Structural Dynamics for Civil Engineers - SDOF Systems

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Lecture – 06
Forced Vibrations Part I

So, far we have been learning about Free Vibrations that is when there is no external force acting on the structure; the structure is given an initial disturbance and it is allowed to freely vibrate. We have learned about damped and undamped systems under free vibrations. So, in undamped systems, the vibration does not decay with time; but in damped systems, the free vibrations decay with time.

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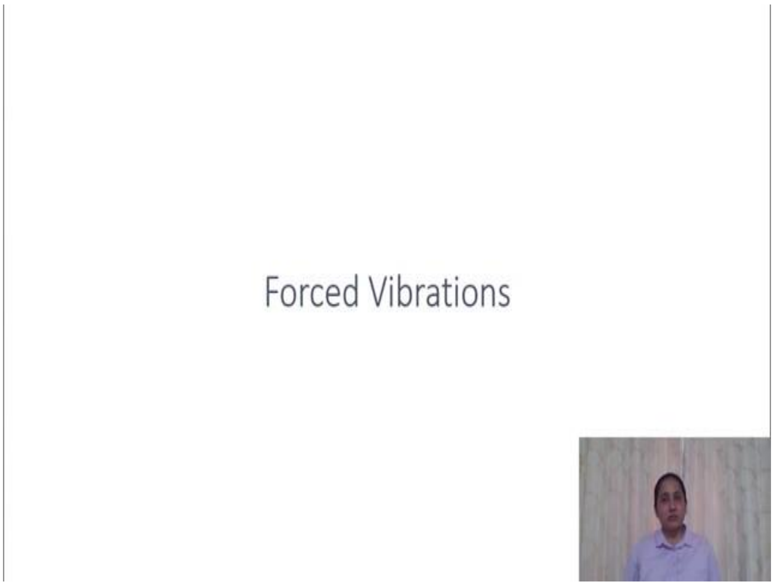

Now, onwards we will be learning about Forced Vibrations, that is some time varying force will be acting on the system and the system will be vibrating under the influence of this time varying force. There will also be some effect because of the initial conditions of the structure.

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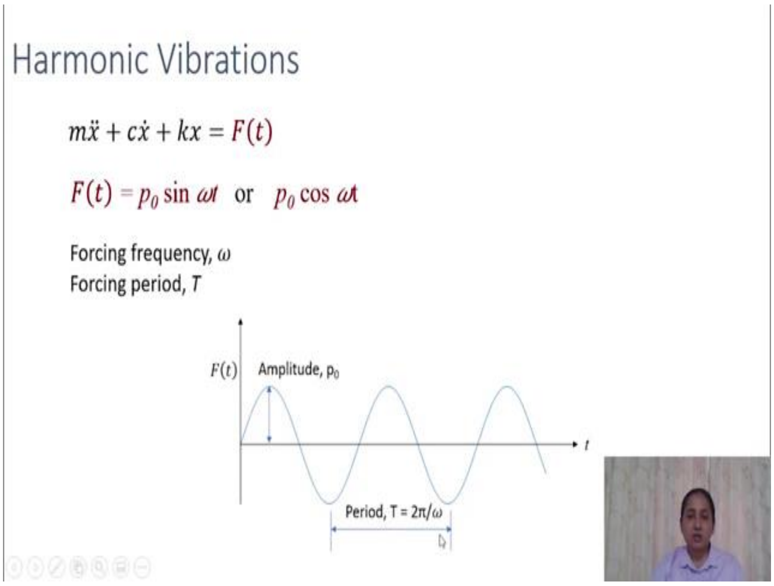

The first category of forced vibrations, we are discussing here is harmonic vibrations that is when harmonic force is acting on the structure. When a rotating machinery is placed on the structure it imparts a harmonic force onto the structure and any periodic force can also be represented as sum of harmonic forces. So, it is important to understand the behavior of structures under harmonic forces. Now let us look into the details of harmonic vibrations, this is the equation of motion of a structure under forced vibrations.

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

So, we have some time varying force acting on the system. So, in harmonic vibrations, a harmonic force is acting on the system and that force can be represented as a sin or cosine function with some amplitude.

That means; the force acting on the system will be varying like this to be varying as a sine or cosine function, and the maximum value of the force which is called the amplitude of the force is p naught (p_0) in this case. So, here is the amplitude of the force and it vary like this; and the period of this force can be calculated as 2 pi by omega, where omega is the frequency of this force. This is the frequency by which this force will vary, so it is called as forcing frequency and we can call this period as forcing period.

$$F(t) = p_0 \sin \omega t \;\; or \; p_0 \cos \omega t$$

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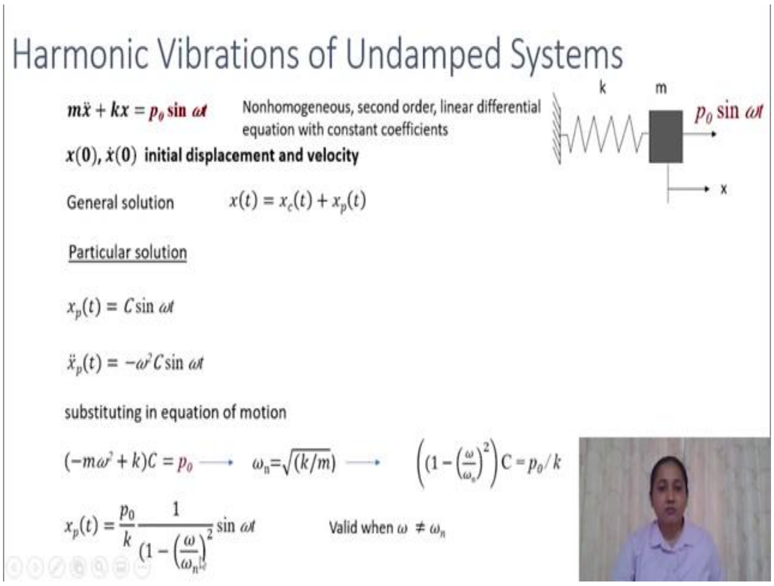

So in free vibrations, we initially learned about undamped systems and then we moved on to damped system. So, here also in harmonic vibrations, first we will be learning about undamped systems.

So, that is when the damping in a system is negligible. So, as we have seen earlier, the components of an undamped systems are it is mass and stiffness. There is no damping in this system. So, now, a force p naught sin omega t($p_0 sin\omega t$) will be acting on this system. So, now, let's write, it is equation of motion. So, the equation of motion of this system can be written as, m x double dot; that is the mass and the acceleration of that mass plus the stiffness of the system multiplied by it is displacement x; and that will be equivalent to the force acting on the system p naught sin omega t($p_0 sin\omega t$).

$$m\ddot{x} + kx = p_0 \sin \omega t$$

And here also there will be some initial displacement and velocity for the system. So, these initial conditions are known to us. Now we can find the response of this undamped system by solving this differential equation. So, we know that this equation is a non homogeneous equation, because the right-hand side is non-zero and this is a second order linear differential equation with constant coefficients. So, from calculus theory, we can solve this equation. And the solution of these type of equations will have two parts; one is a complementary solution and another is a particular solution. So, the complementary solution, will be equal to the solution of the homogeneous equation; that means, the solution of m x double dot plus k x is equal to 0 that is equivalent to the free vibration solution. So, the free vibration solution will be the complementary solution and the particular solution is for this force component.

$$x(t) = x_c(t) + x_p(t)$$

So, now, let us see how to solve this equation of motion. So, first we will look into the particular solution. So, to get a particular solution, first we will assume that the particular solution is of the form C sin omega t($Csin(\omega t)$). So, if you substitute this particular solution in our equation of motion, it should satisfy the equality. So, if you do that, it will satisfy the equality. So, we can choose this as a particular solution. Now we have to find the value of C to get the complete particular solution. So, to do that, we will substitute this in this equation of motion.

$$x_p(t) = C \sin \omega t$$

So, first we have to find out the acceleration, x double dot p. So, we can differentiate this twice and find out the acceleration that would be minus omega square C sin omega t($Csin(\omega t)$).

$$\ddot{x}_p(t) = -\omega^2 C \sin \omega t$$

Now we can substitute these two terms in the equation of motion. So, if you do that, the equation of motion will become this. So, now, in free vibrations we have learned that, the natural frequency of the system is equal to square root of stiffness divided by mass. So, we can substitute this relation in this equation, and that will become this. So, you can divide this equation by k. So, right hand side will become p naught by k, this term will become 1 and this will become omega square divided by omega n square.

$$(-m\omega^2 + k)\,C = p_0 \qquad \omega_n = \sqrt{(k/m)} \qquad \left(1 - \left(\frac{\omega}{\omega n}\right)^2\right)C = p_0/k$$

So, this equation will take this form and we can easily calculate the value of C from this equation, C will be p naught by k divided by this value. So, now, we know the particular solution as, particular solution is equal to p naught by k multiplied by 1 by 1 minus frequency ratio square multiplied by sin omega t($sin(\omega t)$). So, now, we know the particular solution. And if you look at this particular solution, when omega(ω) becomes equal to omega n(ω_n); that is when the forcing frequency is equal to the natural frequency of the system, this particular solution is non-valid; our denominator will 1 minus 1 is equal to 0. So, this is not defined at omega is equal to omega n.

$$x(t) = \frac{p_0}{k} \frac{1}{(1 - \left(\frac{\omega}{\omega n}\right)^2} \sin \omega t$$

So, we have to find another solution, for the condition when omega is equal to omega n; that we will do later. Now we will look into the complementary solution of this equation.

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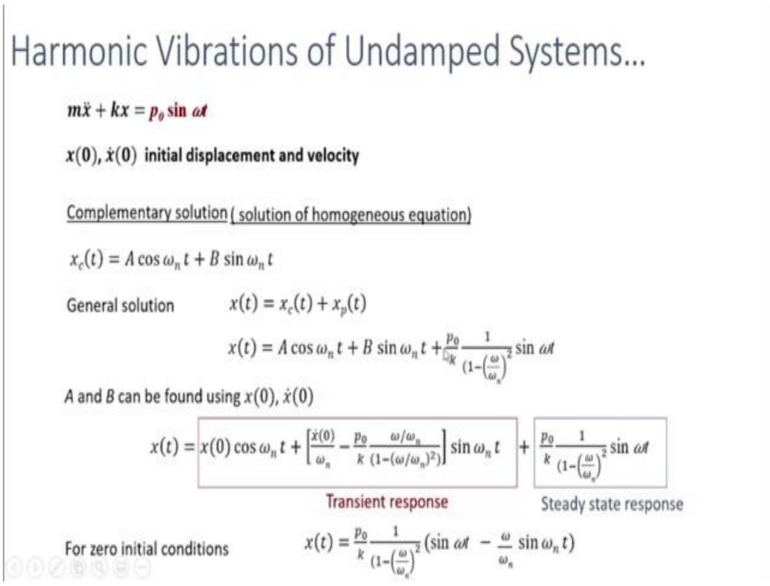

So, again we have the same differential equation; now we will look at the complementary solution. As we have discussed in the previous slide, the complementary solution is equal to the solution of homogeneous equation; that is when this is 0, similar to the free vibration condition. So, in free vibration, in undamped free vibration we have seen that, the solution of the undamped free vibration is of the form, A cos omega n t($A cos \omega_n$) plus B sin omega n t($B sin \omega_n$), where omega n(ω_n) is the natural frequency. So, here also the complementary solution will be equal to this.

$$x_c(t) = A \cos \omega + B \sin \omega t$$

So, the solution of this differential equation can be written as, the sum of the complementary solution and the particular solution.

$$x(t) = x_c(t) + x_p(t)$$

So, we have A cos omega n t plus B sin omega n t plus the particular solution we have derived in the previous slide, p naught by k multiplied by 1 by 1 minus frequency ratio square sin omega t. So, now, we can find out the value of the constants A and B, by using the initial conditions; initial displacement and the initial velocity, this is similar to what we have done during free vibration.

$$x(t) = A\cos \omega t + B\sin \omega t + \frac{p_0}{k}\frac{1}{\left(1-\left(\frac{\omega}{\omega n}\right)^2\right)}\sin \omega t$$

So, we can substitute the value of initial displacement in this equation; and we can differentiate this equation and substitute the value of initial velocity and solve those two equations and calculate A and B. So, we will get the value of A as x 0 is itself that is the initial velocity; value of B would be equal to this. So, it will depend upon the initial velocity, natural frequency, the forcing frequency, and the forcing amplitude this is the particular solution. So, now, this is the total solution of our differential equation, this will give you the displacement response of this forced vibration, of this harmonic vibration.

$$x(t) = x(0)\cos \omega n\, t + \left[\frac{\dot{x}(0)}{\omega n} - \frac{p_0}{k}\frac{\omega/\omega n}{(1-(\omega/\omega n)^2)}\right]\sin \omega n\, t \quad + \frac{p_0}{k}\frac{1}{\left(1-\left(\frac{\omega}{\omega n}\right)^2\right)}\sin \omega t$$

Now, let us look at this solution in a little detail. If we look at the first two terms of the solution, they are of the frequency equal to natural frequency of the system. And this will also depend upon the initial conditions of the system and the forcing frequency will also have some influence in this part of the solution. And since now, we are talking about undamped systems, this part of solution does not decay; but for damped systems, this part of solution will decay with time. So, this part of solution is known as Transient response; that means something which changes or decays with time. And as we have just discussed, the frequency of this motion will be equal to the natural frequency of the system.

And now, we have the particular solution, which is having a frequency equal to the forcing frequency and this part will not depend upon the initial condition. And this part of the solution is known as Steady state response, because this will be present as long as the force is present and this does not decay with time. So, this is a Steady state response. So, now, if the initial conditions are 0, then also the response will be non-zero. This is unlike free vibrations. In free vibrations, when initial conditions are 0; there was no motion, there is no vibration. So, but in forced vibration because of the effect of force, the vibration will be continuing even if the initial conditions are 0. For 0 initial conditions, we will have the response like this, we will have this and this term of the solution.

$$x(t) = \frac{p_0}{k}\frac{1}{(1-\left(\frac{\omega}{\omega n}\right)^2}(\sin \omega t - \frac{\omega}{\omega n}\sin \omega n t)$$

So, we will still have some vibration in the structure.

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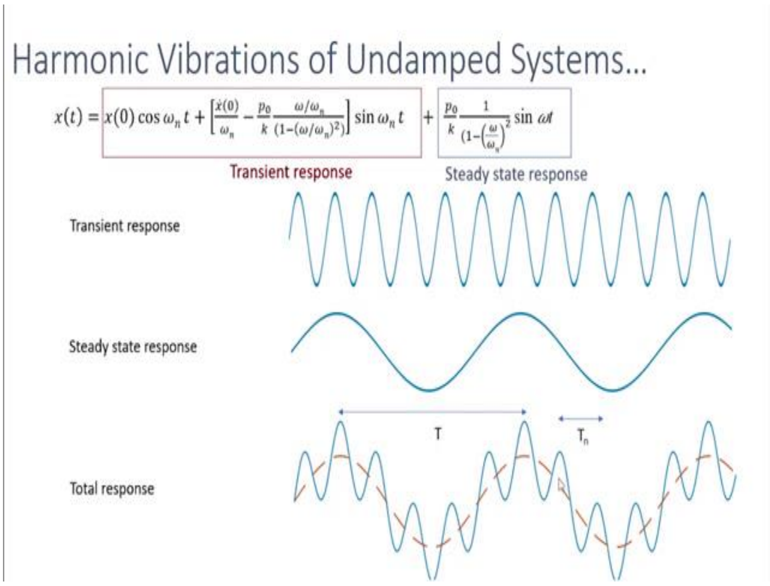

Now, let us look at the response of the system graphically. So, if you plot the transient response this will look like this for undamped systems, the frequency of this response will be equal to the natural frequency of the system and the initial values of the transient response will depend upon the initial conditions as well. So, the transient response will look like this vary as sines or cosines with the frequency equal to natural frequency. Now let us look at the steady state response. The steady state response will vary like this, the amplitude will be depending upon the force amplitude and that value will be equal to this and the frequency of this steady state response will be equal to the forcing frequency. So, this period will be equal to the forcing period.

Now the total response will be the sum of these two, the transient and the steady state responses. So, now, let us see how the total response looks like. This blue line, the blue plot is the total response. So, and the dotted line the same as the steady state response.

So, here you can see this total response will have two frequency components, it is seen in this plot; the period of this total response would be equal to the forcing period. So, as you can see the response will repeat itself after a time equal to t, which is the period of the forcing frequency. And the consecutive peaks or consecutive local maxima of this response will be at a time difference of Tn that is the natural period of the system.

So, we can see from this plot that, the total response is the sum of these two components.

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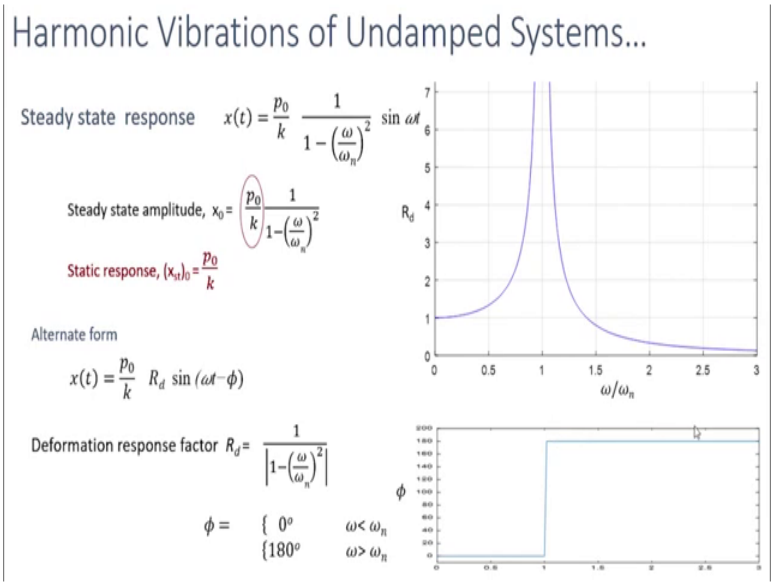

Now, we will look at the steady state response in a little detail. The steady state response will be present even after the transient response dies out.

$$x(t) = \frac{p_0}{k}\frac{1}{(1-\left(\frac{\omega}{\omega n}\right)^2}\sin \omega t$$

So, the steady state response is this, as we have seen earlier, the amplitude of the steady state response is equal to p naught(p_0) by k multiplied by 1 by 1 minus frequency ratio square.

$$x(t) = \frac{p_0}{k}\frac{1}{(1-\left(\frac{\omega}{\omega n}\right)^2}$$

So, if you look at this, this term p naught by k is equivalent to the static response; that means, if our force acting on the system was a constant force, say p naught(p_0), the response of that system would be p naught by the stiffness of the system; that is p naught by k that will be the static deflection of that system.

So, now since our force is dynamic, it is a harmonic force it is equal to p naught multiplied by sin omega t. In that case we have a response, which has an amplitude p naught by k multiplied by another factor. So, if this factor is greater than 1, our dynamic response amplitude will be greater than the static response; and if this factor is less than 1, we will have a dynamic response which has an amplitude even less than the static response. Now let us see how this factor is varying with the value of the forcing frequency. So, the x axis of this curve this plot shows the frequency ratio, the forcing frequency divided by the natural frequency; the y axis shows the factor 1 by 1 minus frequency ratio square.

So, as we can see here, when the forcing frequency is close to 0; the ratio is also close to 0 and that's when this factor will have a value close to 1; that means, this term is vanishing. So, then the steady state amplitude will be equal to the static response of the system. So, as the frequency ratio increases, the value of this factor is also increasing; and when the frequency ratio is near 1, this is increasing at higher rate. So; that means, when the frequency ratio is close to 1, the steady state amplitude will be much higher than the static response. As this factor is very high, when it is close to when the ratio is close to 1.

So, what happens, when the forcing frequency is higher than the natural frequency? So, when forcing frequency is higher than the natural frequency, this term will become higher than 1. So, this factor will become negative. So, when this factor becomes negative, it means that the steady state response and the force acting will be in opposite direction. If the force is acting in one direction, the response will be in other direction; that is what happens when this factor is negative. Now when the forcing frequency is higher than the natural frequency; that is when the frequency ratio is slightly higher than 1, we will have high amplitude for this factor. So, the steady state amplitude will be much higher than the static response; but the response will be in opposite direction of the force acting. And when the forcing frequency increases even more, when the frequency ratio is much higher than 1, this factor will reduce and it will become very close to 0.

So, that means, when this frequency ratio is much higher than 1, the amplitude of the steady state response will be close to 0; it will be much lesser than the static response. This factor will be equal to minus 1, when the frequency ratio is equal to root 2, somewhere here. So, when the frequency ratio is higher than root 2, we will have a steady state amplitude less than the static response. So, until the frequency ratio is root 2, we will have a steady state amplitude which is either equal to or higher than the static response but when the frequency ratio is beyond root 2, our dynamic steady state amplitude will be less than the static response.

Now, we can represent the steady state response in an alternative form like this, p naught by k a factor R d and the sin function omega t minus phi*(φ)*; phi is a phase angle. So, we can define this factor R d as, the deformation response factor and it is value is the amplitude of this factor; that is 1 divided by modulus of 1 minus frequency ratio square. And this phase angle phi will be equal to 0, when omega is less than omega n; that is when the forcing frequency is less than the natural frequency. And the value of this phi*(φ)* will be equal to 180, when the forcing frequency is larger than the natural frequency. We will discuss, what happens when forcing frequency is equal to natural frequency a little later.

$$x(t) = \frac{p_0}{k} R_d \sin(\omega t - \Phi)$$

So, now let us see how this deformation response factor varies with the frequency ratio. So, this factor R d is just a modulus of this factor; the deformation response factor R d varies like this, it increases from 1 to a very high value, when the frequency ratio increases from 0 to 1. When the frequency ratio is higher than 1, it the value of R d decreases from a very high value to very close to 0. So, when the frequency ratio is very high, the value of R d will be very close to 0. And this is the variation of the phase angle phi with respect to the frequency ratio; when the frequency ratio is less than 1, this phase angle is equal to 0; when it is higher than 1, it is equal to 180 degrees as we have defined here. So, the steady state response can be alternatively represented in this form, where R d is defined as, a deformation response factor.

So, this deformation response factor is an indicator of how much the static response is amplified or reduced when the force is dynamic or harmonic in this case

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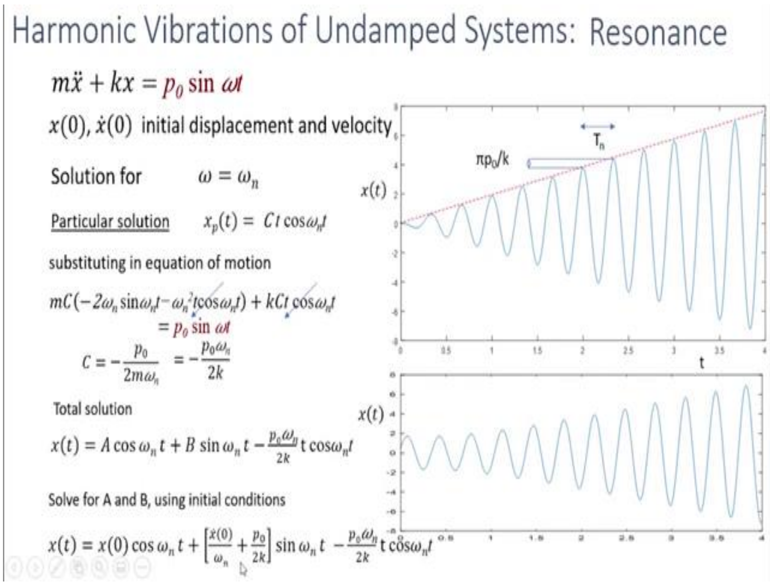

Now let us discuss the concept of resonance. When we were solving for the differential equation, I have mentioned that the particular solution we have chosen was not valid for omega is equal to omega n(ω_n). So, now, we will find a particular solution which is valid, when omega is equal to omega n. So, we will choose the particular solution as C t cos omega n t($Ct\ cos\omega_n t$) and C t sin omega n t($Ct\ sin\omega_n t$) will also work as well. So, now, as we did earlier, we will substitute this expression in the equation of motion and we will find out the value of C.

$$x_p(t) = C t \sin \omega t$$

So, when you substitute this in the equation of motion, we will get an expression like this. Since we are solving for omega is equal to omega n, this omega will be equal to omega n; so, in this equation the cos omega n t terms will cancel out each other, because m omega n square is equal to k. So, this will be cancelled out. So, we can equate the sin terms, sin omega n t *(sin $\omega_n t$)* is equal to omega t *(ω_n)*. So, we can find the value of C as this, minus p naught omega n by 2 k. So, the total solution of this equation of motion for omega is equal to omega n t is equal to this.

$$x(t) = A\cos\omega n t + B\sin\omega n t - \frac{p_0\omega n}{2k} t\cos\omega n t$$

So, this is the transient part of the equation and this is the steady state part of the equation. So, as we did earlier, we can find the value of A and B using the initial conditions. So, if you find the value of A and B, it will be A will be equal to x naught(X_0) and B will be equal to initial velocity divided by natural frequency plus p naught by 2 k.

$$x(t) = x(0)\cos \omega n\, t + \left[\frac{\dot{x}(0)}{\omega n} + \frac{p_0}{2k}\right] \sin \omega n\, t \quad - \quad \frac{p_0 \omega n}{2k} t \cos \omega n\, t$$

So, this is the transient part and this is the steady state part. As you can see, the steady state part has a term t so; that means, as the time increases, the steady state amplitude will also increase. So, after some time of starting the vibration, the steady state part will be the predominant vibration. The effect of this transient vibration will be insignificant after some time and the vibration will be equal to the steady state vibrations. So, the total response will look like this. So, this response is for zero initial conditions; that means, these two terms will be 0 and the effect of this term will die out soon, as the steady state response is increasing with time. So, this is the steady state response; and as you can see in this term, this is the amplitude increases with time, but each cycle the amplitude is increasing.

So, the frequency of this motion is equal to the natural frequency of the system and the time between two peaks will be equal to the natural period. So, in one cycle, the amplitude of the this total response will increase by pi p naught by k. So, if you substitute omega n t plus T n the natural period, you will get an amplitude which is higher than the first amplitude by this much. So, at each cycle, the amplitude increases by p naught by k. So, and since this is an undamped system, this vibration will continue without any decay in time. So, as time goes by, the value of the response will only increase so; that means, the response is unbounded, in this case when omega is equal to omega n and this condition is known as resonance.

So, for undamped systems at resonance, the response grows unbounded and that happens when the forcing frequency is equal to the natural frequency. Now let us see the response when the initial conditions are non-zero. So, if you see in the initial part of this vibration, this response is not growing; as it grows when the time is high, this is because of the effect of the initial conditions. And this effect of initial conditions will die out as the time increases. So, when the time increases, the steady state response will become very prominent and after some time it increases unboundedly. So, hear the responses at each cycle will increase similar to this.