Mechanics Of Solids

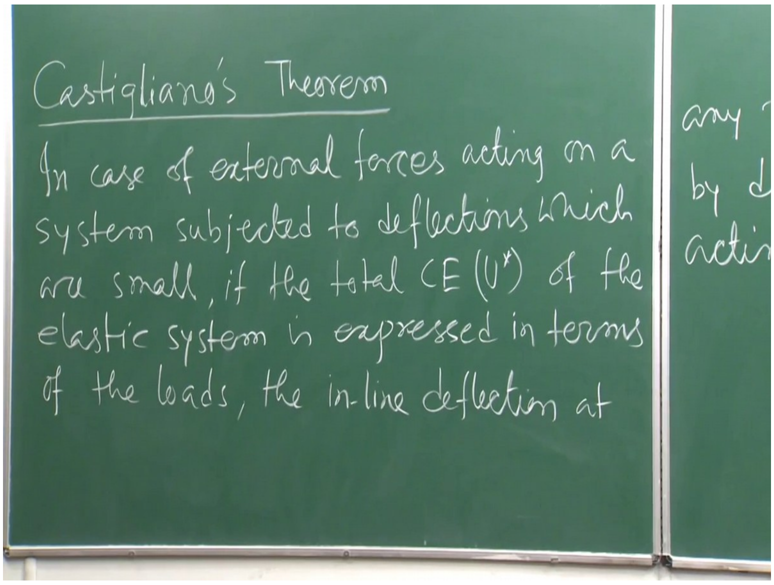
Prof. Priyanka Ghosh
Department of Civil Engineering
Indian Institute of Technology, Kanpur

Lecture - 55
Castigliano's Theorem

Welcome back to the course mechanics of solids. So, in the last lecture if you recall we just talked about the potential energy, complementary energy and then the conservative system. And then based on that we derived one say relation that is del u star del P i is equal to delta i right; that means, if you take the partial derivative with respect to a particular force P i of the of the complementary energy then that will give you the deformation or the deflection, along the line of action of P i at load P i ok.

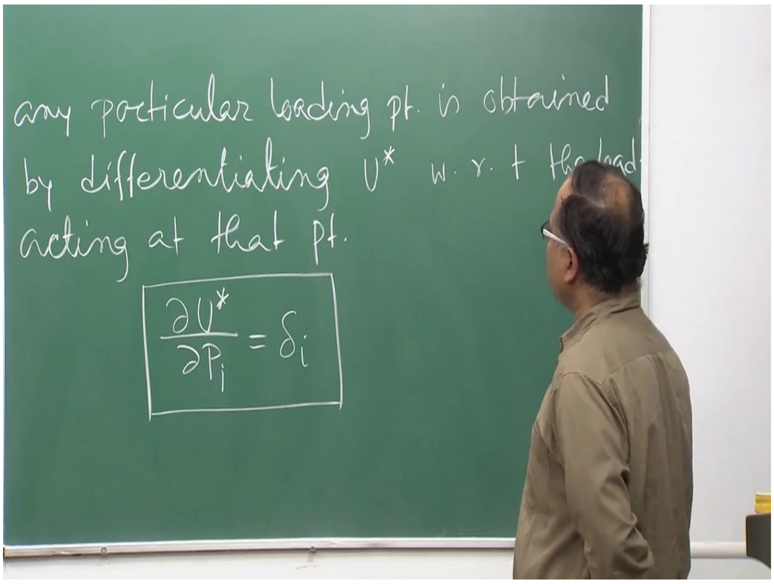
So, based on that actually castiglianos theorem has been developed and that theorem says that, in case of external forces.

(Refer Slide Time: 01:00)



In case of external forces acting on a system subjected to deflections which are small if the total complementary energy u star of the elastic system is expressed in terms of the loads, then the inline deflection at any Particular loading point is obtained by differentiating u star that is a complimentary energy with respect to the load acting at that point.

(Refer Slide Time: 02:48)



So, this is the theorem ok.

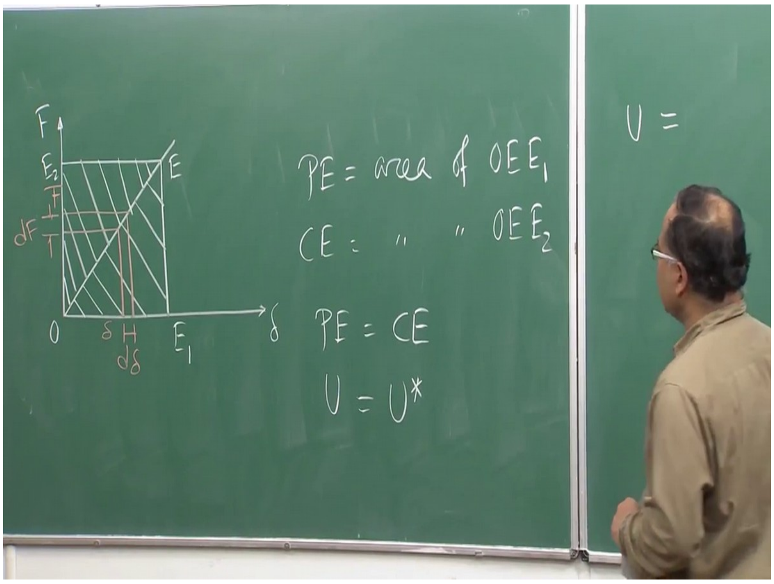
So that means, the theorem says del u star by del P i is equal to delta i. Already we have derived this thing in the last lecture. The castiglianos theorem says that in case of external forces acting on a system subjected to deflections which are small of course, we are talking about the small deflection problem or small deformation problem.

So, if the total complimentary energy of the elastic system is expressed in terms of the loads. So, different loads which are applied on the body on the system conservative system. The inline deflection at any particular loading point is obtained by differentiating the complimentary energy u star with respect to the load acting at that point. So, this is the total complimentary energy of that particular elastic system and we are differentiating that with respect to a particular load say P i and that will give me the deflection at the point where P i is applied. So, this is delta i ok

So, by using this theorem we can find out the deflection as well as deformation under different elastic system under different kind of loading system. So, let us let us see before going to or before jumping to that particular or different cases or situations ok.

Now, we will be talking about the only. So, far we were talking about the non-linear system now. So, far in this particular course we are dealing with the linear system. So, if you consider the linear system, then basically if you consider the linear system.

(Refer Slide Time: 05:39)



Linear elastic system if you consider and then if you try to plot F verses delta curve. So, as you know that is that will be linear ok.

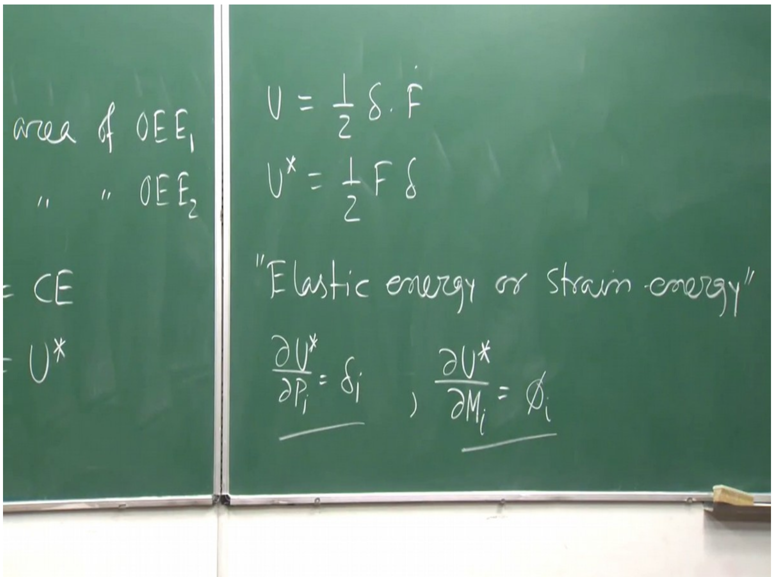
So therefore, this you say O say E this point say E 1 and this point say E 2. So if you consider linear systems O E will be the will be the plot will be the relation between F verses delta for linear system as you have already seen in earlier discussion.

So, your potential energy is nothing but the area of as per our definition O E E 1, area of O E E 1 that is the area under the curve O E. Similarly your complimentary energy is nothing but area of O E E 2 ok.

So, for your linear system as you know this line of course, will be the (Refer Time: 07:52). So, if for a linear system your potential energy must be equal to the complimentary energy. So that means, your u is equal to u star which was not the case in case of non-linear system right. Because at that time this O E line was not straight line that was a non-linear line, but if you deal with linear system then we can very very much right. That u is equal to u star ok.

So now if you write u is equal to u star.

(Refer Slide Time: 08:32)



So therefore From this figure actually we can write u that is nothing but the potential energy which is nothing but half into delta into F if I write that. So, that is the area of the triangle O E E 1. Similarly u star can be written as half into F delta. They are same for a linear system. So, that is the area of triangle O E E 2 ok.

So, this now onward actually because we are dealing with the linear system in this particular course. So, the now onward this we will be using one common term for potential energy or complimentary energy whatever and which is known as elastic energy or the strain energy. So, that is the common term which will be used as instead of potential energy or the complimentary energy, because both are same we are going to use one common term that is elastic energy or strain energy. So, energy stored due to deformation or the elastic I mean energy which is stored in the elastic system. So, this will be your common term.

Now, already you have seen that you have written say I mean in earlier case del u star del P i is equal to delta i. Now what was that? That means, if you take the partial derivative of the total complimentary energy of the system with respect to P i load P i there you will be getting the deflection at that particular point ok.

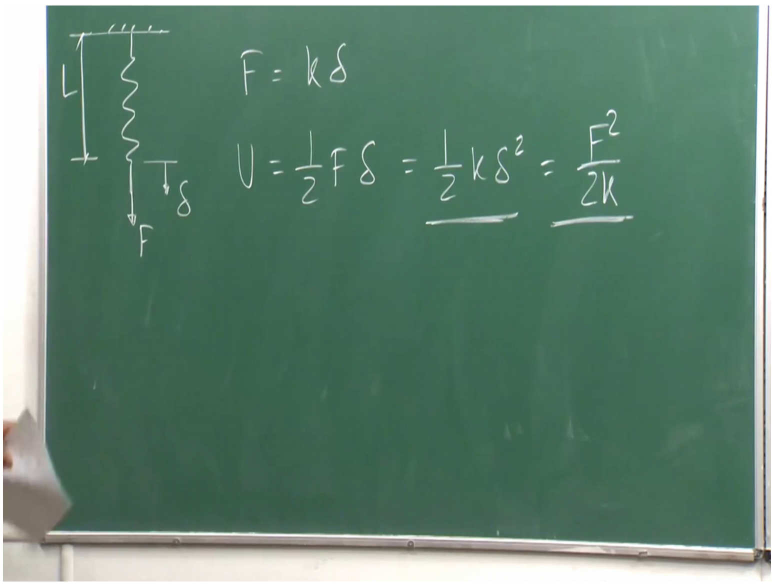
Similarly, if you try to take the partial derivative with respect to a partial derivative of the complimentary energy with respect to say moment m i, some moment because there are say consider the elastic conservative system is experiencing several moments say m one to m i m n ok.

So now if you consider or if you take the partial derivative of the total say complimentary energy with respect to m i. So, that will give you the rotation at that particular point with respect to the same axis of the moment. So, that is your phi i. So, that relation is also applicable because moment only creates rotation and the force I mean axial force will cause the deflection right. In the deformation that already we have seen.

So, this is as this one is valid, this is also valid. And we will be using these things frequently. So now, if you want to find out a rotation at a particular point. So, you find out the moment which is causing the rotation at that particular point and then you take the partial derivative with respect to that particular moment you will get the rotation.

Well, so now onward we will be calling for a linear system we will be calling the complimentary energy or the potential energy by a common term as defined, So that is your strain energy or the elastic energy.

(Refer Slide Time: 12:35)

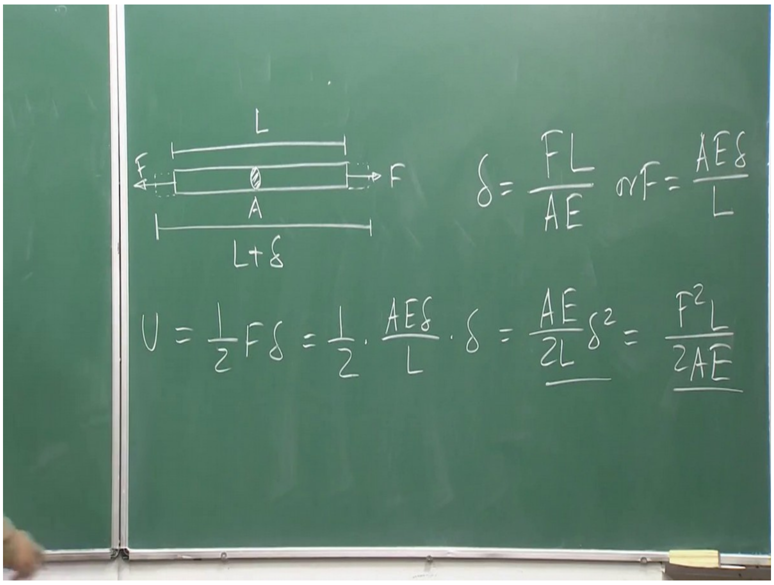


Now, if you consider the basic systems say one spring and you are applying a force F and for that you are observing deflection delta and the length of the spring say L. So, as you know your F is nothing but k delta where k is nothing but the spring constant or the stiffness of the spring. So, your u now will be now onward will be using only u which is nothing but your strain energy or the elastic energy and it will take care of both complimentary energy or the potential energy because we are using the common term.

So, u is equal to half F delta, just now we have seen. Now in place of F if we put k delta. So, I can write this thing half k delta square; that means, the strain energy is expressed in terms of only delta. Or it can be rather expressed as half F square by 2 k; that means, the strain energy can be expressed in terms of only force, agreed fine.

So now if I consider a linear system; so this is the strain energy, what is this? This is the strain energy stored in the strain due to the application of this force F.

(Refer Slide Time: 14:23)

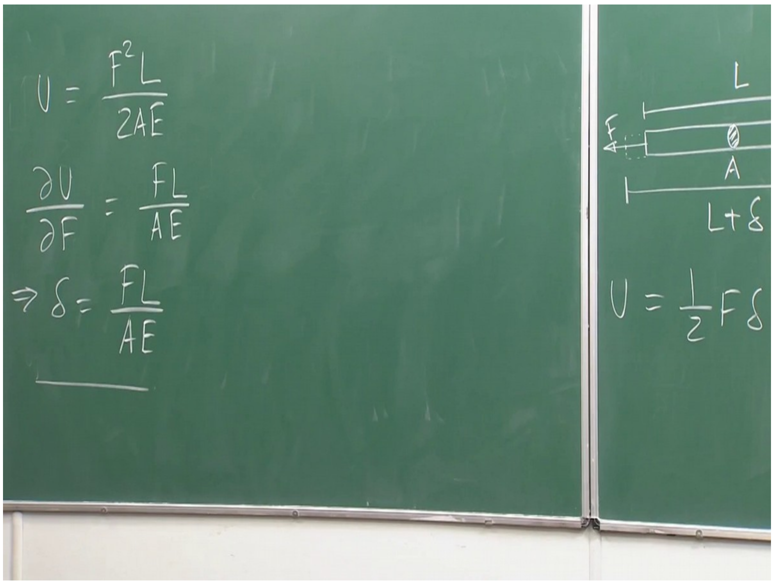


Now if I consider a Uniaxial say member we have seen that thing earlier also the original length of the member is say L. And we are applying a an axial force say F and due to that axial force you are getting some elongation in the member. And now the length is say L plus delta. And cross sectional area of the member you say A.

So, by our previous discussion we can say that is delta is equal to F L by A E. Already we have derived that already we have seen that nothing new actually. So now, or from this I can write F equal to A del T A by L A delta by L. So now, if I want to find out the strain energy for this system which is under the load F then we can write down half F delta that is the area under the curve. So, in place of F I can write this A E delta by L into delta which can be written as A E by 2 L delta square. Or it can be further written as So, this is in terms of deflection I mean deformation delta. So, or it can be written as F square L by twice A E. So, if you want to express interns of force, that is here if you want to express interns of deformation that is here. So, this is a strain energy of the system under this kind of axial force ok.

So now what basically you are getting I mean from the castiglianos theorem? Suppose this is a strain energy right. So, this one is your strain energy.

(Refer Slide Time: 16:59)



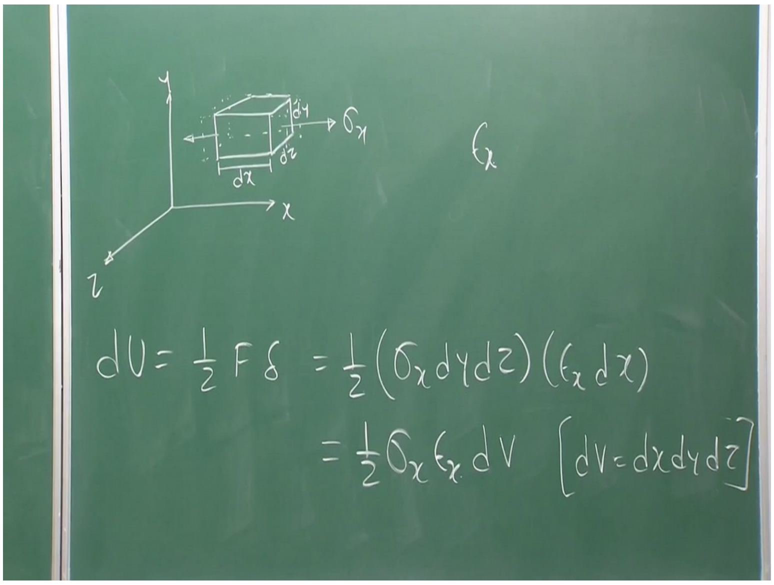
So, for this kind Of system your strain energy has been calculated like that. So, this is my strain energy say F square L by twice A E this is your strain energy ok.

Now, you want to find out delta. And if I want to find out delta means that is the that is the deformation which is along the line of action of force F applied force F right. Tension So now if I do this operation del u del F I should get delta, as per our castiglianos theorem. So, del u del F is nothing but if you see from here and this is nothing but. So, that already we know I mean basically why I am showing this problem because we know the solution the delta is equal to F L b y A E.

So, from the castiglianos theorem we have got it right. So, we if we manage to get the or you manage to find out the strain energy of a particular system then if you take the partial derivative with respect to that force along which we are we are concerned to find out the deflection of the deformation then we can find out. So, this one is your this is coming from your castiglianos theorem.

So now what we will do one thing is very clear that if we manage to get the strain energy for an elastic system under different types of loading different types of loading means, as you know from the previous discussion that you may have the member on the torsion you may have the member under bending. So there are several kinds of types of loading right. So, under this types of loading if we manage to get the or manage to find out the strain energy, then from the strain energy you will be getting other information like deflection or the rotation other things right. So, let us try to determine the strain energy for an elastic system ok.

(Refer Slide Time: 19:22)



Now, we are considering one infinite similarly small element within a linear elastic body in the previously we have considered that we know in when we are talking about the stress distribution of stress and all. So, we are considering what infinite similarly small element inside a inside an elastic body. And that is nothing but like this.

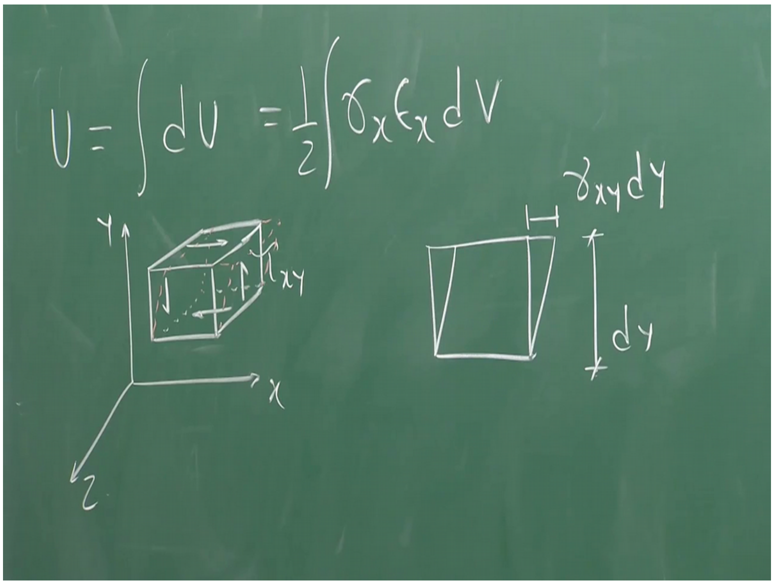
So, this is your elastic body, under the action of say sigma x. So, this is your x y and z. So, under the action of only sigma x; so the original dimension was d x d z d y. And due to this application of sigma x; that means, we are applying on tensile stress due to this application of sigma x you will be getting the elongation of the So, this is the elongated body right. So, you will be getting in some elongation unidirectional elongation of course, the Poisson effect will be there, but you will be getting some elongation in the x direction ok.

So, that elongation is nothing but your epsilon if I if I say the strain in x direction say epsilon x, then what I can write for this infinite actually small element. The strain energy can be written as half F delta. Now what is the force? Half, force is, what is the force? Sigma x into d y d z sigma x into d y d z, that is the force in the x direction. And what is the elongation delta or the deformation the delta is nothing but epsilon x d x, agreed? Epsilon x is your strain, and d x is the original length before deformation before application of sigma x.

So, from this I can write down. So, d u is basically is equal to half sigma x epsilon x d v. Where d v is nothing but the volume d v is equal to d x d y d z, that is the volume of the element small element. So, this is the strain energy due to the application of only sigma x on the element and that you have got sigma x half into sigma x epsilon x d v ok.

Similarly, therefore this if you consider the whole elastic body. I mean, if you consider the whole elastic body is made of this small elements several small elements.

(Refer Slide Time: 22:56)



Then the total strain Energy of the whole body whole elastic body will be the integration of d u right. So, that will be that is the total strain energy of the whole elastic body.

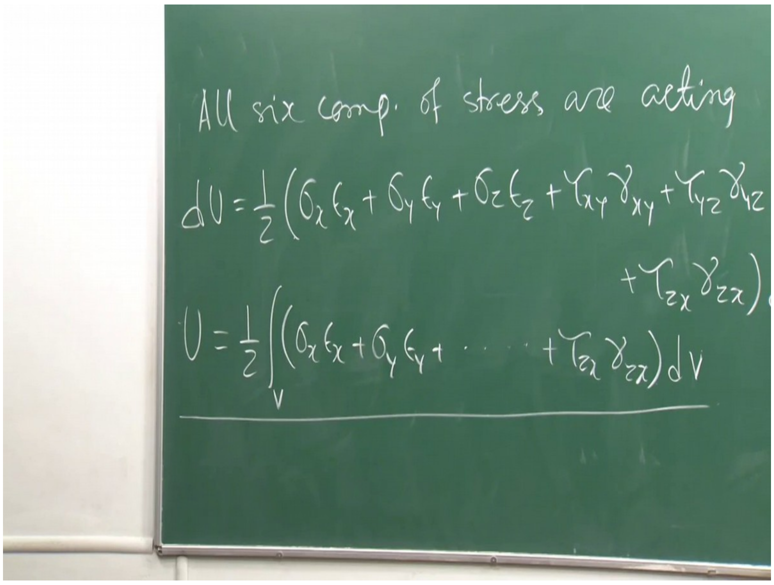
So, you consider one small element in that elastic body and then we have found out the strain energy for that small element, and then if you consider the whole elastic body then this is the strain energy for the whole elastic body just we are taking the integration over the whole volume.

Now, if you consider shear stress is also acting. Now if you consider shear stress is also acting. Suppose again we are considering the small element. So, this is the small element. And you are applying shear. You are applying shear, that is say tau x y only you are you are applying the shear in x y plane. So, so tau x y. So, due to that you will be getting the deform shape like that, something like that. So, shearing So, angular distortion.

So, if you look at only the in the plane. So, this is your shear deformation. So, this was d y say and this is of course, this deformation will be gamma x y d y, agreed? Gamma x y d y why gamma x y is your shear strain ok.

Now, in that situation what will be your strain energy for that particular element? D u, if I write down that.

(Refer Slide Time: 25:39)



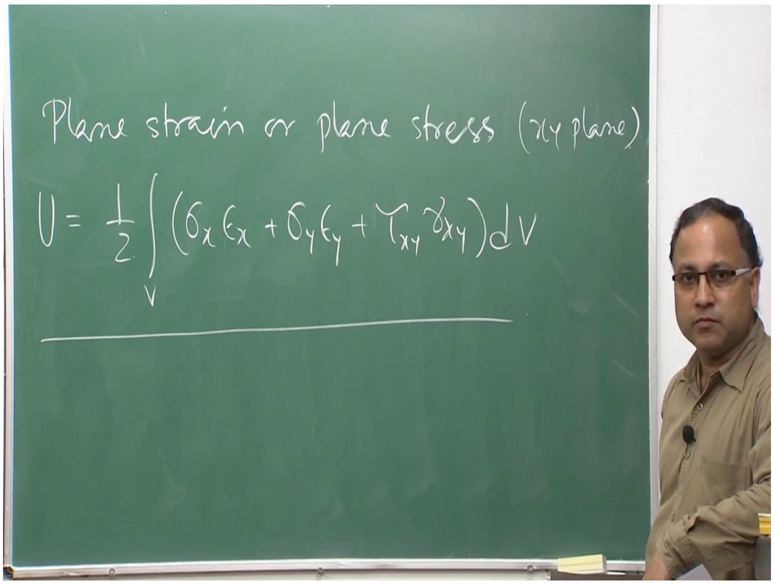
For the small element will be equal to half F delta what is the force is tau x y d x d z this is the force shear force acting on the element multiplied by the deformation delta what is deformation? That is gamma x y d y. So, from this I can write down half tau x y. So, your d u gamma x y d v. So, that is the strain energy for the small element under the action of tau x y.

Now, if I consider the general state of stress; that means, all the say stress components are acting now 3 normal stress 3 shear stresses. So, all stress 6 stress components are acting on this particular element.

Then your d u now, if we consider all 6 components all 6 components of stress are acting. So, if under that situation your d u that is the strain energy for that particular small element will be simply half sigma x epsilon x plus sigma y epsilon y plus sigma z epsilon z plus tau x y gamma x y plus tau y z gamma y z plus tau z x gamma z x into d v. And the total strain energy of this whole body this is a strain energy for that particular element now when the body is consisting of several elements, now the total strain energy will be simply half integration over the whole volume sigma x epsilon x plus sigma y epsilon y and all those things plus tau z x gamma z x whole to d v. That is the total strain energy of the whole body ok.

Now, if you consider the plane strain condition if you consider the plane strain or plane stress condition generally whatever we are dealing with in this particular course if you consider.

(Refer Slide Time: 28:39)



Plane strain or plane stress condition if you consider in x y plane, then this other stress components and strain components will not be there. So, we can simply write the strain energy is equal to half integration over the whole volume sigma x epsilon x plus sigma y epsilon y plus tau x y gamma x y d v. So, this is the strain energy in plane stress or plane strain condition, clear?

So, this is the expression of your strain energy. So, if you one has to get all the stress strain components then basically you can find out the strain energy for a particular system particular elastic body.

So, I will stop here today. So, in the next lecture we will be continuing these discussion and we will see few problems.

Thank you very much.