Structural Dynamics for Civil Engineers - SDOF Systems

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Lecture - 05
Coulomb Damped Free Vibrations

We have seen that the Damping in the structure can be modeled using Viscous Damping. So, in viscous damping, the damping force is proportional to the velocity of the system and the damping coefficient can be chosen, so that the energy dissipated by the viscous damper is equal to the energy dissipated by all energy dissipation mechanism in the structure. So, now, we will learn the different type of damping called Coulomb Damping, so let us see how coulomb damping works.

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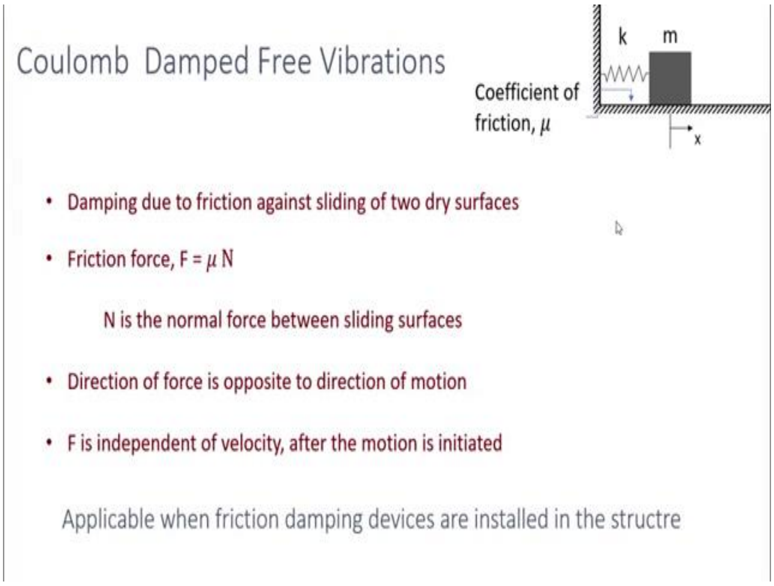

In coulomb damping, the energy dissipation is due to the friction between two sliding surfaces. So, the Coulomb Damped Free Vibration can be modeled as shown here.

So, we have a mass and stiffness and the damping will be due to the friction between this mass and this surface; and the coefficient of friction in the surface is equal to mu(μ). So, the frictional force in the surface is equal to mu(μ) multiplied by N, when N is the normal force acting between these sliding surfaces. The direction of this frictional force will be equal to the opposite direction of the force.

$$F = \mu N$$

So, if the mass is moving in this direction, the frictional force will act in the opposite direction and the value of this frictional force is independent of velocity unlike the viscous damping. So, once the motion is initiated, the value of force is independent of velocity and this type of damping models are applicable when friction damping devices are installed in the structure.

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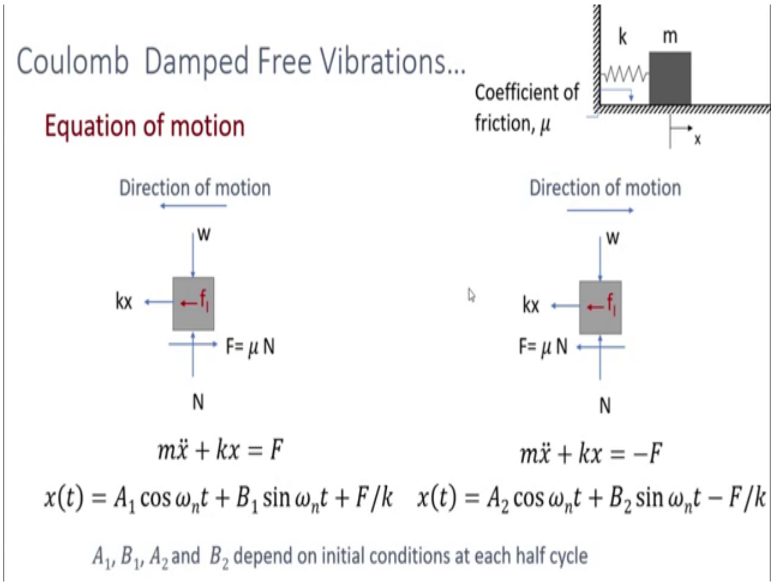

Now, let us write the equation of motion of this coulomb damped free vibration system. So, the free vibration system is this. So, we have x is positive in this direction. So, we have a restoring force k x will be acting on this mass in the opposite direction and an inertia force will also be acting in the opposite direction.

So, now, to calculate the frictional force, it will depend upon the movement of this mass. So, if the direction of motion of this mass is towards left, we have the spring force and inertia force in this direction, weight acting in the downward direction, normal reaction to the weight will acting in the opposite direction, the vertical direction; when the motion is in this direction, the frictional force will be in the opposite direction.

So, now we can sum up the forces in x direction. So, we will have the inertia force plus the spring force is equal to the frictional force. So, m x double dot plus k x is equal to F; where F is the frictional force which is equal to coefficient of friction multiplied by the normal reaction and the solution of this type of differential equation will be equal to A 1 cos omega n t($cos\omega nt$) plus B 1 sin omega n t($sin\omega nt$), this is the free vibration response of undamped system plus the effect due to this constant force that is F by k.

$$m\ddot{x} + kx = F$$

$$x(t) = A1\cos\omega nt + B1\sin\omega nt + F/k$$

So, this is equal to a static displacement, if F is the force and k is the stiffness of the spring this is equivalent to a static displacement. So, omega n(ωn) as we know it is square root of k by m.

So, now let us see how the equation of motion will look like, when the mass is moving in the opposite direction that is towards right. So, the spring and inertia force will still be acting in the same direction but when the mass is moving in this direction, the force will be acting in the opposite direction. So, now, we have our equation of motion as m x double dot plus k x is equal to minus F.

$$m\ddot{x} + kx = -F$$

$$x(t) = A2\cos\omega nt + B2\sin\omega nt - F/k$$

So, the solution of this can be written as A 2 cos omega n t($cos\omega nt$) plus B 2 sin omega n t ($sin\omega nt$), that is the free vibration response minus the static response due to this force F that is minus F by k; the constants A 1 B 1, A 2 B 2 are calculated using the initial conditions at each half cycle.

Because this condition the mass will be moving in one direction in one half cycle; and in the next half cycle the direction of motion will be the opposite. So, these A 1, B 1 and these constants need to be evaluated using the initial conditions at each half cycle. So, this needs to be evaluated at each half cycle. So, we will have a separate solution in each half cycle.

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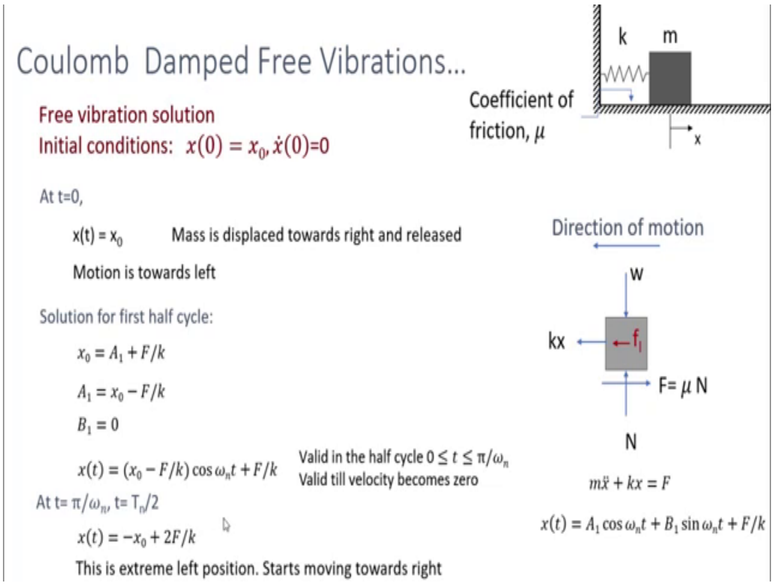

So, now let us see the solution of this free vibration response of a coulomb damped system. So, we will assume initial conditions, displacement initial condition is equal to x naught(*X0*) and we assume that the initial velocity is equal to 0.

So, using this initial condition, we will solve the equation of motions for this system. So, what happens when t is equal to 0? So, we know the initial condition. So, we know that the value of x at t is equal to 0 is equal to x naught(*X0*) and this is the initial condition. So, here the mass is displaced towards right and then it is released.

$$x(t) = x_0$$

So, the initial condition was a displacement towards right. So, when a body is displaced towards right and it is released, it moves to the left. So, our first half cycle will be a motion towards left. So, let us solve the equation of motion for the first half cycle.

So, the motion of the body is towards left. So, this is the equation of motion we have to consider; m x double dot plus k x is equal to the force F. So, the solution is, this with a plus term F by k. So, if you substitute the value of the initial displacement in this, for t is equal to 0; we would get that x naught is equal to A 1 plus F by k, cos omega n t($cos\omega nt$) is 1 at t is equal to 0 and sin omega n t($sin\omega nt$) becomes 0.So, we will get this equation and from this we can write A 1 is equal to x naught(*X0*) minus F by k. Similarly, we can differentiate this and substitute the value of initial velocity, which is equal to 0. So, if you do that, we would get B 1 is equal to 0.

So, we have got both the constants A 1 and B 1. Now we can write this solution for the system in the first half cycle. So, that would be x naught(*X0*) minus F by k cos omega n t plus F by k.

$$x0 = A1 + F/k$$

$$A1 = x0 - F/k$$

$$B1 = 0$$

$$x(t) = (x0 - F/k)\cos\omega nt + F/k$$

So, it is a cosine function with an amplitude x naught(*X0*) minus F by k and it is shifted in the positive y direction by F by k and this solution is valid only in the first half cycle, that is when the time is between 0 and the half of the natural period; that is pi by omega n (ω_n) and this solution is valid only when the velocity will become 0.

So, at the end of this first half cycle, the body will reach the extreme left position and then the velocity will be equal to 0. So, this solution is valid until then, that is during the first half cycle.

So, now let us see the value of this displacement when at the end of this half cycle; that is when time is equal to pi by omega n. So, if we substitute the value of t is equal to T n by 2 or pi by omega n, we would get the value of x as minus x naught plus 2 F by k. So, now, this is the initial displacement for the next half cycle and again at t is equal to this value, we would have the velocity is equal to 0.

At t= $\pi/\omega n$, t= $T_n/2$

$$x(t) = -x0 + 2F/k$$

So, for the next half cycle, we will have initial displacement equal to this and initial velocity equal to 0 and as I have mentioned earlier, this location is the extreme left position. So, after that, the body will move towards right in the next half cycle.

So, at this position the velocity is 0.

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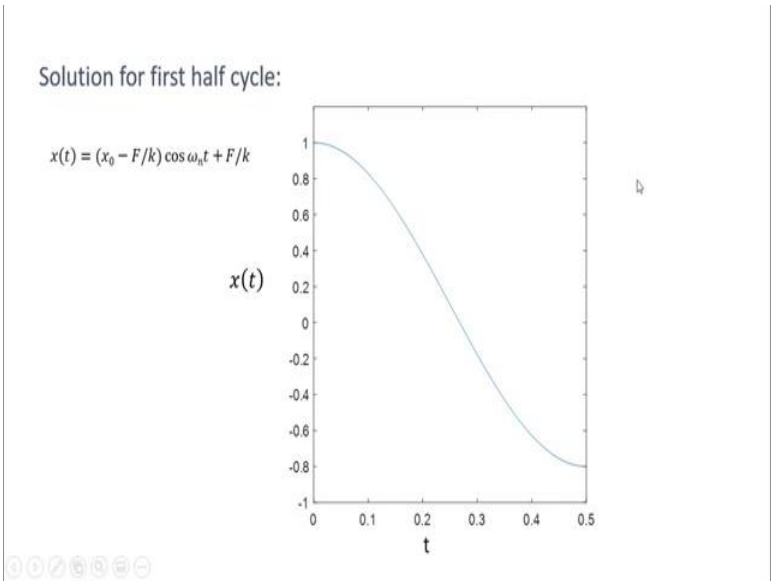

The solution for the first half cycle looks like this; this is a cosine function with an amplitude equal to x naught minus F by k and this cosine function is shifted towards the positive x direction by an amount F by k. So, the velocity was 0 at the starting of the first half cycle, that is the initial velocity and at the end of the first half cycle also the velocity is equal to 0. Now let us see the solution in the second half cycle.

$$x(t) = (x0 - F/k)\cos\omega nt + F/k$$

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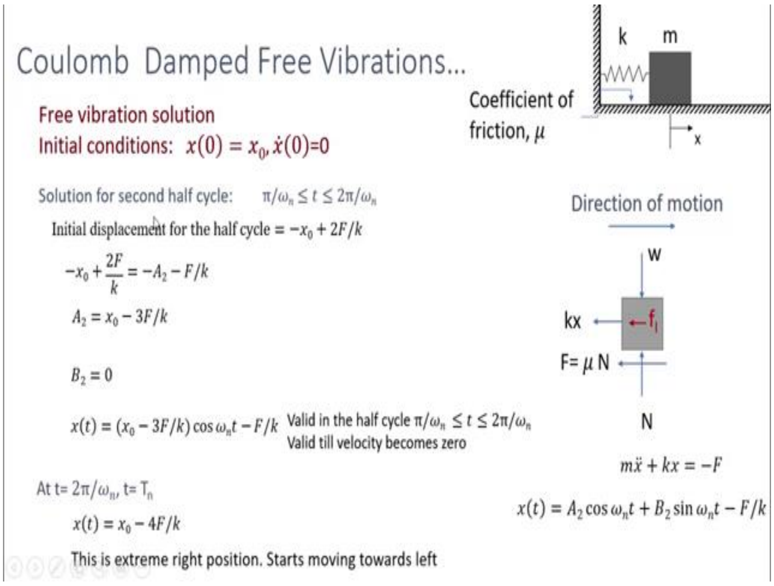

So, in the second half cycle, the time is between pi by omega n and 2 pi by omega n this is the natural period of the system. So, in the second half cycle, the direction of motion is towards right. So, the frictional force will act towards left.

So, the equation of motion is this and the solution is this with the negative F by k term. So, the initial displacement for this half cycle is equal to the displacement at t is equal to pi by omega n, which we found out from the first half cycle. So, that was, is equal to minus x naught plus 2 F by k. So, substituting this in this expression, we can calculate the value of A 2.

$$-x0+\frac{2F}{k}=-A2-F/k$$

So, we can substitute the value of initial displacement and substitute the value of t in this and we can calculate the value of A 2 as equal to x naught minus 3 F by k and here also we can find out using the initial velocity, which is equal to 0 again; we can find out that B 2 is 0.

$$A2=x0-3F/k$$

$$B2=0$$

$$x(t)=(x0-3F/k)\cos\omega nt-F/k$$

So, the total solution for this second half cycle is equal to x naught minus 3 F by k plus cos omega n t minus F by k; and this is valid only in this half cycle, and this is valid only when the velocity becomes 0. So, when the value of t is equal to 2 pi by omega n that is the natural frequency, that is when the second half cycle ends. So, at that point we will have a velocity is equal to 0.

Now let us find the value of the displacement x at t is equal to T n; that is at the end of this second half cycle. So, if you substitute t is equal to T n in this expression, we would get x of t is equal to x naught minus 4 F by k. So, at the end of the first half cycle, the displacement was minus x naught plus 2 F by k.

So, now at the end of second cycle, that is equal to x naught minus 4 F by k and this is the extreme right position and now onwards, the body will start moving towards left and that will be the starting of the next half cycle. So, similarly we can find out the response during the next half cycle.

At t= $2\pi/\omega n$, t= T_n

$$x(t) = x0 - 4F/k$$

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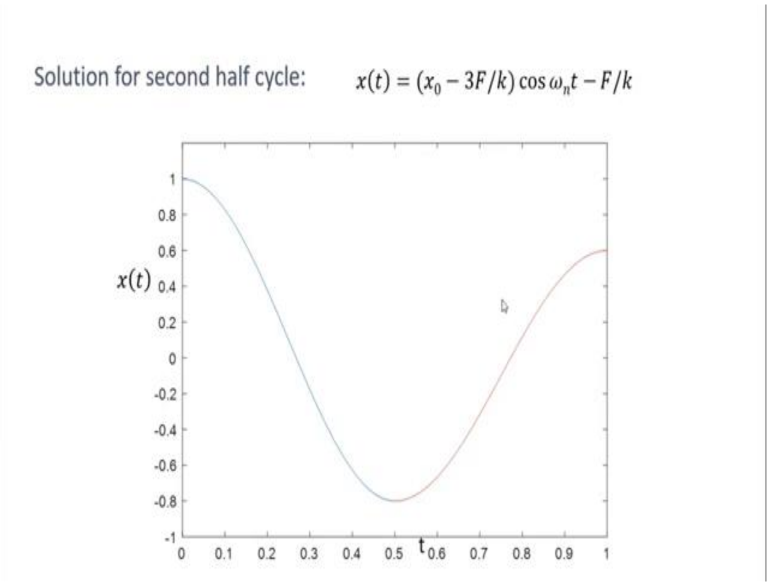

The solution for the coulomb damped system for the second half cycle is expressed like this, x naught minus 3 F by k cos omega n t minus F by k. So, for the second half cycle, the solution is this, the red curve. So, in the first half cycle the response was this, and in the second half cycle it is this. As you can see in this curve, the response at the beginning of first half cycle was this and in one full cycle the amplitude decays.

$$x(t) = (x0 - 3F/k)\cos\omega nt - F/k$$

So, we can calculate how much this decay is; and we can also see that at the end of half cycles the velocity that is the slope of this curve is equal to 0. So, at the end of each half cycle, we will have velocity is equal to 0 and the displacement will have a maximum value. We have solved the coulomb damped free vibration system up to 2 half cycles; now we will solve it for the third half cycle.

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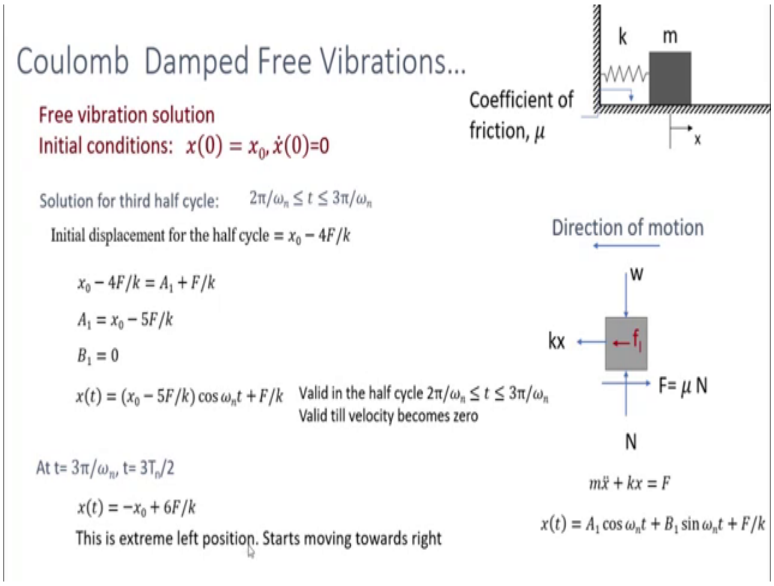

So, the thirds half cycle, the time is between 2 pi by omega n and 3 pi by omega n, so we have calculated the response at the second half cycle. And so, there we have found the displacement response at time is equal to 2 pi by omega n, so that would be the initial displacement for this half cycle.

$$2\pi/\omega n \leq t \leq 3\pi/\omega n$$

So, that is equal to x naught minus 4 F by k and in this half cycle the direction of motion of the mass will be towards left. So, the equation of motion will be as per this relation m x double dot plus k x is equal to F; and we will have the solution like this, plus a positive F by k term.

$$\text{Initial displacement for the half cycle} = x0 - 4F/k$$

$$x0 - 4F/k = A1 + F/k$$

So, substituting the initial displacement value in this, when t is equal to 2 pi by omega n; we can calculate the value of A 1 and that will be equal to x naught(X_0) minus 5 F by k and here also, we can substitute the value of the velocity which is equal to 0. We can differentiate this and substitute the value of velocity is equal to 0, that will give us the value of B 1 and which is again equal to 0.

$$A1 = x0 - 5F/k$$

$$B1 = 0$$

So, this is the solution of the system in the third half cycle, x naught minus 5 F by k cos omega n t plus F by k and this solution this a cosine function with some amplitude and this shifted in the positive x direction by F by k; and the solution is valid only in this half cycle, that is the third half cycle when time is between 2 pi by omega n and 3 pi by omega n and the solution is valid only till the velocity becomes 0 next time; that means, the velocity is 0 when the time is equal to 3 pi by omega n, so until that time, this solution is valid. So, now, let us find out the value of the displacement at t is equal to 3 pi by omega n, that is at the end of the third half cycle.

$$x(t) = (x0 - 5F/k)\cos \omega nt + F/k$$

So, if you substitute the value of t is equal to t pi by omega n, we would get minus x naught plus 6 F by k. So, at the end of the previous half cycle, it was x naught minus 4 F by k. So, the, at the end of third half cycle it is minus x naught plus 6 F by k, this is the extreme left position. So, in the third cycle, the body was moving towards left.

At t= $3\pi/\omega n$, t= $3T_n/2$

$$x(t) = -x0 + 6F/k$$

So, at the end of the third half cycle, it is at the extreme left position; and then onwards it moves towards the right. So, that will be the fourth half cycle. So, we can continue the solution like this.

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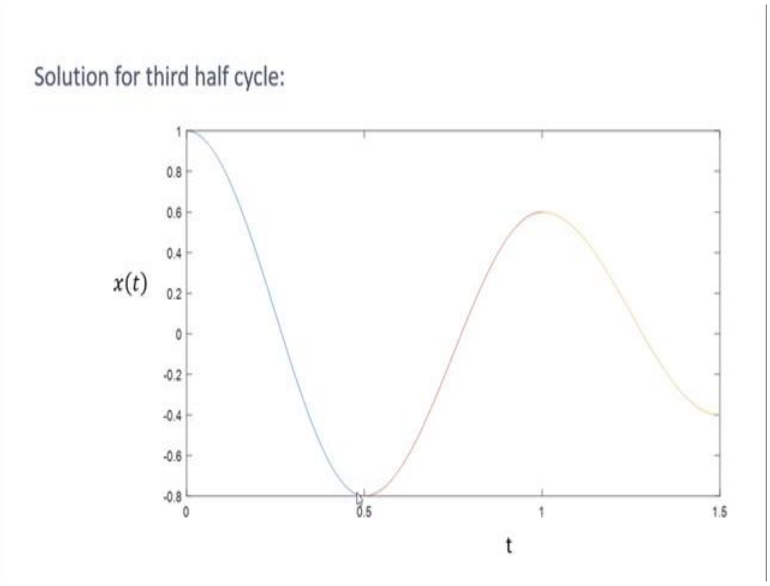

And now let us see the solution in the third half cycle. So, this was the solution in the first half cycle this is the second half cycle.

And now, this is the third half cycle and the solution is equal to x naught minus 5 F by k cos omega n t plus F by k.

$$x(t) = (x0 - 5F/k)\cos\omega nt + F/k$$

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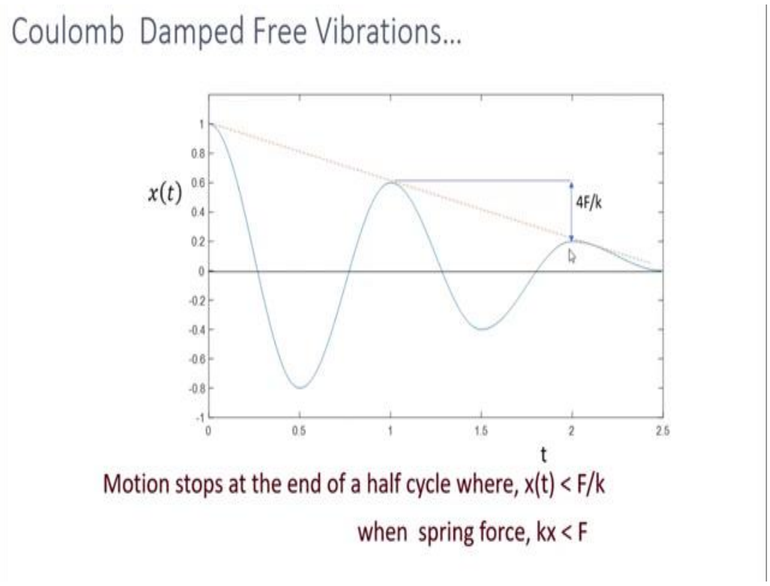

So, this is the solution of the coulomb damped free vibration system for an initial displacement is equal to 1. So, this shows many cycles. So, as you can see the motion decays at each cycle and the decay in a cycle, we can calculate and that would be equal to 4 F by k.

So, in each cycle the amplitude of the response reduces by 4 F by k and when the displacement response is less than the value of F by k; that is equivalent to a static response. So, when the displacement response is less than the value of F by k, the motion stops.

So, when at the beginning of half cycle, if or at the end of a half cycle; if the displacement response is less than F by k, then the motion stops completely, because the next half cycle will not be initiated. So, to initiate the next half cycle, we need a displacement response which is greater than F by k.

So, if it is less than F by k, the motion will not be initiated, so the body stops moving. In another way, we can say that the body stops moving, when the spring force that is k multiplied by x is less than the friction force. So, to continue the motion of the body, we need a spring force which is greater than the friction force; the spring force should overcome the friction, then only the body will move.

So, we can say that, if the spring force is less than the friction force, the body will stop moving.

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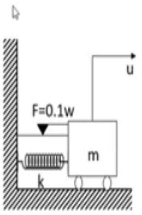
Example

An SDF system consisting of a weight, spring, and friction device is shown Figure. This device slips at a force equal to 10% of the weight, and the natural vibration period of the system is 0.25 sec. If this system is given an initial displacement of 2 *in* and released, what will be the displacement amplitude after six cycles? In how many cycles will the system come to rest?



Given: F = 0.1w , T_n = 0.25 sec

$$\frac{F}{K} = \frac{0.1w}{k} = \frac{0.1mg}{k} = \frac{0.1g}{\omega_n^2} = \frac{0.1g}{(2\pi/T_n)^2} = \frac{0.1g}{(8\pi)^2} = 0.061 \text{ in}$$

The reduction in displacement amplitude per cycle is: 4F/k = 0.244 in.

The displacement amplitude after 6 cycles is: 2.0 – 6(0.244) = 2.0 – 1.464 = 0.536 in.

Motion stops at the end of the half cycle for which the displacement amplitude is less than F/k.

Displacement amplitude at the end of the 8th cycle it is 0.536 – (0.244)2 = 0.048 in. < F/k.

Therefore, the motion stops after 8 cycle.

Now we will solve an example problem in coulomb damped free vibration system. So, an SDF system consisting of a weight, spring, and friction device is shown in the Figure. So, you have a mass and you have a spring with a stiffness k and we have a friction device as shown here.

The friction force is given as 10 percent of the weight. So, this will be equal to 0.1 multiplied by the weight of this mass and the natural vibration period of the system is given and it is equal to 0.25 seconds. If the system is given an initial displacement of 2 inches and released, what will be the displacement amplitude after six cycles and in how many cycles will the system come to rest? Now let us solve this problem.

So, it is already given that the friction force is equal to 10 percentage of the weight of the system. So, that means; F is equal to 0.1 w, w is equal to mass multiplied by gravitational acceleration and the natural period of the system is also given and it is equal to 0.25 seconds. So, we can calculate the value of F by k.

So, F is 0.1 multiplied by the weight, and k is the stiffness of this spring; the stiffness is not given, but we have the mass, we have the weight, and the natural frequency. So, we can rewrite it like, 0.1 mass multiplied by g divided by k and that would be equal to we know m by k is equal to 1 by omega n square or omega n is equal to root of k by m.

So, substituting this value, we know that this is equivalent to 0.1 multiplied by g divided by natural frequency square and this is the circular natural frequency. We know the natural period; and we can calculate the circular natural frequency using the natural period and omega n is equal to 2 pi by T n, so we can substitute that.

Now we know all the value in this expression, so we can calculate the value of F by k as 0.061 inches. So, we have learnt in the theory that, the reduction and displacement amplitude per cycle is 4 F by k. So, F by k is this much. So, 4 F by k is equal to this multiplied by for 0.244 inches. So, now, we can calculate the first question.

So, the displacement amplitude after 6 cycles, we know the initial displacement has 2 inches. So, in each cycle the amplitude will decay by this amount. So, after 6 cycles, the amplitude will be 2 minus 6 times this. So, 2 by 2 minus 1.464 is equal to 0.536; that is the amplitude after 6 cycles and now it is asked, in how many cycles will this system come to rest?

We have just learned, that the system will continue vibrating until the displacement response is less than F by k. So, we know that F by k is 0.061 and in each cycle the displacement response will decay by 0.244 inches. So, at the end of six cycles, the responses 0.536; so we can just guess in two more cycles, in one cycle the decay is 0.244. So, in 2 cycles it will be 0.488.

So, let us calculate the displacement amplitude after 8 cycles. So, after 8 cycles the displacement amplitude would be; the displacement amplitude after 6 cycles minus 2 times the decay in a cycle. So, the displacement amplitude at the end of the 8th cycle is equal to 0.048 inches, which is less than the value of F by k, which is equal to 0.061 inches.

So, at this stage there would not be any motion happening in the next half cycle. So, therefore, the motion stops after 8 cycle. So, this way we can evaluate the response of a coulomb damped free vibration system.