Structural Dynamics for Civil Engineers - SDOF Systems
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Lecture – 04
Damped Free Vibrations

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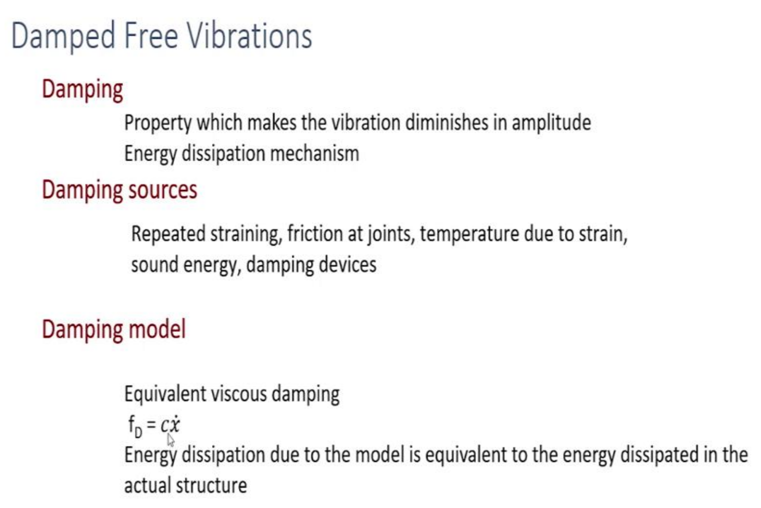

In the previous lesson, we learned about undamped free vibrations; in this lesson we will be learning about Damped Free Vibration. So, what is damping? Earlier, we have seen that when a structure is under free vibration its amplitude reduces with time and eventually the structure will go to rest. And, this property of a structure which makes its vibration diminishes in amplitude with time is known as damping. And, this damping is an energy dissipation mechanism in the structure.

Now, let us see some of the sources of damping. Every material will dissipate some energy when it is under strain. So, when a structure is under repeated straining it will dissipate some energy. Then we have friction at joints, which is another form of energy dissipation and when a structure is getting strained some temperature is also developed. So, that will also cause some energy dissipation. And, when a structure is vibrating sometimes sound is produced, so, that will also cause some energy to dissipate.

So, all these mechanisms will take away some energy from the system and in some structures, there will be additional damping devices installed. So, these devices will be designed in order to absorb some energy when that structure is under vibration. For example, damping devices are installed in structures to save it from earthquake loading.

It is difficult to model all these damping mechanisms in a structure. So, often we assume an equivalent viscous damping model to represent the energy dissipation in a structure. So, in viscous damping it is assumed that the damping force is proportional to the velocity of the structure. So, the damping force can be calculated as c which is a damping coefficient multiplied by the velocity. So, the energy dissipation due to the model represented like this is equivalent to the energy dissipated in the actual structure.

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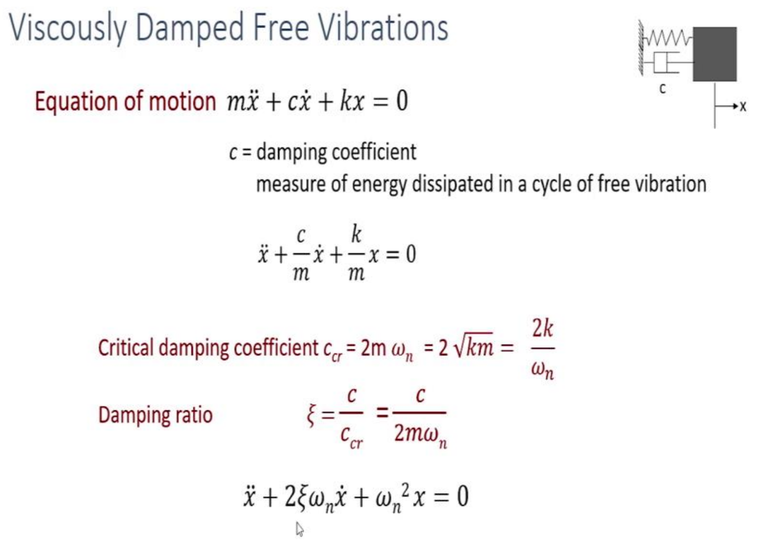

Now, let us look at viscously damped free vibrations. These types of vibrating systems can be represented like this. The system will have spring with some stiffness and a mass and it will also have a viscous damper with a damping coefficient is equal to c. So, this is the equation of motion of this system it is like mx double dot plus cx dot plus kx is equal to 0; c as we have mentioned is the damping coefficient and this is a measure of energy dissipated in one cycle of free vibration. So, the energy dissipated when the structure makes one cycle of vibration is represented by c.

$$m\ddot{x} + c\dot{x} + kx = 0$$

Now, we can rearrange this equation of motion like this just by dividing it by the mass and we know that k by m is equal to omega n square that is square of natural frequency.

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

Now, we can define a critical damping coefficient c cr is equal to 2m omega n. So, this quantity is called as critical damping coefficient. We will explain the physical meaning of it later. Since omega n is equal to root of k by m we can write it like this 2m omega n is equal to 2 square root of km and we can also write it like this 2k by omega n.

So, we just defined a quantity called critical damping coefficient. Now, we can define a damping ratio zeta is equal to c by c critical; that means, this damping ratio is a ratio of the damping present in the system to the critical damping. And, as you can see here the critical damping depends only upon the structural properties, the mass and stiffness because omega n depends only upon stiffness and mass.

So, the critical damping coefficient will also depend only upon the system properties. So, this damping ratio is the ratio of the damping present in the system to the critical damping. Now, we can rewrite this equation of motion in terms of zeta. So, we have x double dot plus 2 zeta omega n x dot plus omega n square x is equal to 0.

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Depending upon the value of zeta the damping in a structure can be classified into three. So, if the value of zeta is greater than 1, that means, the damping is more than the critical damping value that type of damping is called over damping. And, when zeta is equal to 1, the structure is under critical damping; that means, the damping coefficient is equal to the critical damping which is equal to mass multiplied by natural frequency. And, when the value of zeta is less than 1, that type of damping is called under damping. So, in that case the damping coefficient will be less than 2m omega n.

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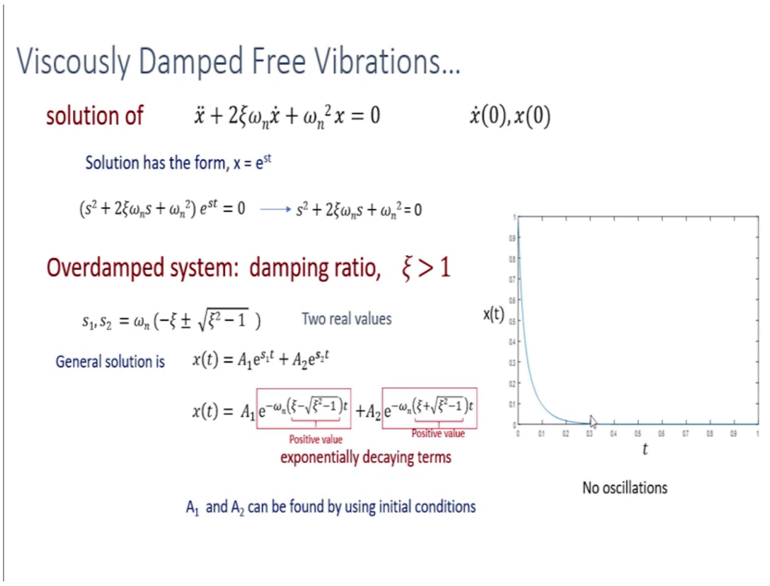

Now, we will solve the equation of motion of viscously damped free vibrations. So, this is a free vibration. So, the force is equal to 0 and we also have initial conditions we know the initial displacement and initial velocity. So, this equation of motion, this is a second order linear differential equation, homogeneous linear differential equation with constant coefficients.

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n{}^2\, x = 0 \qquad \dot{x}(0),\ x(0)$$

So, as we did in our undamped free vibration solution, we know that this equation has a solution of the form x is equal to e to the power st, so that means, we can substitute this in this equation and the identity will hold. So, when we substitute this function in our equation of motion, we get this and we can take this characteristic equation and solve for s. So, we know that for non-trivial solutions this part of the equation should be 0. So, this is the characteristic equation, we can solve this. So, we can solve this for different values of zeta. When zeta is greater than 1, that is, if the system is over damped, we get two values of s that is s 1 and s 2 which is equal to omega n multiplied by minus zeta plus or minus square root of zeta square minus 1.

$$s_1, s_2 = \omega_n (-\xi \pm \sqrt{\xi^2 - 1})$$

So, these are two real values. So, we have two real roots for this characteristic equation. So, we can write our general solution of this equation of motion as x of t is equal to linear combination of e to the power st; so, the linear combination of e to the power s 1 t plus e to the power s 2 t. So, if you substitute the value of s 1 and s 2 we get this.

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$x(t) = A_1 e^{-\omega_n\left(\xi - \sqrt{\xi^2 - 1}\right)t} + A_2 e^{-\omega_n\left(\xi + \sqrt{\xi^2 - 1}\right)t}$$

So, if you look at this equation closely, we can see that this value in the bracket this value this is a positive value because zeta is greater than 1. So, this entire value will be positive and same here also value inside the bracket is positive. So, since this is positive the exponential function this exponent is negative. So, these two terms are exponentially decaying terms.

So, that means, our response our displacement response of this viscously damped free vibration is sum of two exponentially decaying terms. And, as we have discussed earlier the constant in this differential equation solution can be calculated using our initial conditions. So, we have done this use in the free vibration example, undamped free vibration example. So, here also we can substitute the value of initial conditions and solve for A 1 and A 2. So, that will complete our solution.

Now, let us see how this solution will look like. We have already discussed that it will be the sum of two exponentially decaying terms. So, this is how our solution will look like, so it is an exponentially decaying function. It starts with a high value and it decays faster, mathematically this reaches 0 at time is equal to infinity. So, practically that means, this response is 0 after a long time.

So, what does this behavior signifies? It says that an over damped system will not oscillate; that means, this curve this curve will never cross this x-axis. So, after we give this initial displacement or initial velocity, this system will go back to its original equilibrium position slowly and stays there. It does not oscillate this type of damping is used in door closers, automatic door closers.

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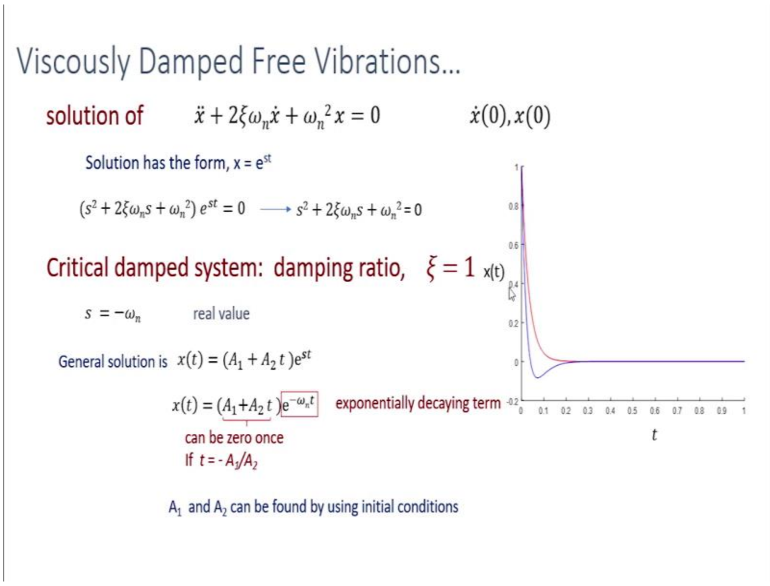

Now, let us see the solution of the critically damped system. So, here we have damping ratio is equal to 1 that is the system is critically damped. So, now, let us see how the solution will look like. So, here we can solve this equation and when zeta is equal to 1, we get the solution to s as minus omega n; that means, we get a single real value solution. So, for these types of problems the general solution as given by x t is equal to A 1 plus A 2 t multiplied by e to the power st. So, here s is minus omega n. So, here we know that this is an exponentially decaying term. So, we will get behavior similar to the previous one the over damped system because of this function.

$$x(t) = (A_1 + A_2\, t)e^{st}$$

$$x(t) = (A_1 + A_2\, t)e^{-\omega_n t}$$

Now, let us look at this part of this solution. So, this is an equation of a straight line. So, if your constants which we can find out using our initial conditions if this A 1 and A 2 are in such a way that this equation can be equal to 0 once. So, if the value of t is equal to this much this can become 0 and the x will have a solution equal to 0. So, this will happen only once because it is a straight line and it will cross the x-axis only once. We can calculate these constants A 1 and A 2 by you make use of the initial conditions.

So, now, let us look at the nature of the solution. So, this equation can have two types of behaviors depending upon the values of A 1 and A 2. The first one is similar to the over damped condition. So, from the initial displacement position it directly goes back to the equilibrium position and stays there that is the behavior we have also seen used in over damped system. The other one is when this part becomes zero; that means, this crosses the x-axis once and then comes back to the equilibrium position and stays there. So, this critical damped system can oscillate once.

So, these types of systems are used in designing weighing scales, where when a weight is put on the scale it deflects and when the weight is removed it goes back to the original position. So, those to design those systems we consider critically damped conditions.

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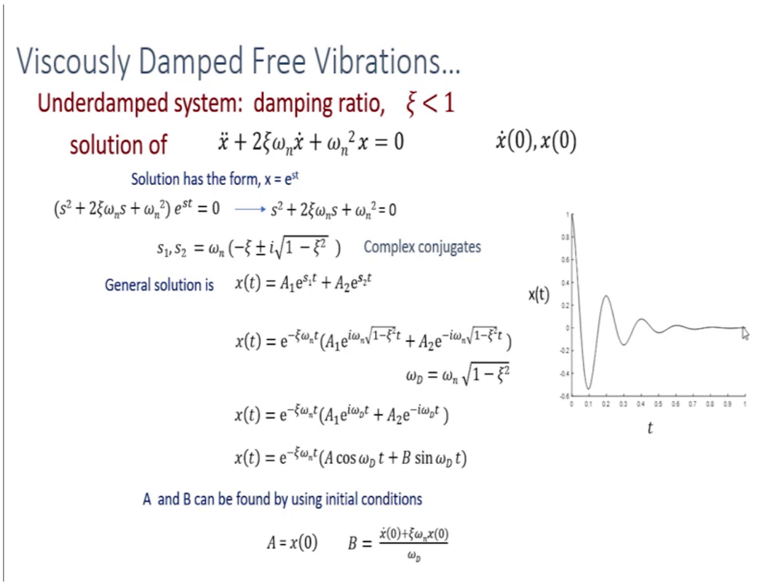

Now, we will look at the solutions of under damped systems; that means, damping ratio is less than 1. So, in all engineering structures in civil engineering structures this condition applies. For all the buildings and other structures, we will have damping ratio much less than 1. So, now, let us look at the solution of these type of vibrations. As we did earlier, we can formulate the characteristic equation and then we can solve this characteristic equation. So, here when the value of zeta is less than 1, we will get two solutions for s, they are complex conjugates.

$$(s2 + 2\xi\omega n s + \omega n2)\, e^{st} = 0 \qquad\qquad s2 + 2\xi\omega n s + \omega n2 = 0$$

$$s1, s2 = \omega n \left(-\xi \pm i\sqrt{1 - \xi 2}\;\right)$$

So, s 1 and s 2 are equal to omega n multiplied by minus zeta plus or minus i square root of 1 minus zeta square. So, we know that the general solution of this differential equation is the linear combination of e to the power s 1 t and e to the power s 2 t. So, we can substitute these values here and here we define that this omega n square root of 1 minus zeta square is equal to omega d. So, that is the damped natural frequency of this system. So, our equation will look like this after the substitution. Here as we did in the undamped system, we can write this e to the power i omega D t terms in sine and cosines.

$$x(t) = A_1 e^{s1t} + A_2 e^{s2t}$$

$$x(t) = e^{-\xi\omega n t}\left(A_1 e^{i\omega n\sqrt{1-\xi 2}t} + A_2 e^{-i\omega n\sqrt{1-\xi 2}t}\right) \qquad\qquad \omega D = \omega n\sqrt{1 - \xi 2}$$

$$x(t) = e^{-\xi\omega n t}\left(A_1 e^{i\omega D t} + A_2 e^{-i\omega D t}\right)$$

$$x(t) = e^{-\xi\omega n t}\left(A\cos\omega D\, t + B\sin\omega D\, t\right)$$

So, we can make that substitution based on Euler's equation and we can write the complete solution as e to the power minus zeta omega n t multiplied by A cos omega D t plus B sine omega D t; here omega D as we have defined it is omega n that is the natural frequency multiplied by square root of 1 minus zeta square. And, so, this part of the equation is similar to the undamped vibration except the fact that we have omega D instead of omega n, but we have an exponentially decaying function multiplied to it. So, that will depend upon the damping ratio and the natural frequency.

Here also we can calculate the constants A and B from the initial conditions. So, we can substitute the initial velocity and initial displacement. Calculate solve for A and B, so, we can find A is equal to x 0 which is the initial displacement and B is equal to this much, initial velocity plus zeta damping ratio omega n and initial displacement divided by omega D. So, if we look at it if we can set zeta is equal to 0, this part will become 0, sorry, this part will become 1 and omega D will become omega n zeta is equal to 0. So, if zeta is equal to 0, this will converge to the undamped free vibration response.

$$A = x(0) \qquad B = \frac{\dot{x}(0)+\xi\omega n x(0)}{\omega D}$$

So, now let us see how the response will look like. This is a sample response. so, the system will vibrate. So, this is the x-axis. So, it will vibrate about its equilibrium position and eventually the vibration amplitude will reduce and eventually it will go to rest. This is similar to what we have seen in the case of a cantilever under free vibration. The cantilever was vibrating and the amplitude of vibration was reducing at each cycle and eventually the displacement became 0.

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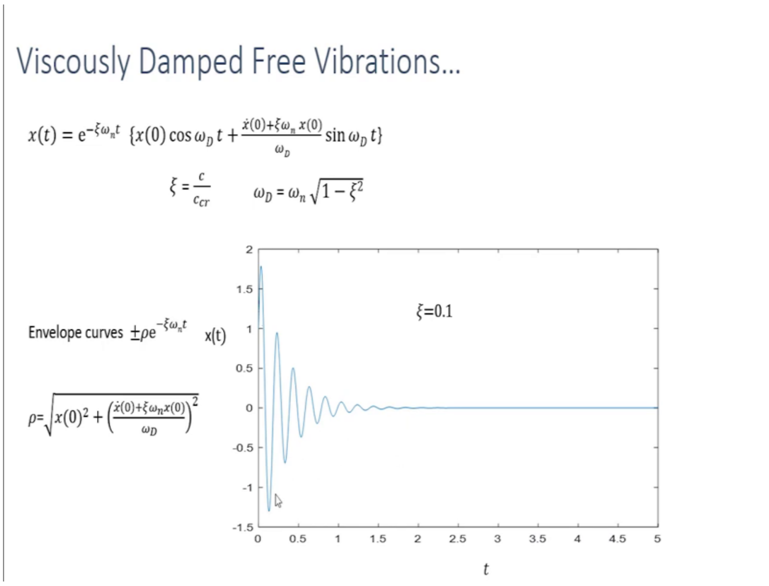

Now, let us see the response of damped free vibrations at different values of damping. So, this is the solution. So, if we take only this part, this is similar to the undamped free vibrations, but the frequency is different instead of omega n we have omega D here. So, the time period, the natural period is also different. So, this is the period for the damped system so, but the behavior is similar to the undamped vibration. So, this part of the solution this does not decay in time. The amplitude is constant throughout the time, but when you look at the complete solution a solution looks like this

$$x(t) = e^{-\xi\omega n t}\ \{x(0)\cos\omega D\, t + \frac{\dot{x}(0)+\xi\omega n\ x(0)}{\omega D}\sin\omega D\, t\}$$

$$\xi = \frac{c}{c_{cr}} \qquad \omega D = \omega n\sqrt{1-\xi 2}$$

This plot is for a damping ratio of 2 percent that is zeta is equal to 0.02. So, this is how the responses is. So, the amplitude value of the peaks decay with time and the equation of this envelope curve is given by rho e to the power minus zeta omega n t. So, this is similar to the, this function and this rho can be calculated like this is square root of this constant a square plus b square. So, this depend upon the initial conditions and the system parameters omega n and omega D. So, this is the solution when zeta is equal to 0.02; that means, when we have 2 percent damping in the system.

So, let us see what happens when the damping is increased. So, when the damping is increasing when the damping is 5 percent, the decay is more faster. Earlier it was this it was decaying slowly, now it is decaying faster. So, when we increase the value of damping this the decay becomes faster still. So, this is when damping is 10 percent.

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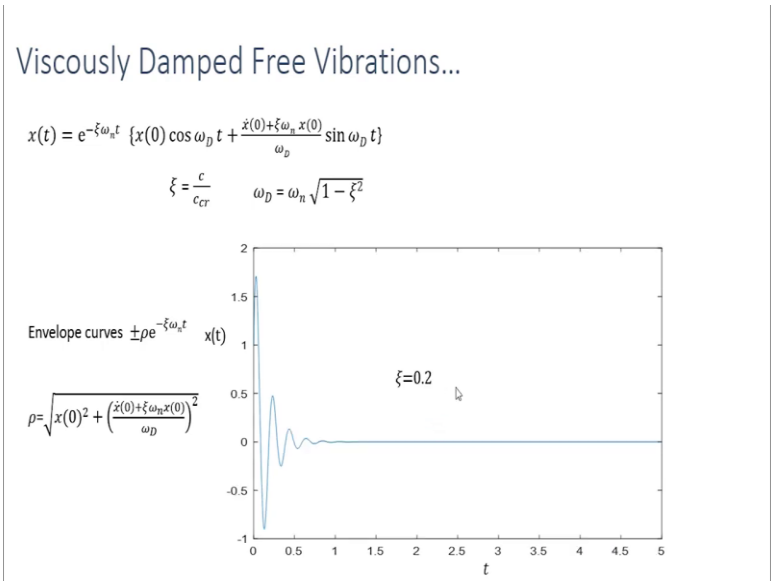

And, when damping is 20 percent it decays much more faster. So, there are only a few cycles remaining before the system comes back to its equilibrium position.

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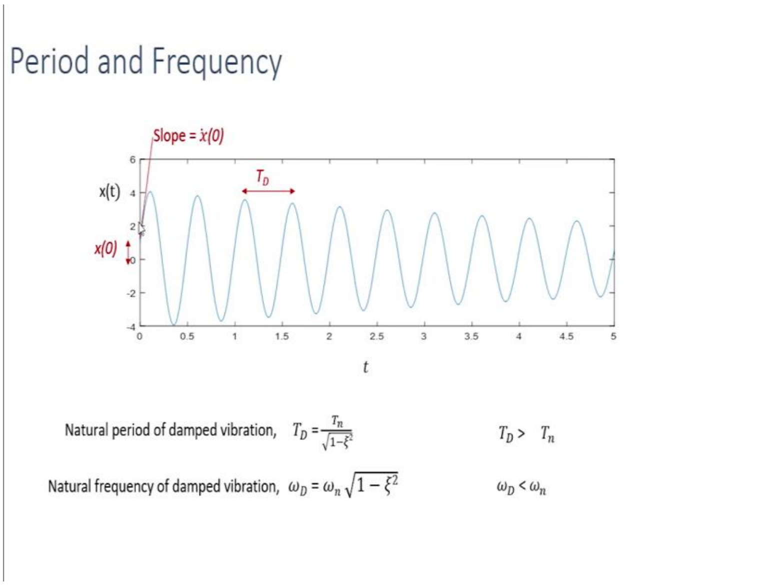

Let us find the period and frequency of an under damped system. So, this is the displacement response of an under damped system under free vibration. So, the initial displacement and the slope are corresponding to the initial conditions given to the system. The natural period of this damped free vibration will be different from that of the undamped free vibrations. So, this damped natural period can be calculated like this. This will be equal to the natural period divided by square root of 1 minus zeta square. Since for undamped systems zeta is less than 1, so, we have T D greater than T n; that means, the period of this damned vibrations will be greater than that of the undamped response.

The natural frequency of damped vibration can be calculated like this is the natural frequency multiplied by square root of one minus zeta square. So, the damped natural frequency will be less than that of the undamped natural frequency. So, in a way we can say that the effect of damping is making the system a little bit more flexible, that is omega D the natural frequency is decreasing when damping is applied.

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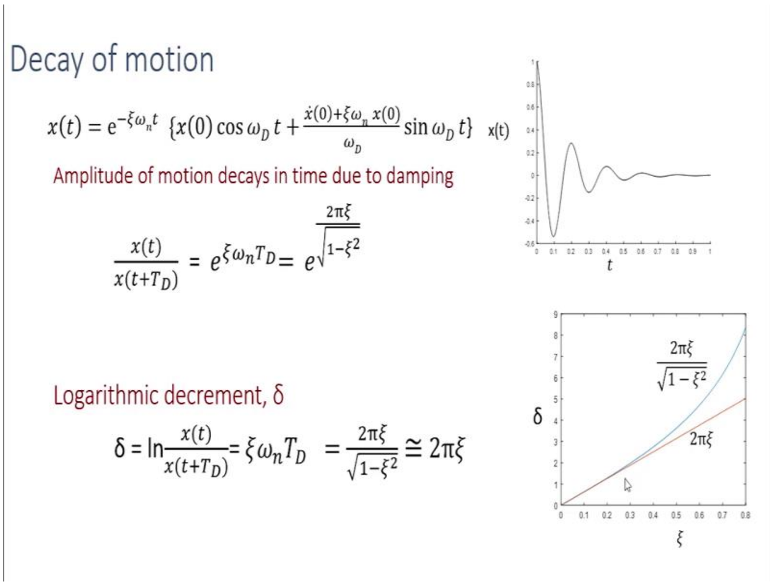

Now, let us see how the responses decay with time. So, this is the equation of the displacement response in under damped system.

$$x(t) = \mathrm{e}^{-\xi \omega n t} \; \{x(0) \cos \omega D\, t + \frac{\dot{x}(0)+\xi\omega n\; x(0)}{\omega D} \sin \omega D\, t\}$$

The amplitude of this response decays in time because of the affect damping. So, let us calculate how much is the decay in one cycle. So, we know that the response at time t is this. So, we can calculate the response at time t plus T D, where T D is the natural period of this damped system. So, this cos and sine terms are periodic. So, this part of the equation will be seen at t plus T D also. So, cos omega D t will be equal to cos omega D t plus T D.

$$\frac{x(t)}{x(t+T_D)} = e^{\xi \omega_n T_D} = e^{\frac{2\pi\xi}{\sqrt{1-\xi^2}}}$$

So, this part of the equation will be same at t plus T D as well. So, the only change is here. So, we can write the ratio x t by x t plus T D is equal to e raised to zeta omega n T D we can rewrite this we can expand this value of T D. So, we will get e to the power 2 pi zeta by 1 minus zeta square under root 1 minus zeta square. So, now, we can take logarithm at right hand side and left-hand side. So, we say logarithm of this ratio is equal to zeta omega n T D which is equal to 2 pi zeta by 1 minus zeta square.

$$\delta = \ln\frac{x(t)}{x(t+T_D)} = \xi\omega_n T_D = \frac{2\pi\xi}{\sqrt{1-\xi^2}} \cong 2\pi\xi$$

So, if the value of zeta is very small; that means, this quantity is equal to 1, in that case we can say that this quantity is approximately equal to 2 pi zeta and this quantity is known as logarithmic decrement. So, logarithmic decrement is the logarithm natural logarithm of the responses at two consecutive peaks. So, this plot shows the variation of logarithmic decrement with respect to zeta, the damping ratio.

So, as you can see this blue line shows the exact value of logarithmic decrement which is 2 pi zeta divided by square root of 1 minus zeta square and this straight line is the approximate value of the logarithmic decrement which is 2 pi zeta. As we can see here for lower values of zeta these are equal. So, for say if zeta is less than 0.2 we can calculate logarithmic decrement using this approximate value itself because the approximate and the exact values are same for lower values of zeta.

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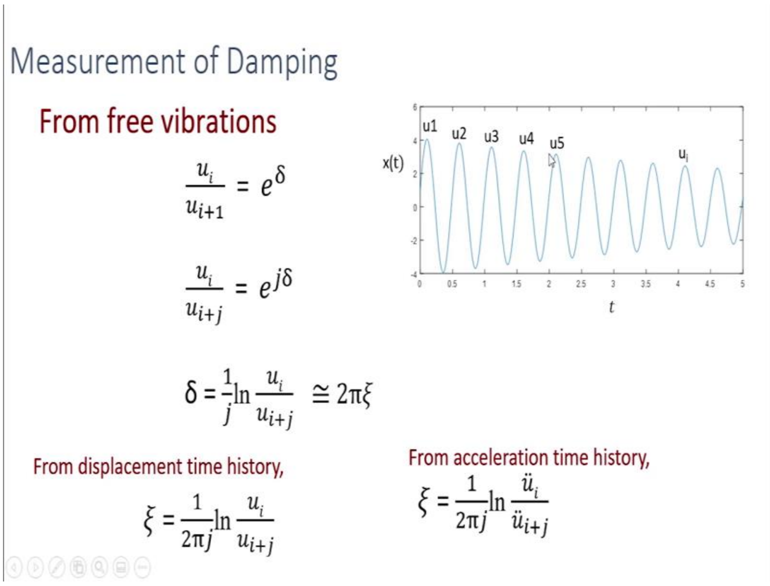

Now, let us see how damping in the system can be measured from it is free vibrations. So, this is a free vibration response of an under damped system we can measure the value of the displacement at each of these peaks. So, now, the decay in displacement in a single cycle can be represented like this, that is the ratio of the peak at the displacement at one of the peaks divided by the displacement at the next peak. So, this ratio is equal to e to the power logarithmic decrement; we just defined logarithmic decrement in the previous slide.

So, similarly you can also we can also calculate the decay in a few number of cycles; that means, you can divide the displacement at one peak by displacement after j cycles. So, if j is equal to 4, we can this ratio will become u 1 by u 5. So, we can equate that to e to the power j delta which is logarithmic decrement. So, you can calculate this ratio from the free vibration response and you can take the logarithm of this and divide it by j and we can calculate the logarithmic decrement and we know that the logarithmic decrement is approximately equal to 2 pi zeta. So, using this relation you can calculate zeta as 1 by 2 pi j logarithm of the ratio of the peaks.

So, j indicates the number of cycles we are considering if we are just considering one single cycle then you can just divide this displacement by this displacement and use this. So, this zeta or the damping ratio can be calculated using the displacement time history as well as the acceleration time history.

So, in all practical situations measuring acceleration is easier than measuring displacement. So, for a real structure the acceleration time history will be more available. So, we can calculate the damping using the acceleration time history also. The procedure is just the same, just calculate the ratio of the peaks, take logarithm and divided by 2 pi j you will get the damping ratio. So, this is how we measure the damping of a system using its free vibrations.

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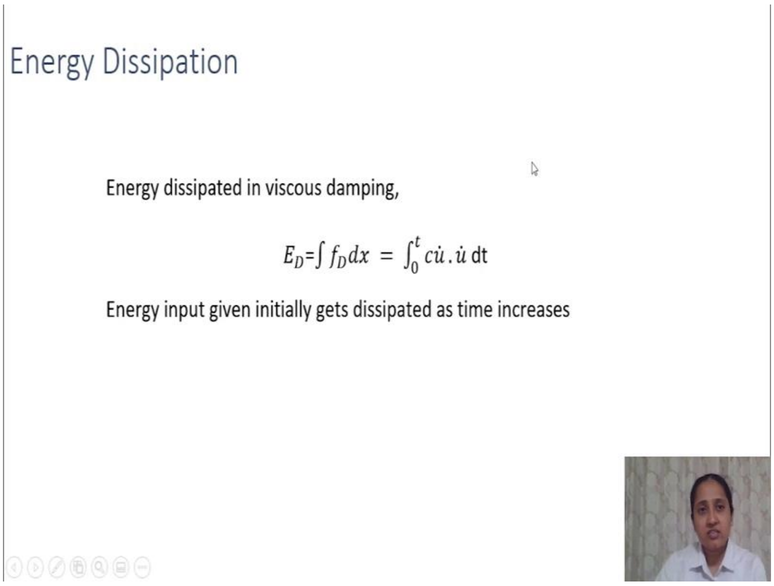

Now, let us see how energy dissipation is done in a viscous damping system. So, energy dissipated in viscous damping is equal to the integral of the damping force dx right. So, so this gives the energy, this dimension is that of energy. This integral is equal to if you integrate from 0 to the time t cu dot that is what f D is. So, f D is equal to cu dot and dx is equal to u dot dt. So, if you integrate this, we will get how much energy is dissipated by the time t. So, we discussed that in free vibration initially we are disturbing the system.

$$E_D=\int f_D\, dx = \int_0^t c\dot{u}\,.\,\dot{u}\, dt$$

So, when we are disturbing the system, we are giving an input energy to the system. So, that input energy which was received at the beginning is getting dissipated because of damping. So, as time increases the dissipation also increases and after some time the all energy is being dissipated. So, that is when the vibration stops. So, that is when the system comes back to its equilibrium position and stays there.