Structural Dynamics for Civil Engineers - SDOF Systems
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Lecture – 12
Response to Arbitrary Excitations

In the previous lectures, we learnt the responses of a single degree of freedom system under free vibrations and under harmonic and periodic forces. From now onwards, we will learn the responses of single degree of freedom structures under any arbitrary excitation.

So, as a first step let us understand the response to unit impulse.

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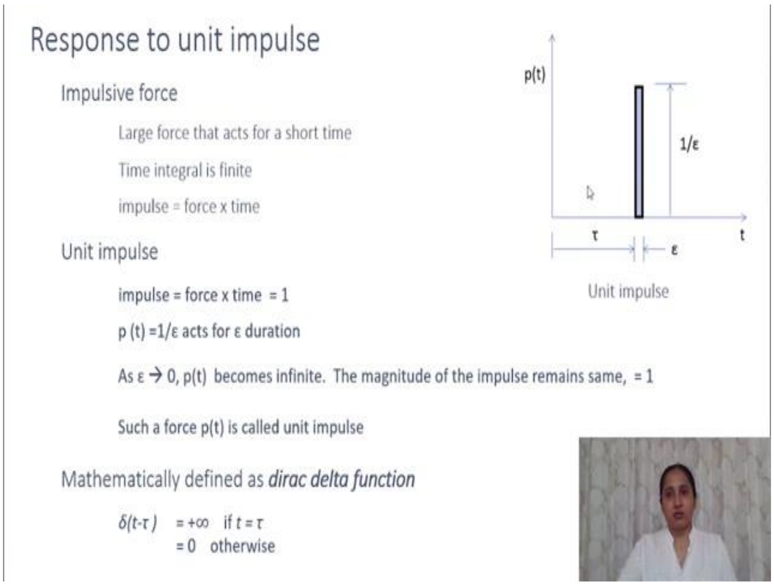

So, what is an impulse? If a large force acts on a body for a very short time that is called impulsive force and the time integral will be finite; that means, the force integrated over time will be the constant it will be a finite value. And, the quantity of this impulse is equal to force times time. So, what is the unit impulse? If, the value of this impulse is equal to 1, we can call this a unit impulse. So, let us understand it in detail.

So, here we have a force acting for a very small duration say at time is equal to tau. So, here this force the magnitude of the force is equal to 1 by epsilon and it is acting for a time epsilon. So, what will be the value of the impulse, it would be 1 by epsilon times epsilon that is 1. So, this is a unit impulse function. So, if this force is acting for a large duration; that means, the value of the force will be less. So, the product should be 1 so, if epsilon is large then force will be smaller. So, what if this epsilon tends to 0?

So, if epsilon tends to 0 for this product to be equal to 1, the force should be close to infinity. So, as epsilon tends to 0, the magnitude of the force becomes infinite, but even then the magnitude of the impulse will be same as 1, because it is force times the duration, when the duration is small force will peak, but again the magnitude of the product will be equal to 1.

So, this is still a unit impulse force. So, such a force is known as unit impulse. So, how will we represent this type of forces in our analysis? So, mathematically we can represent this impulse force using dirac delta function. So, what is dirac delta function? So, this function delta at t minus tau is defined as plus infinity, if t is equal to tau and it is equal to 0 otherwise. So, this function represents this force. So, when t is equal to tau; that means, at this position the value becomes infinity and at all other locations the value is 0.

So, this dirac delta function can be used to mathematically represent this impulsive forces.

$$\delta(t-\tau) = +\infty \quad \text{if } t = \tau$$
$$= 0 \quad \text{otherwise}$$

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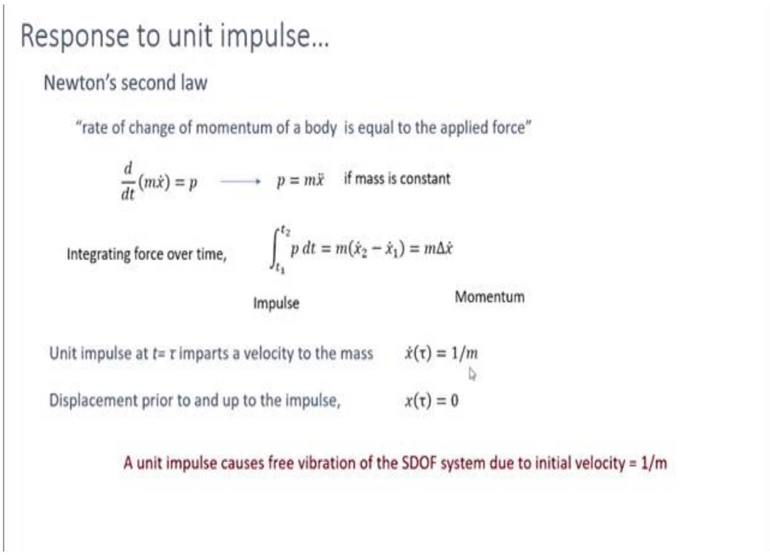

Now, let us see how the unit impulse is initiating a vibration in a single degree freedom system? So, let us go back to our Newton's second law. The Newton's second law states that "rate of change of momentum of a body is equal to the applied force". So, when the forces applied to a body there will be a change of momentum.

So, momentum is mass times velocity. So, there will be a rate of change of momentum. So, d by dt of momentum will be equal to the applied force, this is Newton's second law. So, and if we have a constant mass, if the mass is constant with respect to time we can write this as mass times acceleration. So, the force will be equal to mass times acceleration.

$$\frac{d}{dt}(m\dot{x}) = p \qquad\qquad p = m\ddot{x}$$

So, we can integrate this force this equation over time and we would get the LHS will become integral t 1 to t 2 p dt is equal to mass times, if you integrate acceleration that will be equal to velocity. So, that would be the change in the velocity. So, we would get this impulse is equal to change in momentum.

$$\int_{t_1}^{t_2} p\,dt = m(\dot{x}_2 - \dot{x}_1) = m\Delta\dot{x}$$

So, what does this mean? So, if we apply the unit impulse at time tau that impulse will impart a velocity to the mass and that velocity will be equal to 1 by m. So, for the impulse a change in velocity will be created. So, that velocity will be equal to 1 by m.

So, what would be the displacement? So, the displacement before the impulse and after the impulse will be equal to 0 so, the single degree of freedom system is at rest initially. So, a unit impulse acting on the single degree of freedom system causes free vibration of the system due to this initial velocity 1 by m. So, the impulse acting on that body is giving it an initial velocity and that initial velocity will cause that system to vibrate under free vibrations. It is free vibration because after the impulse there is no force acting on the system. So, the system is under free vibration.

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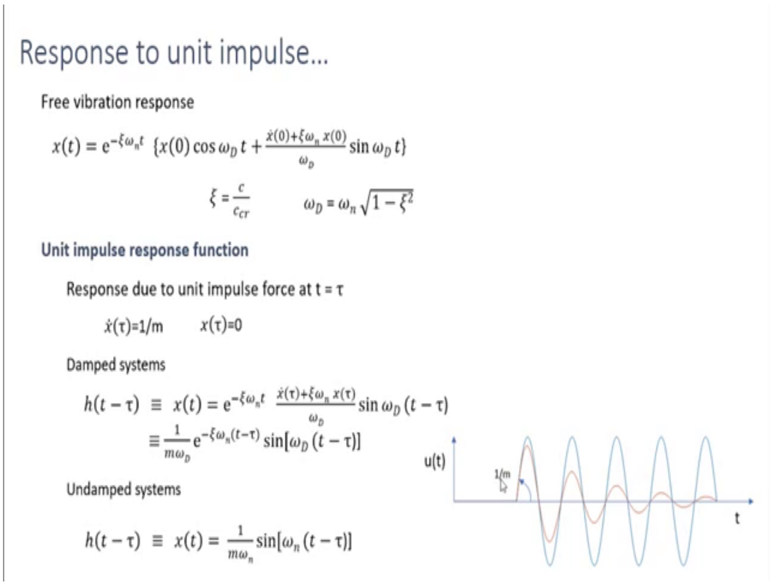

So, now let us calculate the free vibration response. So, we have derived that the free vibration response of a damped single degree of freedom system is given by this and it depends upon the initial displacement and the initial velocity.

$$x(t) = e^{-\xi\omega n t}\{x(0)\cos\omega D\, t + \frac{\dot{x}(0)+\xi\omega n\, x(0)}{\omega D}\sin\omega D\, t\}$$

$$\xi = \frac{c}{c_{cr}} \qquad \omega D = \omega n\sqrt{1-\xi 2}$$

And, in this expression zeta as the damping ratio and omega D is the natural frequency times square root of 1 minus zeta square and this is known as the natural frequency of the damped system. So, we had derived this earlier. So, we will use this to find the response to unit impulse. So, the unit impulse response function is the response due to unit impulse force at say time equal to tau.

So, we just learned that when we apply a unit impulse to the single degree of freedom system, it is equivalent to an initial velocity given to the system and that initial velocity is equal to 1 by m, and that initial displacement is equal to 0. The impulse is acting on the body at time is equal to tau. So, we have x dot tau and x tau known to us.

So, for any damped system using this expression we can write the unit impulse response function, that is represented as function h; so, h t minus tau that is the response to unit impulse when a unit impulse is acting at tau. We can substitute the value of this displacement and velocity in this expression. So, the displacement is 0 so, this term will vanish and we have this term with x naught dot replaced by our x dot tau.

$$h(t-\tau) \equiv x(t) = \mathrm{e}^{-\xi\omega n t}\ \frac{\dot{x}(\tau)+\xi\omega n\ x(\tau)}{\omega D}\sin \boldsymbol{\omega} D\ (t-\tau)$$

$$\equiv \frac{1}{m\omega D}\mathrm{e}^{-\xi\omega n(t-\tau)}\sin[\boldsymbol{\omega} D\ (t-\tau)]$$

So, the expression for the unit impulse response for the damped system is this. So, we can just rearrange the terms and we would get 1 by m omega D, e to the power minus zeta omega m t minus tau sin omega D t minus tau. So, this is the response of a damped system under unit impulse. So, for un-damped system this unit impulse response function is equal to 1 by m omega n sin omega n t minus tau. So, we do not have this term and omega D will be equal to omega n, then it is a undamped system.

$$h(t-\tau) \equiv x(t) = \frac{1}{m\omega n}\sin[\boldsymbol{\omega} n\ (t-\tau)]$$

This is the plot for both these responses so, as you can see we have giving an initial velocity to the system through this unit impulse. So, this initial velocity that is the slope of this response curves at t is equal to tau will be equal to 1 by m, that is equal to the initial velocity. Now, let us find the response to arbitrary force.

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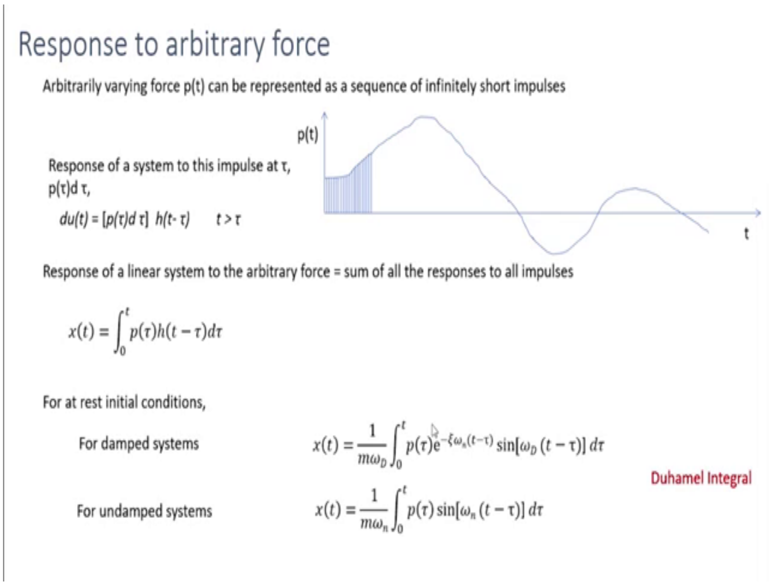

So, we will use the unit impulse response function to calculate the response to arbitrary force. Let us see how it is done.

Arbitrarily varying force can be represented as a sequence of infinitely short impulses. So, we have this force and it is having an arbitrary shape it is varying arbitrarily with time. So, this type of force we can split it into multiple short impulses. So, we can represent this force as the sequence of short impulses. So, it is like one impulse is acting immediately after the other. So, we can represent it like this and we can calculate the response for each of these short impulses.

So, the response of a system to this one impulse at say time tau will be equal to the impulse will be equal to the force times, the duration of the impulse. So, that would be the force at that time tau that is p tau multiplied by the infinitesimal duration that is d tau. So, this will be this short impulse at time tau and the response of the system due to that impulse will be the impulse times the unit impulse response.

$$du(t) = [p(\tau)d\,\tau]\ \ h(t-\tau) \qquad t > \tau$$

So, this h t minus tau is the response to a unit impulse. So, the response of an impulse is equal to p tau d tau will be that times the unit impulse function. So, this is possible since we are dealing with linear system. So, if the system was non-linear this will not work. So, response of a linear system to any arbitrary force is equal to the sum of the responses of all small-small impulses. So, the response due to this arbitrary force can be calculated by integrating this expression over time.

$$x(t) = \int_0^t p(\tau)h(t-\tau)d\tau$$

So, the response x t will be equal to integral 0 to t p tau h t minus tau d tau. So, this p tau and d tau will give the value of the impulse and h is the unit impulse response so, we will get the response due to this force. So, if we have 0 initial conditions that is at time is equal to 0, if our displacement and velocities at 0, then our responses will be for damped systems we can find the responses this where this p tau is the expression for this arbitrary force.

$$x(t) = \frac{1}{m\omega D}\int_0^t p(\tau)\mathrm{e}^{-\xi\omega n(t-\tau)}\sin[\boldsymbol{\omega}D\ (t-\tau)]\ d\tau$$

Similarly, we can find the response for an undamped system by ignoring this damping terms. So, that would be 1 by m omega n integrals over time p tau sin omega n t minus tau d tau. So, we can use these integrals to find the response to a arbitrary force, and these integrals are known as Duhamel Integrals. Now, we will use these Duhamel Integrals to find the response of single degree freedom systems.

$$x(t) = \frac{1}{m\omega n}\int_0^t p(\tau)\sin[\omega n\,(t-\tau)]\,d\tau$$

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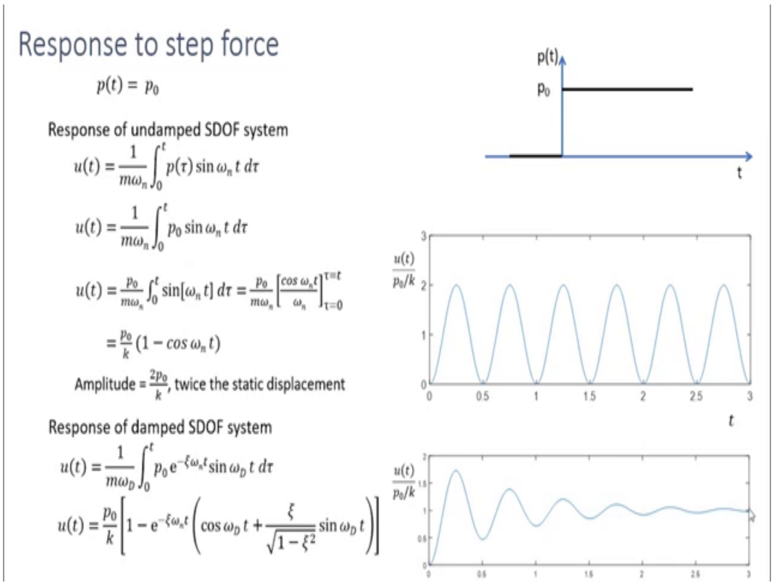

So, first let us find the response due to a step force.

So, in step force constant force p naught is suddenly applied. So, at t is equal to 0 this force p naught is suddenly applied and after that the force is constant. So, it is a constant force, but before it is application the force was 0 so, this was a sudden jump, it was suddenly applied force. So, let us see how the response of single degree of freedom system for this kind of a step forces. So, will this be equal to the static response because in the static case we have a constant force, right.

So, what is the difference here? So, in the static analysis we assume that the load is applied on the structure very gradually. So, that it is dynamic nature can be ignored, but here the value of the force is constant, but it is suddenly applied. So, we will get some dynamic effects and we will see how much that effect is. So, we can find this response using Duhamel Integral. So, the response of an undamped system is given by this so, u t is equal to 1 by m omega n integral over time p tau sin omega n t d tau.

$$u(t) = \frac{1}{m\omega n}\int_0^t p(\tau)\sin \omega n\, t\, d\tau$$

$$u(t) = \frac{1}{m\omega n}\int_0^t p_0 \sin \omega n\, t\, d\tau$$

So, we know that this forcing function is equal to a constant p naught so, we can substitute and carry out this integral. So, if you integrate this will be p naught by m omega n integral sin. So, we would get p naught by k 1 minus cos omega n t. So, this will be the displacement response of a single degree of freedom system under this force of a, this is for a undamped system. So, if we plot this expression the response is like this. So, this is the response divided by p naught by k.

$$u(t) = \frac{p_0}{m\omega n}\int_0^t \sin[\omega n\, t]\, d\tau = \frac{p_0}{m\omega n}\left[\frac{\cos \omega n t}{\omega n}\right]_{\tau=0}^{\tau=t}$$

$$= \frac{p_0}{k}(1 - \cos \omega n\, t)$$

So, we have seen that this p naught by k is the static response. So, if this was a static force p naught then the response would have been p naught by k. So, in this normalized response curve we can see that this single degree of freedom system is oscillating about the normalized response equal to 1. So; that means, it is oscillating about the static deformation position, it is not oscillating about 0 that is it is not oscillating about its original equilibrium position.

But, it shifts by p naught by k, that is the static deformation and it oscillates about that position and the maximum amplitude as we can see from this curve is 2 p naught by k. So, the normalized response is 2, the maximum response is 2 so, the amplitude is 2 p naught by k, that is twice the static displacement. So; that means, if we suddenly apply a constant load p naught the response amplitude will be twice as much as the static load p naught. So, because of this sudden application there will be dynamic effects induced and that will be twice the static displacement.

Now, let us see the response for a damped single degree of freedom system. So, we have this damping term in the Duhamel Integral.

$$u(t) = \frac{1}{m\omega D}\int_0^t p0\, e^{-\xi \omega n t} \sin \omega D\, t\, d\tau$$

So, if you carry out this integral we would get this expression; we are not going to the derivation of this you can do it by yourself.

$$u(t) = \frac{p_0}{k}\left[1 - e^{-\xi \omega n t}\left(\cos \omega D\, t + \frac{\xi}{\sqrt{1-\xi^2}} \sin \omega D\, t\right)\right]$$

So, if you plot this we would get a decaying response. So, this is because of the effect due to damping, this does not go on continuously as in the case of undamped systems, but this decays. After some times the response converges to the static response that is p naught by k so; that means, after some time this dynamic effects die out and the response is equal to static response.

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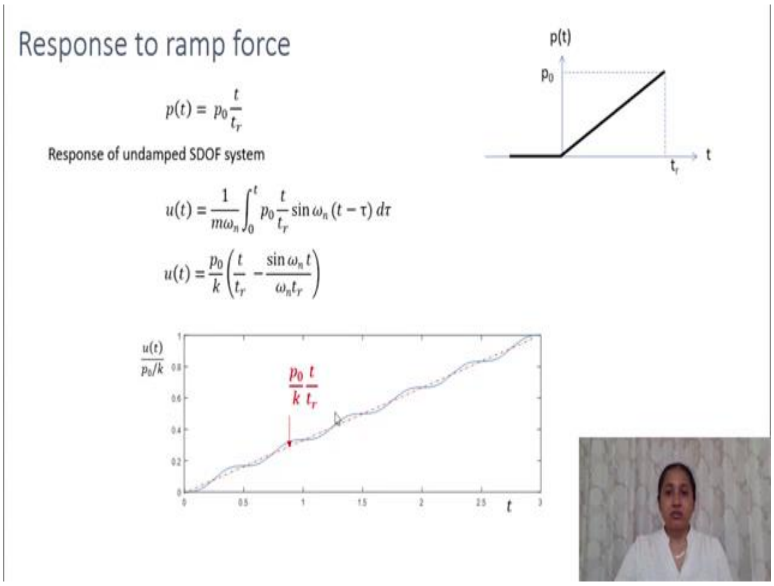

Now, let us see the response to a ramp force; that means a gradually increasing force. So, this force can be represented as p naught t by tr. So, it is linearly increasing and after some time it is reaching the value p naught.

$$p(t) = p_0 \frac{t}{t_r}$$

So, let us see how the response to this force looks like. So, again for an undamped SDOF system, we can use Duhamel Integrals and substitute the value of p t here. So, it would be integral 1 by m omega n integral p naught t by tr sin omega n t minus tau d tau, same as the previous expression we just have to replace the forcing function. So, if we carry out this integral we would get the response as p naught by k that is the static response t by tr minus sin omega nt by omega n tr.

$$u(t) = \frac{1}{m\omega n}\int_0^t p_0 \frac{t}{t_r} \sin \omega n\,(t-\tau)\, d\tau$$

$$u(t) = \frac{p_0}{k}\left(\frac{t}{t_r} - \frac{\sin \omega n\, t}{\omega n t_r}\right)$$

So, let us plot this. So, this is the plot for this expression. So, this blue curve shows total response u t. So, as we can see as the force is increasing, the response also increases, but it has some dynamic effects. So, it is oscillating and that oscillation is about the static response. So, this dotted red line indicates the static response that is p naught by k t by tr. So, as the force is increasing the static response will also increase and our dynamic response will be an oscillation about this static response.

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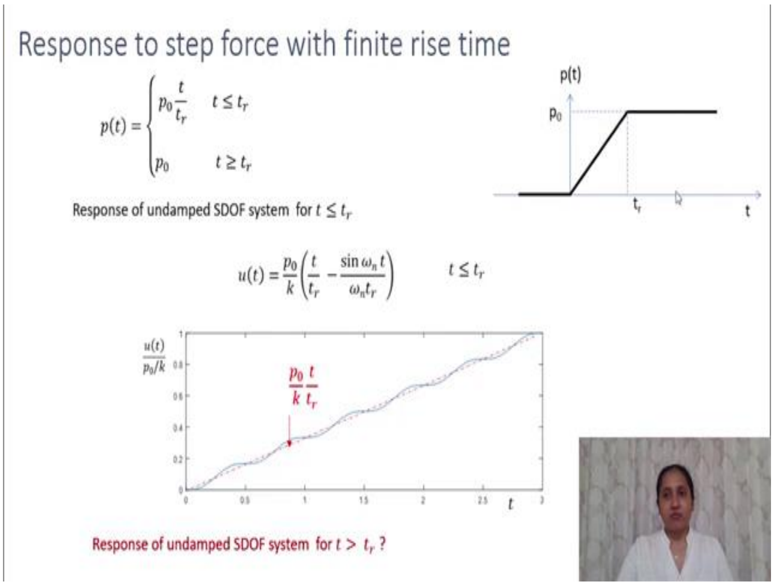

So, now let us see what is the response to a step force with some finite rise time; so, this is the combination of the last two cases we have a step force, but instead of suddenly increasing the force we have a ramp in between. So, it is gradually increasing and then the force is constant throughout the time. So, depending upon the duration of this ramp the effects will change. So, let us calculate the response.

So, in this case the forcing function is equal to p naught t by tr and that is when the time is less than tr. So, after tr the forcing function changes then it is equal to p naught.

$$p(t) = \begin{cases} p_0 \dfrac{t}{t_r} & t \leq t_r \\ p_0 & t \geq t_r \end{cases}$$

So, let us find the response; when time is less than tr that is t less than tr we here in this zone; that means, the ramped force is acting. So, the response in this zone will be equal to what we saw earlier. So, the response will be this p naught by k t by tr minus sin omega n t divided by omega n tr. So, this will be valid when t is less than tr.

$$u(t) = \frac{p_0}{k}\left(\frac{t}{t_r} - \frac{\sin \omega n t}{\omega n t_r}\right) \qquad t \leq t_r$$

So, let us plot this we have seen this earlier. So, this will be oscillating about the static response as the force is increasing, the response will also gradually increased. So, when t is less than tr this is the response. Now, we have to find the response when t is greater than tr. So, what will be the response, when this time is more than tr? So, will that be equal to the step force so, we had derived the response of this step force initially. So, will it be exactly equal to the step force response or it is something else?

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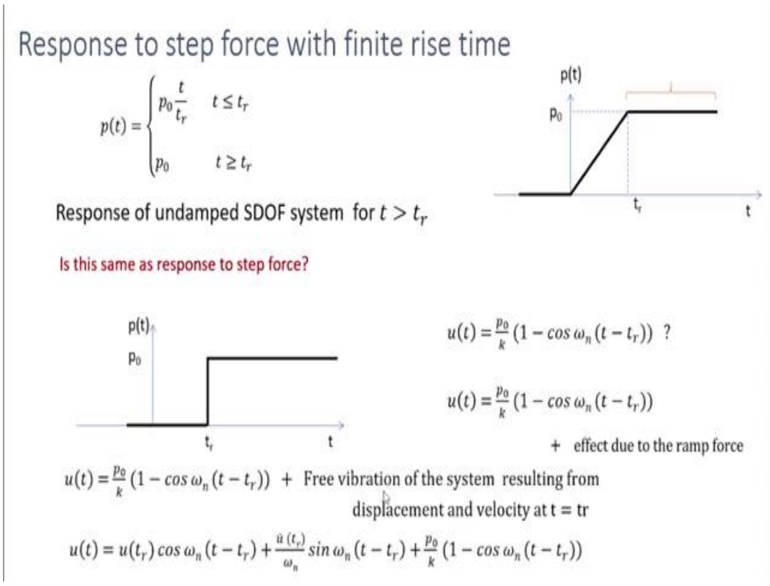

When time is more than tr we have this force that is a constant force similar to a step force. So, will the response be same as the step force. So, this is the step force we have seen earlier. So, if we have a step force at time is equal to tr the response will be this we just derived it earlier. So, when the step starts at tr the response will be this, but in our case here the response of this ramped step function will be this response plus the effects due to this ramp force. So, because of this ramp force in this zone our system is already vibrating. So, at t is equal to tr, it is not at rest, it will have some initial conditions that is the response due to this ramped force at time is equal to tr.

So, the effect of those initial conditions will also affect the response after tr. So, our response of this system of the single degree of freedom system under this force when t greater than tr will be this response plus free vibration of the system resulting from displacement and velocity at t is equal to tr. So, we need to calculate the response due to this ramp function at t is equal to tr velocity and the displacement and using that we have to calculate that free vibration component and add to this step force response.

So, our total response would be, this is the response due to this step force alone and this is the first two terms are the free vibration response, due to the condition at t is equal tr and that is because of the ramped force.

$$u(t) = u(t_r)\cos\omega n\,(t - t_r) + \frac{\dot{u}\,(tr)}{\omega n}\sin\omega n\,(t - t_r) + \frac{p_0}{k}(1 - \cos\omega n\,(t - t_r))$$

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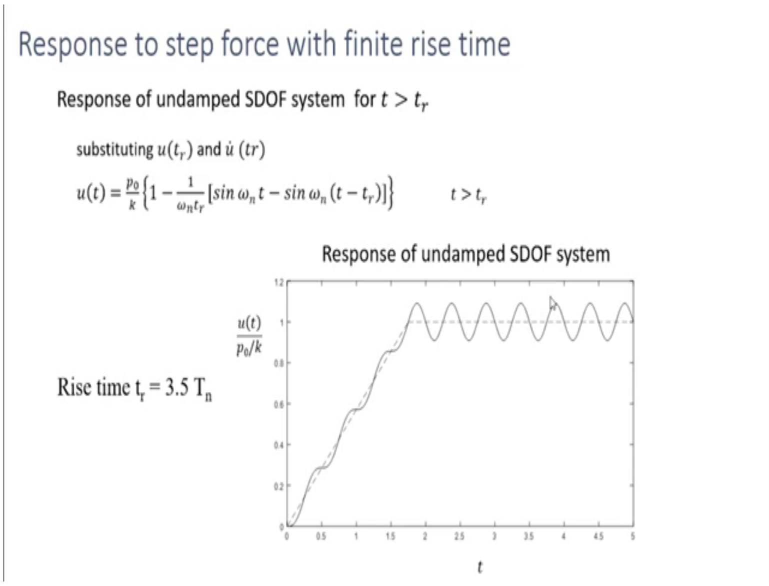

Now, we can substitute the value of velocity and displacement at t is equal to tr and get the expression for the response when t greater than tr. So, if you substitute these values we would get this expression, using this we can calculate the response when t is greater than tr.

$$u(t) = \frac{p_0}{k}\left\{1 - \frac{1}{\omega_n t_r}\left[\sin \omega n t - \sin \omega n (t - t_r)\right]\right\} \qquad t > t_r$$

So, now, let us see the total response of the system. So, when the rise time is 3.5 times Tn the response looks like this. So, this is an undamped system responds; so, this period between these 2 peaks will be equal to the natural period, depending upon the value of the rise time the nature of the vibration changes. So, let us see how it is changing? So, this plot is for a rise time equal to 3.5 times Tn. So, this is a large rise time.

So, here as you can see during the ramped portion the system is oscillating, but that amplitude of oscillation is very less and it is oscillating about the static response as we have discussed earlier. So, after t is equal to tr, this is oscillating about this static response p naught by k.

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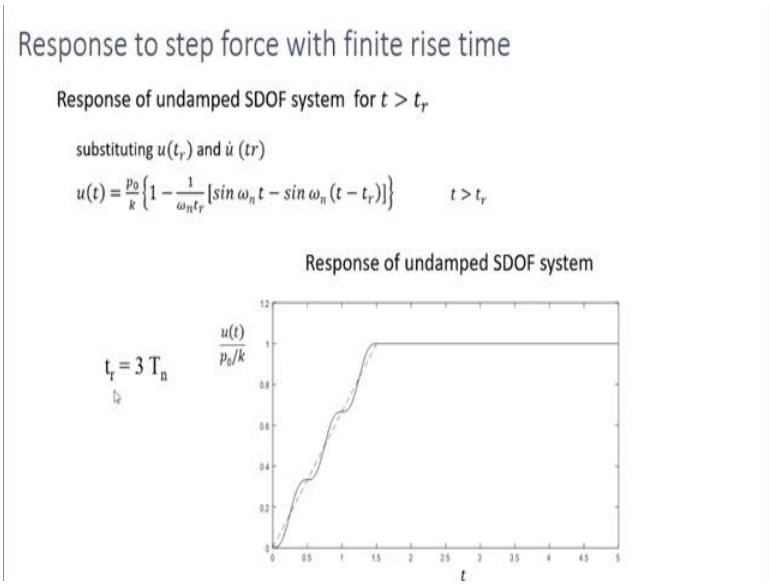

Now, let us see the response for other values of rise time. So, this is when tr is equal to 3 times natural period. So, tr is slightly lesser than what we have seen earlier. So, here this is the response. So, during the ramped portion the system is oscillating, but when t is equal to tr here the velocity of these response is 0, how will we know from this curve? Because, at this point this curve has a flat slope, the slope of this curve at this point is 0.

So; that means, the velocity at t is equal to tr is 0. So, when the velocity is 0, in that case this no oscillation beyond the rise time. So, the force is constant here and the response is also constant. So, this is equivalent to a static response and it is also the value is also equal to the static response. So, this there is no oscillation around this static response, this stays as it is in the static response case. So, now let us see when tr is equal to 2.5.

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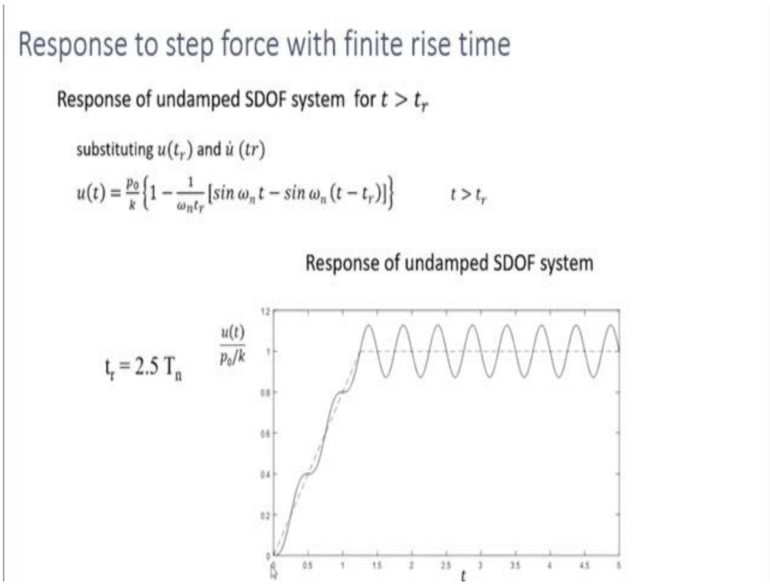

When it is 2.5 the amplitude of the response and this constant force phase is increasing and there is also oscillation. So, here when t is equal to tr the velocity is non-zero we can understand it from this slope. So, this has a positive slope so; that means, the velocity is not 0. So, when the velocity is not 0 we are getting some oscillation at this range and the amplitude of the oscillation is higher compared to 3.5 case.

So, here t is the rise time is 3.5 Tn here the amplitude was low, but here the amplitude is slightly high. And, again the period between 2 peaks will be equal to the natural period because this is a undamped system.

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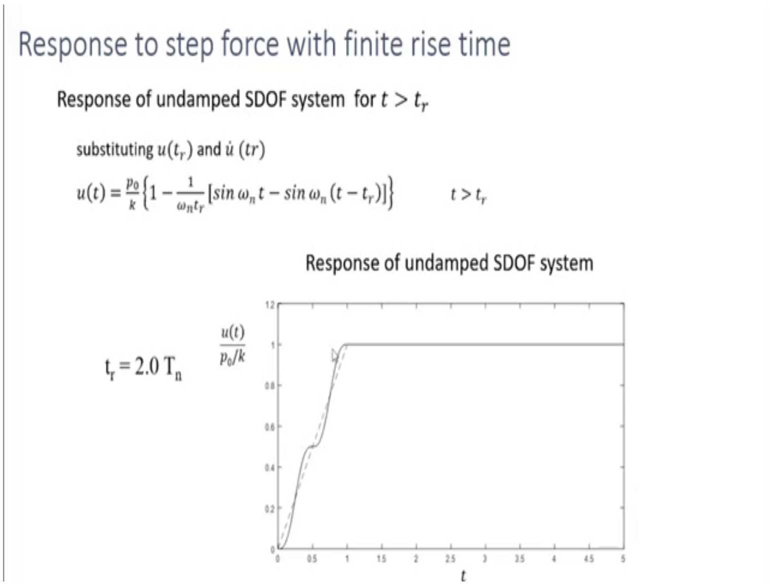

Now, let us see for tr is equal to 2 Tn; here again, we get a condition here that the velocity is 0, and when the velocity at this position is 0 then there is no oscillation, the response is equal to this p naught by k. So, this is again equivalent to static response it does not oscillate.

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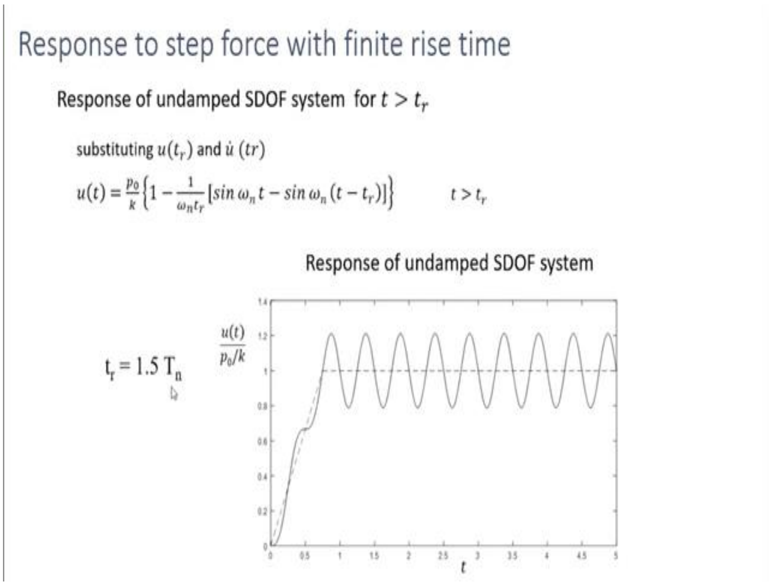

So, when the tr is further reduced when it is 1.5 times the natural period the amplitude of this vibration increases, this will oscillate only if the velocity is non-zero. So, here it is non-zero so, it is oscillating, but the amplitude will be higher than that of tr is equal to 2.5 over t r is equal to 3.5.

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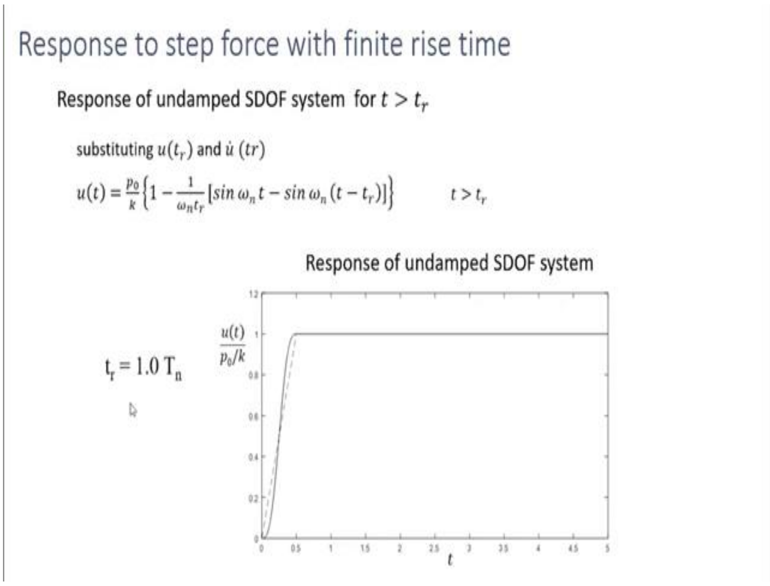

The rise time if you reduce it further here there is no velocity so, there is no oscillation.

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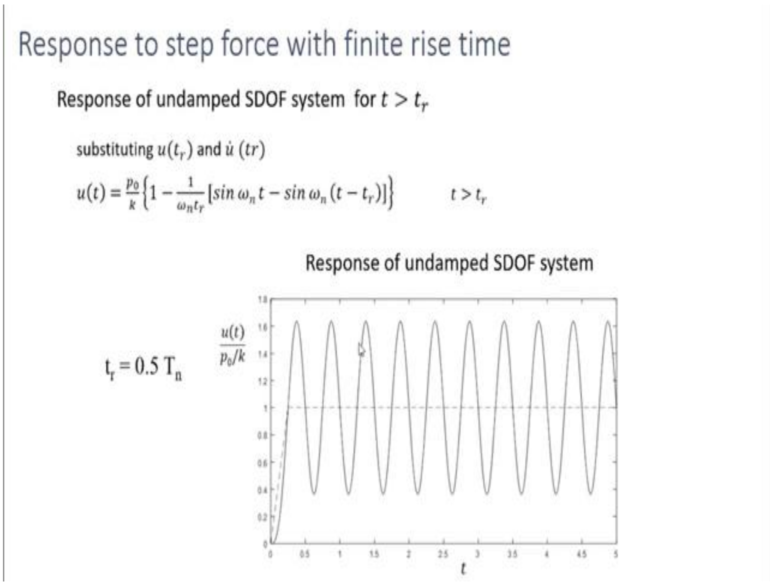

If, you further reduce it there will be oscillation, but the amplitude is increased considerably now. So, the amplitude is 1.6 times the static response that is p naught by k. So, when the rise time reduces the amplitude of the vibration increases. Now, let us see for tr is equal to 0.2 times the natural period.

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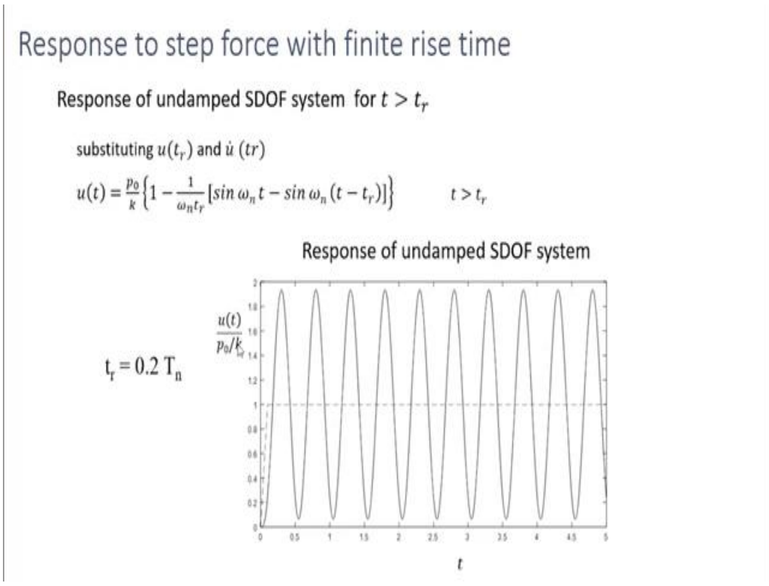

So, in this case the rise time is very low. So, this response is very close to the step force; so, in step force when the forces suddenly applied. So, here in the case of a step force amplitude was twice that of the static response. Here, when the rise time is 0.2 times Tn that is when we have a small rise time the amplitude is very close to 2. So, from this we can understand that if a constant force is applied suddenly, then it can cause some dynamic effects and those effects can make the response of the system twice as much as the static response. And, if the force is applied gradually the response will be close to the static response. So, if we apply the load gradually the dynamic effects will be much less.

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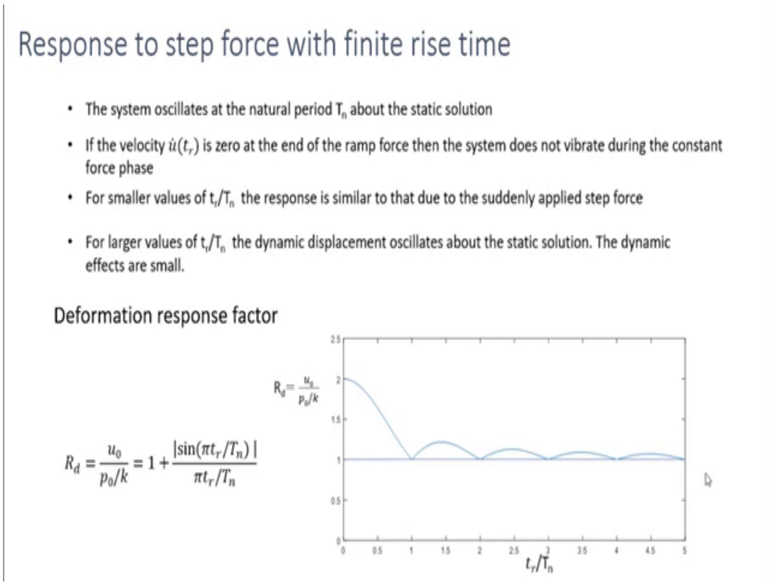

Now, let us conclude our observations regarding the response to step force with finite rise time. So, in this case the system oscillates at the natural period Tn about the static response position. And, if the velocity is 0 at the end of the ramp force then the system does not vibrate during the constant force phase.

So, after the ramp force it will be a constant response, it would not vibrate. For smaller values of tr by Tn the responses similar to that due to the suddenly applied step force. So, when the rise time is less the response will be equal to the suddenly applied force. When tr by Tn is large then the dynamic displacement oscillates about the static solution. And, the dynamics effects are small. So, if you have a large rise time then the dynamic effects are small, but the structure will oscillate about it is static response position.

Now, let us see the deformation response factor. The deformation response factor is the maximum amplitude divided by the static response. So, as we can see this is plotted for different values of rise time that is tr by Tn. So, as we have seen earlier when the rise time is 0, this is equal to the suddenly applied force and this deformation response factor is 2; that means, the response will be twice that of the static response.

$$R_d = \frac{u0}{p0/k} = 1 + \frac{\left|\sin(\pi t_r/T_n)\right|}{\pi t_r/T_n}$$

As the this ratio is increasing and as the rise time is increasing, this Rd is reducing and when the rise time is larger, then the value of Rd is very close to 1; that means, the response will be very close to the static response. So, depending upon the value of the rise time to the natural period, the amplitude changes, in the expression for this deformation response factor that is Rd can be derived from the expression for the displacement by maximizing the displacement. So, you can find out the maximum value of the displacement in terms of tr by n and that is this and this is how we plot this curve.

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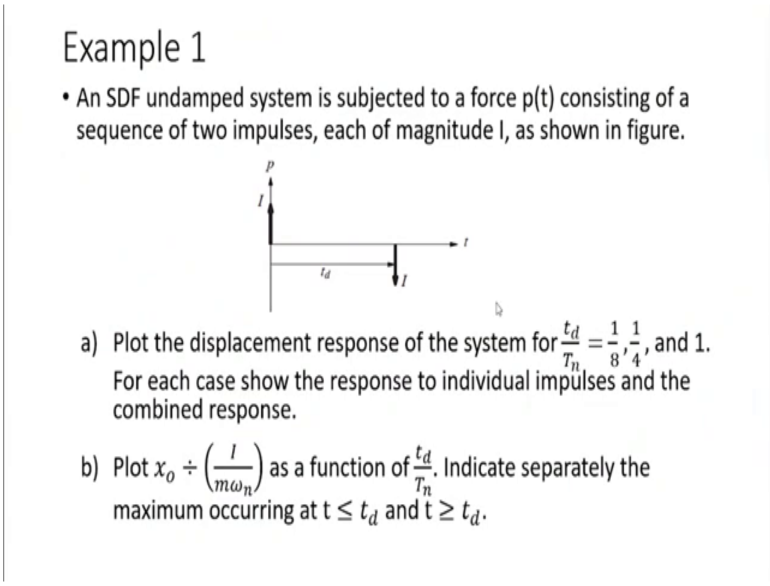

Now, let us look at some examples. So, in the first example we have a single degree of freedom system, undamped system and it is subjected to two impulses. So, the force is consisting of two impulses each of magnitude I has shown in figure. So, at t is equal to 0 we have 1 impulse in the positive direction and which has a magnitude I, then after a duration td this another impulse minus I. And, we need to plot the displacement response of the system for various values of td by Tn. And, for each case show the response to individual impulses and the combined response.

So, we need to find the response due to each of these impulses and then the combined response, also plot x naught by I by m omega n. So, this is like deformation response factor. And, indicate separately the maximum occurring at t less than or equal to td and t greater than equal to td; that means, if the maximum response in this phase and this phase.

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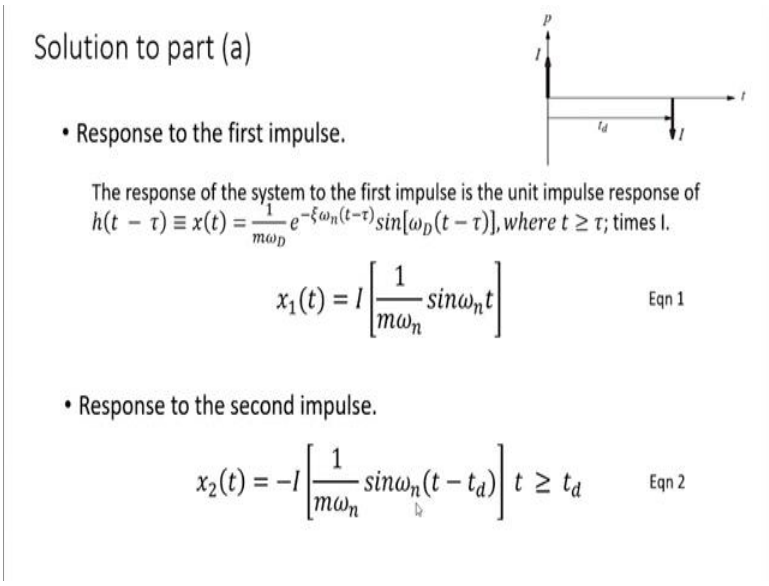

So, let us solve this. The response due to this impulse functions can be calculated by using unit impulse response. So, we can multiply this magnitude of this impulse by the unit impulse response and we get the response corresponding to these impulses. So, the first response that is the response due to this first impulse is the magnitude multiplied by the unit impulse response of the undamped system. So, this is given for the damped system and since ours is an undamped system we can ignore this damping terms. So, for the first response will be I times 1 by m omega n sin omega n t.

$$x_1(t) = I\left[\frac{1}{m\omega_n} sin\omega_n t\right]$$

And, similarly we can calculate for the second impulse and that response would be this magnitude that is minus I multiplied by the unit impulse response at t is equal to td. So, that response will be minus I times 1 by m omega n these terms are similar and sin omega n t minus td. So, that is to locate this position. And, the second response is valid only when t is greater than td. So, if t is less than t d this force is not acting. So, the effect of this will come only when t is greater than td.

$$x_2(t) = -I\left[\frac{1}{m\omega_n} sin\omega_n(t-t_d)\right]\ t \geq t_d$$

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Solution to part (a)

- Response to both impulses.

For $0 \le t \le t_d$:

$$x(t) = I\left[\frac{1}{m\omega_n} sin\omega_n t\right] = \frac{I}{m\omega_n}\sin\frac{2\pi t}{T_n} \qquad \text{Eqn 3}$$

For $t \ge t_d$:

$$x(t) = \frac{I}{m\omega_n}\left[sin\omega_n t - sin\omega_n (t - t_d)\right]$$

$$= \frac{I}{m\omega_n} 2sin\frac{\omega_n t_d}{2} cos\frac{\omega_n (2t - t_d)}{2}$$

$$= \frac{2I}{m\omega_n}\left(sin\frac{\pi t_d}{T_n}\right) cos\left[2\pi\left(\frac{t}{T_n} - \frac{1}{2}\frac{t_d}{T_n}\right)\right] \qquad \text{Eqn 4}$$

So, now let us find the response due to both the impulses. As, we have seen here when t is less than td only the effect due to this impulse is there. So, when time is less than td the response will be equal to x1 and when time is greater than td the total response will be x1 plus x2. So, we can add this sin omega n t then we can add this x2,that is minus sin omega n t minus td term. So, this is the total response when t is greater than td. So, we can simplify this equation and we can get this expression we can reduce this to this expression.

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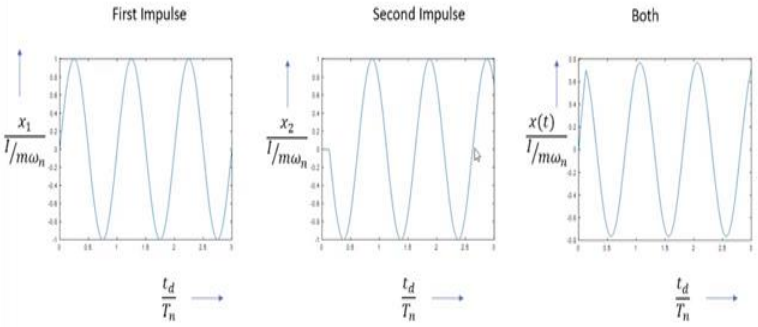
Solution to part (a)

- displacement response plot

Equations 3 & 4 are plotted for $\frac{t_d}{T_n} = \frac{1}{8}$



Now, we can plot this responses the individual responses and the total responses for various values of td by Tn. So, first this result is for td by Tn is equal to 1 by 8. So, the first impulse response is this. So, the amplitude of this first impulse response is x1 by I by m omega n is equal to 1; that means, the amplitude of x 1 is equal to I by m omega n. So, that is equal to the static response.

And, the second impulse response is 0 till t is equal to td, because the impulse starts only when td is equal to time is equal to td. So, till td the response is 0. So, then after that we have a harmonic vibration with amplitude equal to I by m omega n. So, this ratio is 1. So, this is also equal to the static amplitude. So, if we add these two we will get this response. So, when these two get added up the amplitude reduces. So, that this amplitude is only 0.8 this ratio so; that means, the amplitude of this total response is less than the this equivalent static response, that is I by m omega n.

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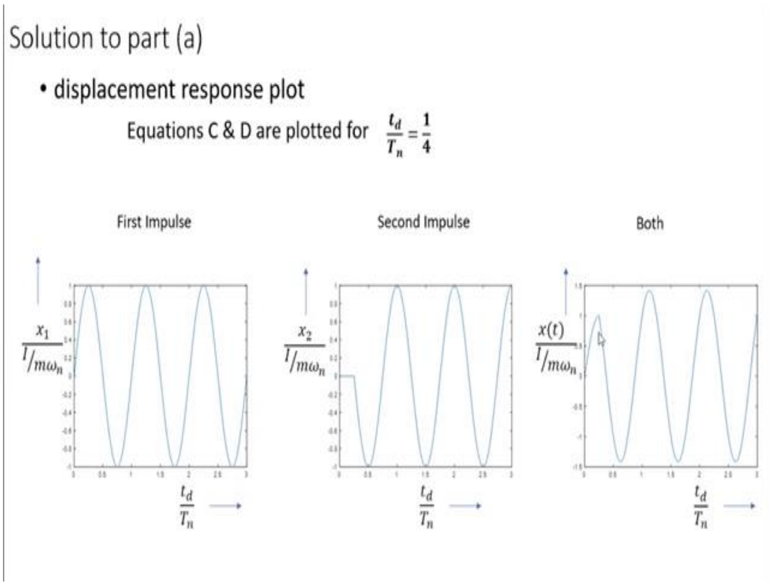

And, this is when this ratio that is td by Tn is equal to 1 by 4.

So, in this case the first impulse response is similar to the previous one. The second one, the second impulse response is little bit more delayed compared to the first one, because here td is more. So, this 0 portion is little longer than the previous one. So, when we add these two the amplitude is now close to 1.5. Both the first and the second impulse responses we are having amplitude one, but when we add together because of this phase difference the total amplitude will be either less or more than the individual amplitudes.

So, that will depend upon the phase difference.

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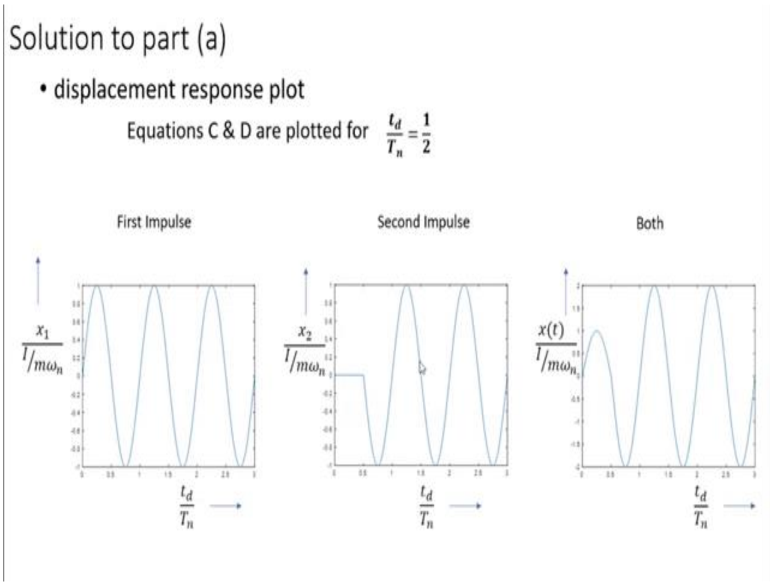

So, when td by Tn is equal to half this phase difference between these two responses is still more. So, this response is same as the previous one and this one is a little bit more delayed response. So, if we can add these together we will get this. And, in this case both the amplitudes add up and we get a combined amplitude is equal to 2; that means, the response of this function x t, the maximum response of this function x t is equal to twice I by m omega n.

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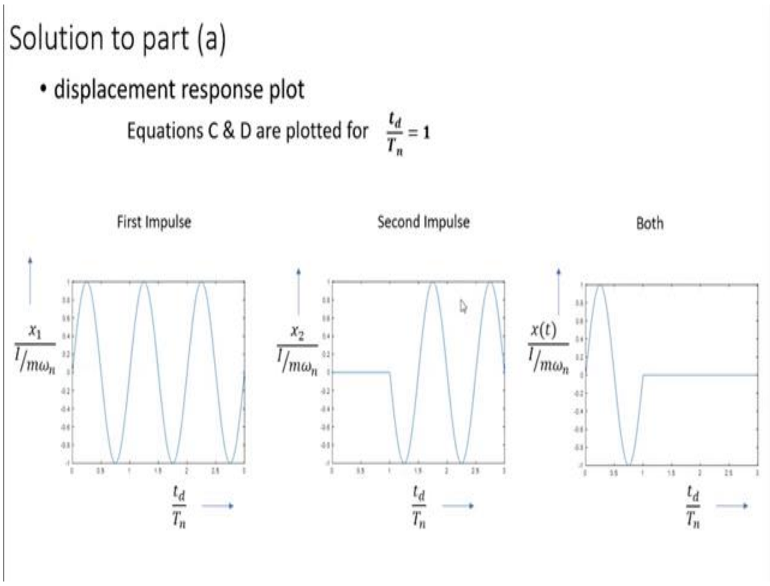

So, let us see this for one more ratio so, when td by Tn is 1 the delay between these two responses is equal to Tn. So, once this second is impulse starts after one cycle of this first impulse. So, if we add these together for t greater than td these responses the second impulse and the first impulse get cancelled out. So, then we get 0 oscillations the response is 0 beyond td.

So, when time is less than td then the total response is equal to the first impulse response, but after td the response is 0 because the response is due to both these cancel out each other.

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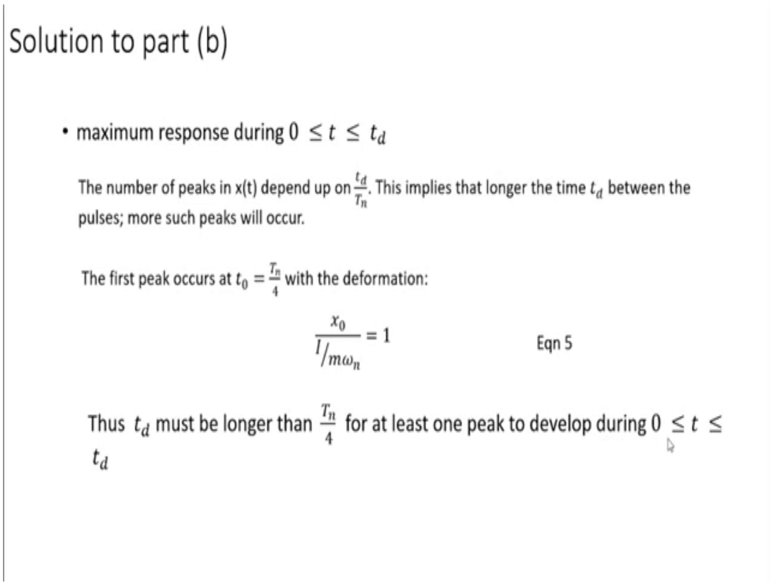

Now, let us move on to the second part of the problem and where we have to find out the maximum response. So, first let us find out the maximum response during time is less than td. So, when time is less than td only this first impulse response is existing.

So, as you can see here this oscillates with natural period Tn and as you can see here it starts with 0 displacement. So, the peak; the first peak appears when t is equal to Tn by 4 that is one-fourth of the natural period. So, in Tn the cycle completes. So, the first peaks appears at Tn by 4. So, the number of peaks in during this time depends upon the value of this td by Tn ratio, if td is much less then we would not be getting this peak. So, if td is less than this Tn by 4, then we will not be getting this peak, but if the value of td is more than Tn by 4, then we will be having this peak.

So, depending upon the value of this td by Tn we can find out the maximum response. So, the first peak occurs at Tn by 4 so, at the peak the response the maximum response is like this x naught by I by m omega m is equal to 1. So; that means, x naught is equal to I by m omega n. So, this is the maximum response when time is less than td and this td is longer than this Tn by 4, otherwise the peak would not happen. So, thus td must be longer than Tn by 4 for at least 1 peak to develop during this duration.

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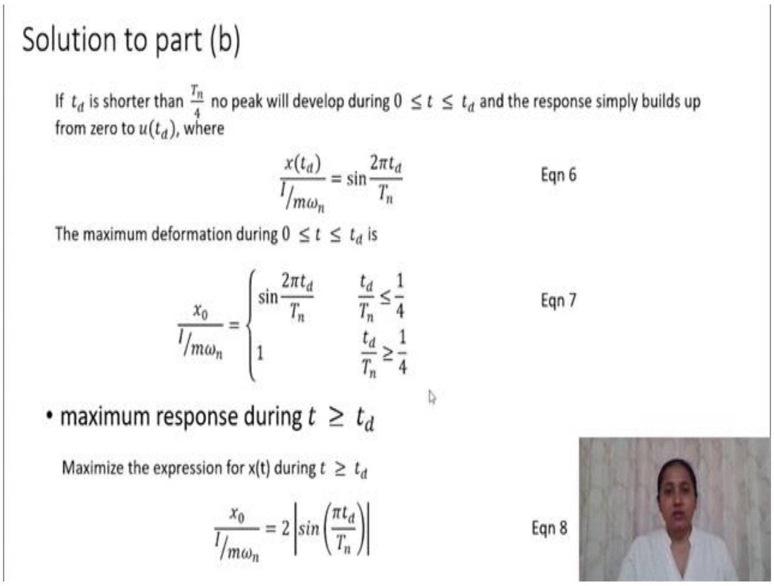

So, if td is shorter than Tn by 4 there would not be any peak developing in this time zone, but the response will increase from 0 to u td. And, the value of that maximum displacement can be calculated using the first impulse response and we can calculate this as this. So, the maximum deformation during time between 0 and td is x naught by I by m omega n is equal to sin 2 pi td by Tn, if td by Tn is less than or equal to 1 by 4 and this ratio is equal to 1, if this ratio is more than 1 by 4.

$$\frac{x_0}{I/m\omega_n} = \begin{cases} \sin\frac{2\pi t_d}{T_n} & \frac{t_d}{T_n} \le \frac{1}{4} \\ 1 & \frac{t_d}{T_n} \ge \frac{1}{4} \end{cases}$$

So, now, we need to find the maximum response when t is higher than td. So, for that we can add the combined response due to the first impulse and the second impulse and maximize that expression and then find the maximum value of this response. So, if you maximize the expression we can find the maxima at x naught is equal to this so, that is x naught by I by m omega n is equal to 2 modulus of sin pi td by Tn.

$$\frac{x_0}{I/m\omega_n} = 2\left|sin\left(\frac{\pi t_d}{T_n}\right)\right|$$

So, this x t when t greater than td has a maximum value of I by m omega n multiplied by this. So, that is the maximum value when t is more than td.

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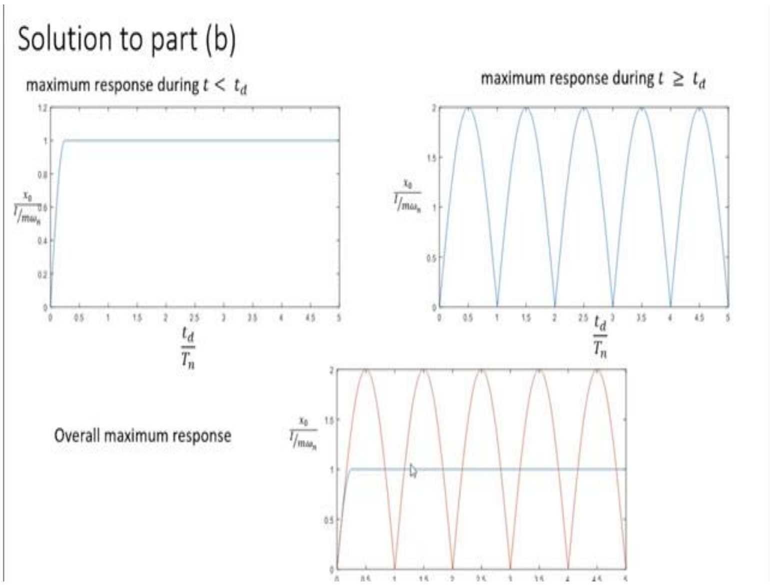

So, let us plot these so, the maximum response during t less than td is given by these two equations this plotted here. So, if td by Tn is less than 1 by 4 the response increases from 0 to 1 and when it is more than 1 by 4 the amplitude is 1. And, maximum response when t is greater than td this follows this curve and that will be according to this expression.

So, now we can find the maximum response for all the values of t, that is when t is less than td and when is when t is greater than td. So, if we combine these two plots we get the overall maximum response and that would be this. So, this is like these two plots plotted together. So, the maximum overall response will be the envelope of these two curves. So, that is the maximum of these two values.

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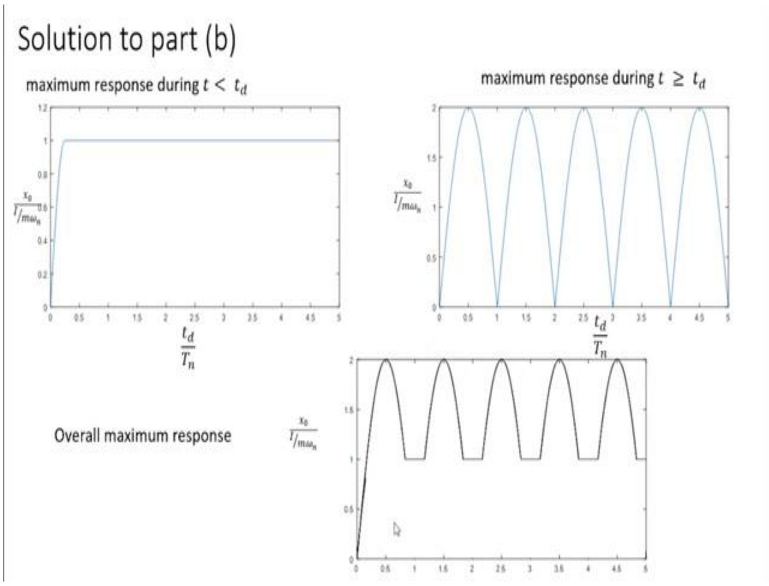

So, if we plot the envelope of these two curves we would get this. So, the overall maximum response of this single degree of freedom system this undamped single degree of freedom system will change like this for different values of td by Tn. This is a deformation response factor and this is also known as response spectra. So; that means, this shows the maximum response of this system for all the values of td by Tn. So, this gives the response spectra.

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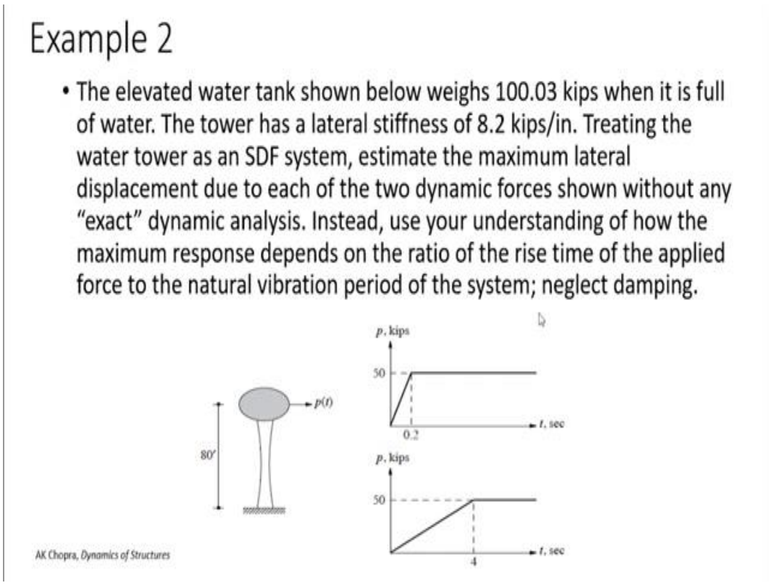

So, in the next example the properties of an elevated water tank is given the elevated water tank shown below weights 100.03 kips when it is full of water. The tower has a lateral stiffness of 8.2 kips per inch. Treating the water tower as a Single Degree of Freedom system, estimate the maximum lateral displacement due to each of the two dynamic forces shown without any “exact” dynamic analysis. Instead, use your understanding of how the maximum response depends on the ratio of the rise time of the applied force to the natural vibration period of the system; neglect damping.

So, we can treat this elevated water tank as an undamped single degree of freedom system, the mass is given the stiffness is also given. We need to find out the maximum lateral displacement. So, two forcing functions are given; so, we do not have to do any exact solution, but depending upon the rise time we need to find an approximate solution, based on our understanding of how the maximum response depends on the ratio of the rise time to the natural period. So, let us find this out.

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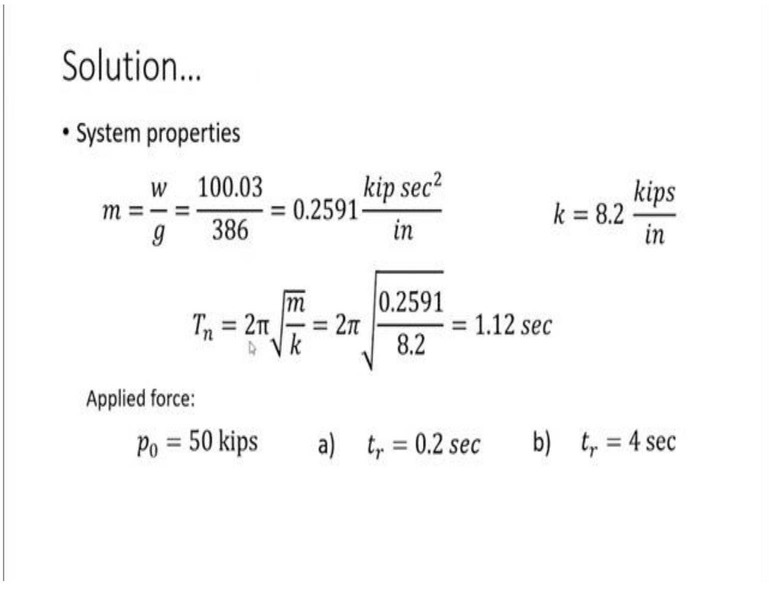

So, the system properties are given mass is given the weight is given. So, the mass can be calculated as weight by gravitational acceleration. So, we can calculate the mass the stiffness is given. So, we can find the natural frequency and the natural period. So, we can calculate the natural period here as 2 pi square root of m by k. So, we get the natural period is 1.12 seconds.

Now, looking at this different forcing function we have two cases, in both the cases the maximum amplitude of the force is same that is 50 kips, but the rise time changes. So, in the first one the rise time is too less that is 0.2, and in the second one the rise time is very large that is 4 and the natural period is only 1.12 seconds.

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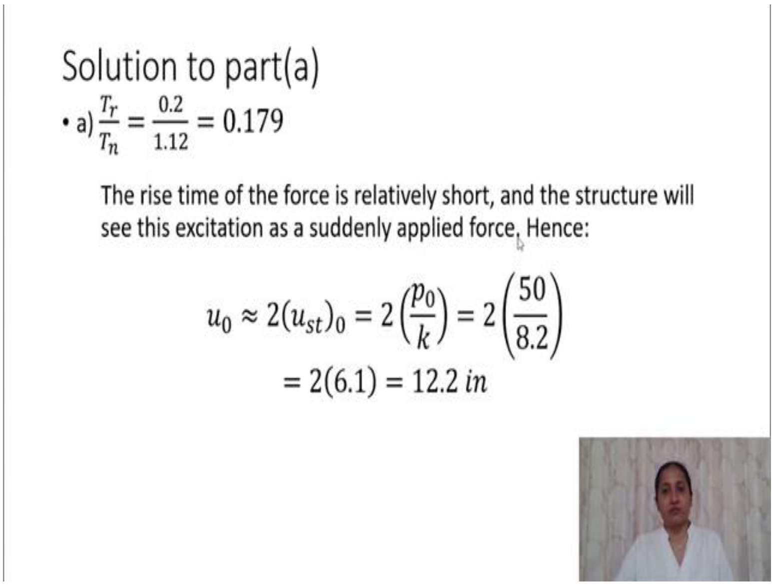

So, to solve this we can find out the ratio of the rise time and the natural period. So, in the first case it is 0.179, so; that means, compared to the natural period this rise time is very low. So, since the rise time of the force is relatively short and the structure will see this excitation as a suddenly applied force. So, since this rise time is too short. So, this force will be like a suddenly applied force.

So, we learned earlier that when a suddenly applied step force is acting on a system the maximum amplitude is approximately equal to twice that of the static displacement. So, we know the static response as p naught by k so, the maximum response will be twice that. So, we can calculate we know p naught we know the value of k so, we can calculate the maximum value of the response as 2 times p naught by k that would be equal to 12.2 inches. So, this is not an exact solution, but this is a approximate solution.

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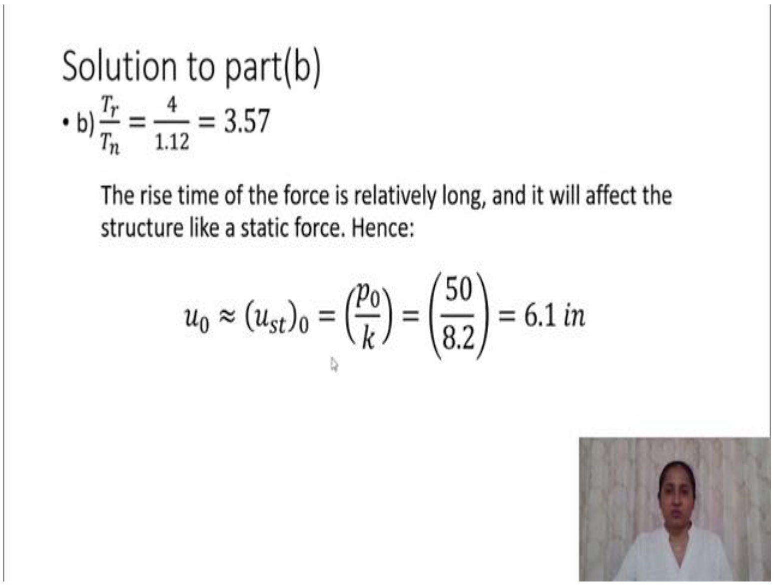

So, for the second case the ratio of the rise time and the natural period is 4 by 1.12 that is 3.57. So, this is very large the rise time of the force is relatively low and it will affect the structure like a static force. So, since the rise time is more the dynamic effect due to the application of the load is less.

So, this is more or less like a static force. So, the maximum response will be approximately equal to the static response that is p naught by k. So, that would be equal to 6.1. So, we can calculate the approximate value of the maximum response using this relationship.