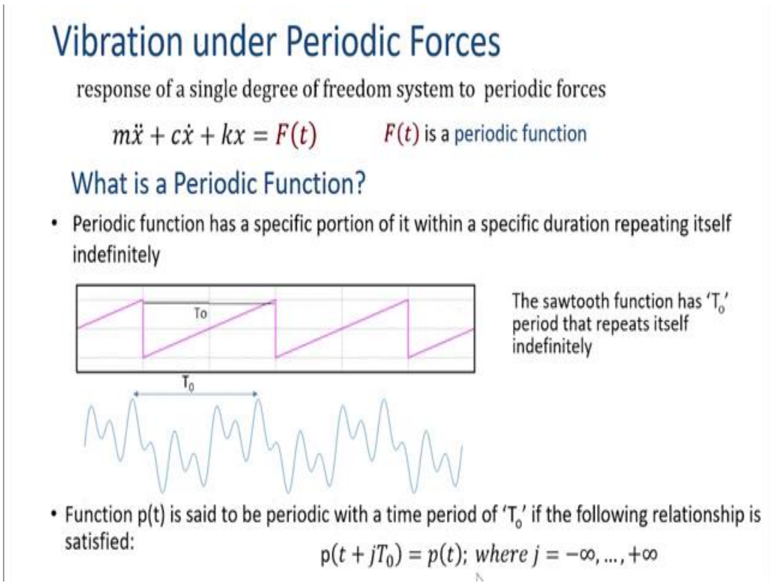
Structural Dynamics for Civil Engineers – SDOF Systems
Dr. Riya Catherine George
Department of Civil Engineering
Hiroshima University, Japan
Indian Institute of Technology, Kanpur

Lecture – 10
Vibration under Periodic Forces

(Refer Slide Time: 00:13)

Now, let us look into Vibration of single degree of freedom systems under Periodic Forces. We have already learnt the vibration under harmonic forces; harmonic force is a special type of periodic force. So, now we will look into the vibrations under any periodic force.

(Refer Slide Time: 00:38)

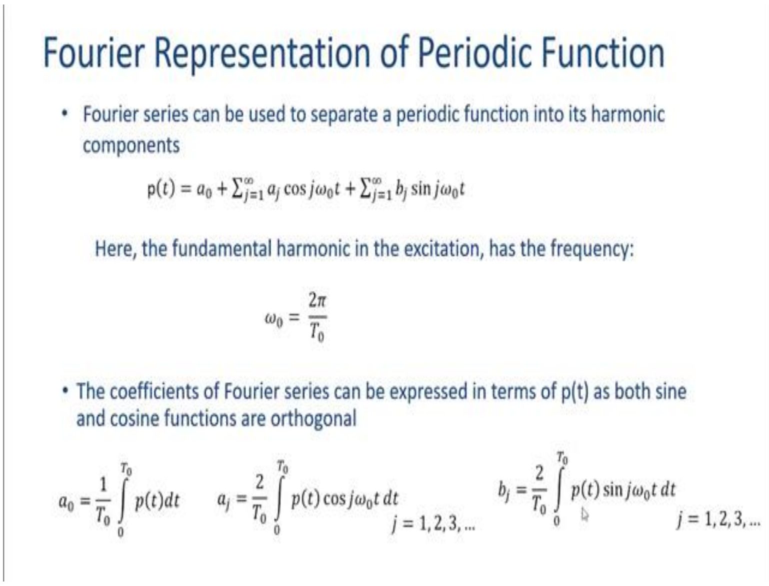


So, in this case the single degree of freedom system is acted upon by a force which is a periodic function in time. So, what is a periodic function? We can say the periodic function has a specific portion of it, within a specific duration repeating itself indefinitely. So, we will understand this using an example. So, this is an example of a periodic function, this function is called sawtooth function and this has a period T naught. So, after the duration of T naught the function repeats itself. So, this much specific portion of this function gets repeated indefinitely.

So, let us see one more example. So, this is also a periodic function and this period is T naught and this portion of this function gets repeated indefinitely. So, the function p t is said to be periodic with the time period of T naught if p t the value of the function at t is equal to p t plus j T naught that is p t is equal to the value of the function after multiples of T naught. So, p t is equal to p t plus j T naught where j ranges from minus infinity to plus infinity. So, this function continues indefinitely. So, now, let us see how a periodic function can be represented in terms of harmonic components.

$$p(t + jT_0) = p(t);\ where\ j = -\infty,\ \ldots,\ +\infty$$

(Refer Slide Time: 02:26)



So, we can do that using Fourier representation. So, any periodic function p t can be represented like this using Fourier representation. So, here a naught aj and bj are constants and we have cosine and sin functions as well. So, this periodic function can be split into multiple harmonic functions that is functions of cos and sins. The series has infinite number of terms and as the value of j increases from 1 to infinity, that particular harmonic will have frequency equal to j times omega naught. Both these harmonics will have same frequency equal to j omega naught.

$$p(t) = a_0 + \sum_{j=1}^{\infty} a_j \cos j\omega_0 t + \sum_{j=1}^{\infty} b_j \sin j\omega_0 t$$

And the fundamental harmonic in this excitation has the frequency omega naught that is when j is equal to 1. So, omega naught has equal to 2 pi by T naught where T naught is the period of the periodic force. And, the coefficients of this Fourier series that is a naught aj and bj can be expressed in terms of the periodic function itself that is p t; a naught the first coefficient is expressed as this this 1 by T naught integral 0 to T naught p t dt so; that means, this is the average value of the periodic function.

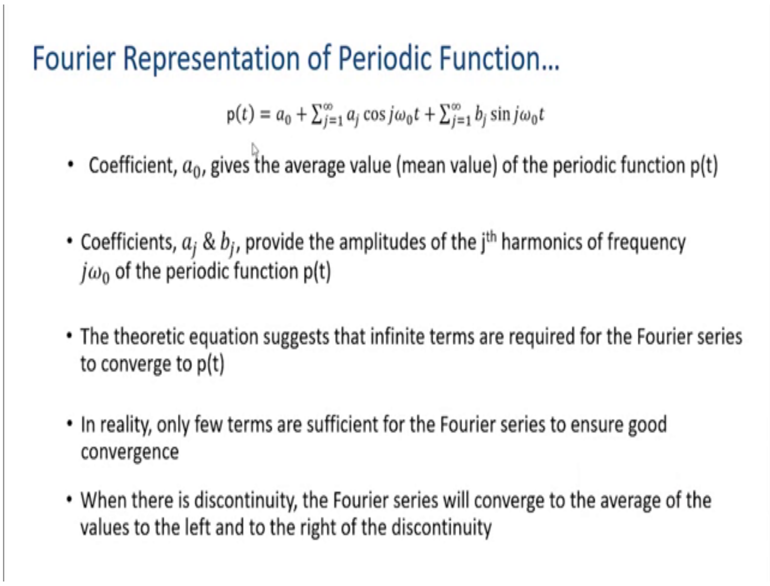
$$a_0 = \frac{1}{T_0}\int_0^{T_0} p(t)dt \qquad a_j = \frac{2}{T_0}\int_0^{T_0} p(t)\cos j\omega_0 t\, dt \qquad b_j = \frac{2}{T_0}\int_0^{T_0} p(t)\sin j\omega_0 t\, dt$$

$$j = 1,\ 2,\ 3,\ \ldots$$

So, if you do this we will get the average value of this function p t. So, that is equivalent to the first coefficient and aj that is the coefficient of the cosine terms are calculated like this 2 by T naught integral 0 to T naught p t cos j omega naught t dt. So, to find a j we have to integrate p t multiplied by the cos harmonic and it can be found out for all the values of j, j ranging from one to infinity.

Similarly, the coefficient of the sin terms can be calculated as 2 by T naught integral 0 to T naught p t multiplied by sin j omega naught t dt. This also can be found out for all the values of j. So, if you can calculate a naught aj and bj we can split the function p t in terms of harmonic components.

(Refer Slide Time: 05:28)

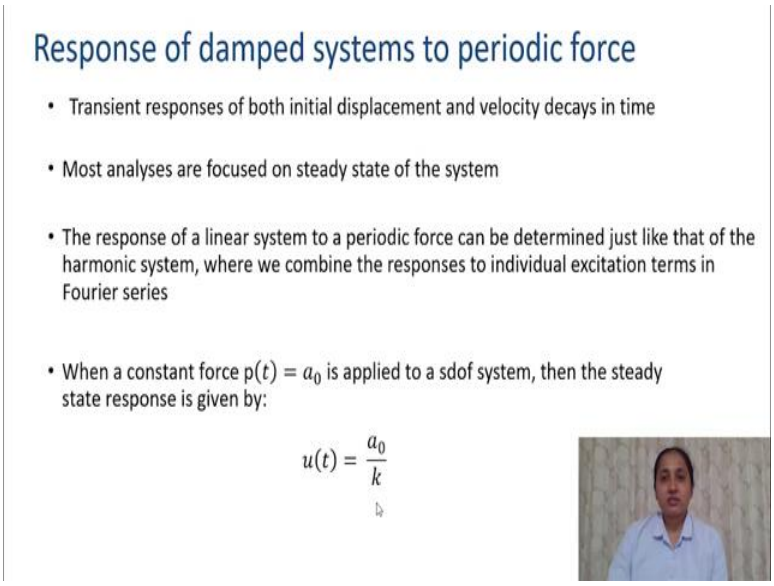


We have seen that the coefficient a naught indicates the average value of p t and the coefficients aj and bj indicate the amplitudes of the j th harmonics and the j th harmonic will have frequency equal to j times omega naught and omega naught is the frequency of the fundamental harmonic that is equal to 2 pi by T naught. Theoretically this series has infinite number of terms; that means, we need infinite terms to represent this periodic function in terms of harmonics, but in reality only a few terms are sufficient for this Fourier series.

$$p(t) = a_0 + \sum_{j=1}^{\infty} a_j \cos j\omega_0 t + \sum_{j=1}^{\infty} b_j \sin j\omega_0 t$$

So, with very few terms this series will converge to the periodic function. If this function has any discontinuity at the discontinuity this Fourier series will converge to an average value that is the average of the values to the left and to the right of the discontinuity that is average of the neighboring values of the discontinuity.

(Refer Slide Time: 06:56)

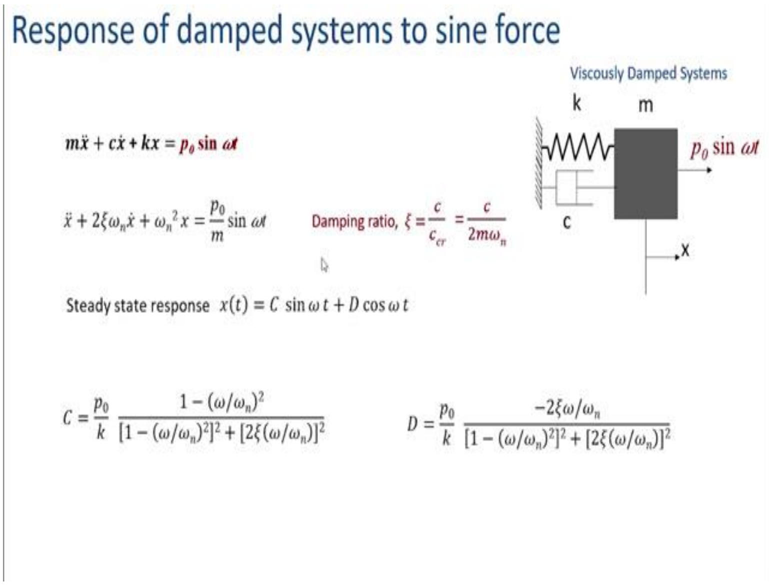


Now, let us find the response of damped systems to periodic force. We have already learnt the response of damped systems under harmonic forces and we learned that transient responses due to initial displacement and velocity decays in time. So, most of the analysis are focused on steady state of the system. So, after some time the transient responses will die out and only the steady state response will be prevalent and the steady state response will be present as long as the force exists. So, we can focus on the steady state response of this system.

Now, the response of a linear system to a periodic force can be determined just like that of a harmonic system and here we combine the responses to individual excitation terms in Fourier series. So, we just have seen that the periodic function can be separated into different harmonics using Fourier series. So, the response of the periodic force can also be treated as the sum of the responses of the individual harmonic forces.

So, the response to a periodic force is equal to sum of the responses to many harmonic forces. So, when a constant force is applied to a single degree of freedom system, the steady state response is given by a naught by k. So, this is like a static response, the force is constant and we are looking at the steady state response. So, this is equal to the static response of the system that is a naught by k.

(Refer Slide Time: 08:56)



Now, let us see: what is the response of damped systems to sin force. This this is exactly what we have seen during harmonic vibrations, this is the equation of motion of a single degree of freedom systems acted upon by a sin force the forces p naught sin omega t.

$$m\ddot{x} + c\dot{x} + kx = p_0 \sin \omega$$

$$\ddot{x} + 2\xi\omega n\dot{x} + \omega n2\, x = \frac{p0}{m}\sin \omega t \qquad \text{Damping ratio, } \xi = \frac{c}{ccr} = \frac{c}{2m\omega n}$$

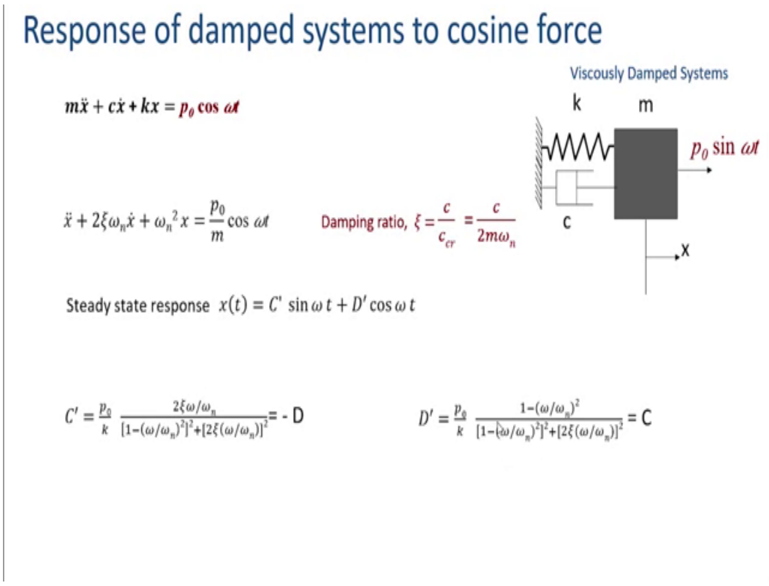
So, we have derived the response of this system. So, we can rewrite this equation like this, where damping ratio is defined as c by c cr and the critical damping coefficient is 2 m omega n.

And, the steady state response of this system is of the form C sin omega t plus D cos omega t, where this omega is equivalent to the forcing frequency. So, and we had derived this coefficients C and D and the coefficients are given by this. So, if you can calculate these coefficients, we can find the steady state response of this system under sin force as this.

$$x(t) = C \sin \omega t + D \cos \omega t$$

$$C = \frac{p0}{k} \frac{1-(\omega/\omega n)2}{[1-(\omega/\omega n)2]2+[2\xi(\omega/\omega n)]2} \qquad D = \frac{p0}{k} \frac{-2\xi\omega/\omega n}{[1-(\omega/\omega n)2]2+[2\xi(\omega/\omega n)]2}$$

(Refer Slide Time: 10:19)

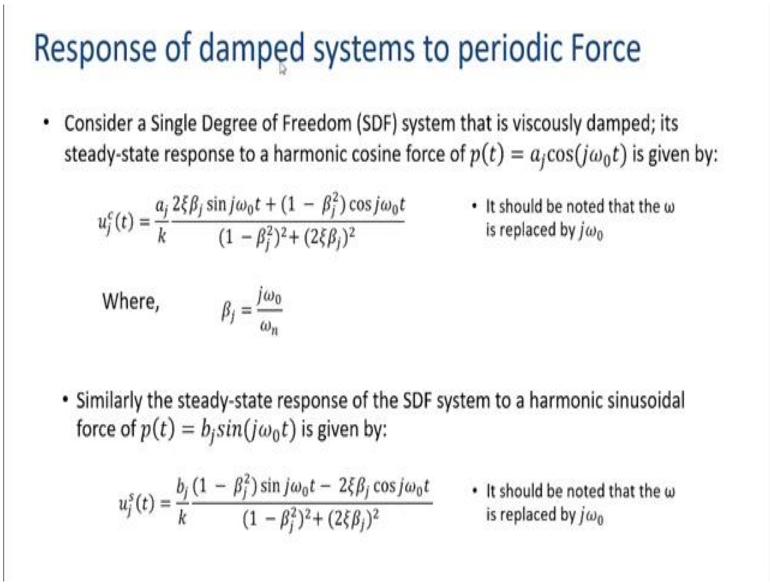


So, now let us look at the response of a damped system under cosine force. If we do a similar derivation as in the case of sin force, we can find the response due to a cosine force. So, we can find the steady state response due to the cosine force is equal to C prime sin omega t D prime cos omega t and C prime and D prime can be calculated in the similar way we did for sin force. So, if you do the derivation we would get the values of C prime and D prime as these and the C prime is equal to minus D. D is the coefficient of the cos omega t term in the steady state response when the acting forces a sin force and we can also calculate D prime as this and that is equal to C.

$$x(t) = C' \sin \omega t + D' \cos \omega t$$

$$C' = \frac{p0}{k} \frac{2\xi\omega/\omega n}{[1-(\omega/\omega n)2]2+[2\xi(\omega/\omega n)]2} = -D \qquad D' = \frac{p0}{k} \frac{1-(\omega/\omega n)2}{[1-(\omega/\omega n)2]2+[2\xi(\omega/\omega n)]2} = C$$

(Refer Slide Time: 11:19)



So, now we can come back to the Fourier representation of the periodic force and let us find the response of damped systems to periodic force. So, in the Fourier series we have this cos terms. So, the response due to this harmonics the cosine harmonics can be calculated using the expression we just saw previously. So, the steady state response due to the cosine force p t is equal to aj cos j omega naught t that is the j th cosine harmonic. So, the response due to this can be found out as aj by k. So, that term is equivalent to p naught by k as aj is the amplitude of this force. So, it is aj by k then 2 zeta beta j and beta j is frequency ratio of this harmonic.

$$p(t) = a_j \cos(j\omega_0 t)$$

$$u_j^c(t) = \frac{a_j}{k}\frac{2\xi\beta_j \sin j\omega_0 t + (1-\beta_j^2)\cos j\omega_0 t}{(1-\beta_j^2)^2 + (2\xi\beta_j)^2} \qquad \beta_j = \frac{j\omega_0}{\omega_n}$$

So, for that particular harmonic the forcing frequency is j omega naught and omega n is the natural frequency. So, this is equivalent to our omega by omega n term, that is the frequency ratio. So, this beta j is equivalent to the frequency ratio for this particular harmonic. So, we have aj by k 2 zeta beta z sin j omega naught t plus 1 minus this frequency ratio square cos j omega naught t, the whole divided by 1 minus frequency ratio square plus 2 zeta frequency ratio the whole square and we can also write the similar expression for the sin terms in the Fourier series.

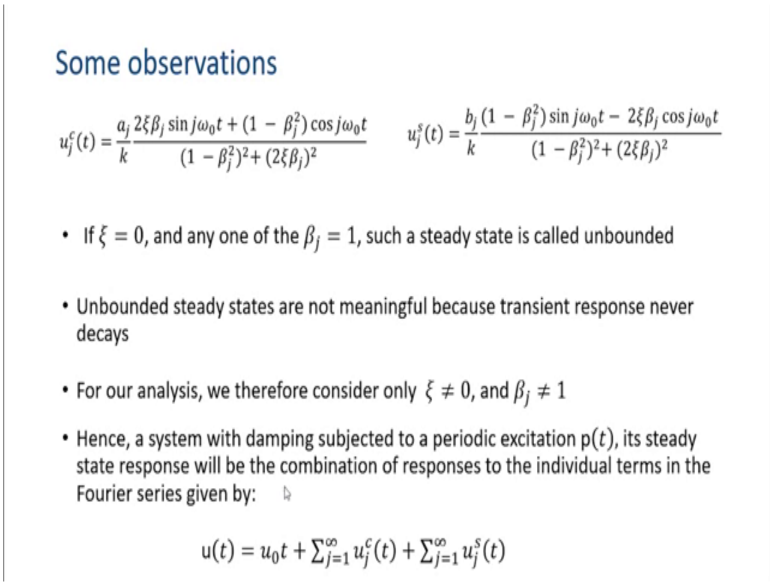
So, the sin harmonics this bj sin j omega naught t; so, the response due to this sin term would be this. So, that would be bj by k again that is equivalent to the p naught by k term and 1 minus frequency ratio square multiplied by sin j omega naught t minus 2 zeta bj cos j omega naught t divided by the same numerator. So, if you just compare these 2 responses that is the responses to sin and cos forces, the expression is quite similar the numerator is same and this constant term is amplitude by k.

$$p(t) = b_j sin(j\omega_0 t)$$

$$u_j^s(t) = \frac{b_j}{k}\frac{(1 - \beta_j^2)\sin j\omega_0 t - 2\xi\beta_j \cos j\omega_0 t}{(1 - \beta_j^2)^2 + (2\xi\beta_j)^2}$$

So, that is aj by k here and bj by k here, both numerators have sin and cosine terms. The coefficient of cosine term in this expression is equal to the coefficient of the sin term in this expression and this coefficient of sin term in this expression as 2 zeta beta j and the coefficient of the cos term here is minus 2 zeta beta j. So, they are like negative of this is the coefficient here. So, now we have the response to you to cosine force and sin force. So, now, we can add all these responses and find the response due to the periodic force.

(Refer Slide Time: 15:04)

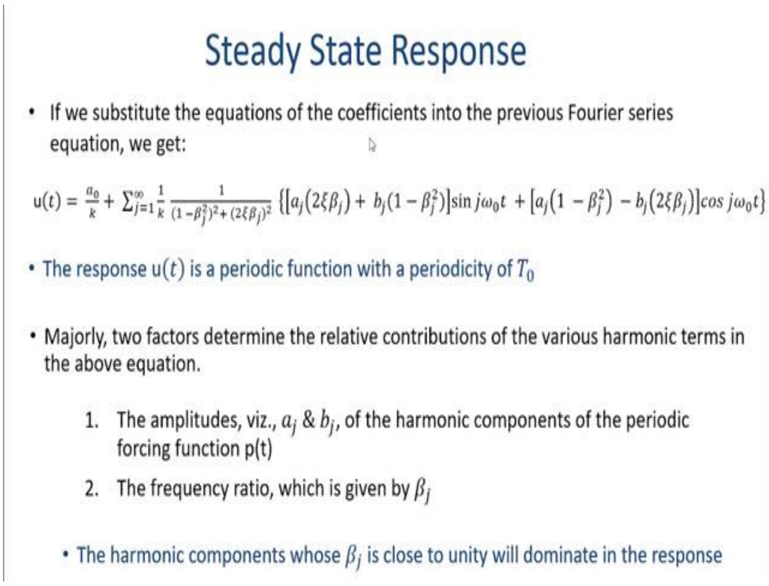


These expressions become indeterminant if zeta is equal to 0 and this in beta j becomes 1. If any of the beta j s becomes equal to 1 and zeta is equal to 0 both these expressions will become indeterminate. So, what does that mean? It means that and that is condition the steady state is unbounded. So, we have seen unbounded steady state in the case of undamped resonance condition so; that means, steady state response was increasing without any bound and this unbounded steady states are not meaningful they are not realistic because in such cases transient response never decays that is because zeta is equal to 0.

All real systems will have some amount of damping. So, the steady state response for any real system will not be unbounded. So, in our analysis we will consider only situations where zeta is not equal to 0 and beta j not equal to 1. So, if this condition is satisfied then we will never get an unbounded solution from this expression. So, the steady state response of the damped system due to a periodic excitation is equal to the combination of responses to the individual terms in the Fourier series. So, the total response is equal to the sum of the responses to the individual terms in the Fourier series.

So, this is due to the constant in the Fourier series and this is the response to the cosine functions and this is the response to the sin functions the sum of all will give the total response due to this periodic excitation p t.

(Refer Slide Time: 17:27)



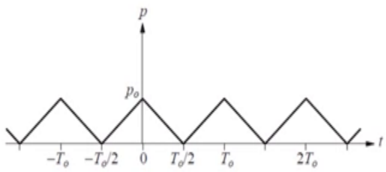
So, if we substitute the equations of the coefficients in the previous expression, we would get the response of the SDOF system due to a periodic force and the expression is this here a naught by k is equivalent to a static response and then we have harmonic responses that is sin and cosine functions. The amplitude of this harmonic is given by these big expressions, you can calculate it if we know the value of the damping and the frequency ratio and the frequency of a harmonic is equal to j times omega naught where omega naught is 2 pi by T naught; that this response will be again a periodic function and its period is T naught.

The total response is the sum of multiple harmonics and the relative contribution of various harmonic terms will depend upon 2 major factors one is the amplitudes aj and bj. So, if the amplitudes aj and bj are high, j th harmonic will contribute more to the response and if the frequency ratio of the j th harmonic that this beta j is near 1, then that harmonic component will also dominate in the response. So, when the frequency ratio is close to 1 the amplitude tends to be very high. So, that harmonic will majorly contribute to the total response.

(Refer Slide Time: 19:34)

Example Problem

- An SDOF system with natural period T_n and damping ratio ξ is subjected to the periodic force shown below with an amplitude p_0 and period T_0.

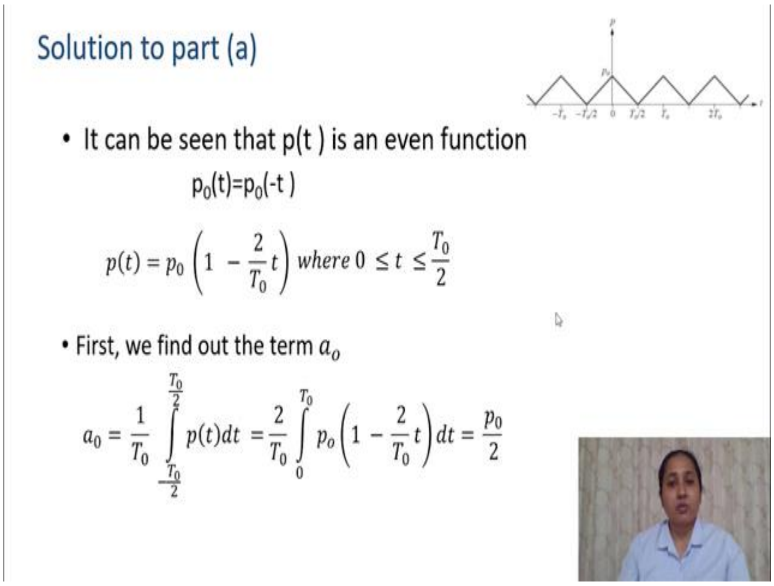


a) Expand the forcing function in its Fourier series.

b) Determine the steady-state response of an undamped system. For what values of T_0 is the solution indeterminate?

Now, let us solve an example problem in periodic vibrations. A single degree of freedom system with natural period Tn and damping ratio zeta is subjected to the periodic force shown below. So, this force has amplitude p naught and period T naught. Expand the forcing function in its Fourier series, determine the steady state response of an undamped system for what values of T naught is the solution indeterminate.

(Refer Slide Time: 20:06)



So, from this forcing function we can understand that, this forcing function is an even function; that means, this function is symmetric about the y axis that is p at time t is equal to p at the time minus t. So, if this condition is satisfied, this function is an even function. Now we can represent this function as a equation.

$$p_0(t)=p_0(-t)$$

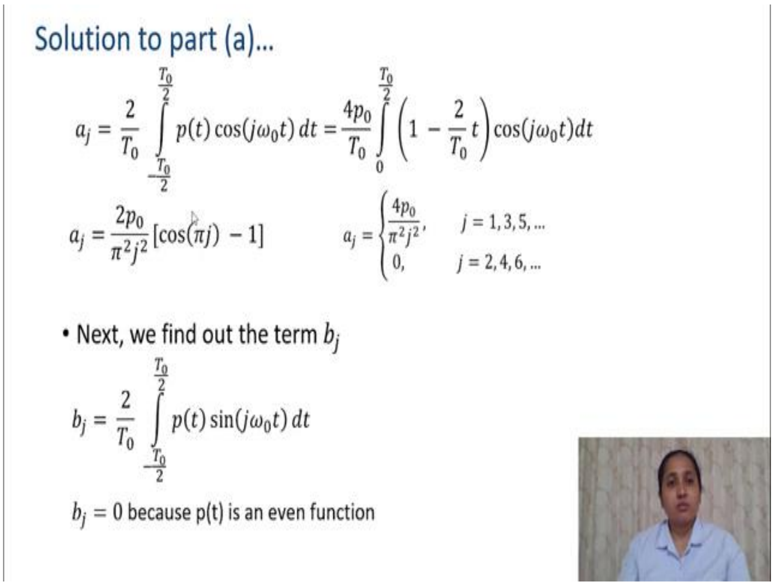
$$p(t) = p_0\left(1 - \frac{2}{T_0}t\right) \quad where\ 0 \le t \le \frac{T_0}{2}$$

So, this is a straight line. So, we can find the equation of this straight line. So, that would be p naught multiplied by 1 minus 2 by T naught t. And this is valid when t is between 0 and T naught by 2 the same equation will be valid when t is minus that is when t is between 0 and minus T naught by 2 and if this function is defined for a duration of T naught that is the period of this function then it is defined everywhere because then its a repetition of the same portion everywhere.

$$a_0 = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} p(t)dt = \frac{2}{T_0}\int_0^{T_0} p_o\left(1 - \frac{2}{T_0}t\right)dt = \frac{p_0}{2}$$

So, this is how we can write this forcing function as an equation. So, to find its Fourier series the first step is to find the coefficient a naught. So, a naught is defined as 1 by T naught integral minus T naught by 2 to plus T naught by 2 p t dt. So, we can substitute the value of p t here and can carry out this integral and we would get p naught by 2. So, a naught or the constant term in the Fourier series is p naught by 2.

(Refer Slide Time: 22:16)



Now we will find aj. So, aj can be found out as 2 by T naught integral minus T naught by 2 to T naught by 2 p t times cos j omega naught t dt. So, this aj is the amplitude of the j th harmonic the cosine function and the frequency at the j th harmonic will be j omega naught. So, if you carry out this integral we would get the value of aj as 2 p naught by p square j square cos pi j minus 1. So, cos pi j will have 2 different values depending upon the value of j; so, j is the value of j is integer. So, for odd integer, this value within the bracket would be 2. So, aj would be 4 p naught by pi square j square.

$$a_j = \frac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} p(t)\cos(j\omega_0 t)\,dt = \frac{4p_0}{T_0}\int_0^{\frac{T_0}{2}}\left(1 - \frac{2}{T_0}t\right)\cos(j\omega_0 t)dt$$

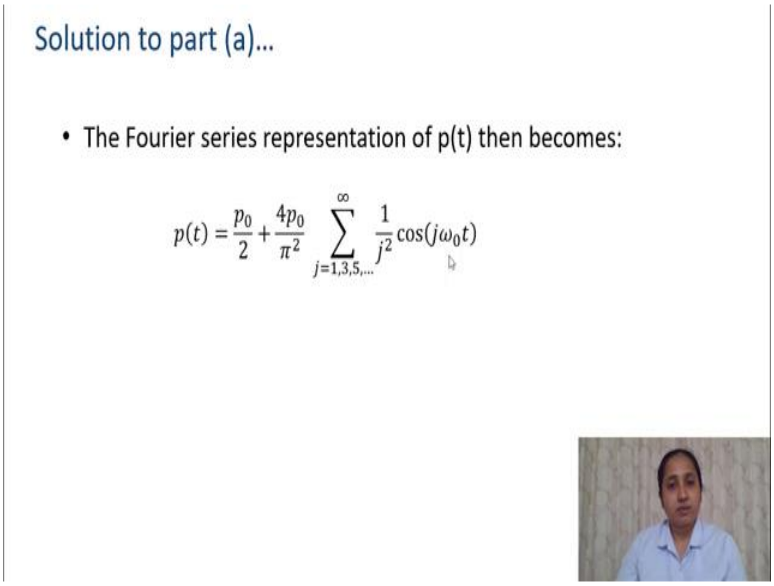
$$a_j = \frac{2p_0}{\pi^2 j^2}[\cos(\pi j) - 1] \qquad a_j = \begin{cases} \frac{4p_0}{\pi^2 j^2}, & j = 1, 3, 5, \dots \\ 0, & j = 2, 4, 6, \dots \end{cases}$$

So, when the value of j is even that is when j is equal to 2, 4, 6 etcetera this becomes 0. So, aj will be 0 now we need to find out the term bj. So, bj is given by 2 by T naught integral p t sin j omega naught t dt. So, we have already seen that p t is an even function and sin we know that it is a odd function so; that means, sin minus x is equal to minus sin x. So, if we integrate an odd function from minus T naught by 2 to T naught by 2 that is over a period that becomes 0. So, this term bj is equal to 0 for all values of j

$$b_j = \frac{2}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} p(t) \sin(j\omega_0 t)\, dt$$

$$b_j = 0$$

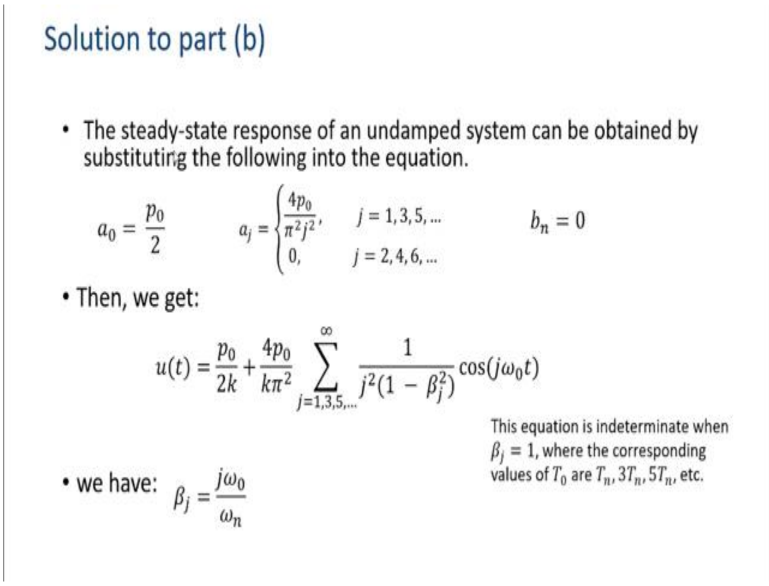
(Refer Slide Time: 24:43)



So, the forcing function p t can be represented as a Fourier series like this. p naught by 2 plus four p naught by pi square summation of odd values of j 1 by j square cos j omega naught t. So, this is how we can represent the forcing function in terms of harmonic functions.

$$p(t) = \frac{p_0}{2} + \frac{4p_0}{\pi^2} \sum_{j=1,3,5,\ldots}^{\infty} \frac{1}{j^2} \cos(j\omega_0 t)$$

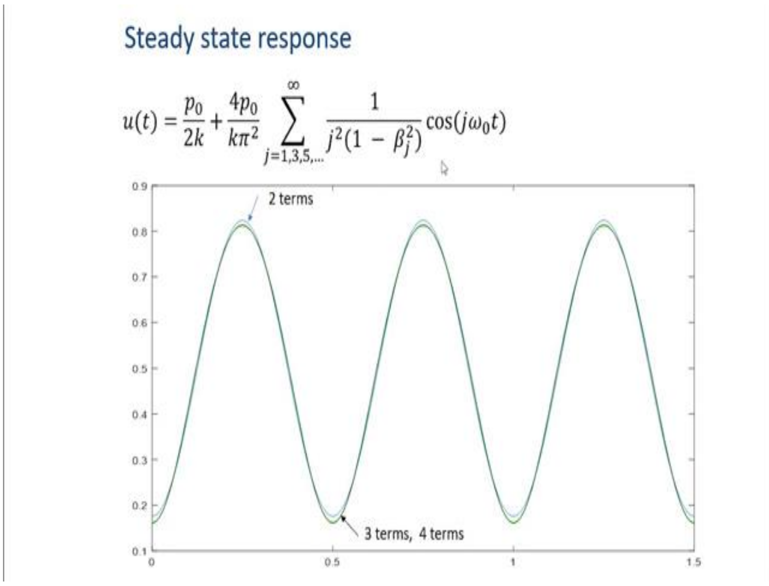
(Refer Slide Time: 25:10)



So, now we can find the response the steady state response of an undamped system due to this periodic force p t. We know the coefficients and we can write the expression of u t by substituting these coefficients in the expression we have derived earlier.

$$u(t) = \frac{p_0}{2k} + \frac{4p_0}{k\pi^2} \sum_{j=1,3,5,\dots}^{\infty} \frac{1}{j^2(1-\beta_j^2)} \cos(j\omega_0 t)$$

(Refer Slide Time: 25:30)



$$u(t) = \frac{p_0}{2k} + \frac{4p_0}{k\pi^2} \sum_{j=1,3,5,...}^{\infty} \frac{1}{j^2(1 - \beta_j^2)} \cos(j\omega_0 t)$$

So, this is our expression for the steady state response and this has infinite number of terms. So, theoretically this response is the sum of infinite number of terms, but if we plot this expression, we can see that only a few terms are necessary to calculate this response.

So, if we consider only the first two terms that is this term and the first of this term we would get this blue curve. So, this is plotted for some values of p naught by k and omega by omega n. If we consider three terms we would get this green curve. And, if we consider four terms we would get this black curve as it is evident from this picture the sum of three terms and sum of four terms are very identical. So that means, the three terms or four terms of the series converges to the actual steady state response.

So, we need not consider infinite number of terms to get this accurate response we can get the correct response with very few terms itself.