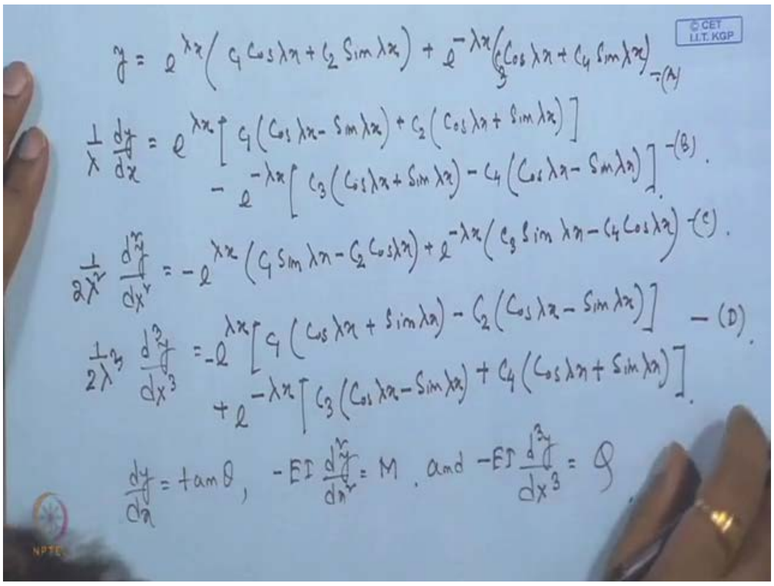

This is the settlement expression that e to the power lambda x into C 1 cos lambda x plus C 2 sin lambda x plus e to the power minus lambda x into cos C 3 cos lambda x plus C 4 sin lambda x. Now, that the slope expression will be 1 by lambda d y d x that is equal to e to the power lambda x into C 1 cos lambda x minus sin lambda x plus C 2 cos lambda x plus sin lambda x minus e to the power minus lambda x into C 3 cos lambda x plus sin lambda x minus C 4 cos lambda x minus sin lambda x.

So, this is differentiate this expression. This once we will get the slope expression. If I differentiate it twice we will get the bending moment expression. Then the shear force expression so, this is del square y del x square that is equal to minus e to the power lambda x but differentiate this expression once and then rearrange that we will get the equation second equation. This is the one equation this is b equation and then the differentiate this expression again b expression again then rearrange. It will get the third expression that is C 1 sin lambda x minus C 2 cos lambda x plus e to the power lambda x C 3 sin lambda x minus C 4 cos lambda x.

So, this will give the moment expression. Then once again differentiate that C expression we will get 2 lambda square so, lambda cube. So, initially this is lambda square the second expression 1 by 2 lambda square. This is 1 by 2 lambda cube then d 3 y d x 3. That will equal to e to the power minus e to the power lambda x C 1 cos lambda x plus sin lambda x minus C 2 cos lambda x minus sin lambda x plus e to the power minus lambda x c three cos lambda x minus sin lambda x plus C 4 cos lambda x plus sin lambda x. This is the shear force expression d. So, here we can write d y d x is equal to tan theta, which is the slope. Then the next one E I d 2 square y d x 2 d two by d x 2 is equal to M minus E I d 2 y d x 2 is equal to M and minus E I d 3 y d x 3 that is equal to Q.

So, that is shear force. So, these are the shear force bending moment and slope expression this b is is A is the deflection expression. B is the slope expression. C is the bending moment expression and D is the shear force expression. So, three... Next one is how to determine this C 1 C 2 and C 3 and C four value. So, depending upon the using the boundary condition of the beam, then how we will calculate this unknown factor that will determine here. So, suppose if I place beam with a beam infinite beam subjected to concentrated load.

Suppose, I have a beam this is x this is the y direction. This is the center where beam is subjected to a concentrated load P. So, this is the concentrated load this is the infinite beam. Now, the deflection equation that we have here here analyzing only the half portion of the beam. As this is a symmetric case. So, the final deflection equation of the beam say y is equal to e to the power lambda x C 1 cos lambda x plus c 2 sin lambda x plus e to the power minus lambda x C 3 cos lambda x plus C 4 sin lambda x. So, this is the basic equation. Now, we have to solve this equation for different condition, so now if this is infinite beam.

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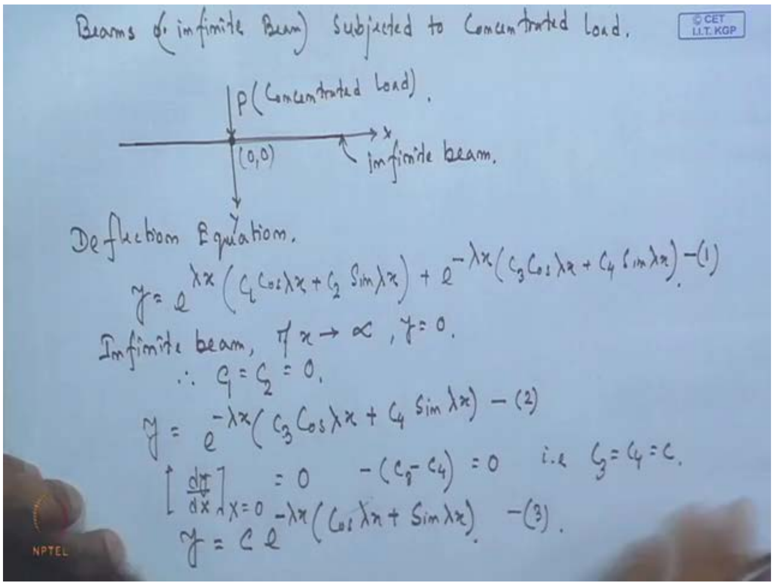

So, if x stands to infinity, then y will be equal to 0, because here we are applying this, concentrated load at 0 0 position. If it is x is infinity then the effect of this concentered load will be negligible. So, there is no effect. So, that means y will be 0 and this is possible only if, the terms related to e to the power lambda x is 0. So, e to the power lambda x and this terms which are attached with e to the power lambda x is 0 then it is only possible. So, in that condition we can write that this is again it is possible if C 1 and c 2 both are 0. So, we can write C 1 is equal to C 2 is equal to 0 then only we will get this condition. So, the expression will be e to the power minus lambda x C 3 cos lambda x plus C 4 sin lambda x. So, we have e to the power minus lambda x and then this expression is further reduced to this expression.

Now, the next condition. Now, we have to apply the another boundary condition or the next condition that is, that will be that our slope at this center is 0 that means d y d x at x equal to 0 is 0 because it is a symmetric problem. So, the slope at the center will be 0. Now, if we differentiate this expression two. Suppose so, this is expression one, this is expression two. Once we differentiate this expression two and then put this condition slope is 0 at x equal to 0. Then we will get this form of expression that minus C 3 minus C 4 is equal to 0. So, then we will get another condition that is C 3 is equal to C 4 is equal to C.

So, from one. First condition infinite beam then we put this condition then it is only possible if C 1 is equal to C 2 and the next condition the slope at the center of the beam is 0 by putting that condition we will get that C 3 is equal to C 4 is equal to C. So, the equation two will be further reduce to e to the power y c e to the power minus lambda x cos lambda x plus sin lambda x. So, that is expression three, because C 3 is equal to C 4 is equal to C. Now, if I write the expression three again, that this y is equal to C e to the power minus lambda x cos lambda x plus sin lambda x.

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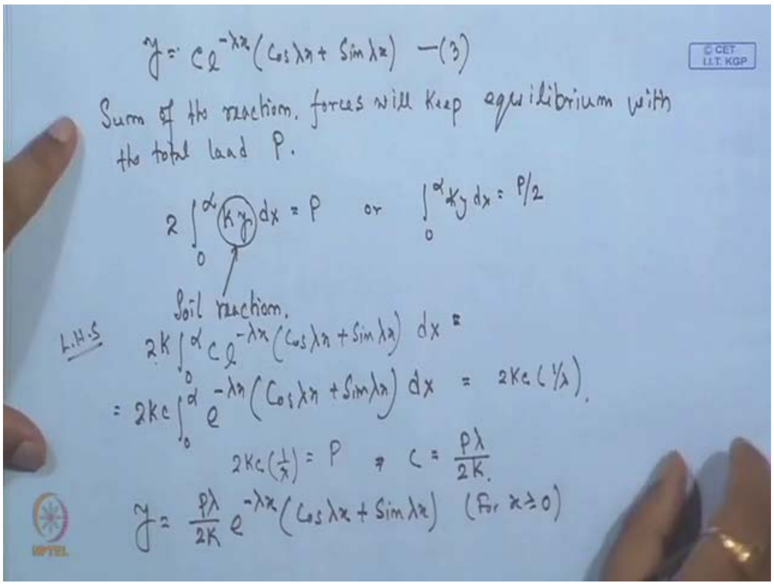

Now, from the equilibrium of the forces that is the sum of the reaction forces, next third condition. The first condition is infinite beam, second condition is slope due to the symmetric slope at the center is 0 the third condition the sum of the reaction forces will be forces will keep equilibrium with the total load. So, that means the total sum of the reaction that will be equal to the force external force, which is applied that is P. P So, that means this thing will be in the equilibrium condition, when only that the sum of the reaction force will be equal to the applied force the P. so in that condition this will be twice zero to infinity K into y into d x that will be equal to P so twice means the we are analyzing the only one half.

So, this is the two half or we can write that is 0 to infinity K y d x that is equal to half of the total external load because the analysis from the half portion. This is 0 to infinity and that soil reaction is K into y, so that K into y term that is the soil reaction. So, finally, once you, once I this is solved, then if I put this expression that is 2 K will be in the outside and 0 to infinity y is C e to the power minus lambda x cos lambda x plus sin lambda x into d x that is equal to . So, if I take this expression left side, left hand side that is two K will be in the outside then if we put the y value here. It is expression 3 here that is C e to the power lambda x and then if I take that c also outside in this two K c 0 to infinity e to the power lambda x cos lambda x plus sin lambda x that is d x

Then we will get this form of expression that is 2 K c into 1 by lambda. So, we will get this expression after solving this up to integrating this equation. So, we will get the expression that 2 K c 1 by lambda that is equal to P because from this expression that is equal to P. So, finally, the third or the final constant that is C will be equal to P lambda divided by 2 K where K is the modulus of sub grade reaction P is the concentrated load and l will be the flexural rigidity of the beam. So, now we will put the final expression will be y is equal to P lambda 2 K e to the power minus lambda x cos lambda x plus sin lambda x and that is for x equal to greater than 0. So, this will give us the final expression or the deformation in this form. So, that means we will get the final expression y is equal to P lambda 2 K e to the power minus delta x cos delta x plus sin delta x. This is for the deflection equation.

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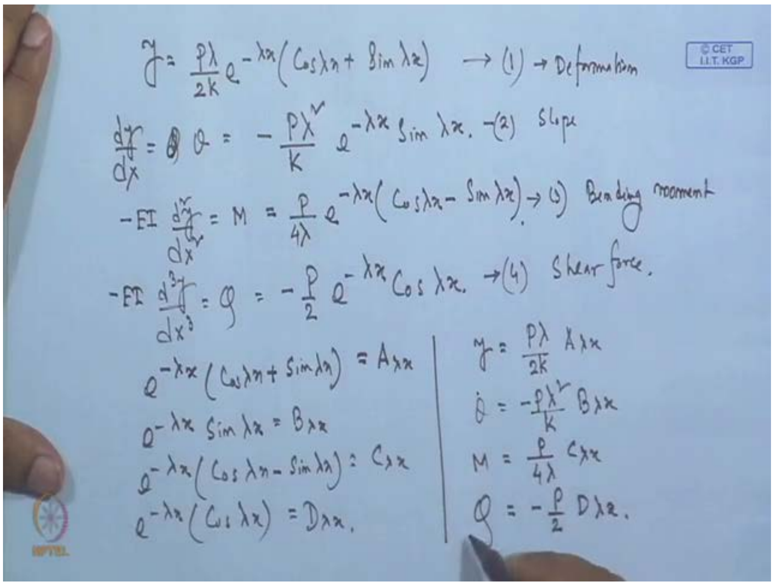

So, for the slope expression d y d x which is equal to theta theta that is equal to minus P del square by K e to the power minus del K del x into sin del x. So, slope will be in this form. So, next expression that is the bending moment expression and the shear force expression because there will be bending moment expression and the shear force expression for this two cases. So, next force expression that minus E I d is to the d square y d x square that is equal to M that is equal to P by 4 lambda e to the power minus lambda x into cos lambda x minus sin lambda x. So, we have this expression into sin minus lambda x. Then further if I differentiate this expression then we will get the shear force expression d 3 y d x 3 that is equal to q. So, this will be minus by 2 e to the power minus lambda x cos lambda x.

So, this is expression one as this for the deformation, this is for the load, this is for the bending moment, four equation number that is for shear force. Now, if I use some different symbol for this case first one, if I use that e to the power minus lambda x and cos lambda x plus sin lambda x is equal to A lambda x and e to the power minus lambda x into sin lambda x, that is equal to B lambda x. e to the power minus lambda x cos lambda x minus sin lambda x equal to C lambda x and e to the power minus lambda x cos lambda x is equal to D lambda x.

Then, these are the symbols then all the expression will change in this form. That y will be equal to P by lambda into 2 K divided by 2 K into A lambda x theta will be equal to minus P lambda square divided by K into B lambda x M will be equal to P by 4 lambda C lambda x and q will be equal to minus P by 2 d lambda x. So, these are the expression of this four quantities, that is deformation slope bending moment and shear force. Now, we have to draw the shape of these four quantities and how these and this shape we can draw so, that we can identify here.

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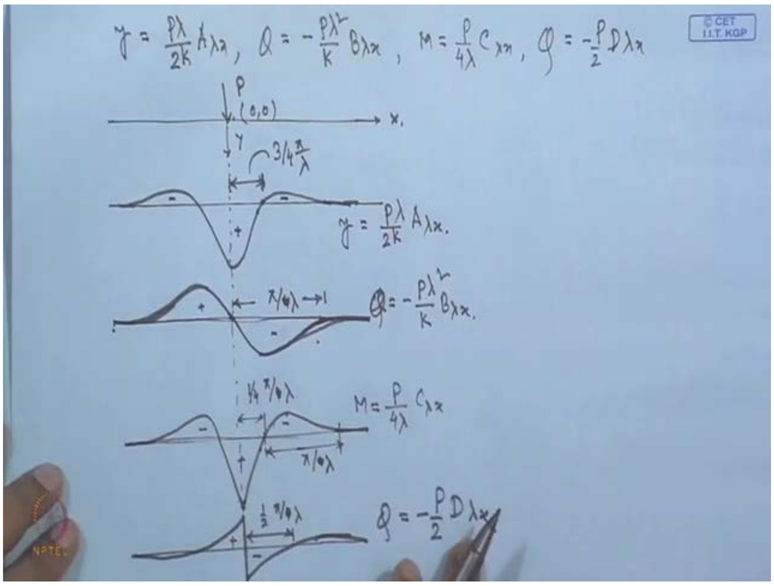

So, first we will get that y is equal to P lambda 2 K A lambda x theta is equal to minus P lambda square by K B lambda x bending moment M is equal to P by 4 lambda C lambda x and shear force q is minus P by 2 D lambda x. So, these are the four expression. Now, first one that we will get, so suppose this is the beam and the concentrated load is applied here. This is x direction and here it is y direction this is 0 zero now first case this will be deformation so, that expression of this curve is y is equal to P by lambda 2 K into A lambda x. Suppose, this is the expression of that form.

Now, here the distance on this point to this point, where the this is 0 is 3 by 4 into pi by lambda. Next one which is for the slope expression this is a straight line, then this will be the slope and slope is 0 at the center it is a boundary condition. So, this is the slope expression. So, this is positive this is negative, this is negative this one is positive. This is negative. So, this expression where this is touching that point actually which is in this form here also, this distance where from here to here this distance is pi by 4. So, this will be minus theta is equal to minus P lambda square by K B lambda x. Next one is the bending moment which we can draw in this form.

To start here then it will go down, then it will go up and again 0 because the center bending moment is maximum. So, we will have the bending moment this is negative, negative positive. So, this expression M is equal to P by 4 lambda C lambda x and this expression from here to here that up to the N rate is again this is the negative part is pi by

4 and this part is 1 fourth by pi by 4. Now, next one is the shear force where at the center then we have this expression this type of variation then it is go down and then it will follow this path. So, Q will be equal to minus P by 2 d lambda x. So, this is positive this is negative. So, that is this part is given by that is this part where this value is given by half pi by 4.

So, we have this is a four diagram. So, the this is the deformation diagram. This is 3 by 4 pi by lambda and this is sorry, this is not pi by 4 this all are pi by lambda. Here also this is pi by lambda, this is pi by lambda. So, here this is this distance where the deformation is the changing from positive to negative, that distance from the center is 3 4 pi by lambda. Hence, for the slope here the slope center is 0 and from center to another point where it is slope is 0 that distance is pi by lambda. Then bending moment that will also follow this expression which is derived where this is where the point is going positive to negative, that is 1 by 4 pi by lambda then from negative to the 0 point that distance is pi by lambda and then again from this point, where this shear force is also changing the sign that is half pi by lambda.

So, these are the different diagram of the one case that I have discussed that is for the infinite beam subjected to concentrated load. In the next class I will discuss with other loading condition for the infinite beam and as well as it possible for the finite beam also.

Thank you.