Water Resources Engineering

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Lecture No. # 08

In the previous lecture we had looked at the groundwater which is the water in the sub surface and we saw that if we want to utilise the groundwater, we have to know the amount of water present, the amount available for withdrawal and how fast we can withdraw it. The other thing which we should look at is how deep the water table is because that will affect the cost of pumping. So the groundwater table has let us say horizontal level initially but when we start pumping it. It will start to go down with time and we should know how this water table will change with time so that we know how we have to lower the pump if the water level goes down very much and we had looked at a mass balance or continuity equation in which we take an element of the aquifer.

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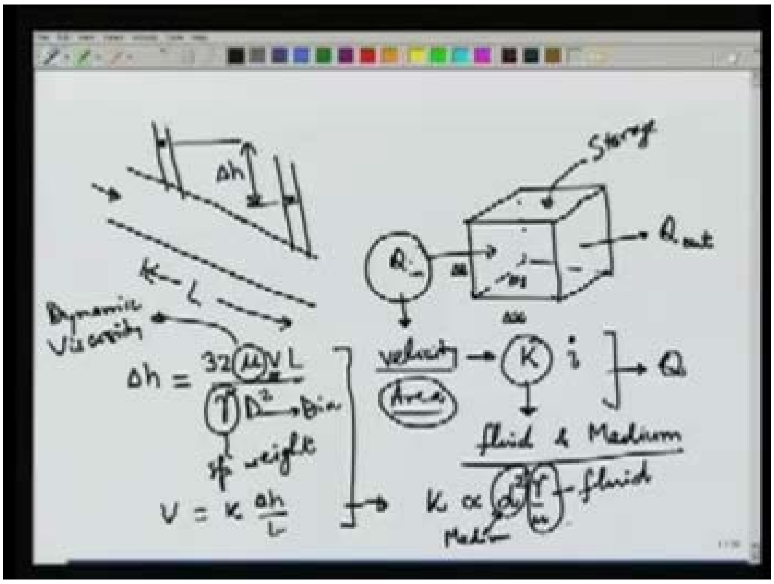

We look at the mass of water flowing into this element and the change of mass within this element and the continuity equation. This tells us that these should be balanced and looking at that mass balance will allow us to determine how the water level or the pressure in the aquifer is changing at that point. So in order to look at the mass balance we should know the amount coming in and the amount going out. If we take an elementary volume like this, we have delta x delta y and delta z then we can find out the storage of water within this volume and how it is changing with time. So we need to know two things. One is the amount of water coming in, amount of water going out. This will depend on the velocity which can be obtained from the Darcy's law and therefore it depends on the hydraulic conductivity K and the hydraulic gradient i. Q also depends on the area of cross section which in this case will be delta x delta y or delta y delta z, depending on the phase we are considering. So combining Darcy law and the area, velocity term, will give us an idea about Q. The other thing which we need to look at is the storage within this volume, the amount of water that comes out of this storage for a unit drop of head. We shall look into these things today. Let us first look at the hydraulic conductivity K. We had discussed briefly that this K is a function of both the fluid property and the medium property. If we take the analogy with flow in circular pipes, in which let us say we have a drop of head equal to delta h in a length of L then for laminar flow, we know the well known equation for the head loss in which ‘mu’ is the, dynamic viscosity of the fluid V is an average velocity, gamma is the specific weight and D is the diameter of pipe. If we want to compare this with flow through porous media, we can think of the D as being the diameter of the particles because that will be related to the diameter of the pores. So in the porous media the equation which we have is the Darcy’s law which says that V = K delta h over L. So if we compare these equations, the K will come out to be dependent on the medium property, the diameter of the particle and it will also depend on the fluid property, specific weight and the dynamic viscosity as gamma over ‘mu’. This is fluid property and d square denotes the porous medium property. So typically gamma over mu is taken out of this equation.

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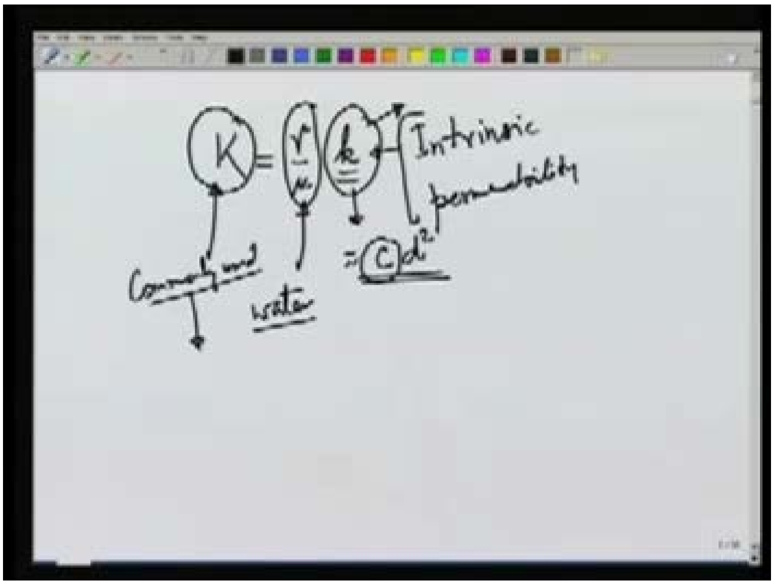

The hydraulic conductivity K is written as gamma over mu times some intrinsic permeability or specific permeability which is only a property of the medium. This K is some constant into the grain size square based on the pipe flow analogy. For different porous medium for example for sand and clay, the value of C may be different and there are a lot of empirical equations which are used to correlate the intrinsic permeability with grain size. Gamma over mu varies with the fluid. If we have flow of water, this value will be different, for oil it will be different, but in general, for groundwater flow, we have flow of water. Even for water, the value of gamma and mu may be different at different temperatures. In general in groundwater flow, we will assume that the temperature variation is very small and water is the flowing fluid. Therefore K is commonly used and not the intrinsic permeability. Commonly used term is the hydraulic conductivity since water is the flowing fluid and temperature can be assumed to be constant. If not then it is better to use the intrinsic permeability because that will not change with fluid properties. The second thing which we want to see is the amount of water that can come out of the aquifer for drop of head.

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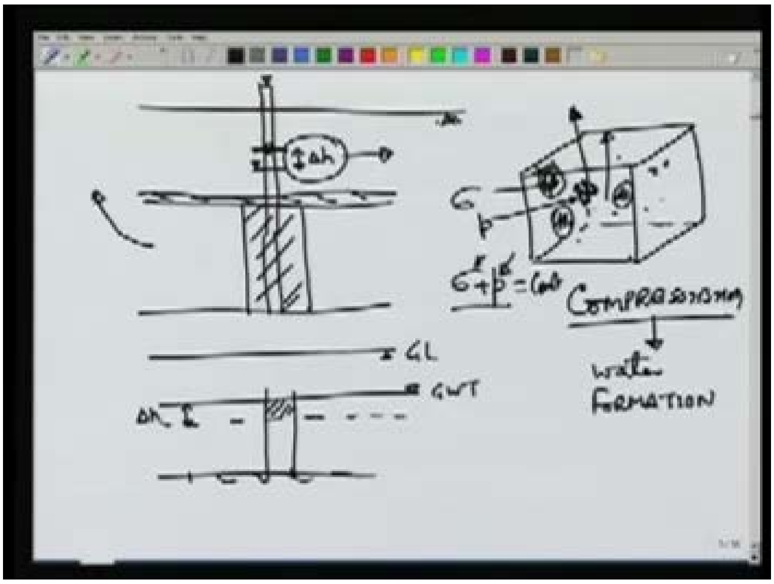

So for a given drop of head, suppose there is this volume which has some solid particles and the rest filled with water. We have already discussed that we will only consider saturated flow conditions. So there is no air inside this. Now this volume is subjected to some pressure and we will start with the confined aquifer case because that is easier to derive. If we put a tube here, a piezometer here, water will rise up to a certain level, which may be below the ground level. It may even be above the ground level. In this case it is called a flowing aquifer. Now suppose the piezometeric head is here, what is required to be seen is the amount of water that will come out, if we lower the piezometeric head by certain amount delta h. This lowering of the head may be because of pumping or it may be because of head drop in some other area where this aquifer is connected. But let us take this delta h as the head drop and if we take an element of the aquifer here, let us say we take some element here; we want to see the amount of water that can come out of this aquifer for head drop of delta h.

Now let us look at the mechanism of how the water comes out for unconfined aquifer. It is quiet straight forward. This is the groundwater table bedrock. If we lower the groundwater table by some amount delta h, water will come out because of drainage of this volume. But in unconfined and in confined aquifers there is no change or not much change in volume. There is little change which we will see. But there is not much change in the volume. Therefore the water which comes out is because of the compressibility and when we talk about compressibility, it would be compressibility of water as well as the compressibility of the medium. So we can call that aquifer or formation compressibility. When we lower the piezometeric head, the water from this element, suppose comes out of the element, this water is under some pressure. It has not been kept constant but now it has been lowered because of this lowering delta h. When we reduce the pressure, water will expand and therefore it will come out of this element.

The other thing which happens is because of this lowering of pressure in the water, since the total stress remains same, the over burden pressure remains same. Lowering of water pressure means increase on the grain pressure. If there is some grain pressure, let us call the pressure sigma. This acts on the grains and will increase if we lower the value of

(Refer Slide Time: 13:09). So the pressure in the water is let us say p, the pressure on the grains is sigma, now sigma + p will be the total pressure which remains constant. By lowering the piezometeric head, we are lowering p and therefore sigma will increase. Due to this, the aquifer or the formation will be compressed (because of this increase in stress) and that compression will lead to further release of water from this element. We shall look at ways to derive these terms which represent release of water due to metrics compressibility or the formation compressibility, release of water due to water compressibility. Let us define a term which is known as the storage coefficient.

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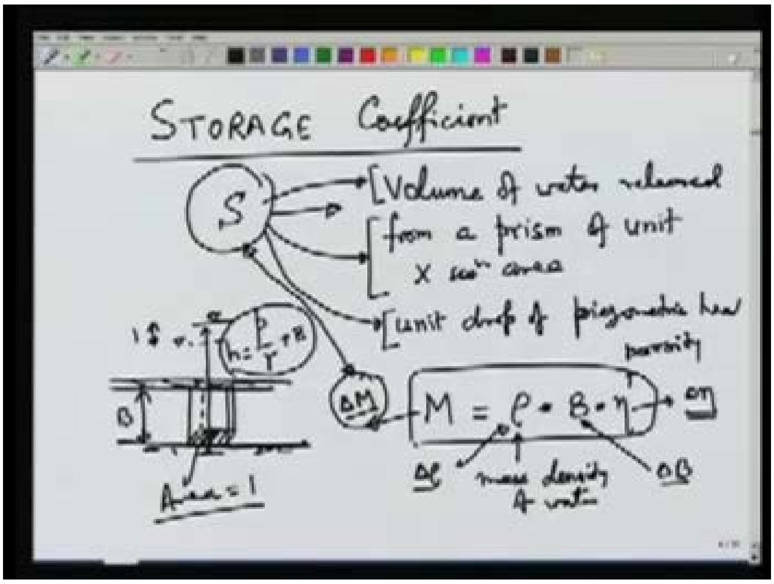

This is denoted by S. S is the volume of water released from the formation from a prism of unit cross section area with unit drop of piezometric head so that 3 since here that S is the volume and it is from a prism of unit cross section area for a unit drop of piezometric head. So if we take this confined aquifer of height B and considered a unit area of (Refer Slide Time: 16:05), this area is 1 and this has some piezometric head which we will call h, the datum can be anywhere. The datum is taken at the base of the aquifer that is at the impermeable rock.

Now this is some value of piezometric head and now if we lower this by a unit amount, we determine the amount of water that will come out of storage. So that term is denoted by the storage coefficient S. Now the mass of water which is present inside this prism M can be related to the volume and the density is rho. Rho is the density of water into the volume which will be (since the area of cross section has been taken as unity); the volume will be equal to B into porosity. The mass of water contained within this confined aquifer, the prism of base area 1 can be written as rho p eta which is the porosity and therefore if this mass is changing, it will change because of change in all these parameters. Suppose we have some change here, delta B, and some change here delta rho, then when we are lowering the piezometeric head, the aquifer thickness B will also change. As we have seen that, by lowering p we are increasing the effective pressure on the grains and therefore the aquifer will be compressed. B will reduce the porosity. Similarly it will also change. So all these three parameters change and their net effect will cause some change in mass delta M. Our aim is to find out this delta M because we can then relate the storage coefficient S with delta M. Let us look at how to find out this delta M for certain change in pressure or the piezometeric head h.

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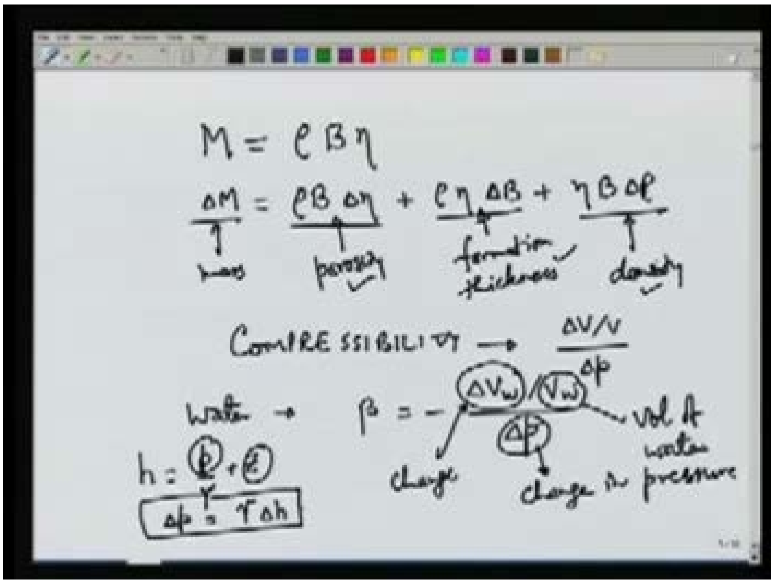

We now start with our original mass. Since all these three can change, we can write this as (Refer Slide Time: 19:17), so change in porosity will cause some mass change in formation thickness. So this is the effect of change in porosity formation thickness and density. Total change in mass occurs because of change in porosity, change in the formation thickness and change in the density of water. We need to evaluate these changes and for that we introduce a property which is known as compressibility. This is inverse of the modulus of velocity and is defined as change in volume per unit volume divided by a change in pressure. This is basically a strain over stress. For water, we can define a compressibility which we can say is represented by beta. In general a positive change in pressure will cause a negative change in volume and therefore we put in negative sign here, where V_w is the volume of water and delta V_w is the change in this volume. Delta p of course is the change in pressure. As we know, the piezometeric head h is p over gama + z, where p is the pressure and z is the elevation. Therefore a change in pressure, delta p can be written as gama into change in piezometeric head, delta h. So we will be using this relation later to find out the storage coefficient. So let us look at the water compressibility equation.

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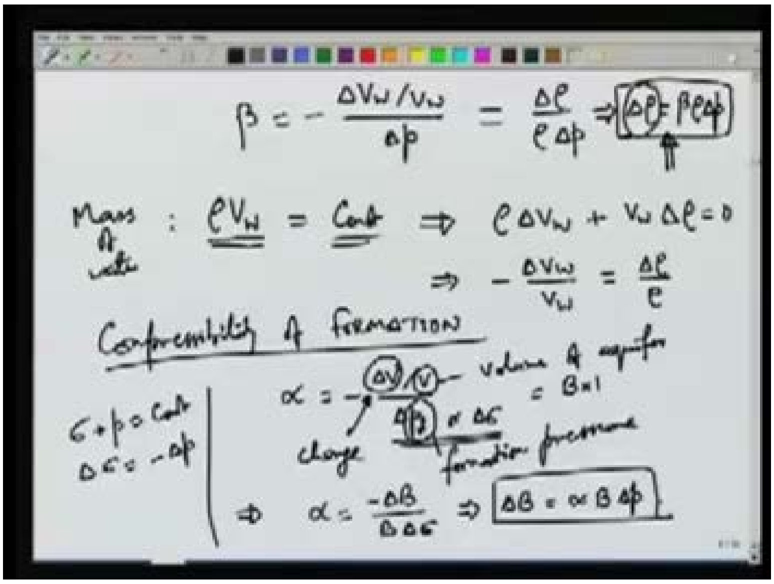

Now V_w is the volume of water and therefore if we write rho V_w, this will be the mass of water, V_w will be the volume within the aquifer. Now if you look at this mass of water rho, V_w continuity tells us that matter cannot be created or destroyed. Therefore this mass should remain constant. So we can relate a change in volume of water with a change in density. So if this is not changing, then the sum of these two changes should be equal to 0 and from here we can find out the term – delta V_w over V_w will be equal to delta rho over rho. We can write this as delta rho over rho delta p and therefore the change in density of water can be written in terms of its compressibility as delta rho, beta rho delta. So using this equation, we have related the change in the density of water with compressibility of water, its density and the change in the pressure. The next thing which we look at is compressibility of the formation of the aquifer material. We define the formation compressibility or aquifer compressibility and use a symbol alpha. In the same way as we did for water, we now denote the volume of the formation or the aquifer. So in this case since we are taking unit area we can say that V will be equal to B into 1 and delta v is the change in V, delta p is the pressure.

Since we are using p as a symbol of pressure in the water, we can use a different symbol here and we can write p_f as delta p in the formation or we can denote this term also by sigma. We can use delta p_f or delta sigma for this term. We can find out delta b from here, delta rho from here. If we look at the equation for delta M, we had delta rho, delta B and delta eta. So we need these three terms to find out the change in mass using this equation. Delta rho can be related with the water compressibility. Delta B can be obtained from here over B delta sigma. We have also seen earlier that sigma + p is constant therefore delta sigma will be equal to – delta and therefore we can also write delta B from here as alpha B delta. The third term which we need is how the porosity is changing with change in pressure and for that we can assume what is generally a very good assumption.

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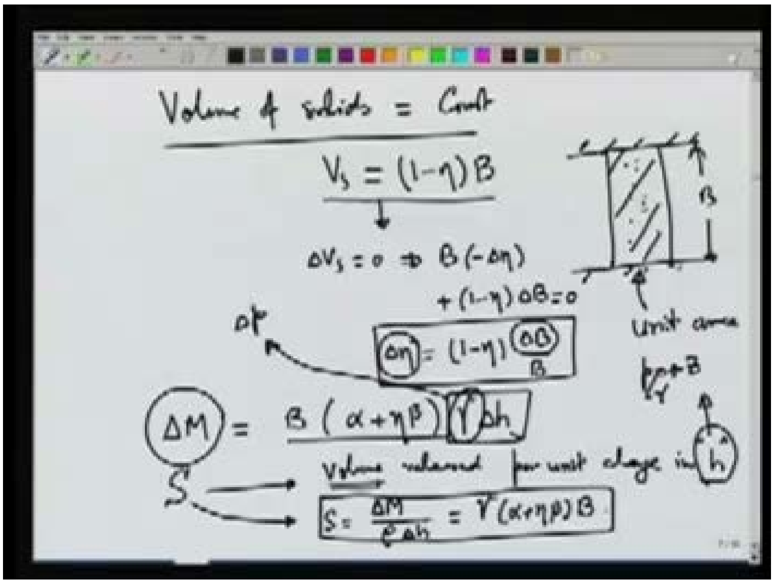

The volume of solids is constant, so if we have an aquifer material here, no matter how much the pressure is changing in this aquifer, water pressure and the grain pressure, the amount of solid material present in this area, volume of solids which we denote as V_s, remains the same because the grains are assumed to be incompressible. Now V_s can be written as 1 – eta. If the thickness as we have taken is B, and cross section area is 1, then the amount of solid volume present inside this unit area and height B will be 1 – eta times B, because total volume is B. Out of that, eta B is the volume of the liquid water in this case. Therefore total volume of solids Vs will be equal to 1– eta B and if this is a constant, then we can use the same for formulation as we use before, that delta V_s will be equal to 0 and this would imply that B times minus (Refer Slide Time: 30:29) equal to 0. From here we can obtain delta eta which is the change in porosity. We can relate it with the change in the formation thickness B.

Now we have all three parts of the equation, formulated delta eta from here which is a function of delta B, delta B we have from here and delta rho, we have from this equation so putting all these three in this equation which tells us total change in mass can be obtained as delta M in terms of the compressibility of the water, alpha and the compressibility of the aquifer formation beta. So we write delta M in terms of the aquifer thickness B into alpha + the porosity into beta multiplied by the specific weight gamma. Now we will need this because we have said that we want to find out a storage coefficient S which is the volume released per unit change in h. This h is the piezometric head and is given by p/gama + Z. We have related all the changes with delta p, so delta p can be replaced by gama delta h. We would write in terms of delta p and then write that delta p as gama delta h, which is nothing but delta p. Then to find out a storage coefficient, we know this is the volume and we have written our equation in terms of mass. S will be equal to the mass released divide by rho per unit change in the piezometeric head. Therefore S will turn out to be gama alpha + eta beta into B. This equation gives us the storage coefficient and represents the volume of water released for a prism of unit cross section area over the entire thickness B.

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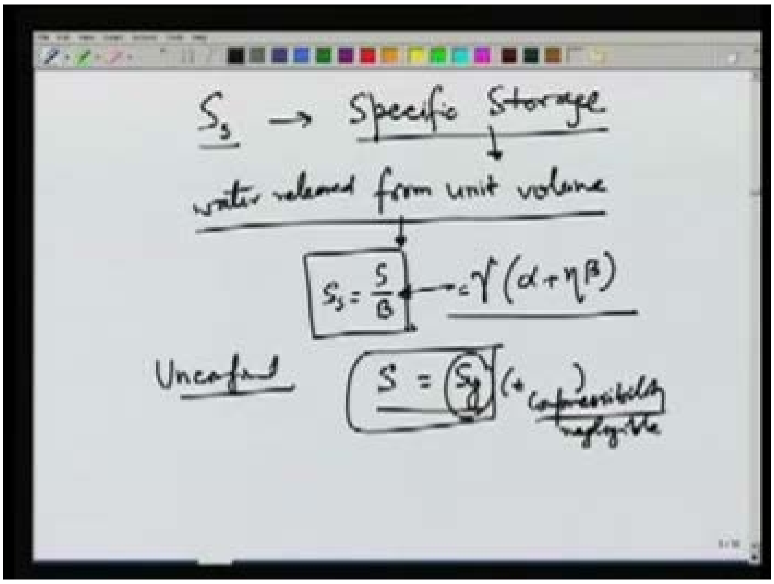

Similar to the storage coefficient S, we have a term, which is known as specific storage S_S and this is the volume of water released from unit volume. so if we compare it with the definition of the storage coefficient, if we divide S by B, we would get S_s because S was the volume released from the entire thickness B and therefore if you take unit volume, the amount of water released will be given by S over B and then we can write this too in terms of the compressibility and the porosity as gama alpha + eta beta.S_S will be used when we derive our continuity equation for a confined aquifer case. For unconfined aquifer, the storage coefficient S is equal to specific yield although there is a small component which comes from the compressibility. It can be ignored with the respect to $S_{y.}$ So compressibility terms are generally negligible and therefore for unconfined aquifer, the storage coefficient S is taken as S_y. Using this storage coefficient S, specific storage S_S, we can derive the continuity equation and that gives us the relation between the head, the peizometeric head in the aquifer and it relates with the amount of water we are pumping out of the aquifer. As we have seen, the storage coefficient can be defined for unconfined and confined aquifers. The specific storage for confined aquifers can also be defined and we will use that, to derive the equation of motion for a confined aquifer.

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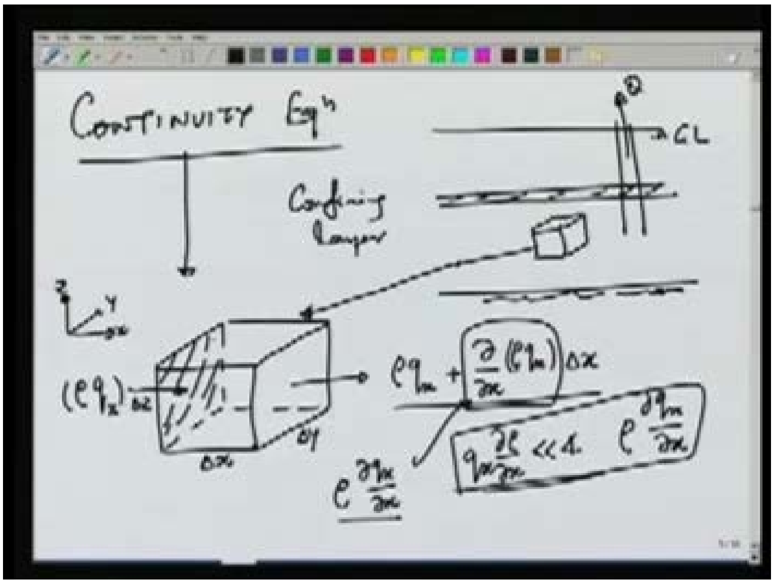

This is the confining layer and at the bottom we have this bedrock. So in this confined aquifer, we can look at an elementary volume and see how the head change affects the amount of storage and the amount of water which is coming in and then by balancing these two, we can derive an equation which will tell us how the head is changing with time and space. So this spatial and temporal variation of the head can be obtained by the continuity equation. So if we take this element, let us enlarge it. We will be using the Cartesian coordinate system but sometimes it is better to use a radial coordinate system. For example if you are pumping from a well, then the piezometeric head will be symmetric about the well in the radial direction and therefore, it would be convenient using a radial coordinate system. But let us start with the Cartesian coordinate system. We take this elementary volume which has size delta x, delta y and delta z. Of course our x, y and z directions are like this.

Let us consider the flow in the x direction. Same thing can be done in y and z direction also. But let us start with the x direction and say that there is some velocity qx which is the Darcy velocity in this case and not the seapage velocity. Because of this velocity qx, the mass which is coming in the mass flux will be rho into qx. What we want to see is what is the net mass flowing into this control volume. But the mass which is going out can be written by using rho qx which is the mass coming in from the left phase + the rate of change multiplied by the distance delta x. Sometimes we assume that this whole term rho is inside the differential here. So when we take del by delta x, we put rho inside the derivative but the variation of rho is very small with respect to the other term which is del qx by del x. If we write rho del qx/del x and write qx, then compared to this term, it can be neglected and therefore generally we write this term only as rho del qx/del x. We will be making an assumption that the change in density can be ignored and therefore it can be taken out of the derivative and we can write as rho del qx/del x.

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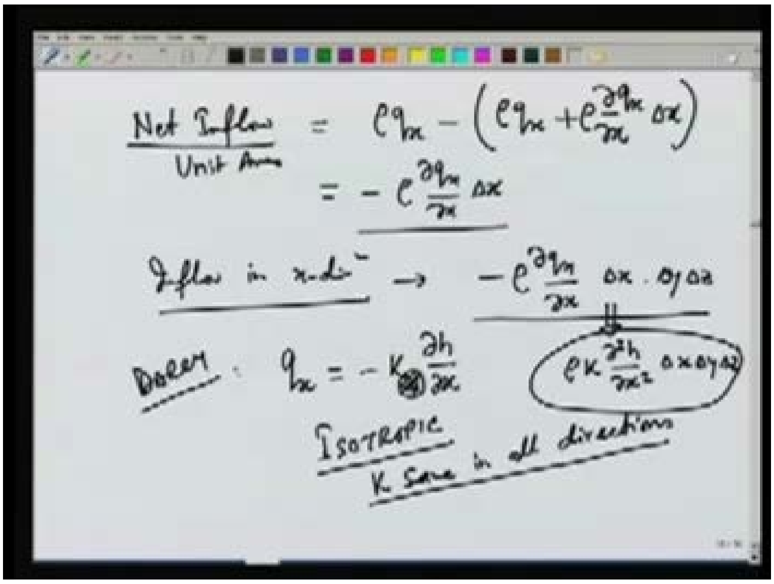

Therefore the net inflow would be rho qx which is the inflow from the left phase minus the outflow from the right phase which is rho qx plus and using our assumption, we have taken rho out of the derivative del/del x of qx into delta x and this will result in minus rho. Now this is the mass flux rate per unit area. We have multiplied it by the area. We can say that the net inflow per unit area is given by this and therefore inflow is in the x direction. Remember that we are talking only about the x direction. Let us do other directions. Inflow in x direction can be written as minus rho multiplied by the area which is in this case, delta y delta z. As you can see that this phase has an area delta y delta z, this term gives us the mass coming in the element in the x direction. We can now use Darcy law to relate it with the head.

As we know qx now, we will make another assumption here. We have a medium which is isotropic. If we make that assumption, then we do not need to use this subscript x. For isotropic medium, we can remove this x because K will be same in all direction. Sometimes the K is not same in all directions. For example if you have a layered formation, then K will be different for flow along the layers and it will be different for flow perpendicular to the layers. But here we assume that the medium is isotropic and therefore we will ignore this subscript x. Using Darcy law, we can write qx as – K delh/del x and therefore from this we can write the inflow in the x direction as rho K. This term represents the mass of water flowing in the x direction. Net inflow of mass into this control volume delta x, delta y, delta z and similar expression can be derived for y and z directions.

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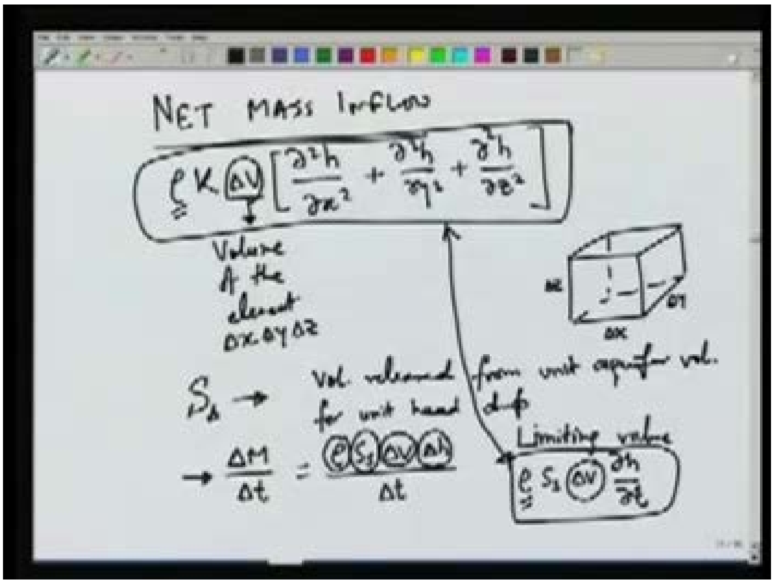

All of them can be added together to get the total or the net mass inflow into the system as rho K. Delta x, delta y, delta z can be combined as delta v. This is the volume of the element into; we have the contribution from the x direction, del two h/del x square and similar terms for y and z direction. Now the continuity equation says that this net mass inflow should be equal to the change of storage within the control volume. Now if we use the storage coefficient S, we have seen that S represents the volume released per unit cross section area using a prism but if you want to use elements with area delta x delta y and delta z we would need to use specific storage rather than a storage coefficient. So we use S_S which is the specific storage that represents the volume released from a unit volume. So volume of water released from unit aquifer volume for unit head drop.

If you want to find out the volume of water released from this elemental volume for a head drop of delta h, we can write an equation which tells us that mass released per unit time from this elementary volume can be written as rho Ss delta v delta h over delta t and the way we write this expression is that Ss is from a unit volume, therefore we have to multiply with delta h. Ss gives us the volume and therefore we multiply with rho to get the mass. The change of mass or unit time in the elementary volume delta m/delta t can be related with the specific storage Ss the density delta v delta h and delta t and in the limit as delta t tends to 0, we can write (Refer Slide Time: 49: 39) and the other limit which we will take is, when this element reduces to a point that delta v tends to 0. So when we equate net mass inflow which is given by this expression with change of mass within the control volume. So these two terms should be equal and therefore delta v will cancel out, rho will also cancel out and we should note that rho will cancel out because we have assumed that change of rho is very small or negligible. If we do not ignore that term, then rho will also be included in the equations.

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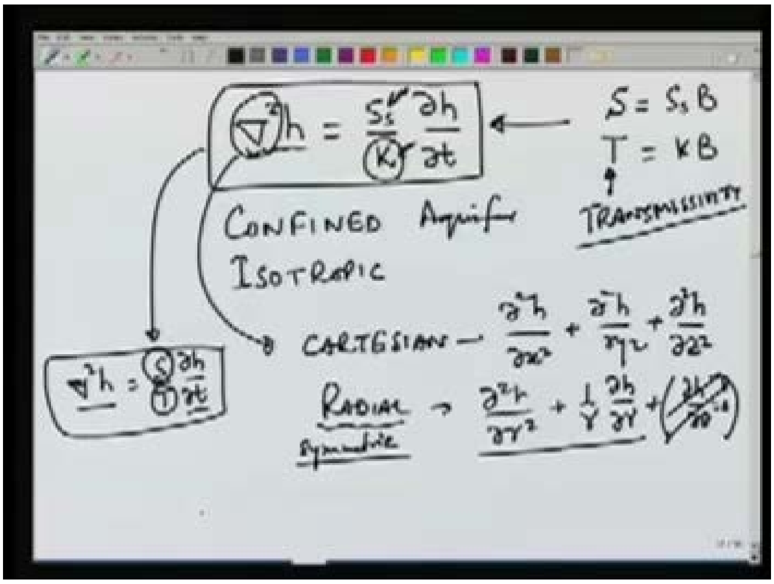

If we equate these two, we finally get an equation which can be written as del square h. This equation represents that we have made some assumptions. So I will write them here. First assumption is that it is a confined aquifer. Second assumption which we have made is that the aquifer is isotropic. That is the reason why we have only K. Otherwise we will have Kx, Ky, Kz, three different terms which will be included in this. The term del square can be expanded in Cartesian coordinates. It has a very simple form and if you have some so situation where you have radial and x is symmetric flow. Symmetric means that theta will not be coming into the equation. Then you get a term in which there is del h/del theta = 0. So that term has not been considered as del 2h/del theta square term which has been neglected because h is not a function of theta. Using this equation we can obtain the variation of h for a given condition.

We can also write this equation in a little different form by noting that the storage coefficient S is specific storage into the depth of aquifer and we have already defined the transmissivity as K into B. So if we use this S and T, we can also write this equation in terms of the storage coefficient and the transmissivity. So either we can use specific storage and hydraulic conductivity or we can use storage coefficient and transmissivity and solve this equation for given conditions. Since this is second order equation space, first order in time, we need one initial condition and two boundary conditions to solve this equation.

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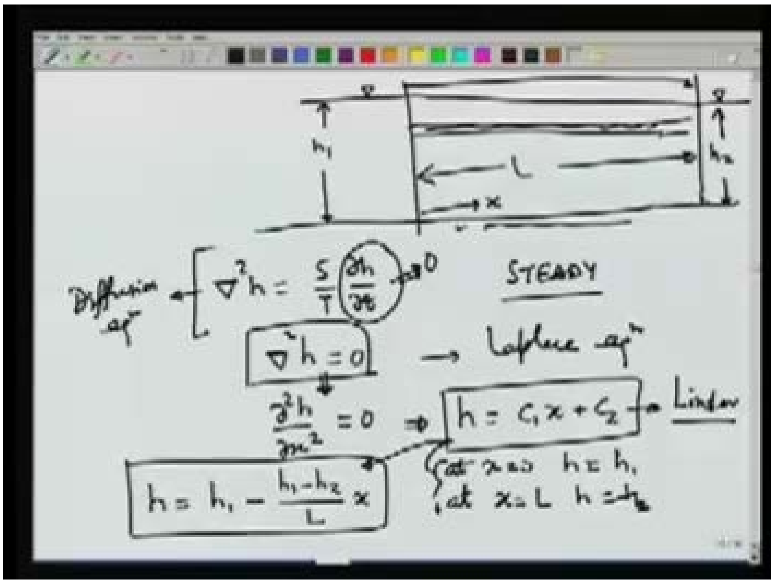

For example if we consider the case of one dimensional flow, we have a confining layer ground level and suppose we are looking at a case where flow is occurring between two water bodies, one of them here is such that the height of water is h_1 and the height of water here is h_2. The piezometeric head in the confined aquifer may be like this, or like this or it may be a straight line like this. So in order to find out which is the variation in the confined aquifer, we need to solve the equation of motion which is again written here as this. Now if we make the assumption that the flow is steady then del h by del t because steady means the parameters not changing with time so del h by del t will be = 0 and therefore we will get a simple equation which is known as the Laplace equation. This equation has a form similar to the head diffusion equation therefore this is also called the diffusion equation. Now let us say that this is a steady state flow condition therefore we will need to solve this and let us also assume that this is a one dimensional flow. So the flow is taking place only in x direction. So we can write since the head is not changing in other directions, this equation simplifies to del 2 h/del x square = 0. The solution of which is h = C_1 x + C_2. C_1 and C_2 are constants which will be obtained by the boundary conditions. In this case, considering the length of the aquifer to be L, the boundary conditions are at x = 0, h = h_1 and at x = L, h = h_2. So using these two boundary conditions, we can obtain the solution of this equation which tells us that h is linear. In this case if we have a flow between these two water bodies, the piezometeric head in the aquifer will vary linearly between h_1 and h_2 and the variation can be obtained directly from applying these two boundary conditions from which we can write h = h_1 – (Refer Slide Time: 58:35). So this is obtained by applying these two boundary conditions in this equation.

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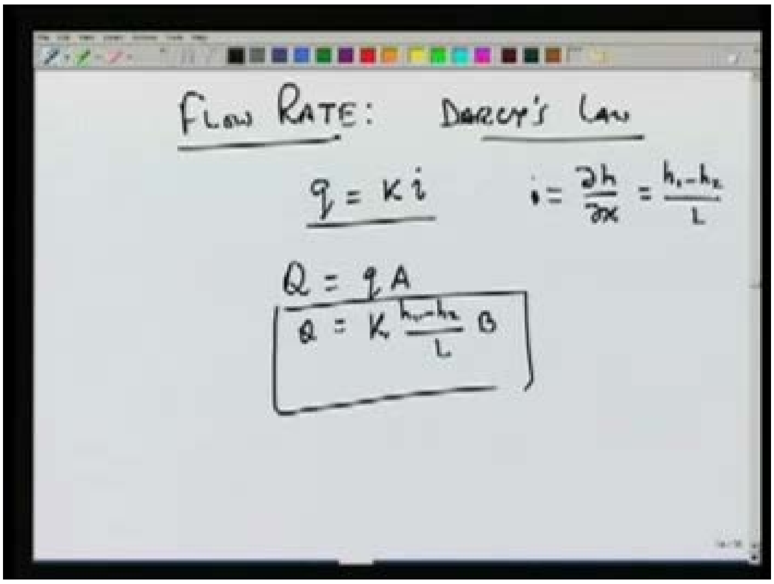

If we want to find out the flow rate, we can use the Darcy's law which gives us q = Ki, i in this case is del h/del x, because we are considering one dimensional flow which is simply $h_1 - h_2$ over L. Since the piezometeric head is linear, i is constant throughout the length and is given by the delta h which is $h_1–h_2$ divided by L and this will give us the apparent velocity or Darcy velocity using which we can find out q into area. If we consider unit width, then area will be equal to the thickness of the aquifer which is b. So we can write this as K $h_1 - h_2$ over L into B. We have seen today how to obtain the equation of motion for combined aquifer, how to solve it for given boundary conditions and the same thing can be done for unconfined aquifer. The thing is that unconfined aquifer is a little more complicated because the water level itself determines the thickness of the aquifer. Thickness of the aquifer is not constant but is varying from place to place. So we need to make some simplifying assumptions in order to derive that equation. Similarly for confined aquifers also we have solved the equation for a one dimensional steady state flow conditions.

If we go for unsteady flow, it will be a little more complicated or if we go for radial coordinate system, the solution will be a little more complicated. So we will look at these solutions for confined aquifer and we will also look at how to derive the equation of motion for unconfined aquifer in the next lecture.