Structural Dynamics for Civil Engineers - SDOF Systems

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Lecture - 03
Undamped Free Vibrations

In the previous lessons, we learned the elements of a dynamic system, we learned about mass, stiffness and damping. We also learned how to formulate the equation of motion of a dynamic system. So, now, onwards we will solve the equation of motion and we will explore the responses of structures under various time varying loads. So, in this lesson we will be talking about free vibrations.

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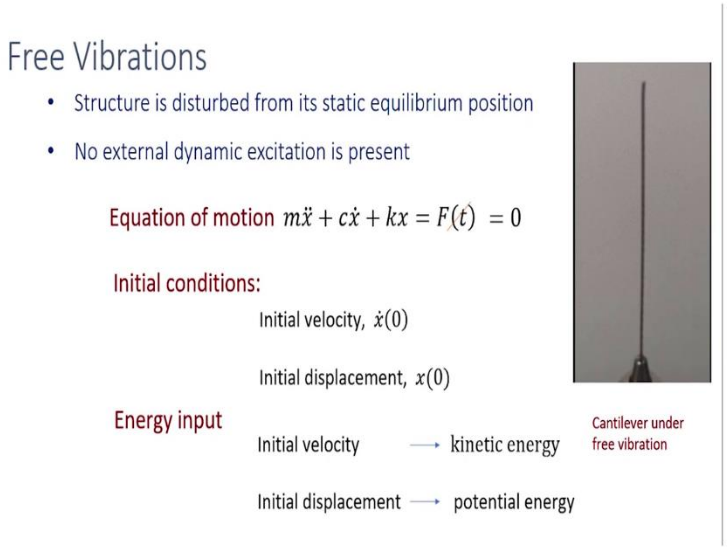

So, what is a free vibration? Before defining it let us see a cantilever under free vibration. So, what was happening here? The cantilever was in equilibrium and I disturbed it; so, it started vibrating and the vibration amplitude reduces with time and the cantilever eventually goes back to its equilibrium position. These types of vibration is called as free vibration. So, what is it;? so, free vibration is when the structure is disturbed from its static equilibrium position and when no external dynamic excitation is present. So, here there was no external force present during the vibration, the structure vibrated on its own after I gave an initial disturbance.

So, in free vibration the structure is free to vibrate on its own; that means, no external force is there to maintain that vibration. These type of vibrations are called as free vibration. So, this is the equation of motion of a dynamic system we have learned it in the previous lessons. So, in the case of free vibrations this force is 0. So, the equation will be mx double dot plus cx dot plus kx is equal to 0, it is a homogeneous differential equation. So, here we know that m indicates the mass of the structure, c is the damping coefficient and k is the stiffness of the structure.

$$m\ddot{x} + c\dot{x} + kx = 0$$

So, now, how this structure is vibrating there is no force acting on it so, how is it vibrating? So, when the structure is disturbed initially we are in fact, giving it some initial conditions; that mean, we are giving it an initial displacement or an initial velocity or both. So, when we are giving this initial displacement or velocity the structure is getting some energy; that means, when we are giving a initial displacement, we are actually giving that structure a potential energy or strain energy and when an initial velocity is given to the structure that is a kinetic energy.

So, we know how to calculate how a kinetic energy and velocity are connected it is like half mv square right. We will come to the energy calculations later, but at this point we have to understand that whenever we are giving an initial velocity $(\dot{x}(0))$ or displacement$(x(0))$ to the structure that is when we disturb a system, we are giving some energy to the system, and the structure is utilizing that energy to vibrate. So, in the video we have seen that, after some time it comes to stop it the vibrations reduces in time.

And then it is coming back to equilibrium position that is because of the damping there is some amount of damping present in this cantilever. So, that is dissipating the energy that we have given it so, that is why the structure is stopping the vibration so, that is the action of damping. Now we will discuss undamped free vibrations.

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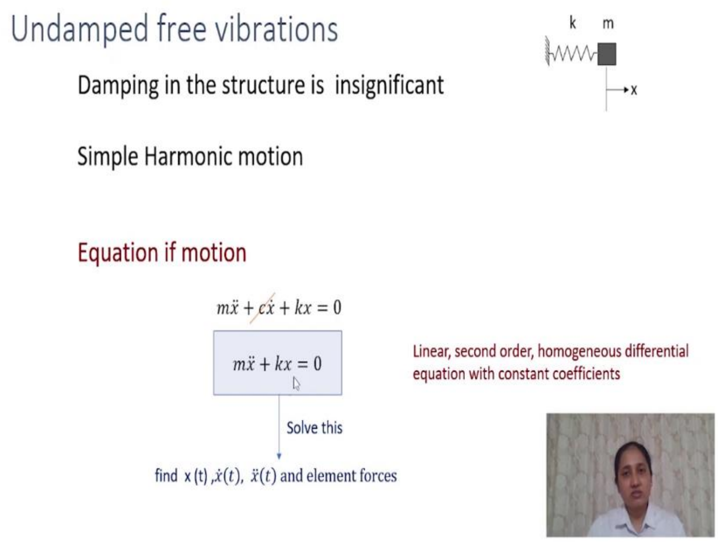

Here we assume that there is no damping present in the system. So, the system has a mass and some stiffness. It is unrealistic to assume that there is no damping present in the system because any real system will have some amount of damping present in it. So, this assumption is valid when the damping in the structure is very insignificant and these type of vibrations is also known as simple harmonic motion. So, now, let us look at the equation of motion. So, this is the equation of motion of a free vibration system, m x double dot plus cx dot plus kx is equal to 0; 0, because there is no forces acting on this. So, now, when we assume that the system is undamped that means, the damping is 0.

So, the correct equation of motion for these types of structures will be mx double dot plus kx is equal to 0. And if you remember your differential equations, this is a linear second order homogeneous differential equation with constant coefficients. Homogeneous because there is no force is acting on this system. So, now, we need to solve this equation; so, if we solve this equation we can find the value of x at each instant using that we can calculate the velocity and acceleration of this system and we can also calculate the element forces present.

$$m\ddot{x} + kx = 0$$

So, the next crucial step is to solve this differential equation. So, now let us see how this differential equation has been solved.

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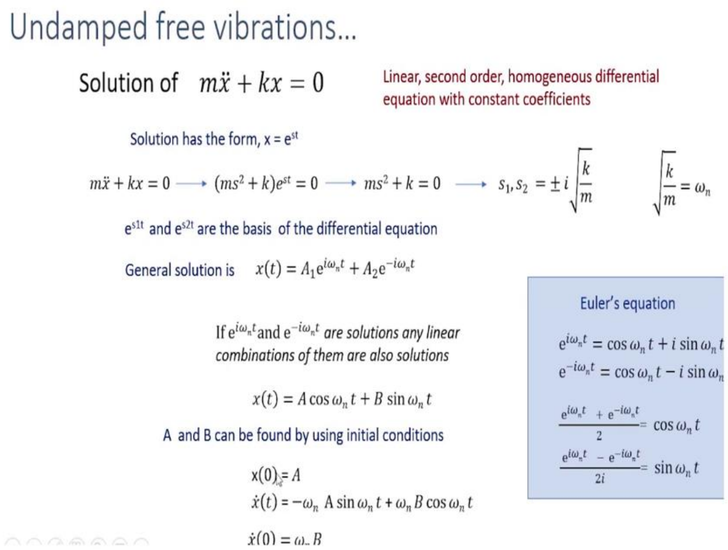

So, we have our equation of motion, this is a second order differential equation linear second order differential equation, this homogeneous and the we have constant coefficients. So, to solve this, the calculus theory tells us that, the general form of the solution is e to the power s t. So, if this exponential function is a solution of this equation, if we substitute this in this it should satisfy the identity. So, let us do that.

So, we have mx double dot plus k x is equal to 0, after substituting the exponential function it becomes this, we can use this characteristic equation solve for it and find the actual values of e to the power st. So, this is our characteristic equation. So, if we solve it we will get 2 values for s, 2 complex conjugate values s 1 and s 2 are plus or minus i square root of k by m. So, at this stage I will substitute this square root of k by m by a variable omega n. We will discuss the physical importance of this value later in this lesson, but as of now we will just make the substitution. So, we have two values of s 1; that means, e to the power s 1 t and e to the power s 2 t both of them are solution of these equation.

$$m\ddot{x} + c\dot{x} + kx = 0$$

So, e to the power s 1 t and e to the power s 2 t are the basis of this differential equation and we can formulate the general solution of this equation by taking linear combination of these two functions, that is the general solution is of the form x of t is equal to A 1 e to the power i omega nt plus A 2 e to the power minus i omega nt. So, A 1 and A 2 are constants which can be calculated using the initial conditions of our differential equation.

$$x(t) = A_1 e^{i\omega_n t} + A_2 e^{-i\omega_n t}$$

Now, we can rearrange this equation in terms of sine and cosine functions using Euler's equation. So, Euler's equation is e to the power i omega nt can be written as cos omega n t plus i sin omega nt. Similarly, e to the power minus i omega nt can be written as cos omega n t minus i sin omega nt. So, if you add these two equations and divide it by 2 we will get cos omega n t, if we subtract these two and divide it by 2 i we would get sin omega nt. So, if we have e i omega nt and e minus i omega nt.

$$e^{i\omega_n t} = \cos \boldsymbol{\omega}_n t + i \sin \boldsymbol{\omega}_n t$$

$$e^{-i\omega_n t} = \cos \boldsymbol{\omega}_n t - i \sin \boldsymbol{\omega}_n t$$

$$\frac{e^{i\omega_n t} + e^{-i\omega_n t}}{2} = \cos \boldsymbol{\omega}_n t$$

$$\frac{e^{i\omega_n t} - e^{-i\omega_n t}}{2i} = \sin \boldsymbol{\omega}_n t$$

If these two functions are the solution of our differential equations, any linear combinations of these solutions should also be a solution of this differential equation. So, this sine and cosine functions are now linear combinations of e raised to i omega n t and e raised to minus i omega n t. So, because of that we can say that sin omega n t and cos omega n t are also solutions of our differential equation. So, we can write the general solution of this in terms of sine and cosine functions, we can represent it as x t is equal to A cos omega n t plus B sin omega n t and A and B are constants we can evaluate these constants using our initial conditions.

$$x(t) = A \cos \boldsymbol{\omega}_n t + B \sin \boldsymbol{\omega}_n t$$

So, let us do that. So, as an initial condition we know the initial conditions. So, we know what is the value of x at time is equal to 0. So, if we can substitute that in this equation and sin omega t will become 0 and we will get that the initial displacement is equal to the constant A. Now, similarly we can find out constant B using the initial velocity. So, differentiate it you will get this and substitute the value of velocity at time is equal to 0. We will get the initial velocity is equal to omega n multiplied by constant B. So, now, we have solved this equation we know the general form of the solution and we know the constants also.

$$x(0) = A$$

$$\dot{x}(t) = -\omega_n A \sin \boldsymbol{\omega}_n t + \omega_n B \cos \boldsymbol{\omega}_n t$$

$$\dot{x}(0) = \omega_n B$$

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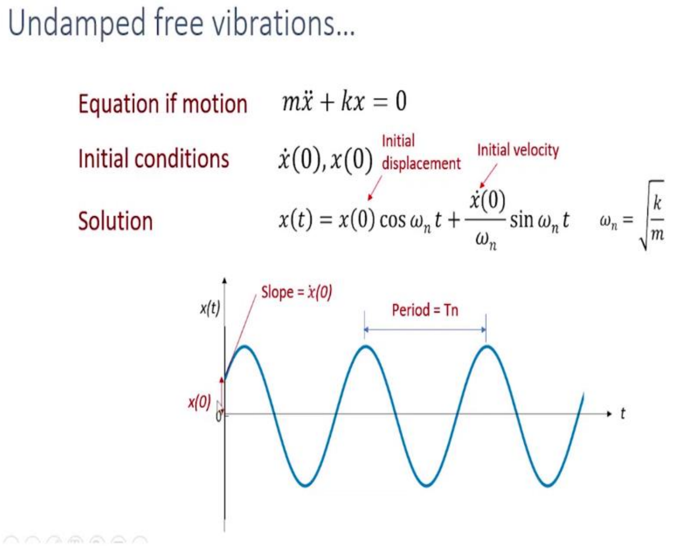

So, this is our equation of motion and we have our initial conditions$(\dot{x}(0), x(0))$. The solution of this differential equation can be represented as this that is the displacement at any time t is equal to the initial displacement multiplied by cos omega n t plus this constant, that is initial velocity divided by omega n multiplied by sin omega t.

$$m\ddot{x} + kx = 0$$

$$x(t) = x(0)\cos \boldsymbol{\omega}_n t + \frac{\dot{x}(0)}{\omega_n}\sin \boldsymbol{\omega}_n t$$

And we know that omega n is equal to root of k by n, k is the stiffness of the structure and m is the mass of the structure. So, all the variables in this equation are known now so, we can plot the displacement. So, this is how the displacement of that structure of the single degree of freedom system looks like the, whatever values we have assumed as initial conditions are seen here. So, the initial displacement is the value of the displacement curve at time is equal to 0 and initial velocity is the slope of this curve at time is equal to 0.

$$\boldsymbol{\omega}_n = \sqrt{\frac{k}{m}}$$

So, if you consider this line tangent to this curve at time is equal to 0 the slope of that line would give you initial velocity that should be equal to whatever we have assumed. So, if we have given only initial velocity to the system our response will look like this, the initial displacement will be 0.

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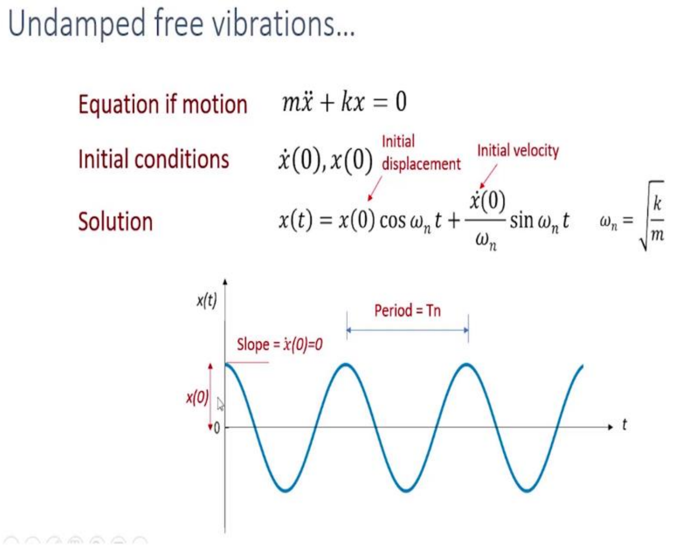

And if there is no initial velocity given and if there is only initial displacement, then the nature of the curve will be like this, slope at 0 will be 0; so, this will be flat.

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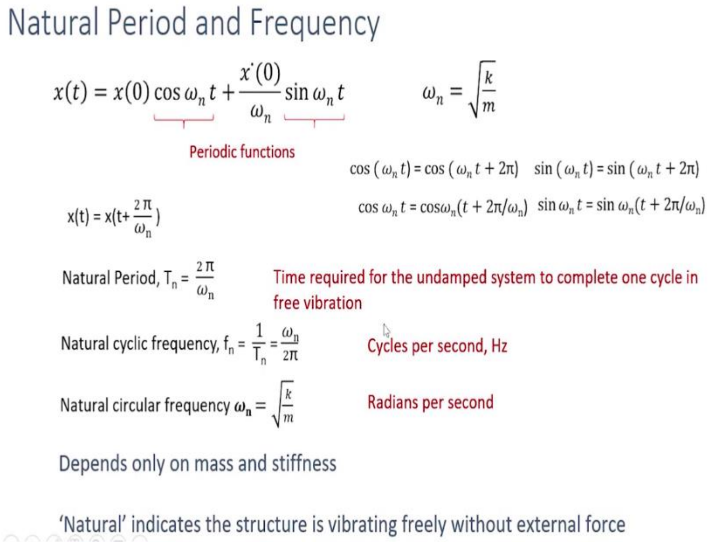

Now we will look at the natural period and frequency. If we look at the solution, we can see that this function x t is a periodic function; that means, the function repeats itself after some time called period. So, if we can measure the duration between two peaks, we can calculate the period of this motion. Now we will see, what is the value of natural period and natural frequency. So, this is the solution; so, these are functions of sin and cos, we know that cosine and sine are periodic functions. So, cos omega n t is equal to cos omega n t plus 2 pi, sin and cos functions have a period of 2 pi.

$$x(t) = x(0)\cos \boldsymbol{\omega}_n t + \frac{\dot{x}(0)}{\omega_n}\sin \boldsymbol{\omega}_n t$$

So, their values will be seen after multiples of 2 pi. So, again sin omega n t will be same as sin omega n t plus 2 pi. If you just rearrange this, we can say that cos omega n t plus 2 pi of by omega n will be equal to cos omega n t; that means, the value of x t at time t will also be equal to value of x at t plus 2 pi by omega n. So, this 2 pi by omega n is the natural period of our vibration that is the time required for our undamped system to complete one cycle in free vibration that is known as natural period. And the value will be equal to 2 pi by omega n and omega n is equal to root of k by m.

$$\cos(\boldsymbol{\omega}_n t) = \cos(\boldsymbol{\omega}_n t + 2\pi)$$

$$\sin(\boldsymbol{\omega}_n t) = \sin(\boldsymbol{\omega}_n t + 2\pi)$$

$$\cos \boldsymbol{\omega}_n t = \cos \boldsymbol{\omega}_n (t + 2\pi/\omega_n)$$

$$\sin \boldsymbol{\omega}_n t = \sin \boldsymbol{\omega}_n (t + 2\pi/\omega_n)$$

$$\boldsymbol{\omega}_n = \sqrt{\frac{k}{m}}$$

So, since we know the period, we can calculate the frequency so; that means, how many cycles of vibration are there in per second. So, natural cyclic frequency can be calculated as the reciprocal of the natural period so, that would be equal to omega n by 2 pi. So, this is called as natural cyclic frequency. And we have one more frequency known as naturals circular frequency, which is equal to omega n that is we have already defined this it is equal to root of k by m, this indicates how much radians are covered in each second.

$$\text{Natural Period, } T_n = \frac{2\pi}{\omega_n}$$

$$\text{Natural cyclic frequency, } f_n = \frac{1}{T_n} = \frac{\omega_n}{2\pi}$$

$$\text{Natural circular frequency } \boldsymbol{\omega}_{\mathbf{n}} = \sqrt{\frac{k}{m}}$$

So, omega n is the natural circular frequency of the system, and this natural period and frequencies depend only on the mass and stiffness of the structure. It does not depend upon the initial conditions we are giving to the system. So, these things are natural to the system and the word natural here it indicates that, the structure is vibrating freely without any external force. So, this natural frequency and period are calculated from free vibrations.

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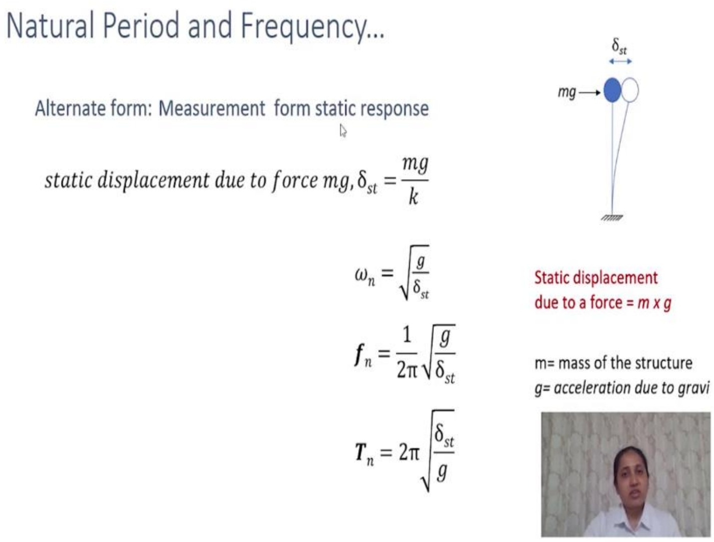

Now we will see an alternate method of calculating natural period and natural frequency that is using static response. So, we have seen that the natural period and natural frequency, they depend only upon the mass and stiffness of the structure. So, we can also calculate these frequencies frequency and period using static response. How do we do that let us say. So, if we have a structure say a single degree of freedom structure, which has some stiffness and some mass we have to calculate the static displacement (δ_{st}) of that structure due to a force equal to mg; that means, mass multiplied by gravitational acceleration.

$$\delta_{st} = \frac{mg}{k}$$

So, we have to apply a force to that structure which is equal to mass times g and measure this static deflection because of this force. And using this static deflection, we can calculate the natural period and natural frequency. So, you know that some this structure will have some stiffness so, the static displacement due to this force mg will be is equal to mg by k, that will be the static deflection.

So, once you know this, omega n can be calculated like this square root of g by delta st. So, if omega n the is this the fn that is the cyclic frequency will be omega n divided by 2 pi. So, we can calculate the cyclic frequency using this formula and natural period will be the reciprocal of this; so, you can calculate the natural period also. So, all these quantities can be calculated from static response itself.

$$\boldsymbol{\omega}_n = \sqrt{\frac{g}{\delta_{st}}}$$

$$\boldsymbol{f}_n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_{st}}}$$

$$\boldsymbol{T}_n = 2\pi\sqrt{\frac{\delta_{st}}{g}}$$

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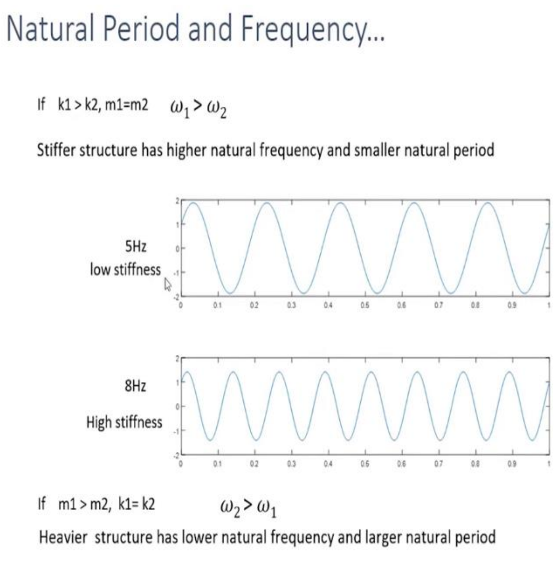

So, now if we have two structures then the stiffer structure will have high natural frequency. So, suppose we have two systems, their masses are equal, but one has higher stiffness compared to the other, then the structure with lesser stiffness will have lesser frequency. So, this is a high Hertz response that would corresponding to the lower frequency; lower stiffness structure and this structure with high stiffness will have high natural frequency.

So, similarly the heavier structure will have lower natural frequency and larger natural period. So, if we have two structures again, if their stiffnesses are same and one is heavier than the other the heavier one will be more flexible the heavier one will be having lower natural frequency. So, this will help in understanding some systems.

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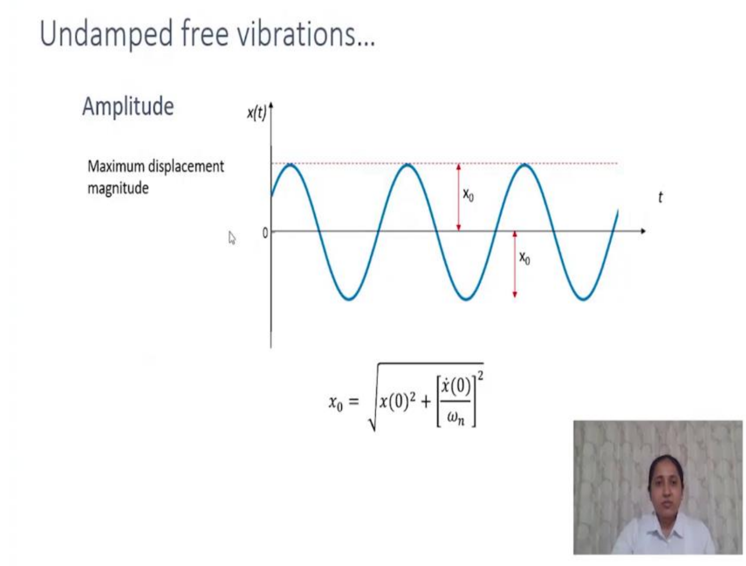

Now, let us look at the amplitude of our vibration. So, by amplitude we mean the maximum displacement magnitude of this vibrations. So, in this we have no damping so, there is no decay in this vibration with time. So, our amplitude will be constant throughout so, this value this x naught is known as the amplitude of the vibration and it can be calculated like this using our constants a and b. So, the amplitude of this vibration will be equal to square root of x 0 square plus x dot 0's by omega n the whole square.

So, these are as we know, these are initial conditions and omega n is the natural frequency of this system. So, if we know the initial conditions and the natural frequency you can calculate the amplitude of this vibration. Now we will see how the energy in the system is balanced.

$$x_0 = \sqrt{x(0)^2 + \left[\frac{\dot{x}(0)}{\omega_n}\right]^2}$$

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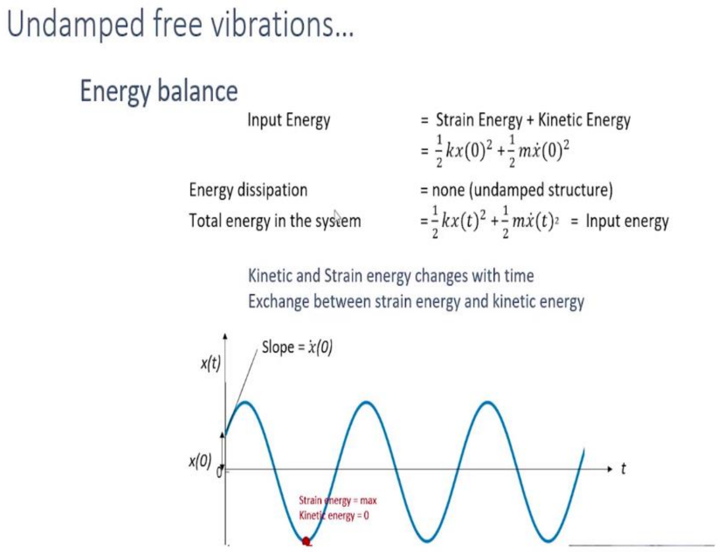

Initially, we have discussed that when we are giving an initial disturbance to the system, we are actually giving it some energy, and this energy is been utilized during the vibration. So, now, let us look into the details of this energy. We are inputting some energy to the system and there are two types of energies, one is strain energy another is kinetic energy.

So, this strain energy is because of the initial displacement which we give. The strain energy is equal to half k x 0 square where k is the stiffness of the structure and x 0 as you know is the initial displacement we gave. And kinetic energy can be calculated as half m velocity square, this is the initial velocity we gave m is the mass of the structure; so, when we disturb it the structure is getting this much of energy.

And currently we are looking at undamped free vibrations; that means, there is no damping present in the system. So, if there is no damping; that means, there is no energy dissipation there is no way this input energy is dissipated in the structure. So, at any instant of time, the structure will have a total energy is equal to the input energy which we have given. But at each instant the amount of strain energy and kinetic energy will change because x t will change with time and x dot t, the velocity will also change with time. So, the strain energy and kinetic energy at each instant will be changing, but their sum will not change, since there is no damping their sum should be equal to the input energy.

$$\text{Input Energy} = \frac{1}{2}kx(0)^2 + \frac{1}{2}m\dot{x}(0)^2$$

So, at any instant there will be an exchange between strain energy and kinetic energy; let us see that in detail. So, this is the response of a single degree of freedom system in free vibration. As we have seen earlier the displacement at 0 is equal to our initial displacement given and the initial velocity is equal to the slope at time is equal to 0. So, the initial input energy is equal to this much that also we have seen. So, now, when the displacement of this system is maximum; that means, the strain energy is maximum because this x not is the displacement is maximum at this position.

So, and that time what is happening to the kinetic energy. So, from this figure we can easily understand that at this location the slope will be 0 slope will be, it will be flat this tangent will be horizontal. So, the velocity is 0; that means, at this point kinetic energy is 0. Now after some time, when at this point the displacement is 0, but the kinetic energy the velocity will be maximum, at this point because if you look at this curve this point will have the maximum slope, right.

Energy dissipation = none (undamped structure)

Total energy in the system $= \frac{1}{2}kx(t)^2 + \frac{1}{2}m\dot{x}(t)^2 =$ Input energy

So, the kinetic energy will have the maximum value but the strain energy will be 0 at this point. Again, when this moves here we have maximum strain energy and 0 kinetic energy, anywhere in between the system will have both strain energy and kinetic energy, but at each instant their values will change, but the sum of the kinetic and strain energy at each instant will be equal to the input energy given. So, again when it moves back here, we have maximum kinetic energy and 0 strain energy because displacement is 0.

Again, at here since velocity is 0, kinetic energy will be 0, but strain energy will be maximum. So, throughout the vibration there will be an exchange between strain energy and kinetic energy and since this is an undamped structure, this response will never decay. Once initial disturbance is given to the system, the system will continue vibrating forever. So, as we have discussed earlier this is an unrealistic assumption and the assumption is valid for structures where the damping is insignificant. So, far we have learned undamped free vibrations. So, in the next lesson, we will learn damped free vibrations.