Structural Dynamics for Civil Engineers – SDOF Systems
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Lecture – 08
Examples

Welcome back to the Structural Dynamics course. In the last week we have learned how dynamic analysis is different from static analysis, we learned how to formulate an equation of motion, we also learnt the responses of free vibrations, we learnt undamped and viscously damped free vibrations. So, in this week initially we will solve a few example problems after that we will explore coulomb damped free vibrations later we will learn forced vibrations. So, now let us move on to example problems.

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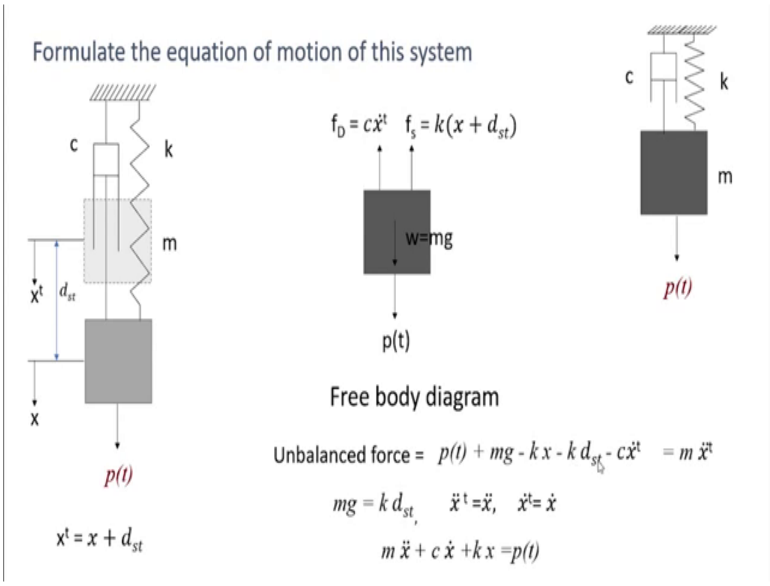

Now, let us formulate the equation of motion of this system, we have a mass, a damper and a spring, a dynamic force p t is acting on this mass. So, for this system because of the self rate the system will have a static displacement. So, the displaced configuration of the system under its own weight is this, d st is the static displacement of this system and X is the displacement of this mass relative to its static equilibrium position, X t is the total displacement of this mass. So, X t will be equal to x plus d st.

Now, let us draw the free body diagram of this mass. So, a dynamic force p t is acting downwards weight of the system is equal to mass times gravitational acceleration is also acting downwards. The damping force is acting upwards and it is equal to c times x dot t the total velocity and this we also have a spring force which is equal to k times x d so, k times x plus d st.

So, now, we can write the total force acting on this body in vertical direction. So, the unbalanced force in vertical direction is equal to p t plus weight that is mg minus k x minus k d st minus c x dot t, and this sum is equal to mass times the acceleration of this body, the total acceleration.

$$\text{Unbalanced force} = p(t) + mg - kx - kd_{st} - c\dot{x}^t = m\ddot{x}^t$$

So, we also know that mg that is the weight of the body is equal to k times the static displacement. And we can find out from this relation that x double dot t will be equal to x double dot because d st is a static displacement which does not vary with time and also we have x dot t would be equal to x dot.

$$mg = kd_{st} \qquad \ddot{x}^t = \ddot{x} \qquad \dot{x}^t = \dot{x}$$

So, we can rewrite this equation like this m x double dot plus c x dot plus k x is equal to p t.

$$m\ddot{x} + c\dot{x} + kx = p(t)$$

So, mg and minus k d st cancel out and we have this equation of motion. So, if you look at this equation of motion, this is in terms of x and its derivatives so; that means, x is the displacement of this mass relative to its static equilibrium position. So, if you write the equation of motion relative to the static equilibrium position, we will not have any weight term in the equation of motion.

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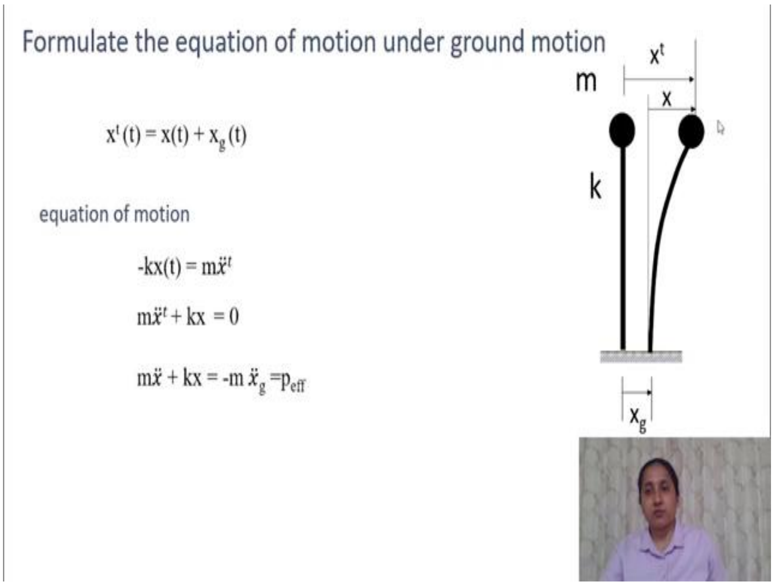

Now, let us look into another system and write the equation of motion. The next system has a lumped mass m and a stiffener k this does not have any damping and the system is affected by a ground displacement Xg. The displacement of this mass relative to the support is indicated as X and the total displacement of this mass has indicated as X t.

$$X^t(t) = X(t) + X_g(t)$$

So, we know that X t is equal to X plus Xg. Now, let us write the equation of motion. So, if you consider this mass there is only a spring force acting on that mass. So, if X is considered positive in that direction. So, the spring force will be in this direction. So, it will be minus k X t. So, that is the force acting on this mass and this should be equal to mass times acceleration. So, that is given by Newton's law. So, this is our equation of motion we can rearrange the term it becomes m x double dot t plus k X is equal to 0. We know that X t is X plus X t. So, we can expand this term. So, we have m x double dot plus k X is equal to minus m x double dot g that is the ground acceleration.

$$m\ddot{x} + kx = -m\ddot{x}_g \text{ -} p_{eff}$$

So, the ground displacement at the support is equivalent to having an effective dynamic force acting on this mass and the effective force is equal to minus mass times the ground acceleration. So, this is how earthquake affects a structure. So, earthquake is a ground acceleration or ground displacement. So, that would be equivalent to having an external dynamic force on this mass and the value of that force will be equal to minus mass time's ground acceleration.

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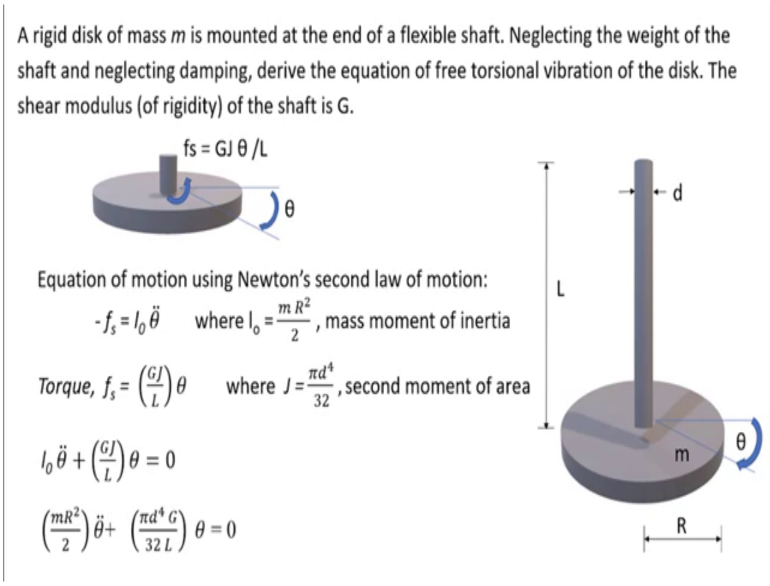

The next example is of torsional vibration. So, we have a rigid disk of mass m mounted on a flexible shaft. So, this is a rigid disk which has a mass m and it is mounted on this flexible shaft with diameter d. Neglecting the weight of the shaft and neglecting damping derive the equation of free torsional vibration of the disk. The shear modulus of rigidity of the shaft is G. So, we can neglect the damping in the system and we can treat the shaft as massless. So, this just offer some stiffness to the system and the disk has some mass. So, this is a undamped free vibration system. So, here it is torsional vibration.

So, let us draw the free body diagram of this disk. So, so far we have been dealing with systems where the mass was lumped, it was concentrated. So, this is an example where the masses distributed the disk has some dimensions some radius and some thickness. So, the mass will be equally distributed across this disk. So, masses distributed here.

So, let us go back to the free body diagram. So, if theta is taken positive in this direction, we can write the restoring force. In this case the restoring torque is equal to the torsional stiffness of this shaft multiplied by the angular displacement that is this angle theta. So, the value of this torsional stiffness of the shaft is equal to GJ by L, G is the shear modulus and L is the length and J is the second moment of area. You would have learned this expressions in the mechanics of solids course. So, the restoring torque can be expressed like this.

$$f_s = GJ\,\theta/L$$

So, now we can write the equation of motion using Newton's second law which says unbalanced force, in this case that is equal to the torque minus fs that should be equal to the inertia. So, that is if it was a translational displacement this would have been equal to mass times translational acceleration.

So, here this is a distributed mass and we are also talking about angular displacement. So, this will be the unbalanced force will be equal to angular acceleration multiplied by I naught which is the mass moment of inertia of this disk. So, for disk like this the mass moment of inertia is equal to m R square by 2, where R is the radius of the disk. So, minus fs is equal to mass moment of inertia times angular acceleration.

$$-f_S = I_0\ddot{\theta} \qquad \text{where } I_0 = \frac{mR^2}{2}$$

So, now, let us see what is the value of this torque. So, the torque fs is equal to GJ by L that is the torsional stiffness of the shaft multiplied by the angular displacement that is theta. J is the second moment of area and for this shaft it is equal to pi d to the power 4 by 32, d is the diameter of this shaft.

$$f_S = \left(\frac{GJ}{L}\right)\theta \qquad \text{where } J = \frac{\pi d^4}{32}$$

So, now we can calculate this torque and the mass moment of inertia, we have the dimensions of this structure. So, now, we can write this equation of motion as I naught theta double dot plus GJ by L theta is equal to 0 this is the equation of motion. We can substitute the value of mass moment of inertia and the second moment of area and we get the expression for our equation of motion as this.

$$I_0\ddot{\theta} + \left(\frac{GJ}{L}\right)\theta = 0$$

$$\left(\frac{mR^2}{2}\right)\ddot{\theta} + \left(\frac{\pi d^4 G}{32}\right)\theta = 0$$

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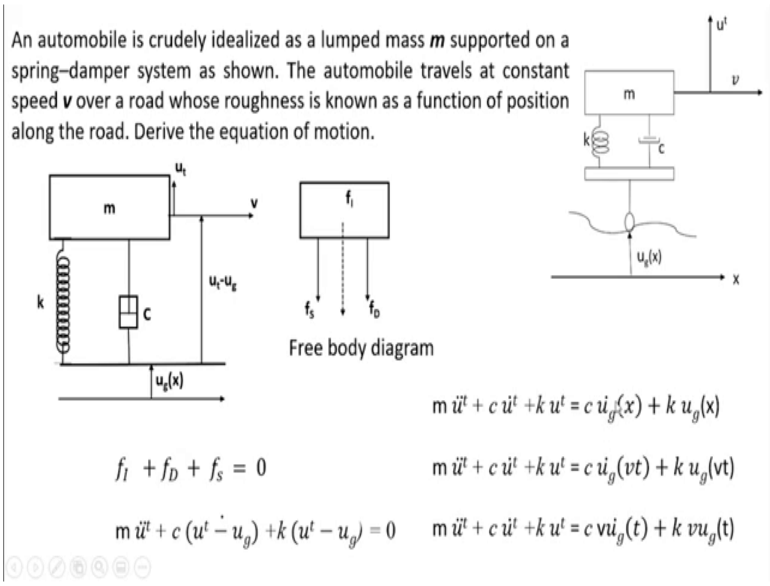

Let us do one more example. In this example an automobile is crudely idealized as a lumped mass m supported by a spring and a damper, and this automobile travels at a constant speed v over a road which has some roughness. So, the roughness of this road can be expressed as a function of X and X is the position of this vehicle on the road.

So, the roughness is represented by a function u g X we have to derive the equation of motion of this system. So, this single degree of freedom system is getting a ground displacement u g at each instant as the vehicle is moving over this road this u g changes. So, this is equivalent to a displacement from the; displacement of the support of this single degree of freedom system. So, let us try to solve this.

So, we have this mass connected to the spring and the damper and it moves with a speed v. So, when it moves it goes up and down and that roughness is given as u g X. The displacement of this mass is represented by the variable u t. So, that would be the total displacement of this mass and the displacement of this mass relative to the base is u t minus u g; so, the total displacement minus the displacement of the base.

Now, let us draw the free body diagram of this mass. So, this is the mass. So, when the mass is moved in this direction there will be restoring force in the opposite direction. So, we have spring force acting in this direction and damping force acting in this direction and there will be inertia force also in this direction.

So, now, let us write the equilibrium equation. So, that would be inertia force plus spring force plus damping force is equal to 0. So, what does inertia force? We have learned that inertia force is equal to mass times the acceleration of the mass that is mass times u t double dot this is the acceleration.

$$f_I + f_D + f_S = 0$$

Now, let us see the damping force. So, the damping coefficient is c. So, damping force would be c multiplied by the velocity of this mass. So, the velocity would be the velocity relative to the base. So, that would be u t minus u g dot, the spring force would be k multiplied by the relative displacement. So, that is u t minus u g and that should be equal to 0.

$$m\ddot{u}^t + c\left(\dot{u^t - u_g}\right) + k\left(u^t - u_g\right) = 0$$

So, now, we can rearrange this terms. So, m u t double dot plus c u t dot plus k u t is equal to take these terms on to the right hand side that would be c u g dot plus k u g and it is given that u g is a function of X and what is X, we know that this automobile is moving with a velocity v.

$$m\ddot{u}^t + c\dot{u}^t + ku^t = c\dot{u}_g(x) + ku_g(x)$$

So, what will be the position of this velocity at each instant? That will be the velocity multiplied by the time. So, we can replace this variable x by v t that is velocity multiplied by time. So, the right hand side becomes c u g dot v t its function of v t and k u g Vt. So, v is just a constant. So, this equation will become c v u g dot that will be a function of t and k v u g t. So, this is the equation of motion of this system.

$$m\ddot{u}^t + c\dot{u}^t + ku^t = c\dot{u}_g(vt) + ku_g(vt)$$

$$m\ddot{u}^t + c\dot{u}^t + ku^t = cv\dot{u}_g(t) + kvu_g(t)$$

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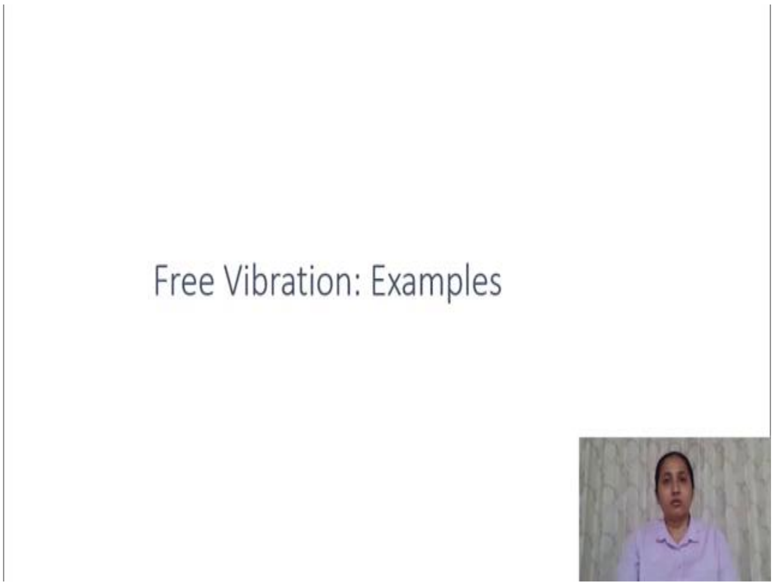

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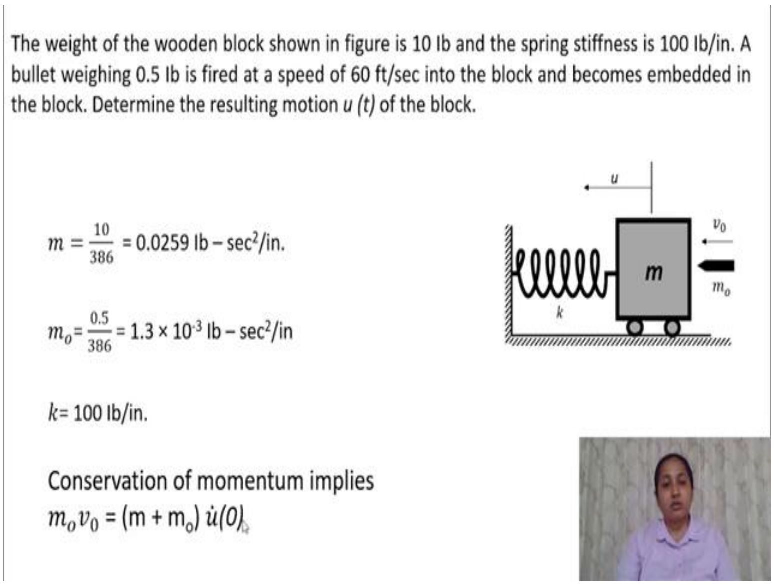

Now, let us look into some free vibration examples. So, in the first example we have a wooden block connected to a spring and a bullet is fired into this block. The bullet has some mass and velocity this bullet gets embedded into this block, we have to find out the resulting motion of the block. So, what happens when this bullet gets fired into this block? So, the bullet was traveling with some velocity and it has a mass. So, when it hits this wooden block the both the masses they will move together with some new velocity.

So, by firing a mass a bullet which has some mass into this wooden block means we are giving some initial velocity to this wooden block. We will find out how much is the initial velocity and using that we have to find out the resulting motion of this block. So, let us see how to solve this. So, in this question, the weight of the block is given; weight of the bullet is given the stiffness of the spring and the velocity of the bullet are also given.

Now, let us calculate. The mass of the block is calculated as the weight which is given divided by gravitational acceleration. So, you will get the mass and the mass of the bullet is also found like this weight is given divided by g you get m naught and stiffness is already given.

So, now, we have to find out the initial velocity of the block. So, how do we calculate it? We can do that using the conservation of momentum. So, the momentum of the system before and after the impact should be same. So, before the impact only the bullet was moving with some velocity v naught. So, the momentum is equal to m naught v naught and that should be equal to the momentum after impact. So, after impact this mass gets embedded into it. So, the total mass will be m plus m naught and it will move with a new velocity u dot naught. So, we can find out what is that initial velocity.

$$m_0 v_0 = (m + m_0)\, \dot{u}(0)$$

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$$\dot{u}(0) = \frac{m_0 v_0}{m + m_0} = 2.8857 \text{ ft/sec} = 34.29 \text{ in./sec}$$

After the impact

Mass = $m + m_0$ = 0.0272 lb –sec^2/in,

Stiffness = k = 100 lb/in.

Natural frequency: $\omega_n = \sqrt{\frac{k}{m + m_0}}$ = 60.63 rads/sec

Initial conditions: $\dot{u}(0) = 34.29\ in./sec$, u(0) =0;

The resulting motion, $u(t) = \frac{\dot{u}(0)}{\omega_n} \sin \omega_n t$ = 0.565 sin (60.63t) in

So, that would be m naught v naught by the total mass.

$$\dot{u}(0) = \frac{m_0 v_0}{m + m_0}$$

So, if you solve this you will get the initial velocity. So, after the impact the mass of our system changes the mass becomes m plus m naught and the stiffness of the spring does not change it is same as the previous.

$$Mass = m + m_0$$

So, we can calculate the natural frequency as root of k by the new mass that is m plus m naught.

$$\omega_n = \sqrt{\frac{k}{m + m_0}}$$

And we just calculated the initial condition in the initial velocity we just calculated initial velocity is this much to do this and the initial displacement of the system of the wooden block was 0. So, the initial displacement is 0.

So, using these two initial conditions we can calculate the resultant motion of the wooden block that is u t is equal to u naught dot by omega n sin omega n t.

$$u(t) = \frac{\dot{u}(0)}{\omega_n} \sin \omega_n t$$

We know that for this free vibration, the response is of the form a cos omega n t plus b sin omega n t and if we use the initial conditions we can find out a and b, and a is equal to the initial displacement here the initial displacement was 0. So, we do not have that cos omega n t term. So, the total response is equivalent to b that is u naught dot by omega n sin omega n t. So, we can substitute these values and write the resultant motion of the block in this format, that would be in the unit inches.

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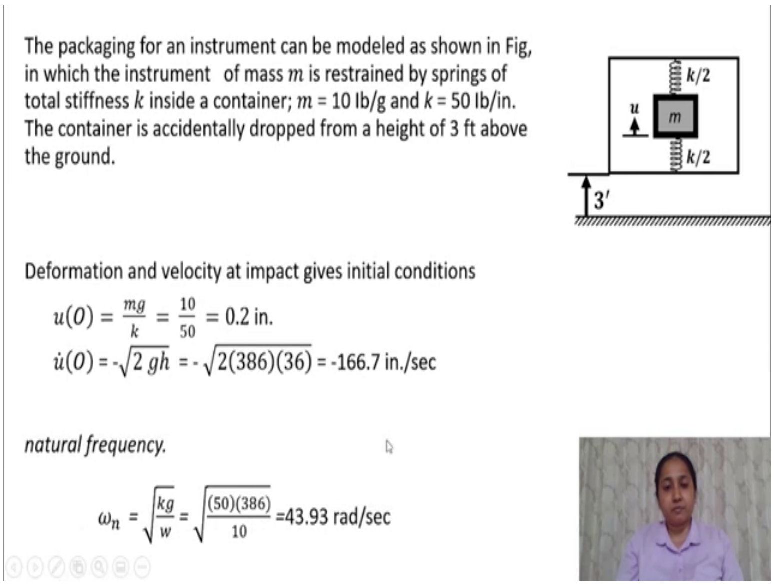

In the next example we have a packaging with an instrument inside it. So, this is an instrument and it is packed in a box and this instrument with this mass m is restrained by springs of total stiffness k by 2. So, there is one string of k by 2 stiffness above this instrument and another one below this instrument. And, mass and mass of the instrument and this stiffness of the springs are given, and this box this container is accidentally dropped from a height of 3 feet above the ground. So, this box was dropped from 3 feet, we have to calculate a few parameter of this system.

So, first let us calculate the initial displacement and velocity of this system. So, when this mass is supported by the springs because of the weight, this spring will get compressed a little and this spring will get extended by some amount and that would be proportional to the weight of this mass weight of this instrument. So, how much that deflection would be the static deflection that we can calculate by the formula m g that is the weight of the instrument divided by the stiffness. So, we can calculate that static deflection. So, that would be the initial displacement of this system when this gets dropped.

So, the initial displacement that is u naught is equal to mg by k, we can calculate the value of it. Now, what will be the initial velocity? So, this package is dropped from a height. So, when it was before dropping it had some potential energy; so, when it is being dropped that potential energy will become converted to kinetic energy. So, this will get some velocity.

$$u(0) = \frac{mg}{k}$$

So, we have learnt in previous courses that that velocity will be equal to minus root 2 g h, the potential energy before dropping is m g h and that will be converted to kinetic energy. So, that is half m v square. So, if you solve it we can find this expression for the velocity. So, using this value you can calculate the initial velocity.

$$\dot{u}(0) = -\sqrt{2gh}$$

So, now let us calculate the natural frequency, the stiffness is given and the mass is also given. So, natural frequency is equal to root of k by mass. So, that is kg by weight. So, you can calculate this value as well.

$$\omega_n = \sqrt{\frac{kg}{w}}$$

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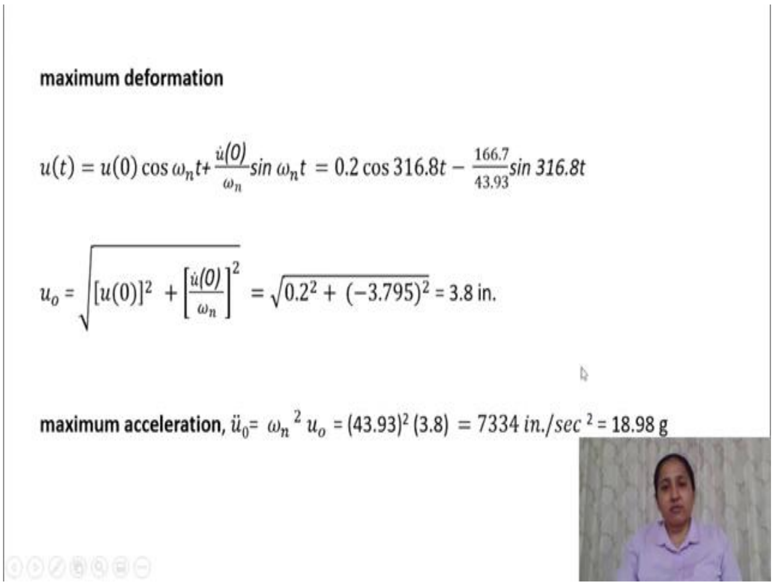

Next we will calculate the deformation of this system. So, you have know the initial conditions, we know the natural frequency. So, you can calculate the displacement as a cos omega n t plus b sin omega t all these values are known. So, you can substitute and get the expression for the displacement of the instrument and you can calculate the maximum deformation as square root of this term square plus this term square. So, you can calculate the maximum deformation of the instrument inside the package.

$$u(t) = u(0)\cos\omega_n t + \frac{\dot{u}(0)}{\omega_n}\sin\omega_n t$$

$$u_0 = \sqrt{[u(0)]^2 + \left[\frac{\dot{u}(0)}{\omega_n}\right]^2}$$

So, if we know the maximum deformation we can also calculate the maximum velocity or even maximum acceleration. So, let us calculate maximum acceleration. So, if this is the displacement what is the acceleration? If you differentiate it twice you will get acceleration. So, if you differentiate it twice, what will happen? You will get a similar expression with omega n square also as a coefficient.

$$\ddot{u_0} = {\omega_n}^2 u_0$$

So acceleration, maximum acceleration is equal to the maximum displacement multiplied by omega n square, we know the value of omega n. So, we can find the maximum acceleration as well just substitute the value and get the acceleration of the instrument.

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Now, let us move on to damped free vibration examples.

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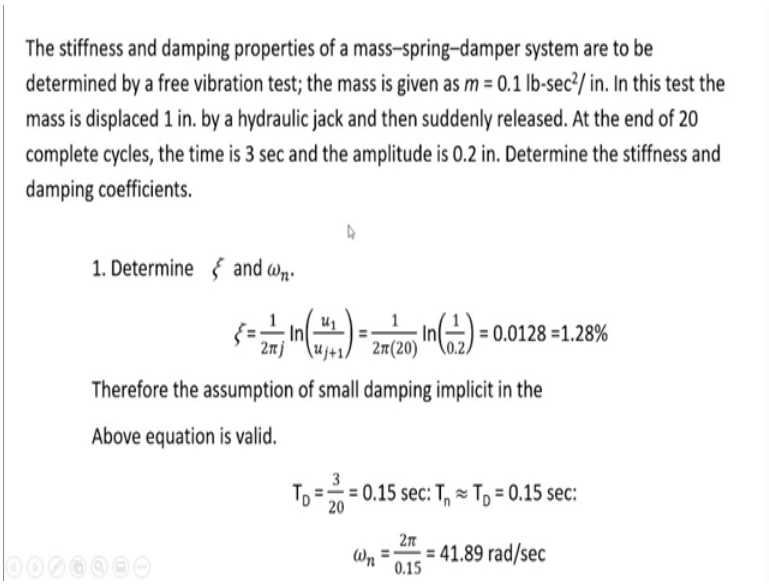

The stiffness and damping properties of a mass spring damper system are to be determined by a free vibration test; the mass is given as m is equal to 0.1 pound second square per inch. In this test the mass is displaced by 1 inch by a hydraulic jack and then suddenly released. At the end of the 20 complete cycles the time is 3 seconds and the amplitude is 0.2 inches. Determine the stiffness and damping coefficients.

So, in this the mass was displaced by 1 inch and then it was released; that means, the initial displacement to the system is 1 inch and it was just released, the mass was just released there was no impact or anything so, the initial velocity is 0. And we also know the amplitude of the motion at 20 cycles and we also know the time taken for twenty cycles. So, we can calculate the damping ratio and natural frequencies.

We learned about logarithmic decrement. So, what is logarithmic decrement? It is the natural logarithm of the ratio of amplitudes at different cycles divided by the number of cycles. So, this is the logarithmic decrement. And the damping ratio zeta can be calculated as 1 by 2 pi of logarithmic decrement. And this is valid when the damping ratio is very small the damping ratio is very high we have to consider the square root of 1 minus zeta squared term as well, but if the damping ratio is small we can calculate it like this.

$$\xi = \frac{1}{2\pi j}\ln\left(\frac{u_1}{u_{j+1}}\right) = \frac{1}{2\pi(20)}\ln\frac{1}{0.2} = 0.0128 = 1.28\%$$

So, here we know 1 by 2 pi j is equal to 20 here. So, we can take j as 20 because we know the information at the first cycle and after 20 cycles. So, u 1 is equal to our initial displacement because that is the amplitude of the first cycle and u j plus 1 is the amplitude after 20 cycles that is given as 0.2. So, you can calculate the value of zeta and it is we get it as 0.0128 that is 1.28 percentage. So, that is very small damping since the damping is very small the assumption that we made here to calculate the zeta is valid. So, we assumed that damping is 0 is small. So, it is actually small. So, the assumption is correct, if it was not small then we had to calculate it using the exact formula.

So, now we can calculate the natural frequency. So, to do that, we will find out the natural period first. So, this system is a mass spring damper system so, it will have a damped natural period. So, we can calculate the damped natural period using the time taken for 20 cycles.

So, it is given that 3 seconds was the time taken for 20 complete cycles. So, period is nothing, but the time taken for 1 cycle. So, the damped natural period would be 3 by 20 that is 0.15 and we just calculated the damping ratio and found that the damping is very less. So, in such cases we can consider that the natural period is equal to the damped natural period. So, the natural period will also be equal to 0.15 seconds.

$$T_D = \frac{3}{20} = 0.15sec: \quad T_n \approx T_D = 0.15sec$$

So, we can calculate the natural circular frequency omega n that would be 1 pi by T n. So, 2 pi by 0.15 is equal to 41.89. Now, we will try to find out the stiffness and the damping coefficients.

$$\omega_n = \frac{2\pi}{0.15} = 41.89\, rad/sec$$

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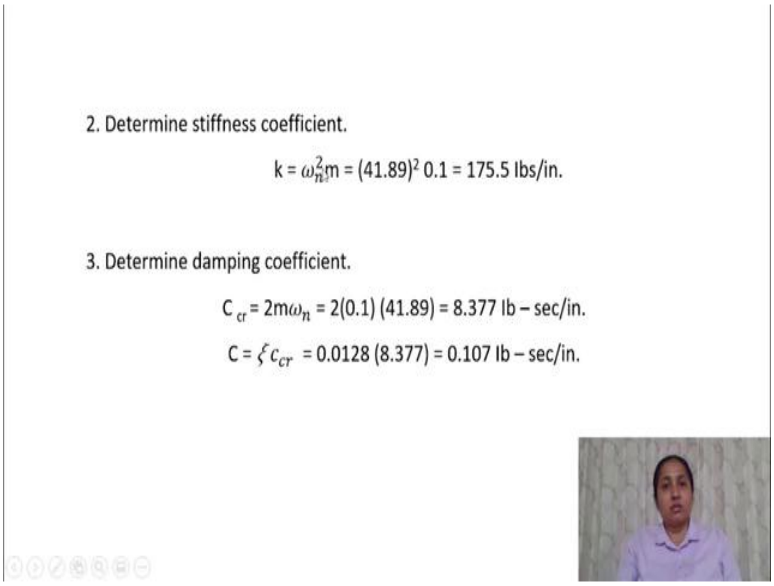

So, the natural frequency square is equal to k by m so, k is equal to omega n square m.

$$k = {\omega_n}^2 m$$

So, we know the natural frequency we know the mass so, we can calculate the stiffness and we know that the critical damping coefficient is equal to 2 m omega n. So, knowing omega n and the mass we can calculate the critical damping coefficient. We know that the damping ratio zeta is equal to the damping coefficient divided by the critical damping coefficient.

$$c_{cr} = 2m\omega_n$$

So, the damping coefficient will be equal to zeta critical damping coefficient. So, you have calculated the zeta already and you can multiply it with the critical damping coefficient we get the damping coefficient.

$$c = \xi c_{cr}$$

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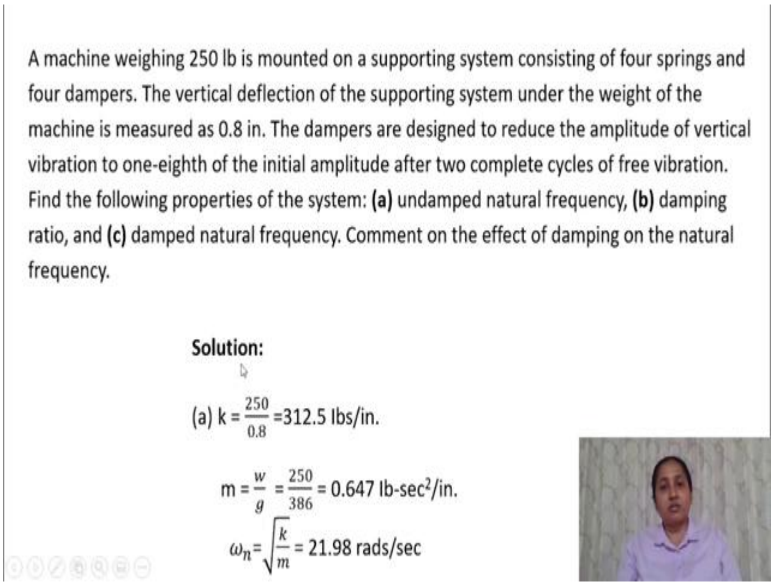

So, in the next problem we have a machine weighing 250 pounds and is mounted on a supporting system consisting of four springs and four dampers. The vertical deflection of the support system under the weight of the machine is measured as 0.8 inches. The dampers are designed to reduce the amplitude of vertical vibration to one-eighth of the initial amplitude after two complete cycles of free vibration. Find the following properties of the system; the undamped natural frequency, the damping ratio and the damped natural frequency. Comment on the effect of damping on natural frequency.

So, it is given the weight of the system and it is also given the deflection of the supporting system under the weight. So, if you know the weight and the deflection we can find the stiffness of the supporting system. So, the weight is the force acting on that springs. So, it is getting deflected because of this force so, the stiffness is the force by deflection. So, the weight is 250 pounds, so this would be 250 by the deflection given as 0.8. So, we can calculate the stiffness of the system. weight is given. So, we can calculate the mass of the system.

$$k = \frac{250}{0.8}$$

$$m = \frac{w}{g}$$

So, masses read by gravitational acceleration. So, now, we know the mass and the stiffness. So, we can calculate the natural frequency of the system which is equal to square root of k by m.

$$\omega_n = \sqrt{(k/m)}$$

The next thing we have to find out is the damping ratio and it is given the amplitudes of two cycles, right; it is given that the dampers are designed to reduce the amplitude of vertical vibration to one-eighth of the initial amplitude. We can calculate the damping ratio using logarithmic decrement.

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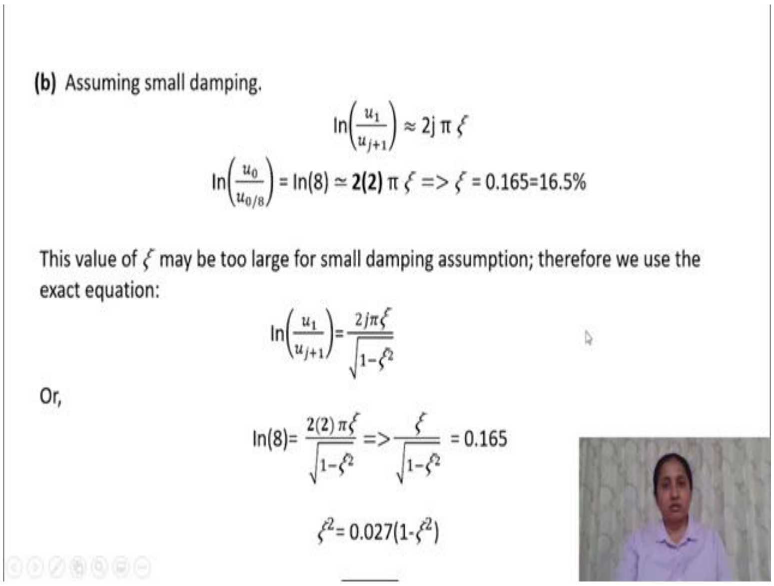

So, as we assumed in the previous example initially we can assume that the damping is small. So, we can equate the logarithmic decrement to 2 j pi zeta this is similar to what we did in the previous problem and we know that the amplitude becomes one-eighth after 2 cycles. So, at the first cycle it is u naught after 2 cycles it is u naught by 8. So, logarithm of 8 is equal to 2 and j is equal to 2 because this amplitude is after 2 cycles and pi zeta which gives the value of zeta is equal to 0.165; that means 16.5 percent. So, that is very high damping.

$$\ln\left(\frac{u_1}{u_{j+1}}\right) \approx 2j\pi\xi$$

$$\ln\left(\frac{u_1}{u_{0/8}}\right) = \ln 8 \simeq 2(2)\pi\xi \Rightarrow \xi = 0.165 = 16.5\%$$

So, since this damping is very high we cannot make this assumption as the damping is small. So, now, we will recalculate the value of zeta using the exact expression for logarithmic decrement that is 2 j pi zeta divided by square root of 1 minus zeta square. So, from this we can evaluate the value of zeta solving this equation.

$$\ln\left(\frac{u_1}{u_{j+1}}\right) = \frac{2j\pi\xi}{\sqrt{1-\xi^2}}$$

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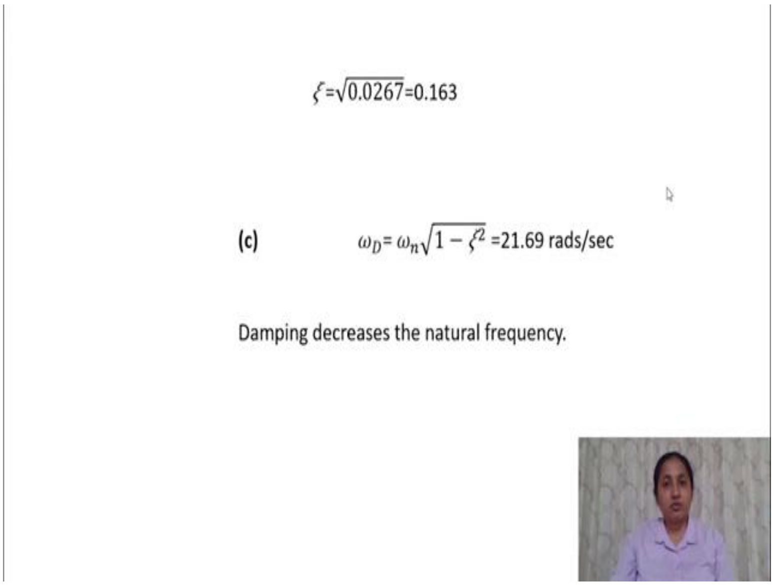

So, we get zeta is equal to 0.163, if you solve the previous equation we will get the value of zeta. Now, we have to calculate the natural frequency of the damped system. So, that would be equal to omega n that is the natural frequency multiplied by square root of 1 minus zeta square. So, we can substitute the value of zeta here and get the value of omega D.

$$\omega_D = \omega_n \sqrt{1-\xi 2}$$

So, how omega n is different from omega D? So, the zeta is not a very small value here. So, this term is less than 1. So, omega D will be less than omega n. So, the damping decreases the natural frequency. So, because of the damping the frequency of the system decreases.