Structural Dynamics for Civil Engineers – SDOF Systems

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Lecture – 09
Harmonic Vibration Examples

Welcome back to the structural dynamics course. We have been learning about the vibrations of single degree of freedom systems under various scenarios. In the first week we learned about free vibrations, undamped and viscously damped free vibrations. In the previous week we learnt coulomb damped free vibration, we also learned about harmonic vibrations, that is when the system vibrates under a harmonic force. We learned about resonance and the influence of damping on resonant responses. We also learned about transmissibility and vibration isolation.

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So, before learning something new this week, we would briefly revise what we have learnt in the previous weeks.

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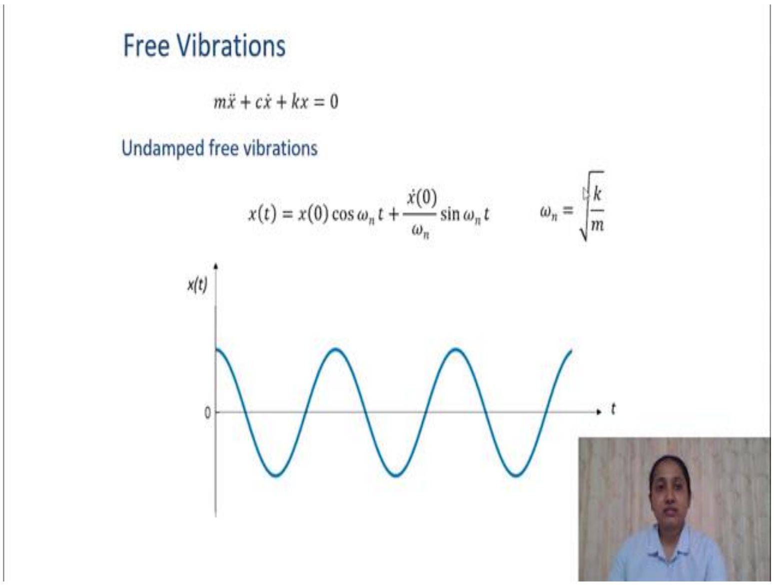

In the first week we discussed free vibrations. This is the equation of motion of a single degree of freedom system under free vibration, m is the mass of the system, c is damping coefficient, k is the stiffness. And since it is free vibration, there is no external force acting on this system so, the right hand side is 0. So, this is the undamped free vibrations response it is undamped when c becomes 0.

$$m\ddot{x} + c\dot{x} + kx = 0$$

So, the displacement response is like this, x naught that is the initial displacement multiplied by cos omega n t plus x dot naught that is the initial velocity divided by omega n multiplied by sin omega n t. And omega n we have seen that it is equal into root of k by m and it is known as the natural frequency of this system. And this is how the response would look like and since there is no damping in the system, this displacement will not decay in time this amplitude will not change at each cycle the amplitude will be same.

$$x(t) = x(0)\cos\omega n\,t + \frac{\dot{x}(0)}{\omega n}\sin\omega n\,t \qquad \omega n = \sqrt{\frac{k}{m}}$$

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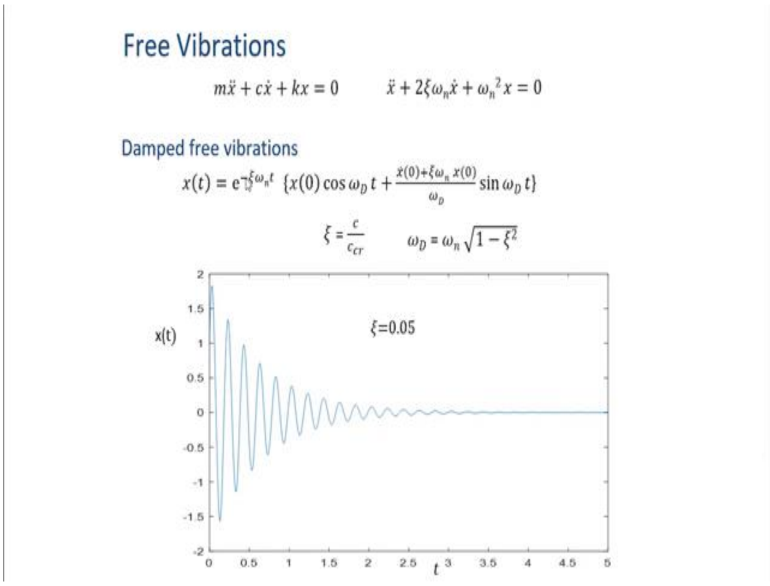

Then we have damped free vibrations. In damped free vibrations damping is non-zero and this equation can be rewritten as this. If you divide this by mass you will get this equation where zeta is defined as the ratio of the damping coefficient and the critical damping coefficient. This is the response of a damped system under free vibrations. So, we have an exponentially decaying term here it is a function of zeta.

$$x(t) = e^{-\xi \omega n t} \{x(0) \cos \omega D\, t + \frac{\dot{x}(0)+\xi \omega n\, x(0)}{\omega D} \sin \omega D\, t\}$$

So, depending upon the damping, there is a decay in the vibration response; so, you can see that in the plot. So, the amplitude of the vibration decays with time and here in this equation this x naught is the initial displacement and omega D is natural frequency multiplied by square root of 1 minus zeta square, where zeta is the damping ratio.

So, if the damping is high, this value will be lower than the natural frequency. So, here we have the initial velocity and the second term is depending upon the damping and the initial displacement. So, this is how the vibration of a damped system looks like and it decays with time and after some time the vibration stops.

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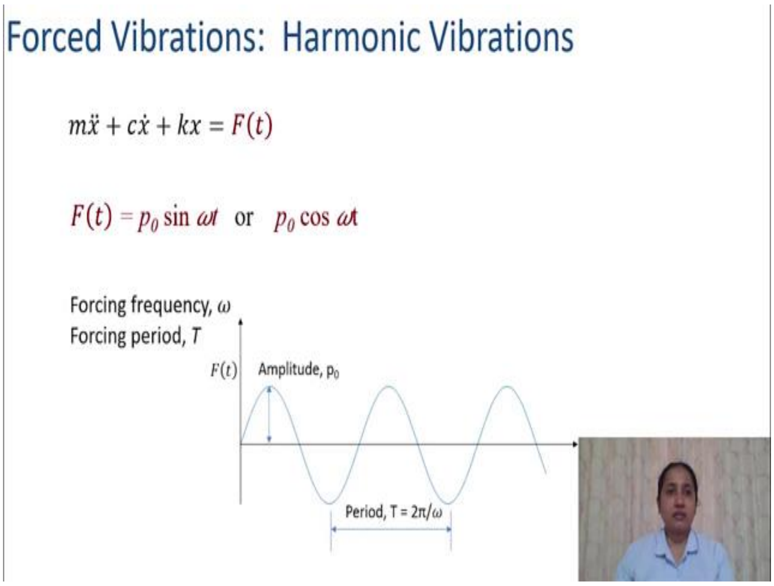

Then we discussed one type of forced vibration called harmonic vibration. It so, in forced vibration, a force a time varying force will be acting on the system and a harmonic force is something which can be written like this that is as a function of sin or cosine. So, this is how a harmonic force will look like and p naught is the amplitude of this force and omega is the forcing frequency. So, this period will be equal to 2 pi by this forcing frequency and this is called forcing period.

$$F(t) = p_0 \sin \omega t \quad \text{or} \quad p_0 \cos \omega t$$

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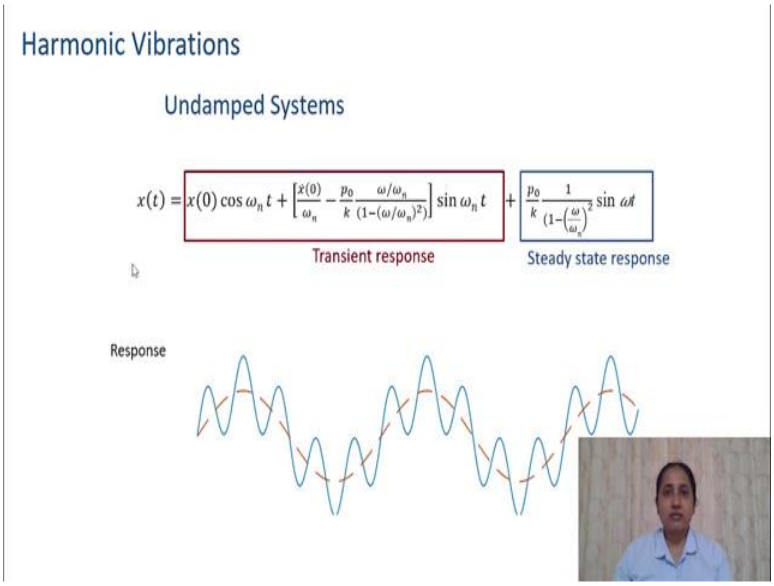

$$x(t) = x(0)\cos \omega n\, t + \left[\frac{\dot{x}(0)}{\omega n} - \frac{p_0}{k}\frac{\omega/\omega n}{(1-(\omega/\omega n)^2)}\right]\sin \omega n\, t \quad + \frac{p_0}{k}\frac{1}{(1-\left(\frac{\omega}{\omega n}\right)^2}\sin \omega t$$

This is the response of an undamped system under harmonic force. So, this response will have two components a transient component and the steady state response. So, the steady state response will be a sin function with frequency equal to the forcing frequency that is omega. And the transient response will have frequency as omega n. So, as you can see in this figure the blue diagram shows this response x t and the dotted line shows the steady state response.

So, this response will have two frequency components the natural frequency and the forcing frequency. So, the time between these two adjacent peaks will be equal to the natural period and the time between two global peaks would be the forcing period.

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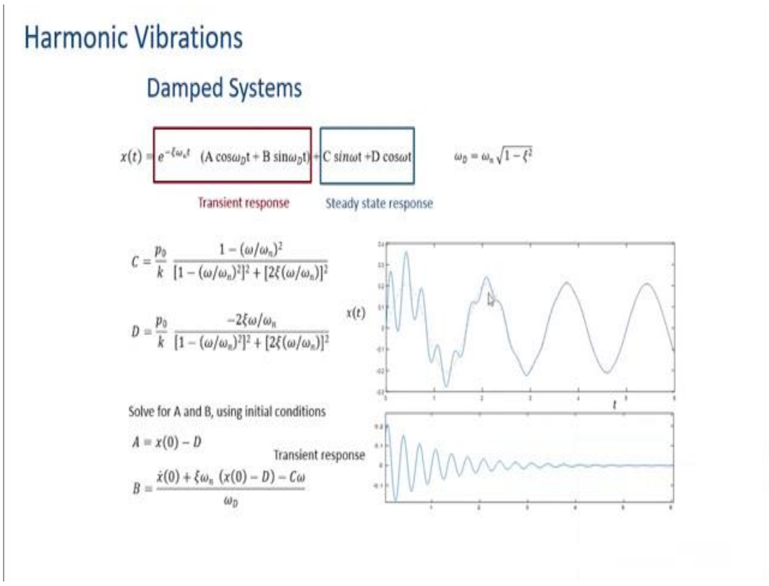

This is how the response of an undamped system under harmonic force. So, again this would have transient response and a steady state response, but in damped system the transient response decays with time depending upon the amount of damping present in the system. So, this transient respond will change with time and decays and becomes 0 and only the steady state response remains after some time.

$$x(t) = e^{-\xi \omega n t} \quad (\text{A} \cos\omega D\text{t} + \text{B} \sin\omega D\text{t}) + \text{C} \sin\omega\text{t} + \text{D} \cos\omega\text{t} \qquad \omega D = \omega n\sqrt{1-\xi 2}$$

So, that is why the name says steady state and transient. Transient is the one which decays and steady state this what stays after some time and the steady state response continues as long as the force is available. So, you can calculate these constants this A and B can be calculated using the initial conditions and C and D can also be evaluated and we have derived this.

$$C = \frac{p0}{k}\ \frac{1-(\omega/\omega n)2}{[1-(\omega/\omega n)2]2+[2\xi(\omega/\omega n)]2}$$

$$D = \frac{p0}{k}\ \frac{-2\xi\omega/\omega n}{[1-(\omega/\omega n)2]2+[2\xi(\omega/\omega n)]2}$$

$$A = x(0) - D$$

$$B = \frac{\dot{x}(0) + \xi\omega n\ \ (x(0) - D) - C\omega}{\omega D}$$

After that we discuss the steady state response in detail.

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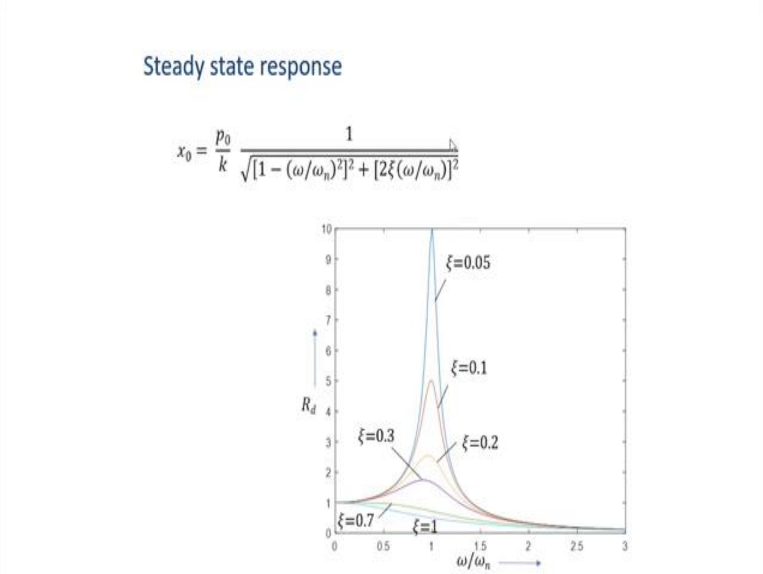

This is the amplitude of the steady state response and here this term p naught by k is equivalent to the static response. If a constant force p naught was acting on the system and this is the deformation response factor. So, this is an amplification or reduction factor depending upon the value of the frequency ratio. So, this figure shows this deformation response factor Rd.

So, as you can see here the value of Rd is close to 1, when the frequency ratio this very less it is very when it is very less than 1 Rd is near 1; that means, when the frequency ratio is much smaller than 1, the steady state amplitude response will be equivalent to the static response. So, x naught will be equal to p naught by k when Rd is 1. And when the frequency ratio is near 1 the response amplifies a lot depending upon the value of damping and when the frequency ratio is much higher than 1 this factor becomes much less than 1, it becomes close to 0; that means, the steady state response amplitude would be much smaller than the static response amplitude.

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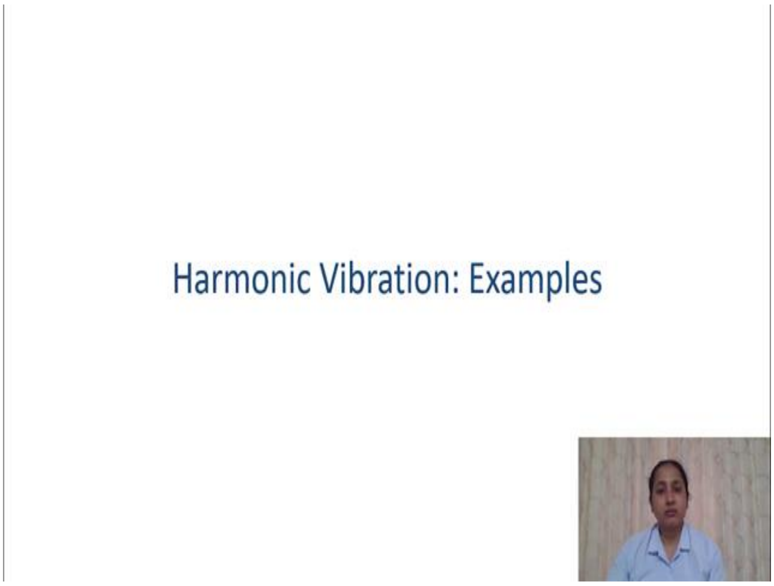

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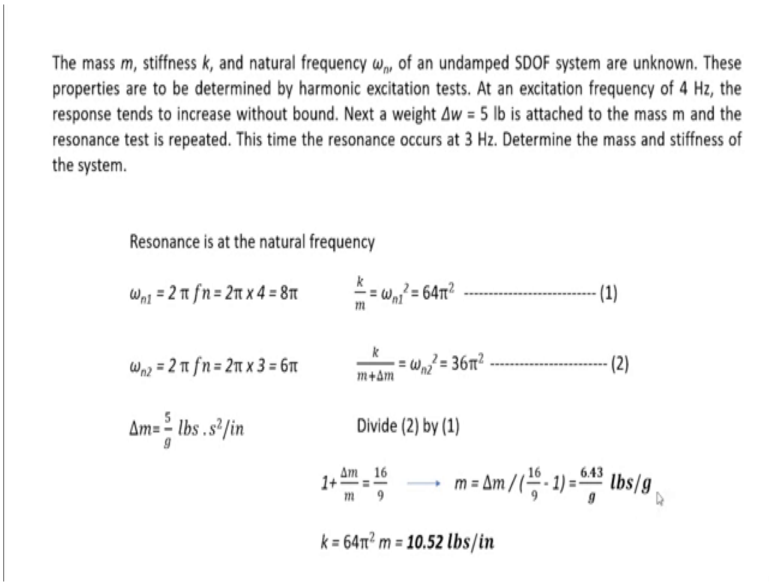

Now, let see some examples of harmonic vibrations. So, in the first example problem the mass m the stiffness k and natural frequency omega n of an undamped single degree of freedom system are unknown. These properties are to be determined by harmonic excitation test. At an excitation frequency of 4 Hertz, the response tends to increase without bound. Next a weight delta w which is equal to 5 pounds is attached to the mass m and the resonance test is repeated. This time the resonance occurs at 3 Hertz determine the mass and stiffness of the system.

So, we have an undamped single degree of freedom system and it is mass and frequency are to be found out using harmonic excitation test. So, how is it done? You excite the structure with multiple excitations with different frequencies and finds when the resonance happens. So, it is given when the resonance is happening. So, let us see how to solve this problem.

So, we have learnt that for undamped system, the resonance is at the natural frequency. So, we can find out that the first resonance frequency which is given as 4 Heartz is the natural frequency of the system. So, we can write omega n 1 is 2 pi fn, fn is given as 4 Heartz here. So, that is the excitation frequency when the resonance happens at the first time. So, omega n 1 is 8 pi and we know that omega natural frequency is equal to root k by m. So, we know this value of k by m now, the square of this. So, in the second time an additional mass was added to the system. So, we also know the second natural frequency that is 3 Hertz, that is 6 pi radians per second.

$$\omega_{n1} = 2\pi fn = 2\pi \times 4 = 8\pi \qquad \frac{k}{m} = \omega_{n1}^2 = 64\pi 2 \text{ ---------------------------- (1)}$$

$$\omega_{n2} = 2\pi fn = 2\pi \times 3 = 6\pi \qquad \frac{k}{m+\Delta m} = \omega_{n2}^2 = 36\pi 2 \text{ ------------------------- (2)}$$

So, now the mass of the system is changed now. So, we have k by m plus delta m. Delta m is the mass of this additional weight. So, now, this is the frequency of the system; so, we have an equation k by m plus delta m is equal to omega 2 square. So, now, we have got two equations solving this we can find the two unknowns, they are k and m. So, just solve this equation and find k and m.

So, we know delta m is equal to the mass of this additional weight. So, we can calculate that the weight is five if you divide it by g you get the mass. So, substitute in these two equations in this equation and if you can divide this equation by this, you will get this and eventually you can calculate the value of the mass. So, the mass comes to be this much and substitute it here, you get the value of the stiffness. So, this is how we find the mass and stiffness of a system using harmonic test.

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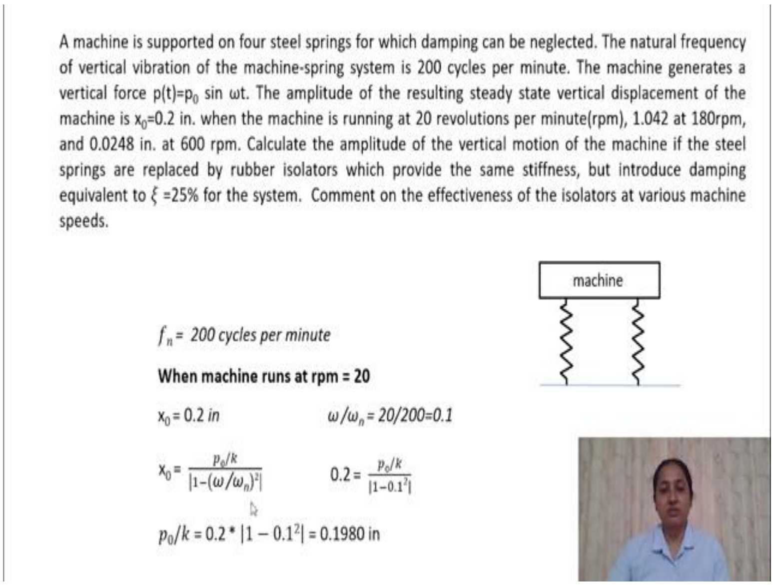

Let us move on to the next example, a machine is supported on four steel springs for which damping can be neglected. The natural frequency of vertical vibration of the machine spring system is 200 cycles per minute. The machine generates vertical force it is a harmonic force.

The amplitude of the resulting steady state vertical displacement of the machine is x naught is equal to 0.2 inches, then the machine is running at 20 revolutions per minute and the amplitude is 1.042 to at 180 rpm and the amplitude is 0.0248 inches at 600 rpm. Calculate the amplitude of the vertical motion of the machine if the steel springs are replaced by rubber isolators, which provide the same stiffness, but introduce damping equivalent to zeta is equal to 25 percent. Comment on the effectiveness of the isolators at various machine speeds.

So, in this we have a machine supported by 4 springs; so, two are in the front and two are at the back. The machine is running at different speeds. So, the amplitude of this vertical vibration is given at different speeds of the machine. And we have to find out how this vibrations will change when these springs are replaced by rubber isolators which provide additional damping to this system. So, let us solve this.

So, it is given that the natural frequency of this machine spring system is 200 cycles per minute. So, when the machine is running at 20 rpm that is 20 revolutions per minute, it is given that the displacement amplitude is 0.2 inches. So, we can calculate the frequency ratio because we know the natural frequency we know the forcing frequency that is 20 rpm. So, the frequency ratio is 0.1 and we know that for undamped systems, the amplitude of the steady state response is equal to p naught by k divided by 1 minus omega n the whole square and then amplitude of that.

So, here we know the amplitude so, we can substitute the value of the amplitude in this equation and find out the value of p naught by k. And we know that p naught by k is the static response of the system if a constant force was acting on the system, a constant force of amplitude p naught was acting on the system. So, this is the steady state response.

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when $\xi = 25\%,\ \omega/\omega_n = 0.1$

$$x_0 = \frac{p_0/k}{\sqrt{[1-(\omega/\omega_n)^2]^2+[2\xi\omega/\omega]^2}} = \frac{p_0/k}{\sqrt{[1-(0.1)^2]^2+[2\times 0.25\times 0.1]^2}} = 0.1997\ in$$

When machine runs at rpm = 180

$x_0 = 1.042\ in$ $\qquad \omega/\omega_n = 180/200 = 0.9$

$$x_0 = \frac{p_0/k}{|1-(\omega/\omega_n)^2|} \qquad 1.042 = \frac{p_0/k}{|1-0.9^2|} \longrightarrow p_0/k = 0.1980\ \text{in}$$

when $\xi = 25\%,\ \omega/\omega_n = 0.9$

$$x_0 = \frac{p_0/k}{\sqrt{[1-(\omega/\omega_n)^2]^2+[2\xi\omega/\omega_n]^2}} = \frac{0.1980}{\sqrt{[1-(0.9)^2]^2+[2\times 0.25\times 0.9]^2}} = 0.4053\ in$$

So, now we will calculate what happens when the springs are replaced by rubber isolators. So, when rubber isolators are kept, you get a damping of 25 percent and the frequency ratio does not change it is same as 0.1. So, we know that the amplitude is expressed like this when a damping is present. So, we can just substitute the value. So, we know the value of p naught by k now and also substitute the value of frequency ratio and damping, you get the response amplitude as 0.1997 inches.

So, now let us move on to 180 rpm. So, when the machine is at 180 rpm the amplitude we know the frequency ratio we can calculate that would be 180 by the natural frequency that is 200. So, the frequency ratio is 0.9 here, again we can substitute in this equation and calculate the value of p naught by k and we should get the same as we calculated earlier. Because this is just a static response it does not depend upon the forcing frequency. So, this should be same as what we have seen earlier.

So, now again let us find what happens when damping is there in the system. So, when the springs are replaced by rubber isolators, we can calculate the response amplitude the displacement amplitude. So, we can substitute the values as we did earlier and we can calculate the response amplitude and that comes out to be 0.4053 inches.

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When machine runs at rpm = 600

$x_0 = 0.0248\ in \qquad \omega/\omega_n = 600/200 = 3$

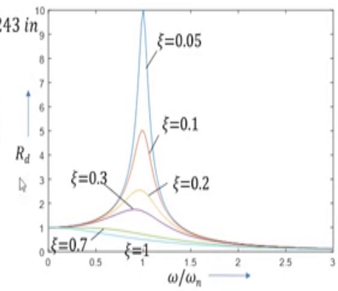
$$x_0 = \frac{p_0/k}{|1-(\omega/\omega_n)^2|} \qquad 0.0248 = \frac{p_0/k}{|1-3^2|} \longrightarrow p_0/k = 0.1980\ \text{in}$$

$when\ \xi = 25\%,\ \omega/\omega_n = 3$

$$x_0 = \frac{p_0/k}{\sqrt{[1-(\omega/\omega_n)^2]^2+[2\xi\omega/\omega_n]^2}}$$

$$= \frac{0.1980}{\sqrt{[1-3^2]^2+[2\times 0.25\times 3]^2}} = 0.0243\ in$$

ω/ω_n	x_0 $(\xi = 0)$	x_0 $(\xi = 0.25)$
0.1	0.2	0.1997
0.9	1.042	0.4043
3.0	0.0248	0.0246



Now, let us calculate for 600 rpm. Again the amplitude is given frequency ratio we can calculate and it comes out to be 3, substitute here you will get the same response static response we can skip the steps by now. I just kept it for clarification. This is the static response and this will be constant irrespective of the rpm. So, when the damping is 25 percent, let us calculate the amplitude and that comes out to be 0.0243. So, this would be the response amplitude, if we replace the springs by rubber isolators.

Now, let us we have calculated for all three cases and now let us summarize the results. So, we run the machine at different speeds. So, we have three frequency ratios forcing frequencies was changing each time. So, these are the forcing the frequency ratios. Now this is the response amplitude when the springs were kept that is when there was no damping. So, these amplitudes were given and we have calculated the response amplitude when the damping is 0.25 that is when the springs are replaced by rubber isolators. So, these are the results.

So, here you can see when the frequency ratio is 0.1, the response amplitudes are very close. When the frequency ratio is 0.9 as we can see the amplitude of response significantly reduces when we replace springs by rubber isolators that is when we introduce some damping to the system. When the frequency ratio is 3, you can see that both the amplitudes are almost same; that means, the rubber isolators did not change the frequency amplitude and this behavior we have already seen when we discussed the response factors.

So, we have seen that the value of Rd that is deformation response factor varies like this with frequency ratio. So, when the frequency ratio was much less than 1 that is in the case here when it was here it is 0.1. So, when it was much less than 1, this factor was close to 1; that means, our responses would be similar that is the trend we are seeing here. That means damping does not have any effect here, but when the frequency ratio is near 1 the damping has effect on this response amplitude.

So, depending upon the damping the response will reduce. So, we had 25 percent damping here. So, our response came down significantly. So, when frequency ratio is very high compared to 1, then again there is no influence of damping this factor will become less than 1, but it does not change much with damping. So, here our frequency ratio is 3. So, as it is seen from this figure damping does not have much control over there. So, here also our damping does not reduce the amplitude of the response.

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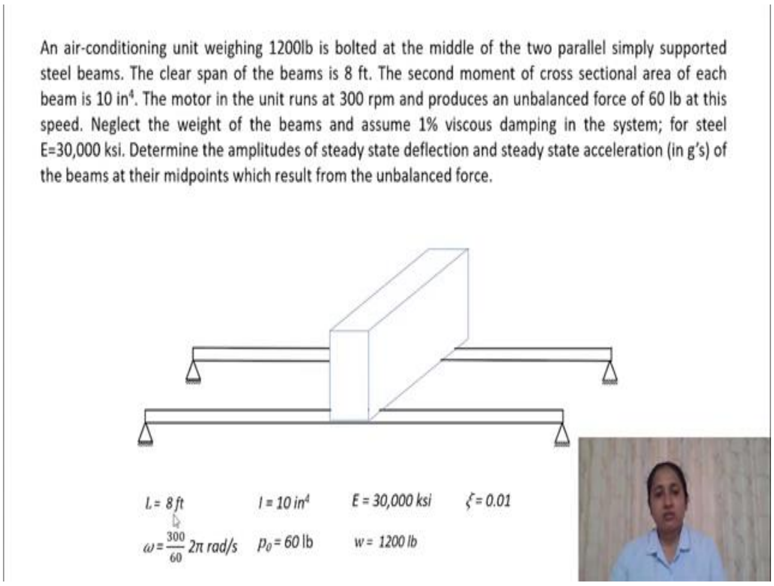

In the next problem we have two simply supported beams and an air conditioning unit is kept on it. So, an air conditioning unit weighing 1200 pounds is bolted at the middle of the two parallel simply supported steel beams. The clear span of the beams is 8 feet, the second moment of cross sectional area of each beam is 10 inch to the power 4 so, that is I. The motor in the unit runs at 300 rpm and produces an unbalanced force of 600 pounds at this speed. Neglect the weight of the beams and assume 1 percent viscous damping in the system.

For steel e is given as 30,000 ksi, that is kips per square inches similar to Newton per meter square millimeter. Determine the amplitudes of steady state deflection and steady state acceleration of the beams at their midpoints which result from the unbalanced force. So, this air conditioning unit is rotating with some speed which is given and that causes a unbalanced force the amplitude of the force is also given. So, this is, this unit is imparting a harmonic force on the beams.

So, the information we already have is the length of the beams is given that is 8 feet and second moment of cross sectional area is given that is I 10 inch to the power 4. Young's modulus is given and it is told that zeta the damping ratio is 0.01 that is 1 percent damping viscous damping. So, we can calculate the forcing frequency that is 300 rpm that is 300 by 60 cycles per second and if you multiply by 2 pi you would get it in radians per second. So, omega is this much, when we know the unbalanced force amplitude is given as 60 pounds and the weight of this AC unit has also given as 1200 pounds.

So, now let us find out the steady state deflection and acceleration.

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Stiffness of this system

$$\text{Stiffness of two beams, k} = 2 \text{ x } \frac{48EI}{L^3} = 32{,}552 \text{ lbs/in}$$

Natural Frequency

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{kg}{w}} = \sqrt{\frac{32552 \times 386}{1200}} = 102.3 \ rad/sec$$

$$\omega/\omega_n = 10\pi/\ 102.3 = 0.307$$

Steady state response

$$\text{Displacement amplitude, } x_0 = \frac{\frac{P_o}{k}}{\sqrt{[1-(\omega/\omega_n)^2]^2+[2\zeta\omega/\omega_n]^2}}$$

$$x_0 = \frac{60}{32{,}552}\frac{1}{\sqrt{[1-(0.307)^2]^2+[2 \times 0.01 \times 0.307]^2}} = 2.035 \times 10^{-3} \text{in}$$

$$\text{Acceleration amplitude, } \ddot{x}_0 = \omega_n{}^2 x_0 = (10\pi)^2 x_0 = 2.009 \text{ in/s}^2$$

So, the stiffness of the system is equal to the stiffness of the two beams. So, we have two simply supported beams and at the mid span the stiffness is equal to 48 EI by L cube. This we you should have studied this in the static course; so, if you do not remember this please refer your static analysis notes. So, this is the 48 EI by L cube is the lateral stiffness of the simply supported beam at mid span.

$$\text{Stiffness of two beams, k} = 2 \text{ x } \frac{48EI}{L3} = 32{,}552 \text{ lbs/in}$$

So, we have two beams. So, the equivalent stiffness of the system is two times that you can calculate this because we have all the values and the natural frequency of the system this omega n that is root of k by m we know the weight of the AC unit. So, we can calculate the mass, mass is weight by g. So, this becomes kg by weight under root. So, again we can calculate the natural frequency.

Forcing frequency is known; so, we can calculate the frequency ratio omega by omega n that is equal to 0.307. So, let us calculate the steady state response now. We know the displacement amplitude for damped system is p naught by k divided by square root of 1 minus damping ratio square the whole square plus 2 zeta damping ratio square the whole square. So, we can just substitute all these values, all these are known to us by now. So, p naught by k is known, damping ratio is known frequency ratio is known. So, substitute it and get the value as this much. So, that will be the amplitude of the displacement of the beams under the effect of a rotating AC unit.

$$\text{Displacement amplitude, } x_0 = \frac{\frac{p_0}{k}}{\sqrt{[1-(\omega/\omega_n)^2]^2+[2\xi\omega/\omega_n]^2}}$$

So, once the displacement amplitude is known, we can calculate the acceleration response acceleration amplitude and the acceleration amplitude is omega n square times displacement amplitude. So, we can calculate that also. So, this acceleration amplitude this 2.009 inches per second square. We can also represent this in terms of g just divide this value by the value of g and we can represent this in terms of g. So, this is equal to 0.0053 times g the gravitational acceleration.

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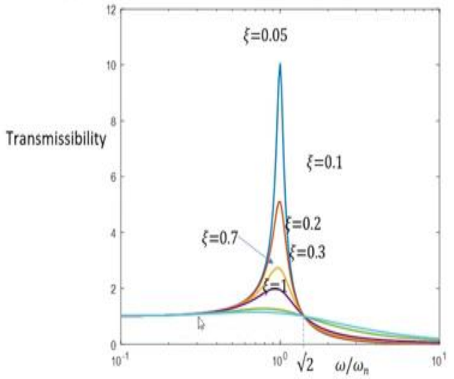
Solution:

Given, $w = 2000$ lbs $\quad f = 1500$ cpm $= 1500/60 = 25$ Hz $\quad E = 30{,}000$ ksi

$$\text{Transmissibility, } TR = \frac{\text{vibration of isolation block}}{\text{vibration of foundation}} = 0.1$$

$$TR = \frac{\text{vibration of isolation block}}{\text{vibration of foundation}} = 0.1 \quad < 1$$



It is given that the weight of the isolation block is 2000 pounds and the forcing frequency is 1500 cycles per minute. It is given that the foundation of the isolation block vibrates in this frequency. So, we can calculate the value in Hertz that is cycles per second; so, 1500 by 60 will give you the value of the frequency in Hertz. It is also given the e value for the steel spring that is 30,000 ksi.

So, we learned about transmissibility. So, that is the ratio of the vibration of the isolation block to the vibration of the foundation and it is given in the question that, that should be 10 percent that is 0.1. So, this transmissibility which is given us which we need to achieve this 0.1 which is less than 1.

So, now let us look at how the value of the transmissibility varies with frequency ratio. So, we have learned this last week. So, this is the transmissibility curve. So, the y axis shows this ratio, the vibration of the isolation block to the vibration of the foundation in this case.

So, as you can see here when the frequency ratio is very less; that means, the transmissibility is equal to 1. So, in that case we cannot achieve this value of 0.1 if the frequency ratio is very less. So, in such case the vibration of the isolation block and the vibration of the foundation will be equal to same irrespective of the damping provided or the frequency ratio because for the whole range the transmissibility value is 1 if the frequency ratio is much less than 1 that is in this range. And if the frequency ratio is close to 1 then the transmissibility will be greater than 1; that means, we cannot reduce the amplitude of the vibration, but it will increase and depending upon the damping the vibration will reduce; that means, transmissibility value will reduce if you have a high damping.

But in that range also the value of transmissibility will be greater than 1 for normal damping levels. So, to get a transmissibility of 0.1 we need to go beyond the root 2; that means, the frequency ratio should be higher than root 2. So, let us find out how much frequency ratio will suffice for our 10 percent reduction in amplitude.

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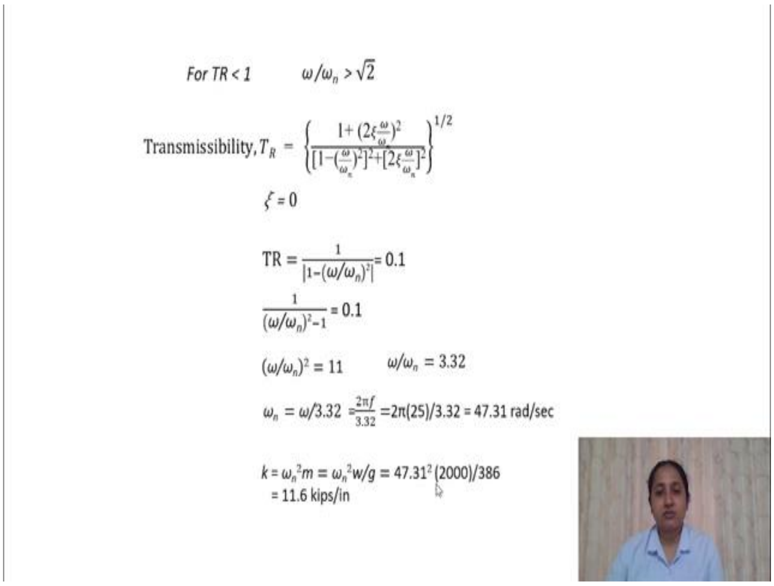

So, for transmissibility less than 1, the frequency ratio should be greater than root 2. So, this is the expression for the transmissibility which we have learnt last week. So, that is 1 plus 2 zeta frequency ratio square the whole square divided by 1 minus frequency ratio square the whole square plus 2 zeta frequency ratio the whole square and the whole thing is under root so, to the power 1 by 2.

$$\text{Transmissibility}, TR = \left\{\frac{1+\left(2\xi\frac{\omega}{\omega n}\right)^2}{\left[1-\left(\frac{\omega}{\omega n}\right)^2\right]^2+\left[2\xi\frac{\omega}{\omega n}\right]^2}\right\}^{1/2}$$

So, when zeta is equal to 0. So, in this case in this question it is given that we can neglect damping. So, zeta is equal to 0 here. So, the transmissibility becomes this, these two terms will cancel. So, our transmissibility comes down to this and we have to make it equivalent to 0.1. So, from this we can calculate the frequency ratio needed for this much amount of transmissibility. So, just solve this equation and we can get the frequency ratio that is omega by omega n and that value is equal to 3.32, this is higher than root 2. So, we will get transmissibility less than 0.1.

So, now we know the forcing frequency omega. So, we can calculate the natural frequency. So, if you substitute all the values we can calculate the natural frequency. So, we know that natural frequency is equal to under root k by m. So, if mass is known and natural frequency is also known, we can calculate the stiffness. So, just substitute the value and just get the stiffness. So, if we keep the effective stiffness of that isolation system as this much this 11.6 kips per inch, we can make sure that the response of the isolated system is only 10 percent of the foundation response.