Structural Dynamics for Civil Engineers - SDOF Systems

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Lecture - 02
Modeling of Dynamic System

In the previous video, we have seen how dynamic analysis is different from static analysis. So, now, we will understand the Modeling of a Dynamic System.

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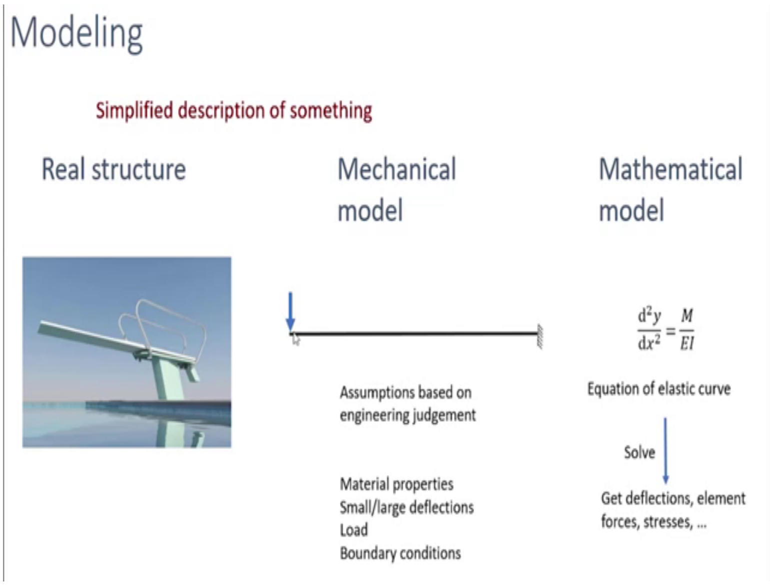

So, what is modeling in the context of structural analysis? So, it means a simplified description of something. So, for example, if we have a real structure, a real life problem so, this is diving board. Suppose, we have to understand the behavior of this structure so, I have to do an analysis of this. So, what is the first step in analyzing this type of system?

So, first we have to make a simplified description of this using mechanical principles. So, we will make a mechanical model. So, this structure can be idealized as a cantilever beam with some load. So, in making this model I have to make some assumptions based on some engineering judgment. Here we can see that this board is rigidly clamped to this support system. So, I can assume that this structure has a fixed support at this end because it cannot rotate or translate in any direction. So, I can assume it as a fixed support.

imilarly, we can make some assumptions regarding the material properties of the structure. So, we can use homogeneous isotropic properties for the material and we also should know the strength of this material. So, we will assume some value for the strength of this material. We should also decide whether we are considering small or large deflection theory while doing this analysis. So, we can decide that the deflections are very small. So, I can consider it as a small deflection problem, we can do a linear analysis to do that.

We can also assume that when a person is standing on it the load acting on this structure can be treated as a concentrated load. In reality, this force is not concentrated because a weight of the person will be distributed over the area of his feet, but with respect to the dimension of the structure that dimension is very small. So, we can treat this as a concentrated load. So, we can make some assumptions based on our engineering judgment we can simplify this structure in a way which we can handle it using our mechanical principles.

So, once this mechanical model is set up we can make a mathematical model of this. So, this equation the equation of elastic curve, this is a representation of this structure mathematically. So, now, we will be able to solve this equation and get the deflection of the structure and once the deflection is found out we can calculate the stresses, strains, element forces etcetera. So, we can calculate all those parameters and we can take decisions about this system. So, if the stresses are high we know that this thickness is not sufficient so, we can design it better.

So, in analyzing any kind of a structure we follow these procedure; first we understand the real structure make some assumptions and create mechanical model, then we will represent this mathematically, solve the mathematical model, get the values of the parameters, based on those parameters and we decide on the structure. So, even in analyzing the dynamic structure we have to make simplified models, we have to describe the elements of a dynamic system in simplified terms and we have to develop a mathematical model of the system, so that we can solve it.

$$d^2y/dx^2 = M/EI$$

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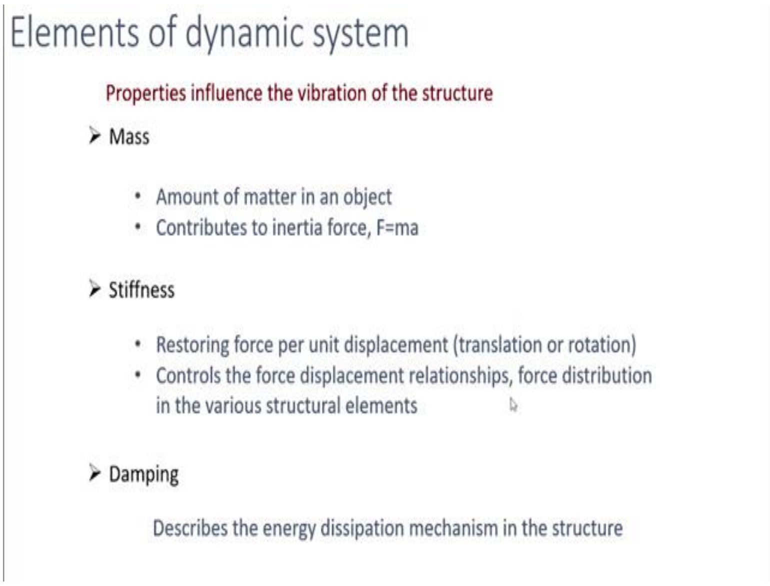

So, now let us see the elements of a dynamic system. By elements, I just mean that the properties influence the vibration of the structure. So, what are the properties which affect the vibration of the structure? Their mass, stiffness and damping. In the previous discussion, we have seen that mass will play a very important role in dynamic systems because it is contributing to the inertia force. So, there will be unbalanced force acting in the structure. So, that will lead to inertia force which will be equal to mass times acceleration of the structure. So, we have to consider the mass of the structure in the dynamic analysis.

$$F=ma$$

So, the next is stiffness; you are familiar with stiffness in static analysis. Stiffness is the force per unit displacement. It is the restoring force developed in a structure and it is the force per unit displacement. So, the stiffness is the property of the structural element and that controls the force displacement relationship in a structure and it also controls the force distribution. So, if two structural elements are deforming in a similar way, then stiffer element will have higher force it will attract higher force. So, the force distribution and force displacement relationship are controlled by the stiffness of the structure.

The next property is damping. Damping is a description of the energy dissipation mechanisms in a structure. So, in a vibrating system some part of energy is absorbed by the system due to many mechanisms so, damping describes all that.

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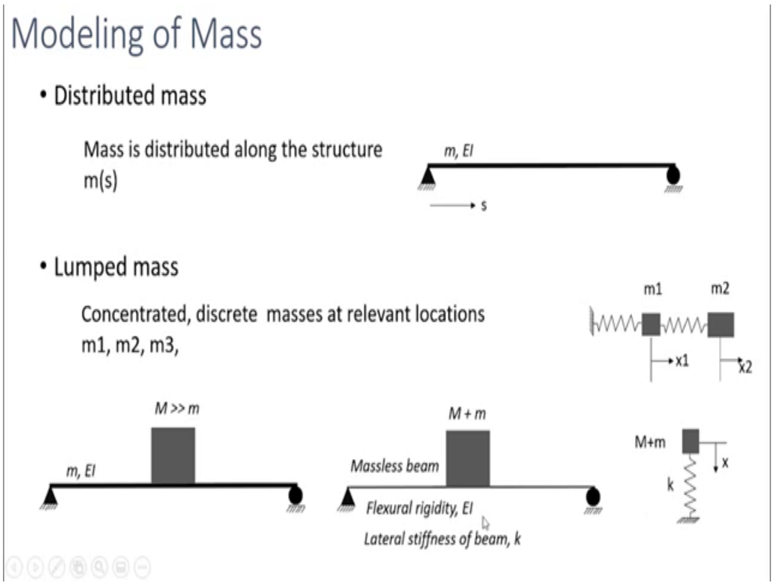

Now, let us see how we will model the mass in a dynamic system. So, it can be treated as a distributed mass or lumped mass. So, what is distributed mass? So, every structure the mass it is the amount of material it has so, it is distributed along the structure, right. Each part of this beam for example, will have some mass. So, the mass is distributed along the structure and this is the real scenario in case of any structure. Every structure has a mass and that would be distributed along the structure. So, in this assumption, this is the most realistic treatment of mass. So, here we can treat mass as a function of the position along the structure. So, in this case it indicates the position along the structure so, the mass is a function of say s.

So, in lumped mass system what we do is we assume mass as concentrated discrete masses at relevant locations. So, instead of treating it as a function of position, we can treat it as discrete quantities like mass 1, mass 2, mass 3 etcetera. So, for example, this is a system where lumped mass assumption is used to model mass. So, in this; it is assumed that a mass a concentrated mass this moving is a concentrated mass is attached to a massless spring. So, this spring gives stiffness and the mass is assumed to be concentrated at this particular position and again another mass is treated as the concentrated mass at this particular position.

So, when can we use this type of assumption, is it realistic to use this assumption? Well, in some cases this is realistic. For example, the same beam if we had a very heavy machine kept on this beam; this machine is causing some dynamic force to this beam. Let us assume that condition and the mass of this additional say machine is much higher than the mass of the beam. So, this has very high mass compared to the beam. So, in such situations we can neglect the mass of this beam with respect to this additional load on the structure.

So, what can we do? We can treat this as a massless beam and we can lump the mass of the beam also with the mass m. So, we can treat it as a single mass acting here and a massless beam. This assumption is valid when the dynamic effect due to the small mass m is negligible compared to the dynamic effect due to the actual mass, the load mass. So, in such conditions we can treat this as a massless beam and we can lump all the masses at one location. So, this is equivalent to a system like this.

So, we know that the beam has some flexural rigidity; so, it will offer some lateral stiffness. So, we can calculate that in static analysis we have learnt how to solve and find out the lateral stiffness at this particular location; so, we can do that. So, this entire system can be represented like this. So, we can treat it as a mass M plus m sitting on a spring with it is spring constant k, and that k can be calculated from the property of this plane, ok. So, the mass is modeled in two different ways distributed mass and lumped mass.

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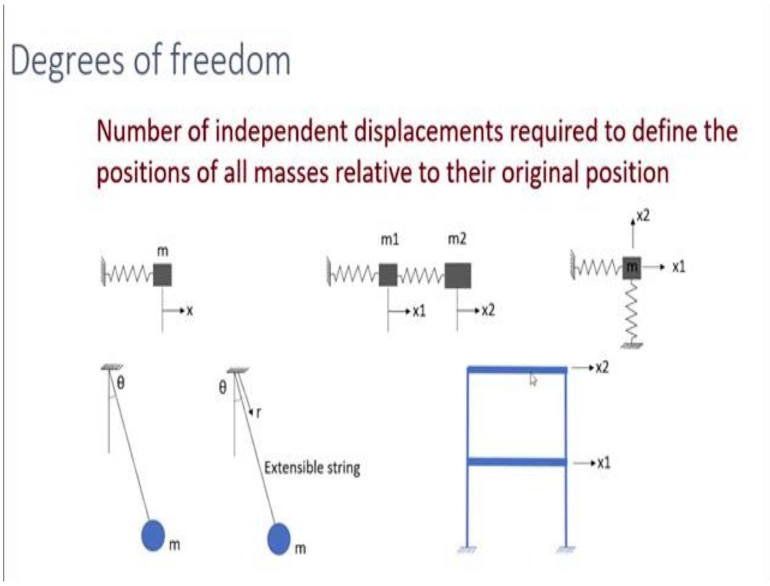

So, now let us see what a degree of freedom is because our course is about single degree of freedom structures. So, now, let us see what a degree of freedom is. The degree of freedom is the number of independent displacements required to define the positions of all masses in a system relative to their original position.

So, let us understand this statement a little in detail. Suppose, I have a system like this a spring connected to a mass and this will be acted upon by some vibrating load say in x direction. So, this is free to vibrate in this direction. So, at any instant of time, if I want to describe the system all I need to know is the position of this mass in this direction because this cannot move in that direction. So, if I know the displacement of this mass in one direction, I can calculate the remaining properties of the structure. So, this is a single degree of freedom structure.

Similarly, here we have two masses and both the masses can move in this horizontal direction. So, this mass can move independently and this also can move independently. So, we need two independent displacements or two coordinates to define this system. So, at any instant of time to know the displaced configuration of this structure we need two coordinates, x 1 and x 2. So, this is a two-degree of freedom system. So, in this case we have only one mass, but this is free to move in two directions. So, this can move in this direction some stiffness is attached to it and this can also move in the x 2 direction; so, this mass at any point can move in this entire plane.

So, to describe the system completely at any instant we need to know the two coordinates. So, even though the mass is just one we need two coordinates to describe the system. So, this is also a two degree of freedom structure. So, now, let us see this simple pendulum. We have a string and we have a mass attached to it. So, what is the degree of freedom of this structure? How many degree of freedoms it has? If you are assuming that the string is not extensible we can say that it is a single degree of freedom structure because to describe the position of this mass at any instant we only need to know this angle theta. So, if the theta is not we know where the mass is. So, this is a single degree of freedom system.

So, now, what if the string is extensible; that means, this can rotate, but also the mass can move along the string axis right. So, that means, in addition to theta we need to also consider r that is the position of mass from this origin. So, the string is extensible this becomes a two-degree of freedom system. This is a two-storey frame. So, we can assume that the old masses on the floor is lumped at the floor and if the floor is rigid, this frame will move only in this horizontal direction. So, in this case all we need to know is the movement of this mass along this direction and the movement of the second mass in this direction. So, this is again a two-degree of freedom system.

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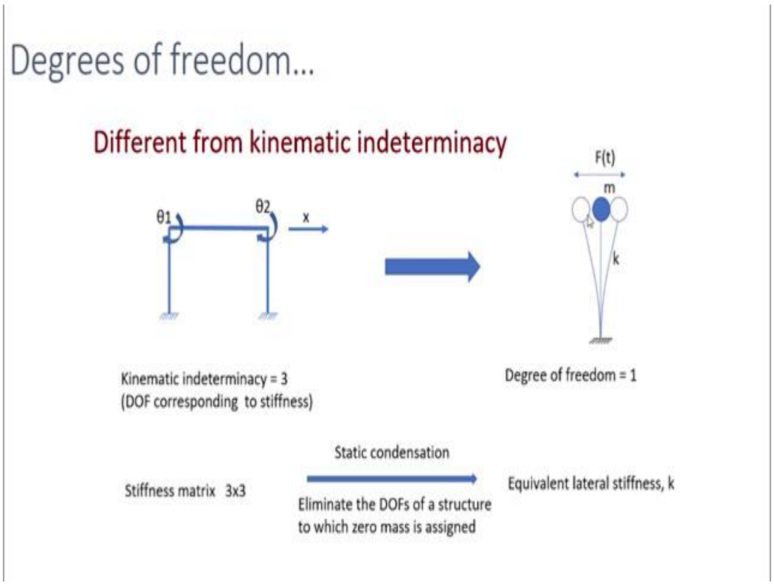

The name degrees of freedom should not get confused with kinematic indeterminacy. So, in static analysis you would have analyzed these type of frames. So, you know that we have two join rotations – theta 1 and theta 2 and one translation. So, we also refer these quantities; these rotations and translation as degrees of freedom also, but in this course when we say degrees of freedom which means the number of independent displacement the masses in the structure is having. So, you should not get confused with kinematic indeterminacy and the degrees of freedom corresponding to the mass.

So, in dynamic analysis we can treat the system as this. So, we can find an equivalent stiffness k and we can lump all the masses of the frame here. So, we can treat it as vibrating in this direction. So, in this case we only need the displacement of this mass in this horizontal direction. So, this system has only one degree of freedom. So, we had kinematic indeterminacy is equal to 3, but for dynamic analysis we need only 1 degree of freedom.

Now, how do we do this dynamic analysis? To do that we need an equivalent stiffness of this frame in the lateral direction, right. So, we have to calculate the lateral stiffness of this frame. So, you know that for this frame you have kinematic indeterminacy is 3, stiffness matrix will be 3 by 3. So, you will have stiffness coefficients corresponding to these three degrees of freedom. So, in the dynamic analysis we need only the stiffness in the horizontal direction that is, this stiffness equivalent and this degree of freedom.

So, we have to calculate that equivalent lateral stiffness. So, how do you do that? In static analysis you might have learnt a technique called static condensation. So, that is to eliminate the unwanted degrees of freedom of a structure in which zero mass is assigned. So, if we have more degrees of freedom corresponding to stiffness that is the kinematic indeterminacy, we can eliminate those additional degrees of freedom where no mass is assigned using this technique called static condensation. We will discuss this in detail when we do example problems, but now we just have to understand that using some statics techniques we can calculate the equivalent lateral stiffness for this frame and we can use that in dynamic analysis.

So, we just learnt how to model the mass in dynamic analysis, we learned about degrees of freedom.

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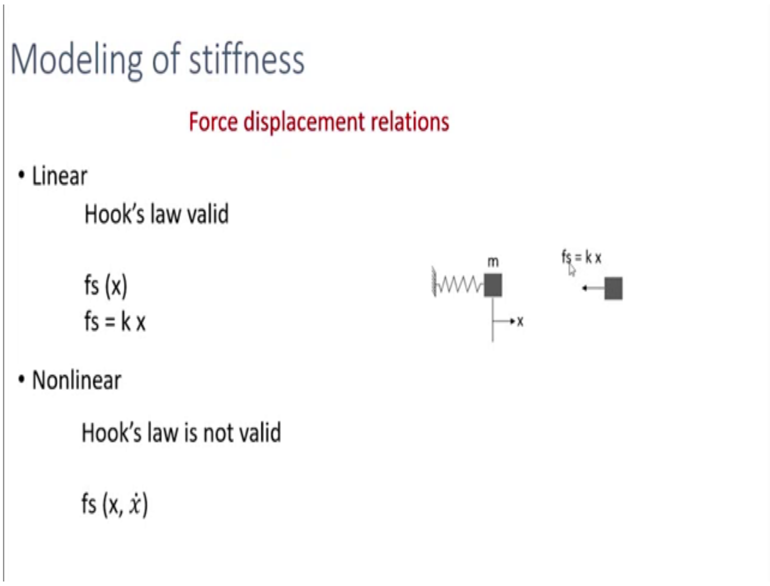

Now, let us review how stiffness is modeled. This is similar to the modeling of stiffness in static analysis. So, we are not going into much detail. So, stiffness defines the force displacement relation in a structure. So, it can be modeled as linear or non-linear stiffness. So, in case of linear stiffness we assume that the Hook's law is valid the force is proportional to the displacement produced. So, the stiffness is defined as the proportionality constant. So, the restoring force acting on the structure will be function of the displacement given because stiffness is constant. So, restoring force will be equal to the stiffness multiplied by the displacement given.

$$fs\ (x)$$

$$fs = k\ x$$

So, if we have a mass attached to a spring and if we are moving this mass a little bit in x direction, say if you move it by x the spring will apply a restoring force on this mass like this and that force is equal to k multiplied by x, where x is the displacement given. So, this is the restoring force due to the effect of this spring. So, in non-linear case Hook's law is not valid and this restoring force, it is not only just a function of x it is also a function of x dot, because this restoring force will be depending on the rate of loading also. So, this restoring force will be a function of displacement and the velocity of this body.

So, in our course we will be dealing with linear's material so, our stiffness model will be linear. So, we will be dealing with systems where fs is equal to k x.

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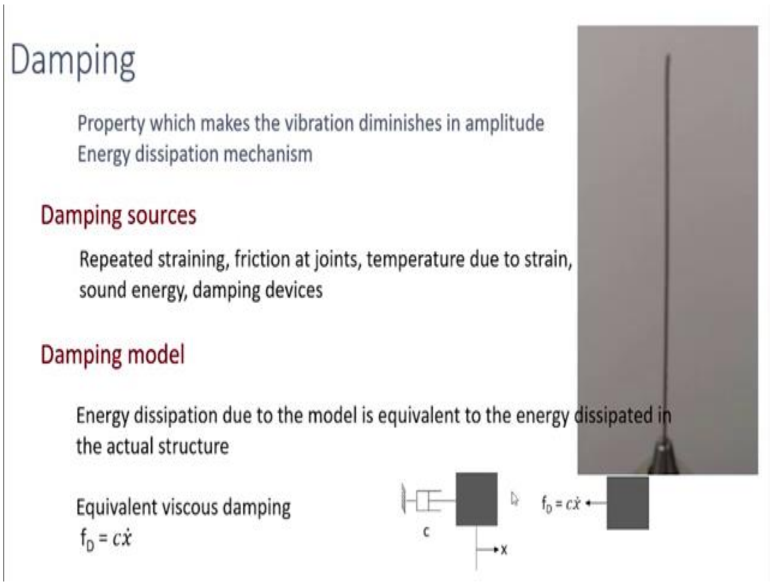

Now, let us understand what damping is. So, before we define damping let us look at the vibration of a small cantilever. So, in this the cantilever is vibrating and as you can see with time the amplitude of vibration reduces and finally, the cantilever is coming to rest. So, the property by which the vibration in a system is diminishing in amplitude is known as damping. So, damping is referred as the energy dissipation mechanisms present in a structure.

So, what are the sources of damping? Any mechanism which is contributing to dissipation of energy in the system is a source of damping. So, when a structure is repeatedly straining because of that straining some energy is absorbed. So, that causes damping and then there will be friction at joints; so, because of friction some energy is lost, that is also a source of damping. So, when we stress a structure because of the strain sometimes some temperature is developed so, that needs energy. So, that is also a cause of energy dissipation.

In some systems when vibrating sound energy is produced so, that is a cause of damping. And, in concrete structures because of repeated straining there will be opening and closing of cracks that will dissipate some amount of energy and in some structures there will be additional damping devices installed to absorb some energy during vibration. So, all these mechanisms will contribute to the energy dissipation in the structure.

So, while modeling damping it is very complex to model each of these mechanisms exactly in detail. So, what we do is we consider an equivalent damping. So, in that the energy dissipation due to the model is equivalent to the energy dissipated in the actual structure; that is, the energy dissipation in the model is equivalent to the sum of the energy dissipations due to all these different sources.

So, in most of the time we use a very simple model called viscous damping. So, in viscous damping we assume that the damping force acting on a structure in one cycle is equal to a damping coefficient multiplied by the velocity in that cycle. So, we can represent this viscous damping system like this and if you move this mass in x-direction by an amount x, restoring force is applied by the damper on this mass and the force is equal to c x dot.

$$f_D = cx$$

And, the value of this damping coefficient is chosen, so that the energy dissipated by this damping mechanism is equivalent to the sum of the energy dissipated by all these mechanisms. So, we will learn about damping in detail in this course. So, at this moment we just have to understand that we are using an equivalent damping and that viscous damping is a function of velocity.

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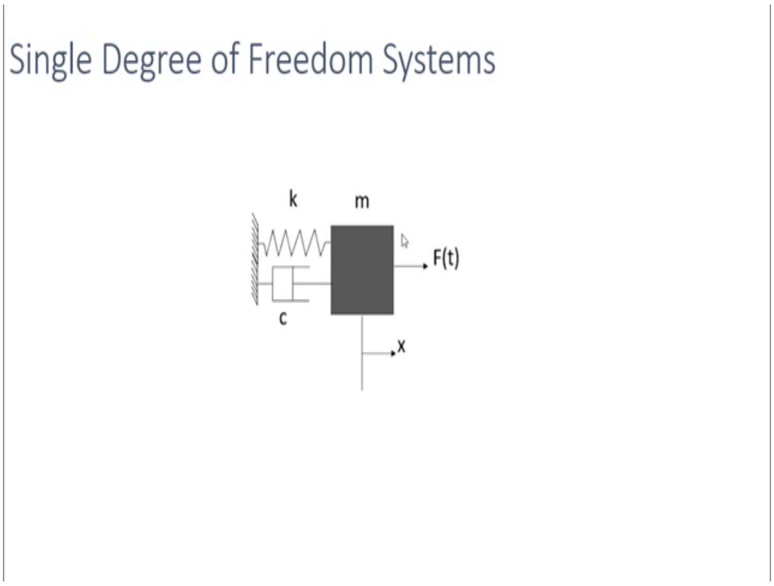

So, the single degree of freedom system can be represented like this – mass can be lumped as a single mass single concentrated mass and there will be a massless spring which is representing the stiffness in the system and the damping. The energy dissipation mechanism can be represented using a viscous damping system with the damping coefficient is equal to c and there will be a force a dynamic time varying force acting on the system.

So, a single degree of freedom system is represented like this. In our course we will be learning about single degree of freedom systems. So, we will be understanding these types of system, we will find out how it is behaving with various type of dynamic loading and many real life structures can be simplified into a single degree of freedom system. For example, a water tank can be treated as a single degree of freedom system for a very simple analysis. So, you can treat that the mass at the tank is we can lump it as a concentrated mass and the columns can be modeled as a spring we can also assume some amount of damping in the system.

So, many real life structures can be simplified as a single degree of freedom system.

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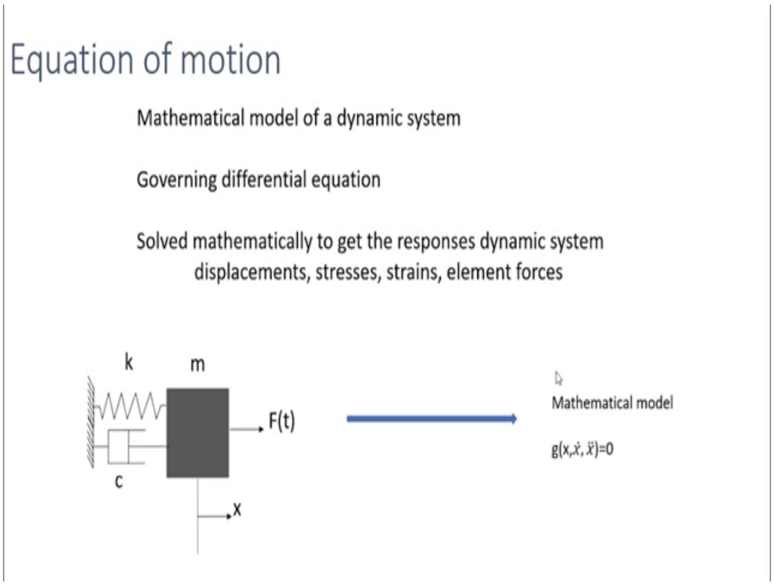

Now, we will see what an equation of motion is, ok. So, earlier we have discussed that to make a model we have to make a mechanical model first and then make a mathematical model. Equation of motion is a mathematical model of a dynamic system. So, it is a governing differential equation and we can solve this equation and get the responses of the dynamic system like displacement, stresses, strains etcetera. So, if you have a single degree of freedom system like this; so, this is a mechanical model. So, we have identified the elements of a dynamic system, mass, stiffness and damping and we know that the system is interacting like this. The each element is interacting in a way described by this diagram.

So, now we need to form some equation which is a function of all these parameters, all these elements. So, now, we will look into how an equation motion is formulated.

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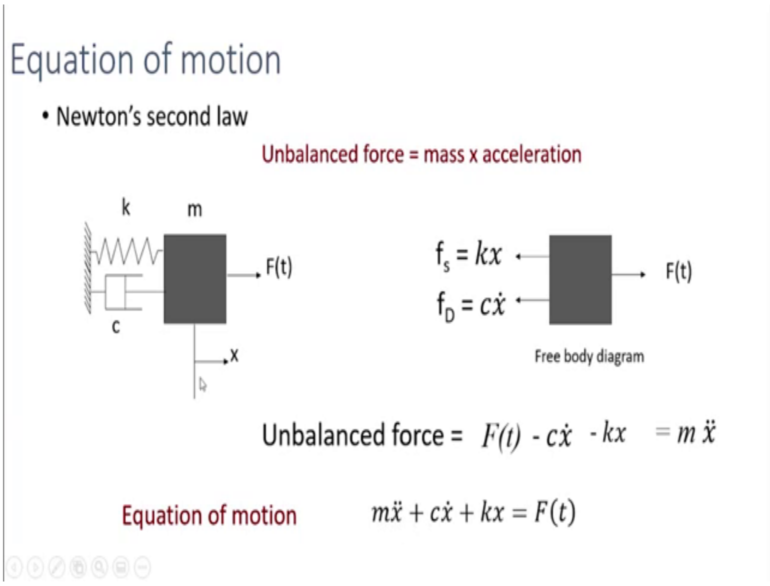

So, first we will make use of Newton's second law to find equation of motion of a system. So, the Newton's second law says the unbalanced force on a body is equal to the mass of the body multiplied by the acceleration it gains. So, this is our single degree of freedom system. So, now, let us isolate this body which is the mass and let us draw the free body diagram of the mass. So, if the system is getting vibrated in this direction because we have a dynamic force in this x-direction.

So, when a force is acting in this direction this mass will move in the direction right, force is in this direction. So, it will try to pull this mass into the x-direction into the right side. So, what happens when a force is trying to pull this in to the right direction what will happen is this stiffness and the damping will exert some restoring force on the body to make it in equilibrium. So, the spring will exert a restoring force is equal to stiffness times the displacement and the damper will exert another force called damping force and that is equal to the damping coefficient multiplied by the velocity of this mass particle. So, these are the forces presently acting on this body.

So, now we will calculate the total force acting on that body which is equal to the unbalanced force. So, the unbalanced force on this body will be equal to let us take this direction as the positive direction. So, it will be F t and this is in the opposite direction so, it is minus negative damping force we need to add this one. So, that is also in the opposite direction so, it will be minus k x. So, this is the unbalanced force acting on it and what is Newton's law say? It says the unbalanced force is equal to mass times acceleration. So, we can equate this to mass, mass of our body and the acceleration of the body at that particular instant. So, it is mx double dot.

$$m\ddot{x}+c\dot{x}+kx=F(t)$$

So, now, just rearrange this equation we will get m x double dot plus kx plus cx dot is equal to F of t and this equation is known as the equation of motion of this single degree of freedom structure. And, this equation represents all properties of a dynamic system it is so, this term talks about the inertia force developed and this is the damping force and this is the spring force contributed by the stiffness of the system. So, this is known as the equation of motion.

So, we can solve this for different values of force and calculate the value of x that is the displacement at any instant, then we can calculate velocity we can calculate acceleration and element forces stresses strains everything we can calculate. And, this equation of motion can be formulated in many ways.

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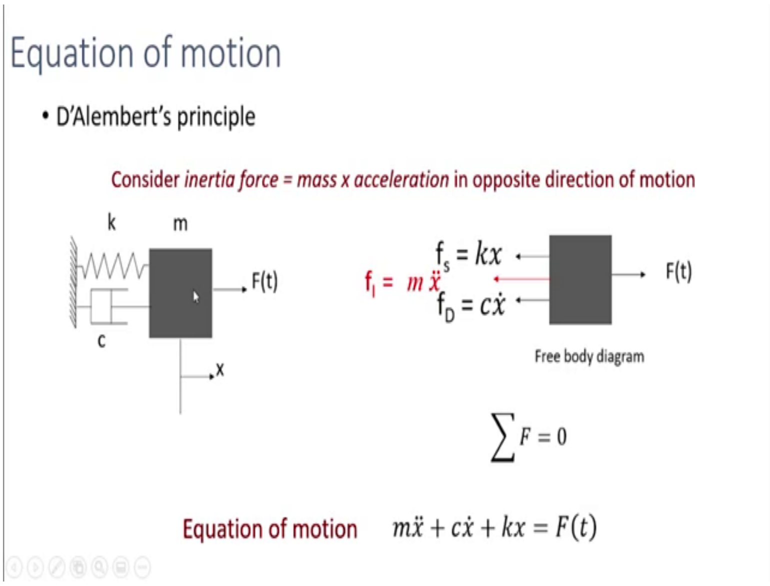

We will discuss one more method that is D'Alembert's principle. We have already discussed D'Alembert's principle in the introduction. So, what is D'Alembert's principle say? It says that if you consider an inertia force which is equal to mass and acceleration in the opposite direction of the motion of the body, we can treat this as a equilibrium. We can assume that the body is in equilibrium. So, this is our single degree of freedom system.

$$f_I = m\ddot{x}$$

So, let us again draw the free body diagram. So, here as we have drawn earlier we have the force acting in right direction and we have the restoring forces in the opposite direction that is stiffness force, that is equal to k x and the damping force equal to c x dot. So, according to D'Alembert's principle we need to apply one more force called inertia force which is equal to mass times acceleration in the opposite direction of motion. So, force is in this direction. So, our motion is also in that direction that is towards right. So, our inertia force needs to be applied in the opposite direction that is leftwards and that is equal to m x double dot. So, now, this body is in equilibrium. So, we can just treat it as a body in equilibrium. So, just write the sum of all forces and that will be equal to 0.

$$f_S = kx$$

$$f_D = c\dot{x}$$

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

So, what are the forces here? We have in this direction in left direction we have mx double dot then cx dot and kx and that will be equal to the force in the right direction that is equal to F t here. So, we will get the same equation as we got in the previous slide. So, this equation is the equation of motion. So, this is the governing differential equation of this single degree of freedom system. So, now, we have made the mathematical model of a single degree freedom system so, now, we need to solve it. So, we can solve this equation for various types of forces and get the behavior of structures in those dynamic forces.