Environmental Air Pollution

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Lecture No. 30

Derivation of Gaussian Model

We start from where we left. We were trying to develop the relationship between emission and its impact.

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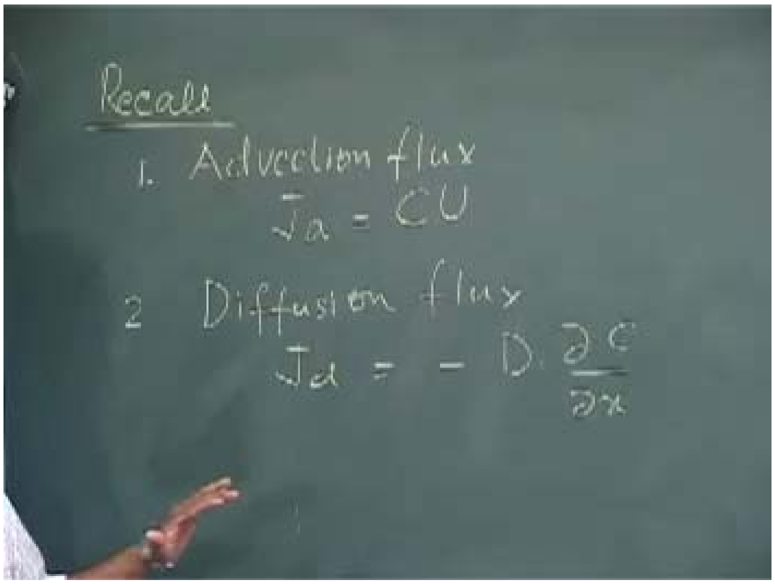

If you recall, we were talking about flux – flux of the pollutant and the first thing we said about the flux of the pollutant, number one was… which flux? Advection flux – flux due to… flux of what? Pollutant. We will define that as J_a and that was equal to…? CU. CU and there was another kind of flux that we have defined more because of diffusion –that we call flux due to diffusion and that we say J, I do not know J_d, that was negative diffusion coefficient. Did we say D_X or did we simply say D? D. We will write D. Then, del C…. divided by del x. We want to utilize them now to develop some kind of relationship or something called equation of continuity or advection diffusion equation. So for that, let us suppose I have a strip.

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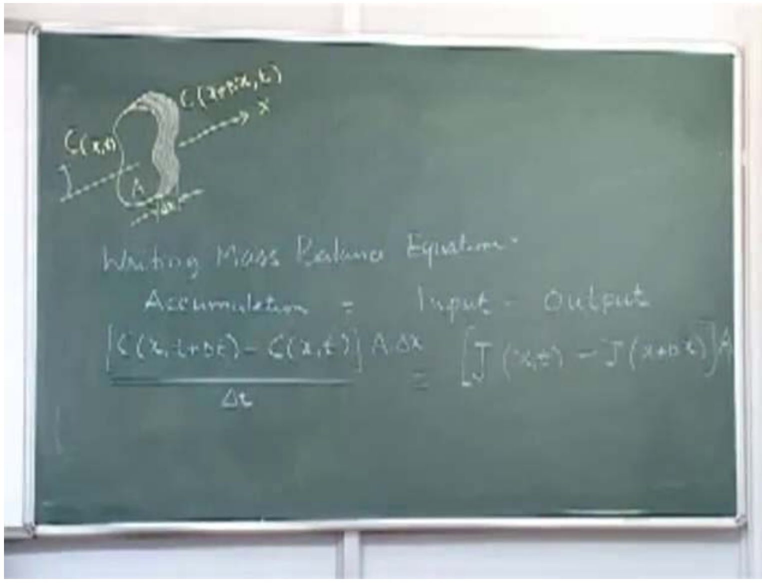

We will try to do it in one dimension. My flux is in this direction and the area is A and the direction I am calling this as X, the width of the strip let us call that as delta X. Then, C is the concentration at this point, C here – that is at x and t, concentration C has two domains – it can be function of x and it can be function of t, here on this side of the strip, C will be at time x plus delta x and t. Just to make it clearer, if you like I can even make it look a little solid. Now, I want to write the mass balance about this strip, writing mass balance, so I can say the accumulation inside the strip is input minus output and the flux that I am talking about here is let us say J. What I can say about the accumulation is the concentration at x times t plus delta t minus concentration at x, t – that is the change in concentration you have. I must convert this into mass per time, so what do I do? Area times del x and time is delta t, that is the accumulation part.

All right. This is again something similar to del C by del t, but we are talking about this strip the concentration, we are talking about accumulation. How much is the accumulation rate? Concentration at t plus delta t minus concentration at t divided by delta t, so concentration per second or per time, but we are interested in mass balance, so area times delta x. Is that all right? This must be equal to… I can take the advantage of J now. What do I do? J at the time at x, t minus J at x plus delta x. What do I multiply here to get the same units as this one? What is flux?

Mass per time per area, right? What I want is mass per time, so I have multiplied by area here. See here if you agree with this. Is it all right?

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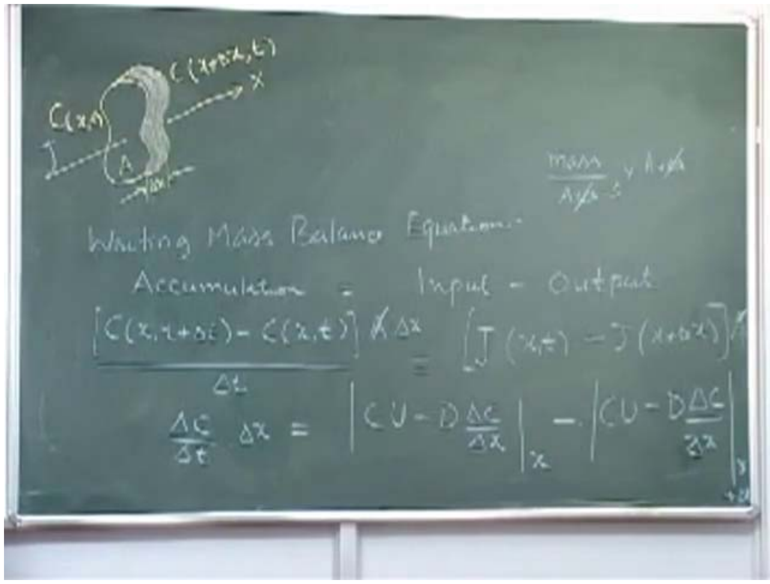

We will do it once again. What are the units of J? Mass per area second, so if I multiply by this area I am getting mass per second; mass per second in, mass per second out is mass per second accumulated, right? That is also mass per second (Refer Slide Time: 08:00), mass per second, mass per second. Is that right? All in agreement with me? This area, area will go off from here and this one if you agree, I can write this as del t times let us say I am still maintaining the delta x here. For J, what can I write? This plus this – that is the total flux, flux due to advection and flux due to diffusion. If you agree with me, I can write here CU minus D and since we are writing still in deltas, I will write this as delta C over delta X. This is evaluated at what? At x, so at x minus CU minus D del c over del x, delta x rather evaluated at x plus delta x. Any problem so far? Yes?

[Conversation between student and professor - Not Audible (09:42 min)]

In this, in here (Refer Slide Time: 09:52)? What are you saying?

[Conversation between student and professor - Not Audible (09:54 min)]

At the outlet?

[Conversation between student and professor - Not Audible (09:59 min)]

x plus delta x

[Conversation between student and professor - Not Audible (10:05 min)]

Where is the C I am talking about? Concentration is in the strip, concentration is in the strip – not at the phases, right? The concentration is in the strip, not at the phases; I am not talking about the phases, the concentration is in the strip, so how will concentration in the strip change? Concentration of the strip will only change because of change in time, time domain, so this is the concentration, this accumulation in the strip, if you like in strip. Then, suppose you take the limit delta t tends to 0, delta x tends to 0, let us see what we can write.

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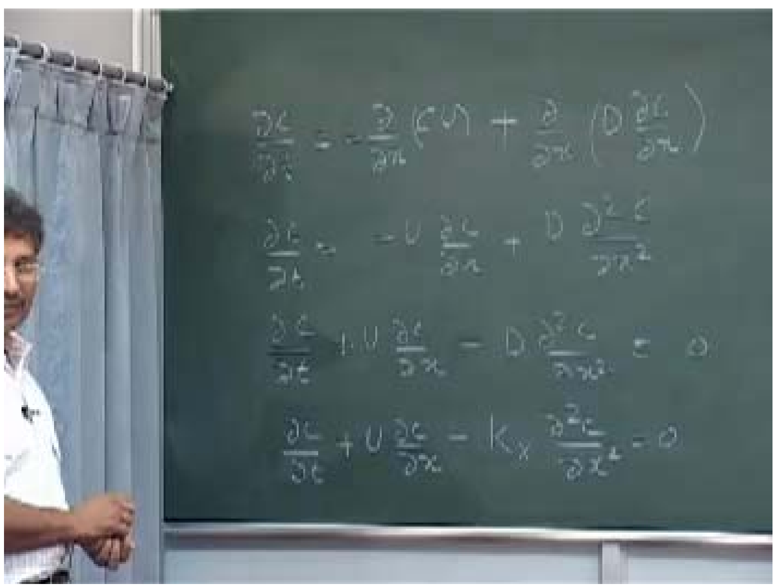

This I have this minus this, so I am getting this as minus, that will be plus, you know del x of D, this by del x. See if you are getting the same expression or not. You also have to make sure what I have written is correct. Is it minus sign? You see everyone how the minus sign comes here. For derivative, it is derivative at x plus delta x minus the concentration at x whatever, but this you see here (Refer Slide Time: 12:37), this term is plus minus this. We got it the other way, we got it the other way, we will do it again, let us see, should be minus. Does everyone see that that will be minus or do we need to do some expansion? You see that is minus, very good, so this will be minus. Or minus suppose I say U is not a function of x, it means if I am going, I am travelling along the direction x, the wind speed does not change in my case, so that is the approximation I am taking or that is the assumption I am making, so U is out del c by del x and I am also saying the diffusion constant is not function of position, so this also can be taken out – D or plus U del c by del x minus is equal to 0.

If you agree, I can write a three-dimensional form of this one but before we do that, you recall we are taking this D as the molecular diffusion, so now we will replace this D with the turbulent diffusion because most of the time, we encounter turbulent diffusion in the atmosphere. Hoping that the theory remains the same, I can write this plus U del c by del x minus… normally, the turbulent diffusion we are talking in the x direction only, so I can write the turbulent diffusion coefficient K_X. Note that K_X is the turbulent diffusion or the diffusion term in the atmosphere and the units of K_X will be the same as that of D, because we are simply replacing the D by K.

Now if you agree, I will generalize this equation in the form of a three-dimensional equation because our air thing is always three-dimensional. This equation is valid even for a water system, this will be valid even for the river system – one-dimensional thing, but we will make it three-dimensional and try to write in three dimensions.

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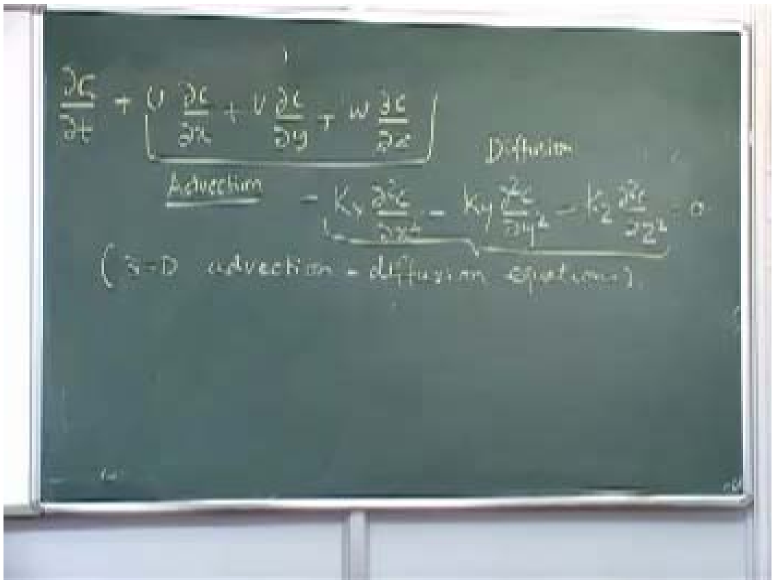

Minus K_X del square C by del x square minus K_y del square C over del y square minus K_z and this is what our three-dimensional advection diffusion equation is – this is our three-dimensional advection diffusion equation. If you agree, I can write that these terms together are my advection term or what is causing advection diffusion at advection transport and this term I can write as diffusion or turbulent diffusion – I am not using the word turbulent diffusion [18:03]. Here, we have not added the terms source and sink – to keep life simple, we have not added the term source and sink; we are not adding any more emissions as a source but soon we will do that – the same thing.

Some of you.… Now, let me explain. U is the velocity in the x direction, V is the velocity in the y direction and W is the velocity in the z direction. Similarly, the turbulent diffusion coefficient in the x direction (Refer Slide Time: 18:49), in the y direction and in the z direction. Clear?

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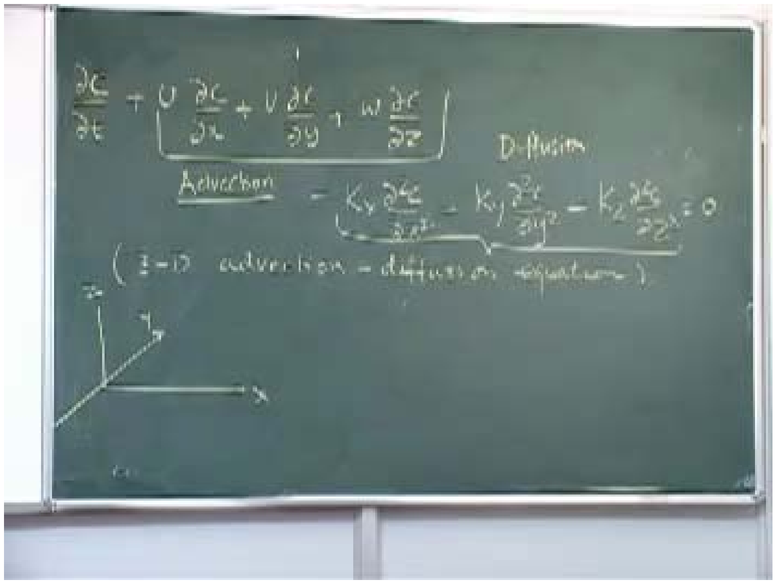

To give you little feel of the system of coordinate system, this is the system, this is our x direction, this is our z direction, and this is our y direction. Now, what I want to do is… some of you know about the… have you heard about plug flow reactor? Have you heard about plug flow reactor? In plug flow reactor, there is no diffusion at all – it is only the advection. If I want to write the diffusion equation or the equation for the plug flow reactor, I will say this term is 0 and suppose I have taken a big box and I have put some ink or something to disperse. So there is there is no movement of the fluid from the box and I am shaking the box, so I am causing turbulent diffusion but there is no movement of the fluid as such, the velocity is 0. Suppose I take a box and put some let us say ink and I am shaking it, so then my equation that will be valid will be… all these terms will be 0, the advection terms will be 0. So you can modify this equation, which is a very generalized form of the equation; you can modify depending on what situation you are talking about.

Having said that, I can say this is the mother of all equations that we need to know and from there, we will make many assumptions and try to solve it for the concentration part. But then if you agree with me, these equations do not have an analytical solution. This partial differential equation does not have an analytical solution. Once the equation is not solvable with your conventional differential equations, the methods that we use, how do we solve the differential equations once we do not have the analytical solution or closed form solution? How do we solve them? We solve them numerically, so one way people went about solving this equation is to solve it numerical solution.

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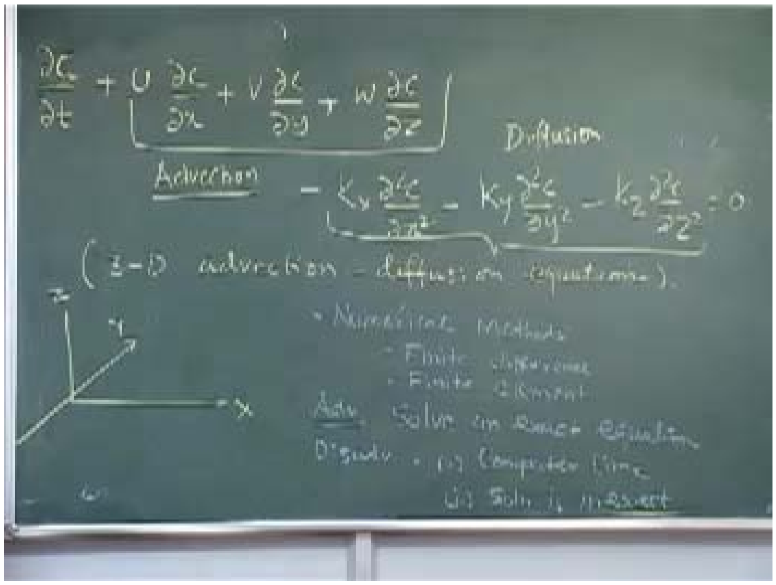

One way to solve this equation is the numerical method. There are many methods through which you can solve it, for example, finite difference – you can use the finite volume or finite element that is more commonly used. What is the advantage of numerical method? The advantage of the numerical method is you are using the exact equation, the equation that you derived from basic fundamentals – you are not tampering with the equation in any form, you are not making any assumptions, so you solve an exact equation. The disadvantage is you need sophisticated computers – computer time and the second solution will always be approximate, numerical methods will never be 100 percent true. The solution is inexact, then I can write here solution of… equation solved through numerical means is inexact, you will always agree with me. If that is the case… but then they are analytical methods also, if we can solve it.

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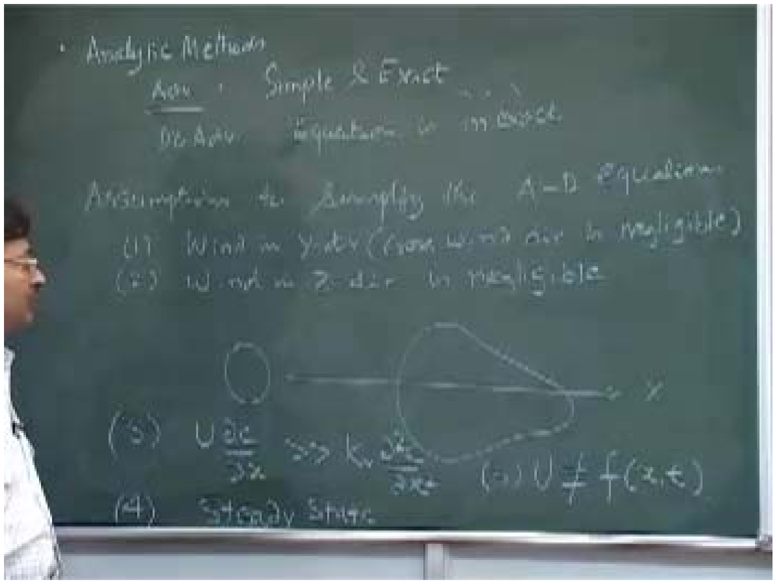

In this case, there is no analytical solution but I can make some assumptions, I can simplify life and then solve it but once I simplify the equation, is the equation exact? The advantage of this one, the advantage of a such method or the analytical method is – we are talking in context of this equation, do not forget that – advantage is simple, well sometimes it may be so simple that you do not even need calculators to solve if you have the closed form solutions; advantage is very simple but the disadvantage is to start with, the equation become inexact; simple and exact but you started with equation is inexact.

So there are two ways to solve it. This one (Refer Slide Time: 25:48) – we will talk little later; after few things which we have covered we will do one example and show how we can solve this equation numerically – you should have an idea how this can be solved numerically but the focus of the coming lectures will be on how to get the analytical solution of an equation that we can simplify and can get an analytical or sometimes also called the closed form solution. Then I have to make certain assumptions, so now our objective is to get the analytical solution; to get the analytical solution, I must simplify and how can I simplify? I make some reasonable assumptions, so we make some assumptions.

Assumptions to simplify the advection, I can write this as advection diffusion equation and you see the assumptions we will make: assumption number one that I will make and I will see the impact of that assumption in my equation that I have written on my left hand side. Compared to U, what I am saying now is the cross-wind direction, my x is the wind direction, so compared to my wind in the x direction what do you think the wind will be in the perpendicular direction? Almost negligible. So I can say the wind in y direction or more commonly we write cross-wind direction is negligible.

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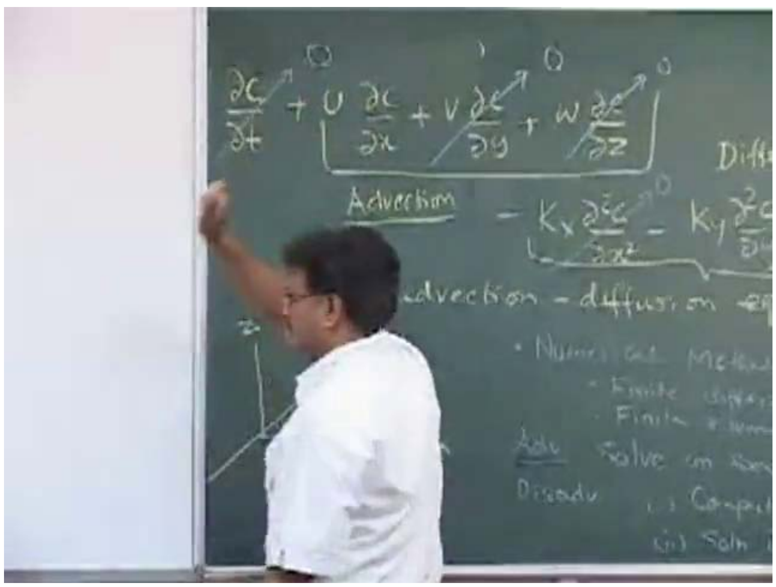

What do I do? Drop this term. What do you say about W? I can also say the vertical winds will also be negligible, so assumption number two is wind in z direction is negligible. These are the important things to understand as to which I get I will get a solution that we will use very frequently or if you stay in the air pollution field, you will use it all your life but then you should know where that equation has come from. Then, if that is the case, I make this equal to 0. I make the other assumptions. A body is being advected in the x direction and at the same time, the body is also diffusing in the x direction. Suppose I have air pollution in the form of this circular thing and it is diffusing in all directions. So this will become bigger because of the diffusion – as it travels it becomes bigger and bigger because it is diffusing, but when we actually see it, it remains same in this one but you see here, it diffuses much more in the x direction and that diffusion is because of the advection, because we have dropped the advection terms in y and z direction but here you see this term, which is the advection in the x direction (this is my x direction), so the term advection is overwhelmingly large compared to diffusion in the x direction. Did you understand what I said? The diffusion due to advection in the x direction is overwhelmingly large compared to the diffusion because of the turbulent diffusion in the x direction.

So the third assumption I can say looking at those things if you agree with me is U del c by del x is much much bigger than what? K_x del c by del x square. It is much bigger than K_x, agreed? I will apply my assumption there and drop which term? This term compared to this, so this term I put this equal to 0.

The fourth assumption… because I must keep on simplifying until I can solve it, the fourth assumption I can say steady state, steady state condition and you all know the derivative with respect to time should go to 0. The other implicit assumption – you all know about this one – is that dispersion coefficients are not function of time, they are not function of space; another important assumption that is implicit in this one is the U – the wind speed in the direction X is neither function of time nor is it a function of space. Do not forget, let us write so that you have it with you. I will write little mathematically, U is not function of x and T and so when there is a constant wind that you are facing all the time.

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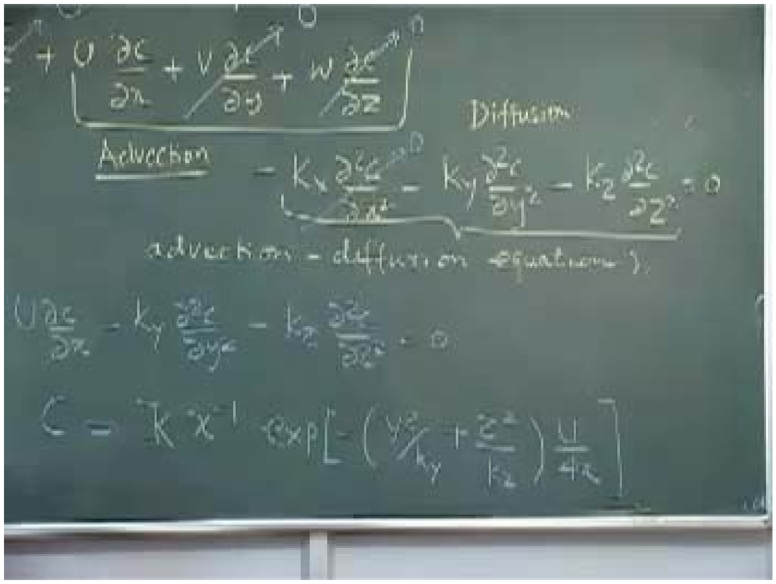

Once I make these all these assumptions, I will rewrite that equation. That term is gone, so I have minus, this term is gone, K_y del square c by del y square minus K_z is equal to 0 – a simple equation. There are ways to solve this equation as you can figure out from books or from the various methods that are available. I will write a general solution for this one so that…. What I want to find out is… do not forget my interest is to find out the ground-level concentration. The general expression for this one is C equals to a big K let us say – this is a constant, exponential minus y square K_y plus z square K_z times… – that is what is a general expression I get. I promise you that we will make some assumptions and solve this equation analytically.

Suddenly, life becomes very simple, you can do anything and everything with little calculator or with your log tables and forget about solving it numerically, but do not forget we have made many assumptions to get to that thing – sometimes, the assumptions are correct, sometimes they are not so correct, but anyway we have simplified the equation. Now the challenge is to find out the K. For any differential equation when you want to find out the constants, we need some information on the initial boundary condition or the boundary conditions. I will discuss the boundary condition, which will help me to find K. Now I am making something [35:14] or let me draw the picture – that is even better.

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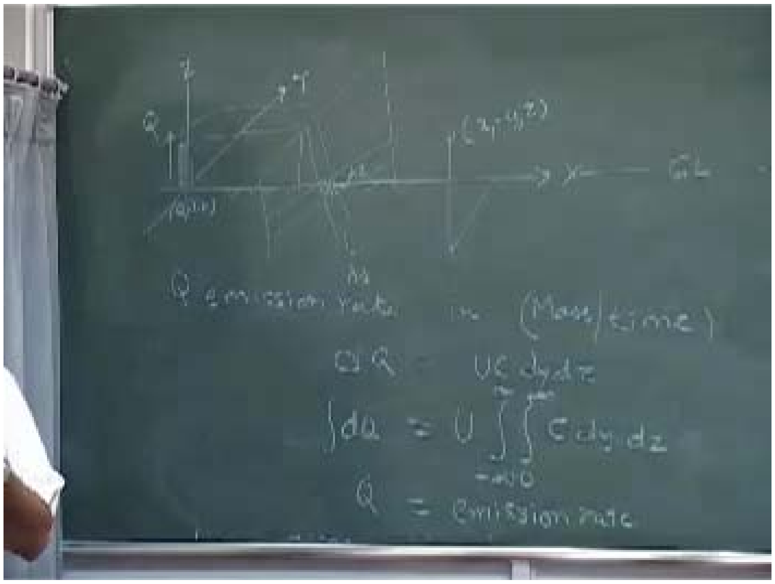

Suppose I have a chimney and this coordinate is 0, 0, 0. To make the things little more clear I can also say… What are the coordinates of this point? Do not forget this is X direction, this is Y and this is Z, so this coordinate of this point just for the sake of understanding so that we understand our coordinate system very well because we will use this several times, if you agree, I can write this is equal to… just to give you a little feel of this one. Let us say this is my chimney and now I am introducing a source. So far we had not talked about the emission source but now, I am introducing a source and I have a plane perpendicular to the X direction. Imagine a plane that is perpendicular to the X direction and that is extending infinitely in the Y direction and infinitely in the Z direction but cannot go negative of Z direction because this is the ground level; do not forget that this is my ground level.

If that is what is my plane, infinite plane and if I take one strip this way in the plane, the other strip this way, this will be my dy and this will be my dz. The amount of the… the Q, the emission rate here I am calling this as Q – Q is the emission rate in mass per time and so you will agree with me that dQ passing through this strip will be nothing but U times C dy dz. If I integrate from both the sides… U as I am saying is not the function of anything, why I integrate from minus infinity to plus infinity is because it can go in this direction and the plume can go in this direction, so this is minus infinity to plus infinity and this is double integration and z I can integrate because nothing will go below the ground from 0 to minus infinity let us say, 0 to minus infinity, so I have to write C dy and this will be equal to nothing but

[Conversation between student and professor - Not Audible (39:20)]

In Y direction what?

[Conversation between student and professor - Not Audible (39:25)]

This will be nothing but equal to Q – this will be the entire amount of that Q that will flow through that plane; that is my total emission rate, which is known to me – I know how much is the emission that is coming out from the source, I can say so much is the emission coming out from the Panki power station for example, so Q is known to me and if I put… in place of this C. I can put this expression (Refer Slide Time: 40:13), right? This is my C. Integrate this one, find the value of Q, find the value of K in terms of Q. Is everyone seeing what I am doing? I take the value of C from that equation, put that expression here and I can find the value of the constant large K or big K in the form of Q and that constant comes out to be…

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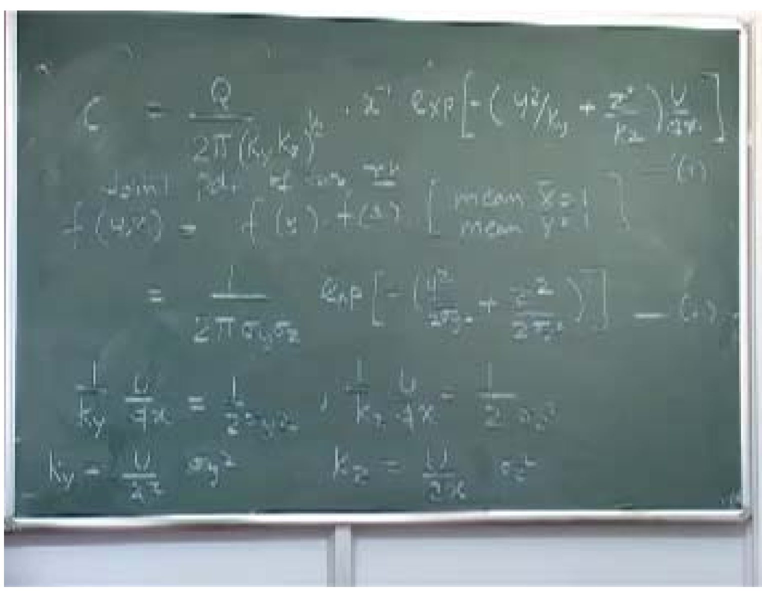

With this boundary condition, my big K is equal to… we will check the same because we are not deriving it fully. If you agree with me, I will take little advantage of what I am writing here and call this as C and multiply with this one (Refer Slide Time: 41:45). You write the exact same expression that we are getting minus y square… I can write this way. I want to do define something else. I want to take a little bit of statistics and we say normal distribution. Leave this one for the time being.

Suppose I want to find out the probability density function or joint probability density function of two random variables, I am just giving you an example, y and z, how do I do it? How do I get the joint probability density function? I simply do joint probability density function or let us say variable y and z is simply…. If you will agree with me, what I can write is…. Another assumption I am making is that mean…. What I am writing is the joint probability density function of two rv – rv stands for random variables. Now what I do is…. This is my equation (1), this is my equation (2). I compare my equation (1) and (2) and let us see if I can do something. I look at this part inside the exponential term.

Let us see if I can… if you agree with me about a few things, K_Y is coming from there or rather 1 upon K_Y times U upon 4 x, We are comparing that, so I just need some space here, so 1 upon K_Y times U upon 4 x is equal to 2 sigma y square. I am just comparing that and these can be just scaling parameters here, scaling parameters here, but then it is essentially joint probability density function and suppose I quit this and correspondingly I will change my other things also, so I can do that in place of this (Refer Slide Time: 46:35)… and similarly if you agree with me, I can say 1 upon K_Z times U by 4 x is equal to… so this K_Y will be equal to U upon 4 x 2 sigma y square and K_Z will be equal to 4 x 2 sigma z square or…. Now, based on this understanding, based on some statistical comparison, can I write rewrite my equation (1) in terms of C?

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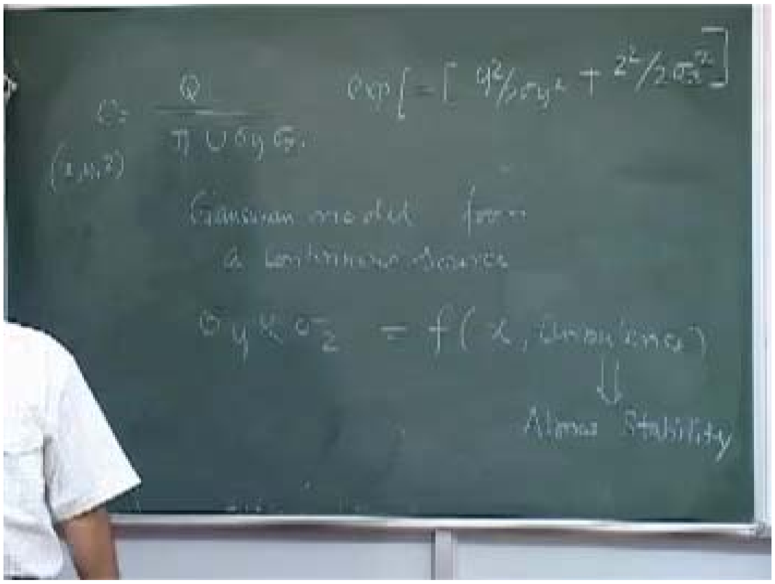

C is equal to Q upon 2 pi K_y times K_z and then the square root, so what do I get? You see here the x, x will cancel, right? The 2 will come out and then sigma y, sigma z and U will be square and root of U will be U, so what I can write here is.…

[Conversation between student and professor - Not Audible (49:03 min)]

The x will cancel and the 2 also will come, let me write 2 here and then exponential, the 2 will go up, exponential minus and this is what is called your Gaussian model from a continuous source. This 2, 2 will cancel, so let us take this one out – this is a Gaussian model for continuous…. Why do we call it Gaussian? Because our expression was in the Gaussian form; this (Refer Slide Time: 51:01) is just equivalent to the Gaussian distribution, this came in the Gaussian form.

Somebody very quickly tell me what will be the units of sigma y and sigma y because dimensionally, this is an analytical solution and so the units must match. Units of sigma y and sigma z? Meters. Meters, length units, good. We will go a little bit more into this one and explain a few things and where am I trying to find out C? I have not lost the feature of three dimensionality in the whole thing – I have made some assumptions but my equation or my model is valid in three dimensions, so my C will be at any value of x, y and z. What you see here is this one. Now, we will explain the meaning of what is sigma y, sigma z but quickly tell me where has the x gone because here I want to find out the concentration with respect to x. Where is x?

[Conversation between student and professor - Not Audible (52:23 min)]

x somehow is… the way we defined sigma y, sigma z, x is inbuilt in sigma y and sigma z, so always remember that the sigma y and sigma z will be function of x – do not forget; sigma y… x has disappeared from here, but x has disappeared because my sigma y and sigma z will be function of x. Similarly, my dispersion is lost; where has it gone? My dispersion is in some formation of sigma y and sigma z, so that we will see.

Can you quickly… before we end the class, sigma y and sigma z will be a function of x and…?

[Conversation between student and professor - Not Audible (53:15 min)]

K_y and K_z and K_y and K_z are function of turbulence, right? In our case so far what we have done is we are defining turbulence as…? What are we defining turbulence as? Atmospheric stability. Recall? See the connection here: x is inbuilt between sigma y and sigma z, sigma y also represents K_y and K_z, K_y and K_z are the turbulent diffusivities and as per my earlier classes, I am defining turbulence in the form of atmospheric stability. So we conclude that sigma y and sigma z will be a function of downwind distance x, including z also – sigma z also, function of x and turbulence and this turbulence so far we have discussed and so far the way we have talked in the class is dependent on or is defined by atmospheric stability.

That is why we spend so much time explaining stability. If there is any question, I would like to answer. The turbulence that we are getting incorporates both mechanical turbulence and thermal turbulence? Everything. Sir, how do we separate that? How we separate that is either we need to separate that or we somehow find a composite way to find that one, because ultimately we want to find out the C. How to find out the turbulence, which is mechanical, thermal or because of the kinetics – how to do that one? That we will see but by and large, that will be governed by the atmospheric stability. We will see how it is governed later, but as I always say in the class, we have the qualitative and quantitative. At least in the qualitative sense what function it is we do not know, but you will agree with me that the sigma y and sigma z are the function of x and turbulence; we have defined turbulence as the stability so far in the class and we can find out the stability through many ways – one of the ways was the Turner method to get the data from the airport and find out the stability from any place in the world. We will stop there, thank you very much.