Structural Dynamics for Civil Engineers - SDOF Systems
Dr. Riya Catherine George
Department of Civil Engineering
Hiroshima University, Japan
Indian Institute of Technology, Kanpur

Lecture – 11
Energy and Damping

Welcome back to the Structural Dynamics course. In the last few lectures we have been learning about harmonic vibrations, that is the vibrations of a single degree of freedom system under harmonic forces. In the last week we also saw that, any periodic force can be represented in terms of multiple harmonic forces. So, a response to a periodic force of a linear system is equal to the sum of the responses to it is harmonics. Now, let us see how energy is balanced during harmonic vibrations.

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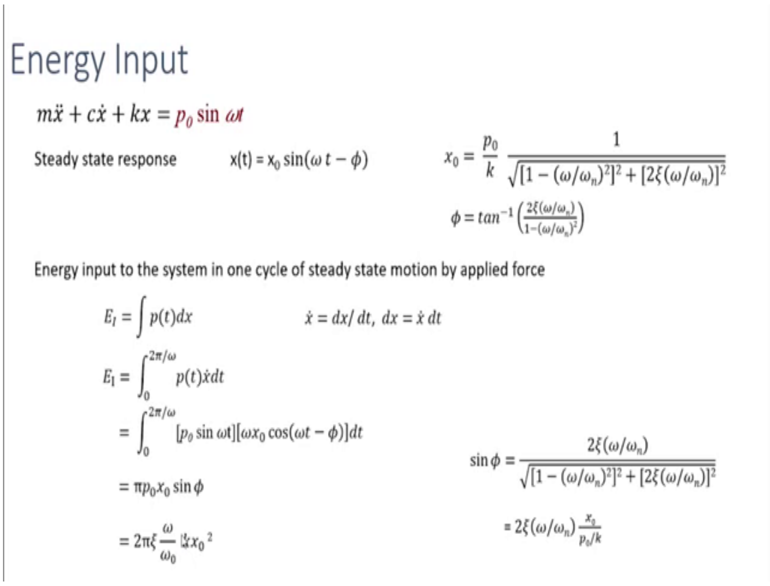

Now, let us see the energy input to a single degree of freedom system under harmonic force. This is the equation of motion of a single degree of freedom system; we have mass damping and stiffness. So, this is the spring force, this is the damping force, this is the inertia force and the sum is equal to the harmonic force acting on the system.

$$m\ddot{x} + c\dot{x} + kx = p_0 \sin \omega t$$

We also learned that this system has two types of response, transient response and steady state response. Transient response decays in time so, after some time only steady state response will be dominant and the steady state response is equal to x naught sin omega t minus phi. So, we learnt the expression for x naught that is the amplitude of the steady state response. So, we have derived this expression and this p naught by k is equivalent to the static response that is if this force was a static force the response would have been p naught by k that is force by stiffness.

$$x(t) = x_0 \sin(\omega t - \phi) \qquad x0 = \frac{p0}{k} \frac{1}{\sqrt{[1-(\omega/\omega n)2]2+[2\xi(\omega/\omega n)]2}}$$

$$\phi = tan^{-1}\left(\frac{2\xi(\omega/\omega n)}{1-(\omega/\omega n)2}\right)$$

So, this p naught by k is the static response of the system and if you multiply it with this factor which is called dynamic response factor we will get the amplitude of this steady state response. And, the value of phi was derived as this 2 zeta frequency ratio divided by 1 minus frequency ratio square, this we have derived in our previous lectures. So, we can calculate the value of the steady state amplitude.

Now, let us look at the energy inputted to the system by this harmonic force. So, the energy input to the system in one cycle of the steady state response, by this applied force is equal to integral p t dx p t is this force and dx is the infinitesimal displacement of the system. So, if you integrate p t dx you will get the energy. And, we know that x dot is dx by dt. So, dx can be written as x dot dt.

$$E_I = \int p(t)dx \qquad \dot{x} = dx/\,dt,\; dx = \dot{x}\,dt$$

So, our energy will become integral 0 to 2 phi by omega that is equivalent to one cycle p t x dot dt. So, we can substitute the values p t is equal to p naught sin omega t and x we know we can differentiate this and get x dot that is the velocity. So, that would be omega x naught cos omega t minus phi and dt. So, if you can integrate this you will get the energy.

So, if you integrate this we would get pi p naught x naught sin phi. And, we know that phi is tan inverse this ratio. So, we can find out what is the expression for sin phi? So, tan phi is 2 zeta this ratio divided by 1 minus frequency ratio square. So, we can calculate sin phi is equal to denominator will be square root sum of squares of these two terms and the numerator would be 2 zeta omega by omega n. And, we can simplify this we know that x naught divided by p naught by k is this expression.

$$E_I = \int_0^{2\pi/\omega} p(t)\dot{x}dt$$

$$= \int_0^{2\pi/\omega} [p_o \sin \omega t][\omega x_0 \cos(\omega t - \phi)]dt$$

$$= \pi p_0 x_0 \sin \phi$$

$$= 2\pi\xi \frac{\omega}{\omega 0} \; k x_0 2$$

So, we can substitute that and we would get sin phi is equal to 2 zeta omega by omega n x naught by p naught by k. So, this we can substitute in the expression for energy and you will get the expression for energy as 2 pi zeta omega by omega naught k x naught square. Now, let us see how much is the potential and kinetic energy in one cycle of the steady state motion.

$$\sin \phi = \frac{2\xi(\omega/\omega n)}{\sqrt{[1-(\omega/\omega n)2]2 + [2\xi(\omega/\omega n)]2}}$$

$$= 2\xi(\omega/\omega n)\frac{x0}{p0/k}$$

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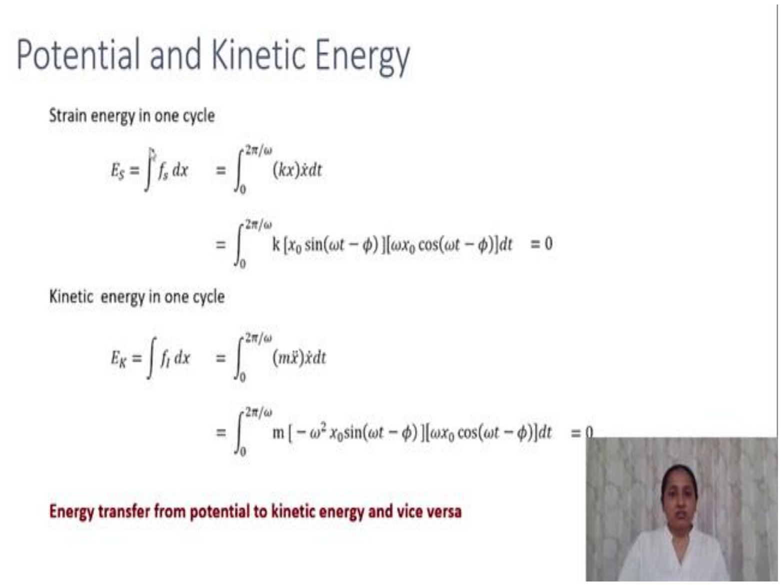

So, potential energy is equal to the strain energy of the system. So, that is equal to integral fs dx fs is the spring force. We know that this fs is equal to k times x where, k is the stiffness of the spring and x is the displacement and x dot is the velocity dx is equal to x dot dt. So, we can integrate it over one cycle that is from 0 to 2 pi by omega.

$$E_S = \int f_s \, dx = \int_0^{2\pi/\omega} (kx)\dot{x} \, dt$$

$$= \int_0^{2\pi/\omega} \mathrm{k}\,[x_0 \sin(\omega t - \phi)][\omega x_0 \cos(\omega t - \phi)] dt \quad = 0$$

So, we can substitute the value of x that is the displacement and the velocity and we will get this expression, we can integrate it to get the total strain energy in one cycle. As you can see this expression is a product of a sin and a cos. So, if you multiply the sin function and this cos function and integrate it over one cycle it would become 0. So, the total strain energy in one cycle of the steady state motion is equal to 0.

So, now let us calculate the kinetic energy in one cycle. So, the kinetic energy would be integral the inertia force dx. And, we know that this inertia force is equal to mass times acceleration and dx is x dot dt. If, we substitute the value of acceleration and velocity we would get this this expression for acceleration, we would get if we differentiate this expression for displacement twice and this is the velocity.

$$E_K = \int f_I \, dx \quad = \int_0^{2\pi/\omega} (m\ddot{x})\dot{x} \, dt$$

$$= \int_0^{2\pi/\omega} \mathrm{m}\,[-\omega^2 x_0 \sin(\omega t - \phi)][\omega x_0 \cos(\omega t - \phi)] dt \quad = 0$$

And, this is again product of sin and cos terms. So, if you integrate this over a cycle this integral will become equal to 0. So, the total strain energy and the total kinetic energy in one cycle of the steady state motion is equal to 0. So, this means energy is transferring from potential and kinetic energy during the vibration, but the total energy in a cycle is 0.

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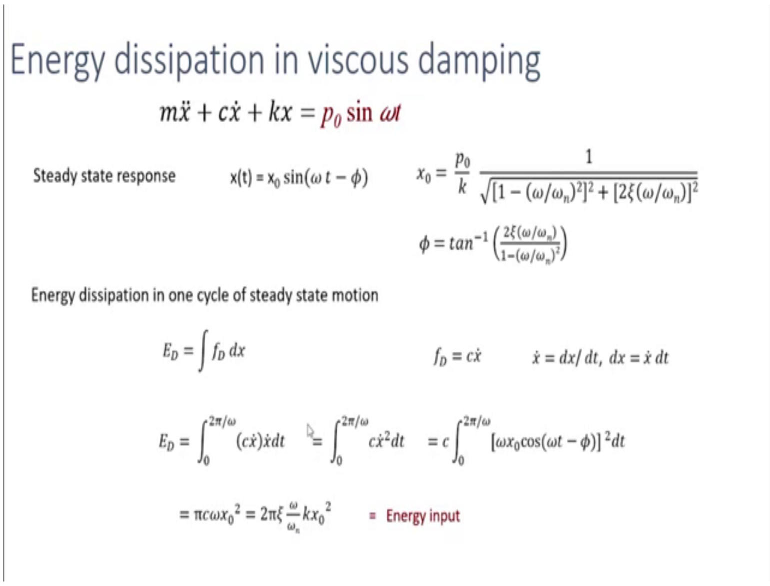

Now, let us see the energy dissipated because of damping. So, if we consider discuss damping, this is the equation of motion and we know the expression for steady state response.

$$m\ddot{x} + c\dot{x} + kx = p_0 \sin \omega t$$

$$x(t) = x_0 \sin(\omega t - \phi) \qquad x0 = \frac{p0}{k} \frac{1}{\sqrt{[1-(\omega/\omega n)2]2+[2\xi(\omega/\omega n)]2}}$$

$$\phi = tan^{-1}\left(\frac{2\xi(\omega/\omega n)}{1-(\omega/\omega n)2}\right)$$

The energy dissipation in one cycle due to this viscous damping is equal to integral f D dx. And, f D is the damping force and that is equal to c that is the damping coefficient multiplied by the velocity. And, so, this energy dissipated due to damping is equal to integral 0 to 2 pi by omega c x dot x dot dt.

$$E_D = \int f_D\, dx \qquad f_D = c\dot{x} \qquad \dot{x} = dx/\,dt,\ dx = \dot{x}\, dt$$

So, we can substitute the expression for the velocity and we would get this and if we integrate this we would get pi c omega x naught square. And, we know that the coefficient of damping c can be expressed in terms of the damping ratio zeta and so, if you substitute the value of c in terms of zeta you would get this expression 2 pi zeta omega by omega n k x naught square. And, this is equal to the energy input due to this harmonic force. So, this is the expression of the energy input we have derived earlier.

$$E_D = \int_0^{2\pi/\omega} (c\dot{x})\dot{x}\,dt = \int_0^{2\pi/\omega} c\dot{x}^2 dt = c\int_0^{2\pi/\omega} [\omega x_0 \cos(\omega t - \phi)]^2 dt$$

$$= \pi c \omega {x_0}^2 = 2\pi\xi \frac{\omega}{\omega n} k {x_0}^2 = \text{Energy input}$$

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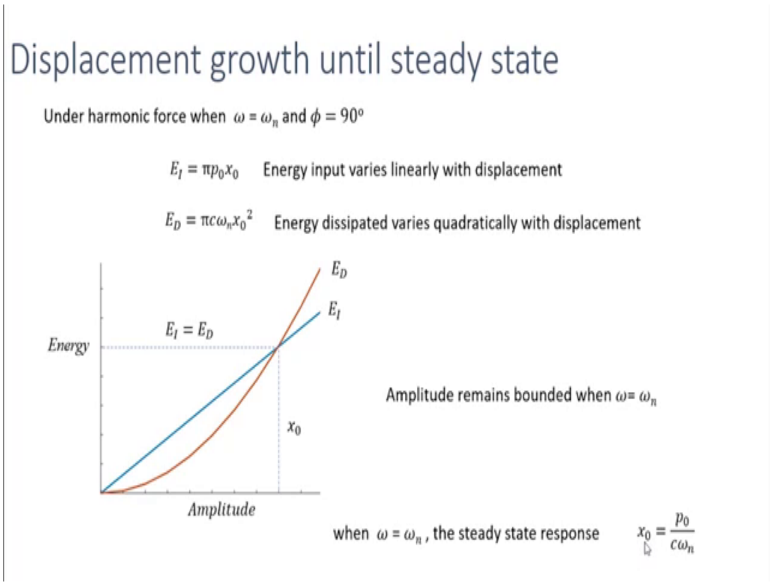

Now, based on the energy input and the energy dissipation, we will try to understand the displacement growth of a single degree of freedom system, when the forcing frequency of the harmonic force is equal to the natural frequency. That is when omega is equal to omega n and phi is equal to 90 degree. So, we have learnt earlier that for an undamped system, when omega is equal to omega n the displacement will keep on increase with time that is the displacement will increase unboundedly. But, in damped systems, because of the damping present in the system at omega is equal to omega n the displacement will not increase unboundedly.

The displacement will grow until a steady state and after that the displacement amplitude is constant throughout the time. So, now, let us understand how damping is helping to have a bounded response at omega is equal to omega n. So, the input energy in the system, when omega is equal to omega n and phi is equal to 90 degree is equal to this pi p naught x naught where x naught is the steady state amplitude of the system. So, from this expression we know that this energy input varies linearly with the displacement and this is the dissipated energy.

So, when omega is equal to omega n the dissipated energy will be equal to this and we can see that the energy dissipated is proportional to x naught square; that means, this varies quadratically with the displacement. Now, let us plot these two expressions. So, E I varies linearly with x naught. So, this is the displacement amplitude in the x axis and we have energy in the y axis. So, this E I is a linear and the energy dissipated is it varies quadratically. So, this is the curve for the red one is the curve for the dissipated energy.

So, when the amplitude is very low the dissipated energy will also be very low, this increases quadratically. So, when the amplitude is low the input energy will be higher than the dissipated energy. So, the energy in the system is high. So, it will vibrate more so; that means, the displacement will grow. As the displacement grows the dissipated energy keeps on increasing quadratically, the rate of increase in the dissipated energy is more than, the rate of increase in the input energy.

So, at amplitude is equal to x naught that is the steady state amplitude both the energies become equal. So, at steady state, input energy is equal to the dissipated energy. So, if the amplitude increases beyond x naught, that is beyond the steady state, then the dissipated energy would have been more than the input energy. So, if the displacement increases beyond x naught more energy will be dissipated so; that means, the amplitude will come down.

So, when the amplitude goes below the steady state again the input energy will be more. So, that amplitude will tend to increase. So, the amplitude of the damped system will be bounded at this steady state. So, because of the damping present in the system, the amplitude of the damp system remains bounded when omega is equal to omega n. It would not keep on increasing, because after the steady state, if the amplitude becomes more than the steady state, the dissipated energy will be more than the input energy, that will force the amplitude to reduce. And, when omega is equal to omega n the steady state response can be written as p naught by c omega n.

$$x_0 = \frac{p_0}{c\omega_n}$$

So, this we can find out from the expression of the amp steady state amplitude we have calculated earlier.

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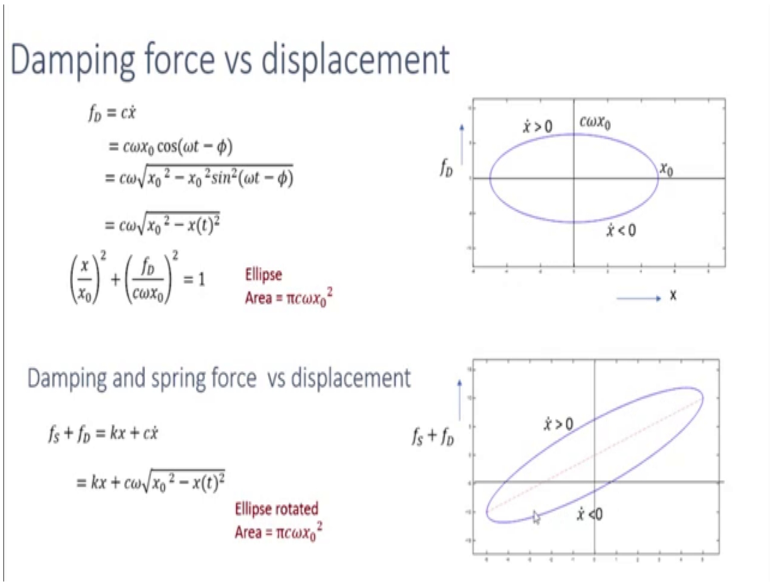

Now, let us understand, how the damping force is related to the displacement. So, we know that the damping force is equal to c x dot that is damping coefficient times velocity. So, we can substitute the steady state velocity and this cos function can be written in terms of sin function. So, we can write this like this. So, this is the displacement at any time t. So, that is x t square.

$$f_D = c\dot{x}$$

$$= c\omega x_0 \cos(\omega t - \phi)$$

$$= c\omega\sqrt{x_0{}^2 - x_0{}^2 sin^2(\omega t - \phi)}$$

$$= c\omega\sqrt{x_0{}^2 - x(t)^2}$$

So, the damping force is equal to c omega square root of x naught square minus x t square, x naught is the steady state amplitude, displacement amplitude and x t is the displacement at any time t. So, if you square this expression, we can rearrange it in this fashion.

$$\left(\frac{x}{x_0}\right)^2 + \left(\frac{f_D}{c\omega x_0}\right)^2 = 1$$

So, this is an equation of an ellipse with it is major axis as x naught and minor axis as c omega x naught. And, the area of this ellipse is equal to pi times major axis and minor axis. So, pi c omega x naught square is the area of the ellipse.

So, let us plot the ellipse. So, this is how the function the damping force looks like. So, x axis is the displacement and y axis is the damping force. So, this damping force is a double valued function. So, for each value of x this will have 2 values, when the velocity is positive the damping force is positive and when the velocity is negative the damping force is also negative. Now, let us see how damping and the spring force varies with displacement. So, we know that the spring force is equal to k x and the damping force is same as c x dot.

$$f_S + f_D = kx + c\dot{x}$$

$$= kx + c\omega\sqrt{{x_0}^2 - x(t)^2}$$

So, the c x dot can be represented like this. So, this represents again an ellipse, but it is rotated and the rotation is determined by the term k x. So, this fs plus f D is represented by this blue ellipse. So, this is also an ellipse, but it is rotated. And, the area inside the ellipse will be same as the previous ellipse; because this is only get rotated. And, the area enclosed by this this curve that is this line k x is 0 because this is just a line.

So, the area enclosed by that is 0 and the area inside this ellipses same as the previous ellipse. This force displacement curve is also known as hysteresis loop and the area of this loop will be proportional to the forcing frequency. So, if the forcing frequency is high the area enclosed, that is the energy dissipated will also be high.

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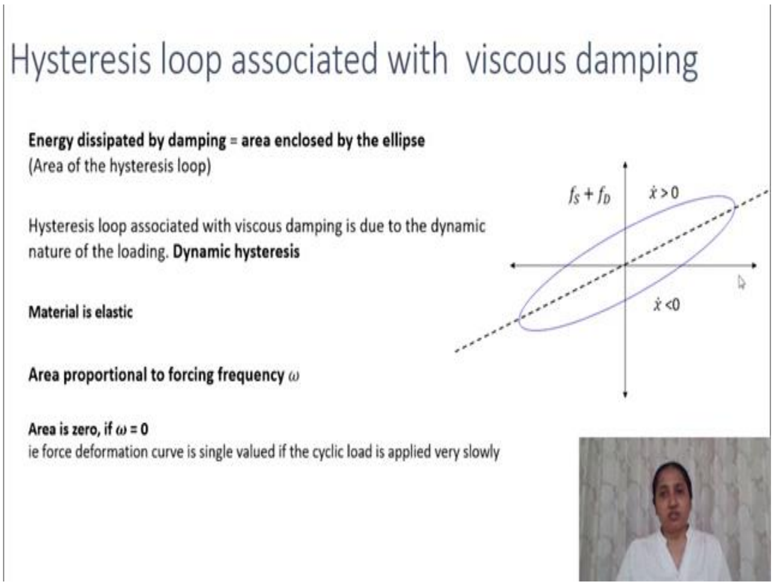

Now, let us look at some properties of this hysteresis loop associated with viscous damping. So, this is the loop we just saw energy dissipated by the damping force that is the viscous damping is equal to the area inside this ellipse or the area inside this hysteresis loop. And, the hysteresis loop associated with viscous damping is due to the dynamic nature of the loading and this is called dynamic hysteresis.

So, this particular force deformation behavior is due to the dynamic nature of the loading. And, this hysteresis should not be confused with the hysteresis when a cyclic load is acting on an inelastic material. So, when a cyclic load is acting on an inelastic element, then also we get some hysteresis loops and the shape of those loops will be different from this shape, but that will also dissipate some energy, but that will be due to plastic deformation. In this case in the case of dynamic hysteresis the material is elastic.

So, even when the material is elastic some energy will be dissipated because of the damping action. And, this is also called hysteresis loop and this is due to dynamic hysteresis. And, the area inside this loop is proportional to the forcing frequency. So, if the forcing frequency is 0 the area will be 0; that means, this force deformation curve will be a single valued function. So, when will the forcing frequency is 0? So, forcing frequency is 0 when the force acting on the system is static.

So, at that time the force would not be varying with time, so this frequency is 0. So, if a static load is acting on the system, then we will not have this dynamic hysteresis happening. So, at that time the force deformation relation will be a single valued function.

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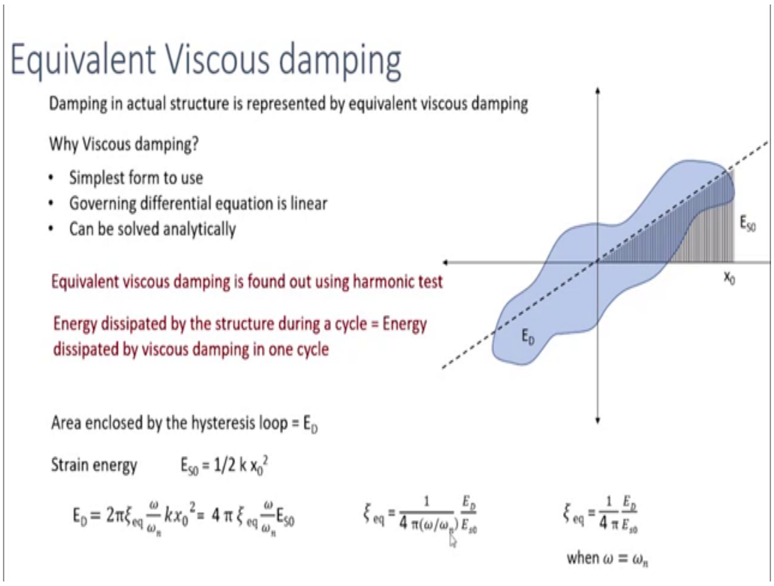

Now, let us use this dynamic hysteresis loop to calculate the equivalent viscous damping in a system. So, in the beginning we discussed that the damping in an actual structure is due to many different energy dissipation mechanisms. And, these mechanisms can be represented by an equivalent viscous damping.

So, why do we choose viscous damping? Viscous damping is the simplest form of damping which can be used and the governing differential equation, if we use viscous damping is linear. So, it is easy to be handled. So, since this is a linear equation we can solve it analytically. So, the equivalent viscous damping is found out using harmonic test, we have already learned that using a harmonic test, we can find the half power bandwidth and using the half power bandwidth we can calculate the damping ratio.

So, now we will find out how to calculate the equivalent damping using the energy dissipated. So, the energy dissipated by the structure during a cycle will be equal to the energy dissipated by the viscous damping in the cycle. So, we can equate these two and find out an equivalent damping ratio. So, we can do a harmonic test, we can excite the system using harmonic forces with different forcing frequency and we can find this dynamic hysteresis loop then we can measure its area.

So, that will give us the energy dissipated. And, the strain energy of that system will be equal to half k x square k x is the stiffness of the spring and x naught is the amplitude. So, this shaded portion indicates the strain energy and this curve the area of this, area inside this curve indicates the energy dissipated. So, we can equate this energy dissipated to this expression, which we have derived earlier and we can find the zeta equivalent that is the viscous damping ratio equivalent to this amount of energy dissipated. So, this energy dissipated would be due to many different energy dissipation mechanisms, but we can represent it equivalently by using a viscous damping system.

$$E_{S0} = 1/2\ k\ x_0^2$$

$E_D = 2\pi\xi_{eq}\frac{\omega}{\omega n}k{x_0}^2 = 4\ \pi\ \xi_{eq}\frac{\omega}{\omega n}E_{S0}$ $\quad \xi_{eq} = \frac{1}{4\ \pi(\omega/\omega n)}\frac{ED}{Es0}$ $\quad \xi_{eq} = \frac{1}{4\ \pi}\frac{ED}{Es0}$ when $\omega = \omega n$

So, if this expression can be written in terms of strain energy. So, that would be equal to 4 pi zeta equivalent omega by omega n strain energy. So, from this we can calculate the zeta equivalent that is damping ratio equivalent as 1 by 4 pi omega by omega n multiplied by E D that is the area of this enclosed curve divided by the strain energy. So, if we do the harmonic test at resonance, then the forcing frequency is equal to the natural frequency, then we can calculate the damping ratio using this expression, that is when omega is equal to omega n.

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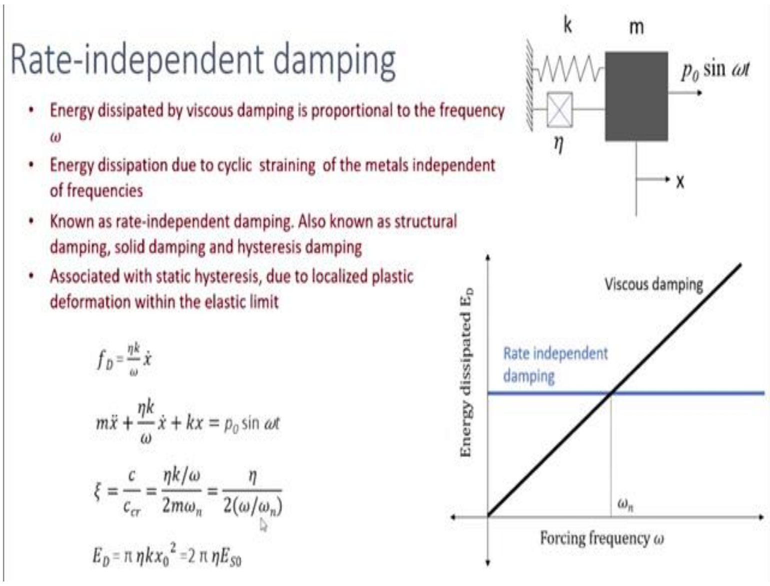

Now, let us see some examples of non-viscous damping. So, the first one we are considering is rate independent damping. So, we have learnt that energy dissipated by a viscous damper is proportional to the forcing frequency omega, but some experiments on metallic structures showed that, the energy dissipation due to cyclic straining of a metal is independent of the forcing frequencies.

So, this type of energy dissipation or this type of damping is known as a rate independent damping, because it is independent of the frequencies. And, this damping is also known as structural damping, solid damping, and hysteresis damping. And, this damping is due to localized plastic deformation in the material within the elastic limit. So, this is a static hysteresis. So, at very localized positions plastic deformation is happening and because of that some energy is getting dissipated and that energy dissipation is independent of the forcing frequency.

So, because of this mechanism, the damping is called rate independent damping. So, the damping force in this case is modeled as eta k by omega, where omega is the forcing frequency and it is proportional to the velocity of the system single degree of freedom system. So, this eta is the measure of the damping in this system. This shows the energy dissipation with respect to the forcing frequency in two different cases of damping. In viscous damping as we have learnt earlier the energy dissipation depends on the forcing frequency, but in the case of rate independent damping this energy dissipation does not depend on the forcing frequency, but these two energy dissipation becomes equal at omega is equal to omega n that is the forcing frequency is equal to the natural frequency.

$$fD = \frac{\eta k}{\omega}\dot{x}$$

$$m\ddot{x} + \frac{\eta k}{\omega}\dot{x} + kx = p_0 \sin \omega t$$

So, this is the equation of motion when we consider rate independent damping. So, the viscous damping coefficient c is replaced by eta k by omega where omega is the forcing frequency. So, we know that zeta the damping ratio is equal to c by c cr where c is the viscous damping coefficient. So, here in this case we can replace the value of c as eta k by omega. So, we would get zeta is equal to eta by 2 multiplied by the frequency ratio. The energy dissipated due to this rate independent damping can be calculated by replacing the value of zeta in the expression for E D in the case of viscous damping.

$$\xi = \frac{c}{ccr} = \frac{\eta k/\omega}{2m\omega n} = \frac{\eta}{2(\omega/\omega n)}$$

$$ED = \pi\, \eta k {x_0}^2 = 2\, \pi\, \eta ES0$$

So, we have derived the expression of energy dissipated during viscous damping. So, in that expression we just need to replace the value of zeta by this. So, if you do that we would get the energy dissipated in independent damping as pi eta k x naught square, x naught is the steady state displacement amplitude. So, this can be again represented in terms of the strain energy in a cycle. So, that would be 2 pi eta E S naught. So, E S naught is equal to half k x naught square; so, this is the expression of the energy dissipated due to rate independent damping.

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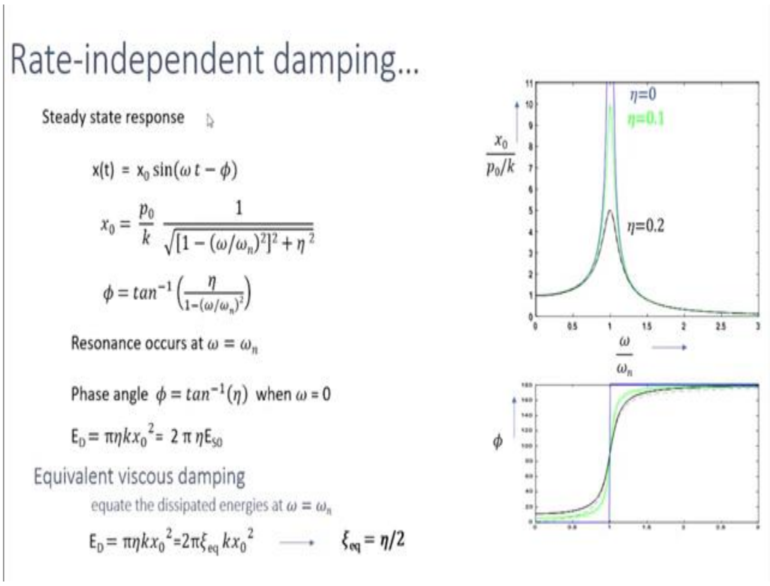

So, let us find out the steady state response. So, this is the expression for steady state response, we need to find the amplitude and the phase angle. So, the amplitude and the phase angle can be calculated again by replacing the value of zeta in the expression for viscous damping response. So, this is the amplitude of the steady state and it can be obtained like p 0 by k. So, this is the static response due to a constant force p naught.

$$x(t) = x_0 \sin(\omega t - \phi)$$

$$x0 = \frac{p0}{k} \frac{1}{\sqrt{[1-(\omega/\omega n)2]2 + \eta\, 2}}$$

$$\phi = tan^{-1}\left(\frac{\eta}{1-(\omega/\omega n)2}\right)$$

So, this is the static force and this is the deformation amplitude factor. And, in the case of rate independent damping this will become 1 by square root of 1 minus frequency ratio square the whole square plus eta square. So, in the case of viscous damping if you remember instead of eta it was 2 zeta frequency ratio. Similarly, we can calculate the value of phi by substituting zeta in terms of eta and we would get it as tan inverse eta by 1 minus frequency ratio square. And, here if phi, look at it, the maximum response that is resonance occurs when omega is equal to omega n.

So, when this omega is equal to omega n we get the maximum response. So, in the case of viscous damping the resonance used to occur when omega was slightly less than the natural frequency, but in the case of rate independent damping resonance occurs when omega is equal to the natural frequency. And, the phase angle used to be 0 when omega was 0 in the case of viscous damping, but in the case of rate independent damping if omega is equal to 0 then this phase angle phi will become tan inverse eta. So, these are the two differences comparing to the viscous damping.

So, now let us see how the deformation response factor looks, when we use this rate independent damping? So, this is the deformation response factor that is the steady state amplitude divided by the static amplitude x naught by p naught by k and this is how the phase angle varies. So, I have plotted the values for 3 different values of eta; one is when there is no damping in the system that is eta is equal to 0. So, in that case when omega is equal to omega n the response grows unboundedly.

So, it is very large and when eta is equal to 0.1 and 0.2 the amplification at omega by omega n is finite and the amplitude reduces as the damping increases. This is similar when we considered viscous damping also. As the damping increases the amplitude decreases and this is how the phi angle the phase angle changes. So, when there is no damping in the system, the fee angle is this it follows this curve.

So, if omega is less than omega n the phi angle is 0 and if omega is greater than omega n the phase angle is 180. The variation of phase angle with respect to the frequency ratio changes as the damping increases. So, the green curve is for eta is equal to 0.1 and this black one is for eta is equal to 0.2. So, as we have discussed now when the forcing frequency is 0 the phase angle is not equal to 0, but it is equal to the tan inverse eta value.

So, here when omega is 0 we have nonzero value of phase angle, this is the property of this rate independent damping. So, in the previous slide we have seen the energy dissipated and that is equal to pi eta k x naught square. So, using this expression we can calculate the equivalent viscous damping. So, what is the equivalent viscous damping value corresponding to the rate independent damping? So, we can represent this rate independent damping in terms of viscous damping.

$$E_D = \pi \eta k {x_0}^2 = 2\,\pi\,\eta E_{S0}$$

So, to do that we have to equate the dissipated energies at omega is equal to omega n. This is the expression for the energy dissipated when omega is equal to omega n for rate independent damping. So, we can equate this to the expression of energy dissipation due to viscous damping. So, that is 2 pi zeta k x naught square. So, if you equate these 2 we can find out the equivalent value of zeta that is the equivalent viscous damping corresponding to the rate independent damping.

$$E_D = \pi \eta k {x_0}^2 = 2\pi \xi_{eq}\, k {x_0}^2 \qquad \xi_{eq} = \boldsymbol{\eta/2}$$

So, if we solve this we get the zeta equivalent is equal to eta by 2; that means, the effect or the energy dissipation due to rate independent damping can be represented by viscous damping, if we consider an equivalent viscous damping value and this equivalent viscous damping ratio is given by eta by 2. And, now we will see how this deformation response factor and the phase angle changes when we use this equivalent viscous damping. So, we have plotted the values of the deformation response factor and the phase angle using dotted lines. So, as you can see here both the curves are very close; that means approximation in using this equivalent viscous damping is very accurate.

So, in this figure the dotted line, which is representing the response due to the equivalent viscous damping at a frequency, which is slightly lower than the natural frequency. And, when we are considering equivalent viscous damping we get phi angle is equal to 0, when omega is 0, but in the case of rate independent damping at omega is equal to 0 the phase angle will be tan inverse eta. So, we will get a non-zero value, but other than that the phase angle is approximate when we use this equivalent viscous damping.

So, in the case of rate independent damping we can use viscous damping to represent this energy dissipation.

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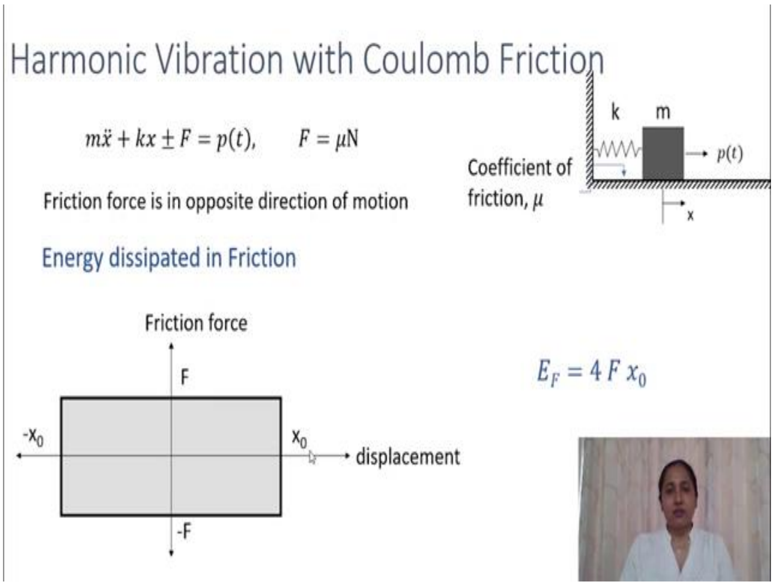

Now, let us see another example of non-viscous damping. So, this is called as coulomb friction damping and it is due to the friction between two surfaces. So, when this mass moves over the surface because of the friction there will be some energy dissipation, we have seen this type of damping during the free vibration case. So, in the case of harmonic vibration the equation of motion as this, m x double dot plus k x plus or minus F is equal to p t the harmonic force.

$$m\ddot{x} + kx \pm F = p(t), \quad F = \mu \mathrm{N}$$

Here, we know that this F is the frictional force and that is equal to the coefficient of friction times the normal force acting on this surface. So, in this case this could be the weight of this mass and the friction force is in the opposite direction of the motion of this mass. So, if the mass is moving towards right, the force will be acting towards left and when the mass is moving towards left the force will be in the right direction.

So, because of that we have this plus or minus sign in this equation of motion. So, in each half cycle this sign will change. So, we had solved this equation for the free vibration case. So, now, we will not be solving this, but we will just examine some of the results. So, the energy dissipated in this friction is represented by this force displacement curve. So, as we discuss now when this mass is moving towards the right, the force will be acting in the opposite direction. So, if we consider this x to be positive when it is towards right. So, when the mass moves from right to left then the force will be acting towards right

So, when the displacement changes from plus X naught to minus X naught the frictional force will be positive F and when it goes right wards when the mass goes right wards; that means, when the mass moves from minus X naught to X naught the friction force will be in the opposite direction that is minus F. So, this is the force displacement curve due to this friction damping. So, the energy dissipated due to this friction will be equal to the area enclosed in this curve and that will be 4 times F x naught. So, this is the energy dissipated due to friction.

$$EF = 4\,F\,x0$$

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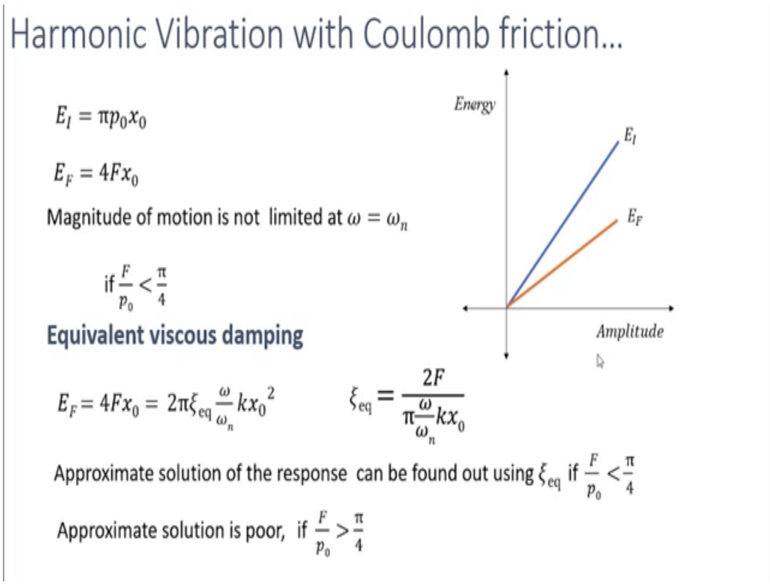

So, we have seen that the input energy to the single degree of freedom system, due to this harmonic force is equal to pi p naught x naught, when omega is equal to omega n that is when the forcing frequency is equal to natural frequency. And, x naught is the steady state amplitude and we also saw that the energy dissipated by friction is equal to 4 F times x naught.

$$E_I = \pi p_0 x_0$$

$$EF = 4Fx0$$

So, now let us plot this expression. So, this is the energy amplitude plot and this input energy is linearly proportional to this amplitude and the energy dissipated due to friction is also linearly proportional to the amplitude. So, both these curves are linear and if the frictional force divided by p naught is less than pi by 4 then the input energy will be higher than the energy dissipated due to friction this is when omega is equal to omega n.

So, because of this at omega is equal to omega n when this condition is satisfied the magnitude of the motion will be unbounded this will not be limited, because the input energy is more than the energy dissipated. So, if this F by p naught is higher than pi by 4 in that case the motion of the single degree of freedom system will be bounded. So, the response at omega is equal to omega n will be limited if this value is greater than pi by 4, otherwise it will be unbounded.

So, now let us calculate the equivalent viscous damping in the case of coulomb friction. So, we can equate the energy dissipated due to friction to the energy dissipated due to viscous damping. So, if we do that we would get zeta equivalent is equal to 2 F by pi frequency ratio k x naught. We can use this equivalent viscous damping to calculate the solution of this coulomb friction dam system approximately. And, this approximate solution is accurate only if this F by p naught ratio is less than pi by 4.

$$E_F = 4Fx0 = 2\pi\xi_{eq}\frac{\omega}{\omega n}k{x_0}^2 \qquad \xi_{eq} = \frac{2F}{\pi\frac{\omega}{\omega n}kx0}$$

So, if this friction in the system is large and if this F by p naught is higher than pi by 4 then this approximate solution calculated using this equivalent viscous damping will be over. So, this gives an accurate result when the damping in the system is low.