Advanced Hydrology
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Lecture – 39

Good morning and welcome to this video course on advanced hydrology. In the last class we looked at different types of aquifers that is confined and unconfined aquifers. What is a burst aquifer and what is a leaky aquifer? We also looked at various characteristics of these aquifers such as how the specific yield or the how the yield is taken out of a particular aquifer. We said that in the case of unconfined aquifer when we pump the water out there are changes in storages. But in a confined aquifer when we take the water out the water does not come from the storage, but there is a pressure difference that is absorbed. What we would like to do today is we will start with defining, what is an anisotropic aquifer? In which the properties or the aquifer properties will be different in different directions and how we can deal with such a situation.

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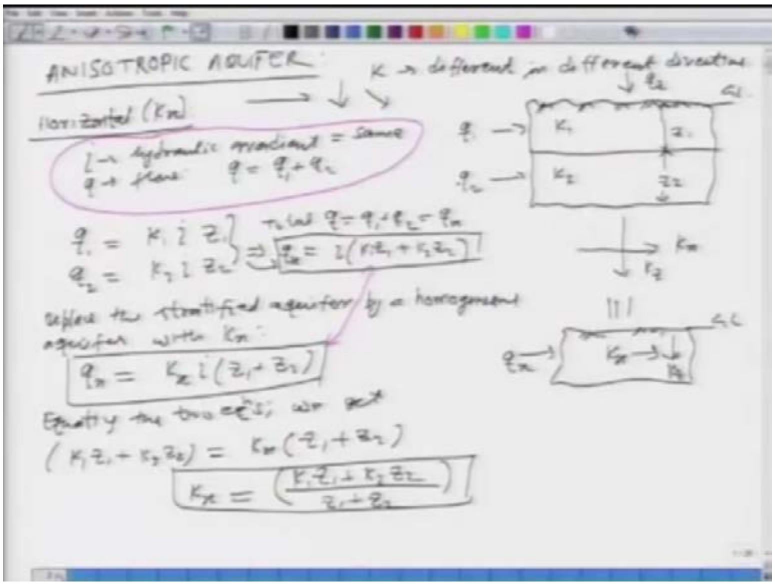

So, let us see here what is an anisotropic aquifer? We have seen that in an idolize aquifer we assumed that it is isotropic, that is the properties of the aquifer are same in all the directions. So, in the anisotropic aquifer which is opposite of an isotropic aquifer different property such as the hydraulic conductivity which is the most important one is let say different in different directions. When can that happen? Well when we have a situation where we need to model things horizontally, and also vertically or in any other direction. But the value of K would be different, there may be many reasons due to which the aquifer properties maybe different due to or in different directions, and let us see how it can be possible.

So, if we see here if we have layered soil this is your ground level. And then if you have layered soil that is to say upper layer is of hydraulic conductivity K 1 and the bottom layer or the second layer if is of hydraulic conductivity K 2. And let us say the thickness of the upper layer is z 1 and the thickness of the second layer is z 2. And let us say that there some flow taking place q 1 from the upper layer and q 2 from the bottom layer and few may also be taking place in the vertical direction. Let us say that is q z, z is your vertical direction here. Now, if we look at this system where there are 2 different layers sitting on top of each other, if we have a horizontal flow if the flow is taking place due to certain gradient. Then what will be the hydraulic conductivity of this combined aquifer, you have K 1 on the top and K 2 from the bottom. So, for horizontal case it will it will have certain hydraulic conductivity, because of that stratification.

The same aquifer or the same type of soil may experience flow in a vertical direction during infiltration. So, if you want to model that what value of K you would use? If you take this sample you can determine K 1 and K 2. But how do we model how we account for these special variations in the hydraulic conductivity in this case? So, what we will do is we will try to look at the mechanism by which we can find what is called the equivalent hydraulic conductivity in stratified soils. So, let us say the hydraulic conductivity is K x in the horizontal direction and it is K z in the vertical direction. So, first we will look at the horizontal case the flow is taking place in the horizontal direction, what will be K z which is the composite hydraulic conductivity.

Now, when we look at these things what we will do is in order to derive the equation or compute or estimate this K x we will look at 2 things; 1 is the i which is the hydraulic conductivity sorry the hydraulic grade line or gradient. And the other thing is the q. What do you think in this particular case, when the flow is horizontal what will the q v in terms of q 1 and q 2? It is clear or it is obvious to see that the total q would be nothing but q 1 and q 2, the sum of q 1 and q 2 what about i when you have a horizontal flow taking place you have the stratified soil 2 different soils flow is taking place in the horizontal direction.

So what is the driving force? Driving force is the hydraulic gradient which will be actually same for the horizontal flow situation. So, this is important to realize. So, what we are going to say is that i is same in the 2 soils. So, we use this important results these two for the horizontal case in which we say that the hydraulic gradient will be same and the q will be the sum of the individual q sum the individual layers. So, what will be q 1 and q 2? So, you have q 1 as K 1 i is same z 1 times 1 you take the unit width. And similarly, you have q 2 as K 2 i z 2. The total q you are saying is the q 1 plus q 2 let us say it is q x total flow in the x direction. So, what will be this q x; this q x actually will be let us say i times what if you put the values individual q 1 and q 2's it will be K 1 z 1 plus K 2 z 2. I have just put these two things in this equation so that is the total q.

Now, what we do is we replace the stratified aquifer, we replace the stratified aquifer by a homogeneous aquifer with K x. So, what we are saying is basically this whole thing is equivalent to an aquifer like this, this is ground level. And this has a hydraulic conductivity of K x in this direction and q x is going through this a single soil. And this vertical direction you say is K z and we will work on the K z little later. Then what will be q x for this a homogeneous aquifer? It will be nothing but conductivity is K x i is same and then you will have z 1 plus z 2. So, it will be K x i z 1 plus K x i z 2. So, we now we have 2 expressions for q x and then you say we equate these two, equating the 2 equations, what do we get? We will have i will cancel out. So, you will have K 1 z 1 plus K 2 z 2 is equal to what K x i is gone and then z 1 plus z 2, is that clear?

So, that it will give you the expression for K x which will be nothing but what K 1 z 1 plus K to z 2 divided by z 1 plus z 2? So, this way you see that when we have a stratified soil or aquifer in which the stratification is in the horizontal direction. And the flow is taking place along that direction. In such a situation which is the composite or the equivalent hydraulic conductivity. Well it is nothing but the wetted average of the individual hydraulic conductivities. The expression h you have just seen is what K K 1 z 1 plus K 2 z 2 divided by the total depth z 1 plus z 2? So it is nothing but the weighted average if there are more than 2 layers we can extend the same concept when we have multiple layers. So, we may have 3 4 5 6 or maybe n different layers. So, we can write a general expression as follows.

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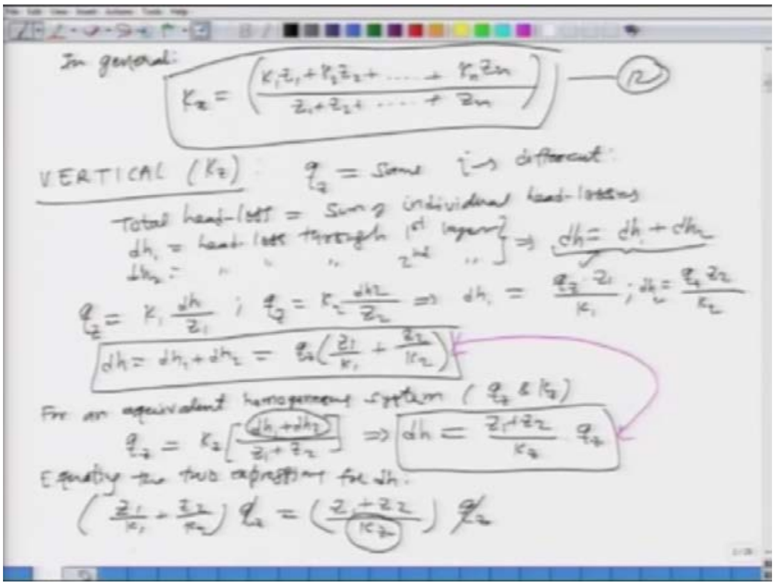

So in general we can say that your K x is going to be equal to K 1 z 1 K 2 z 2 plus K n z n. This whole thing divided by z 1 plus z 2 plus z n. And I am going to number this equation as 12. So, this was the horizontal case. So, if you have the same stratification what will be the hydraulic conductivity in the vertical case K Z? For the vertical flow for the vertical flow if I go back to my figure the flow is taking place in the downward direction let us say like this. So, what will be same the q will be same or the hydraulic conductivity the hydraulic gradient will be same; obviously, the q will be same as we are going down the same q will be passing both the layers.

So, you say your q z will be same and I would be different and this i will depend upon what the individual thicknesses. What would that be, what would be the total head loss? So, there will be certain amount of head loss in the first layer, certain amount of head loss in the second and third and so on. And the total head loss for passing that q through all the layers will be what will be the sum of the individual head losses. So, if we understand this physical concept then we can apply that to determine your equivalent hydraulic conductivity in the vertical direction.

So, let us write that expression so we say that the total head loss will be the sum of individual head losses. So, if you say that your d h 1 is the head loss through the first layer the top layer we have and d h 2 is the head loss through the second layer. Then we have total head loss d h is d h 1 plus d h 2 that is what we are saying. Now, we said that the q is same so what is this q z through the first layer? Well it is nothing but q 1 the head loss through this or the gradient through the first layer. So, it will be d h 1 over z 1 d h 1 is the total head loss in what distance z 1. And similarly, you also have q z this K 2 times d h 2 is the head loss through the second 1 over a distance of z 2.

So, now if we use this equation what will that be? So, these 2 equations actually give you what your d h 1 is your q z times z 1 over K 1 is not it? And then d of h 2 will be q z times z 2 over K 2. Let me see if that is correct so d h 2 is q z Z 2 over K 2. Now, we use these 2 relations so that way you will have d h 1 is equal to d h 1 plus d h 2 which will be equal to q z outside of your what z 1 over K 1 plus z 2 over K 2? So, this is the total head loss through the all the layers. Now, what we do is we write the equivalent expression for an equivalent homogeneous system, for which you have flow is q z. And conductivity is K z in the vertical direction that is what we are talking about this one. This is the equivalent 1 what would that be? You have q z is equal to K z times total head loss is what d h 1 plus d h 2 divided by the total distance which is z 1 plus z 2. Now, what does this give you? This is equal to you d h. So, you write the expression that d h is equal to what it will be z 1 plus z 2 over K z times q z.

So you have another expression for d h or the total head loss again you do the same thing, compare these 2 or equate these two. Equating the 2 expressions for d h what are we going to get? We will have z 1 over K 1 plus z 2 over K 2 of your what did we have q z is equal to what we will have z 1 plus z 2 over K z of your q z? You see that q z will cancel out and what you need to find is this K z which you can do easily and I will give you this.

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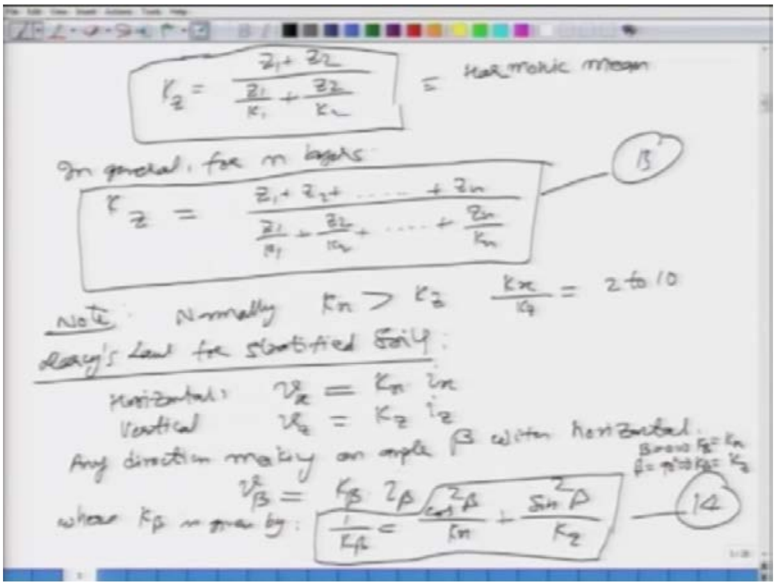

So, it will be your K z finally will be equal to what z 1 plus z 2 over z 1 over K 1 plus z 2 over K 2. So, this is your equivalent hydraulic conductivity in a stratified soil which is what which is the nothing but the harmonic mean with respect to your z's. And earlier one this one we had seen this was what this was your weighted mean, weighted simple average? So, in general we can extend this expression actually for n layer like we did for the earlier case, what will be K z? K z is going to be equal to your z 1 plus z 2 plus. All the way to your z n some of all the individual layer thicknesses divided by z 1 over K 1 plus z 2 over K 2 and plus z n over K n. So, you see that how we can determine the conductivity in a direction which is important for us. So, we have looked at 2 different directions horizontal and vertical which are most predominant in the ground water flow situations.

So, this way we have an anisotropic aquifer or we have a mechanism of modeling the aquifer in which the hydraulic conductivity or the aquifer properties are varying in different directions. So, if we have horizontal flow we use K x, if I have vertical flow we use K z. So, we do not have to use a single value of hydraulic conductivity. Now, what do we do if we have a flow taking at an angle or in some other direction? We will not go through the derivation but I will I would like to give you the final expression for that. So, let us look at that. Before we go to that just an important note that normally your K x is greater than K z that is what has been found in various aquifers. And in fact that ratio of K x over K z has been found in the range of 2 to 10 for most of the aquifers.

Now, once we have done this analysis what we can so is we can apply the Darcy's law stratified soils or aquifers which is which are certified like this. So, you have horizontal flow, you will say the horizontal component of the velocity is going to be K x times i x. And if you have vertical flow only then you will have v z is K z i z, but if we have flow taking place in any direction beta. So, if we have the flow taking place in any direction making an angle beta with the horizontal. Then we say your v beta is equal to K beta times i beta means i in that direction along the direction of flow. Remember that hydraulic grade line i is always taken along the direction of flow if it is horizontal in the horizontal direction between 2 points we find out, what is the head loss or what is the difference in the heads? Similarly, for the vertical case also we do the same thing.

But if the flow is at an angle we see along that direction of flow how much is the difference in the altitude of the head between those that distance along that path. So, that is what we mean when we say your v beta is equal to K beta times i beta where i is taken along the direction of the flow, where the K beta that is what I was talking about is given by this expression which we will not derive. We will take it graph or granted 1 over K beta is cos squared beta over K x plus sin square beta over K z. This is a general expression which can be used let me number these equations I think I forgot this is 13 and this is 14.

So, if you look at this expression you have 1 over K beta is given by this. So, if you have the horizontal flow what is beta for horizontal flow beta is 0. So, if the beta is 0 then what which term will be 0? Sin 0 is 0. So, the second term is 0 cos beta is 1 so K beta will be equal to K x. Similarly, if theta is 90 degree then the cos term will be 0 then K beta will be equal to K z. So, it is easy to see that for maybe I can write it beta is equal to 0 means K beta is equal to what K x horizontal flow beta is equal to 90 degree means what vertical flow? So, you have K of beta is equal to what K z, but if this beta is between 0 and 90 or any other value then you can calculate the hydraulic conductivity accordingly.

So the next thing what we are going to do is we will look at the modeling of the movement of ground water in the saturated zone. Until now we looked at certain types of aquifer the aquifer properties and you know how we can do this anisotropic business. Now, we will move into the derivation of the basic governing differential equations which will be the partial differential equation for different types of aquifers. And first we will take the confined aquifer case which is a simpler one.

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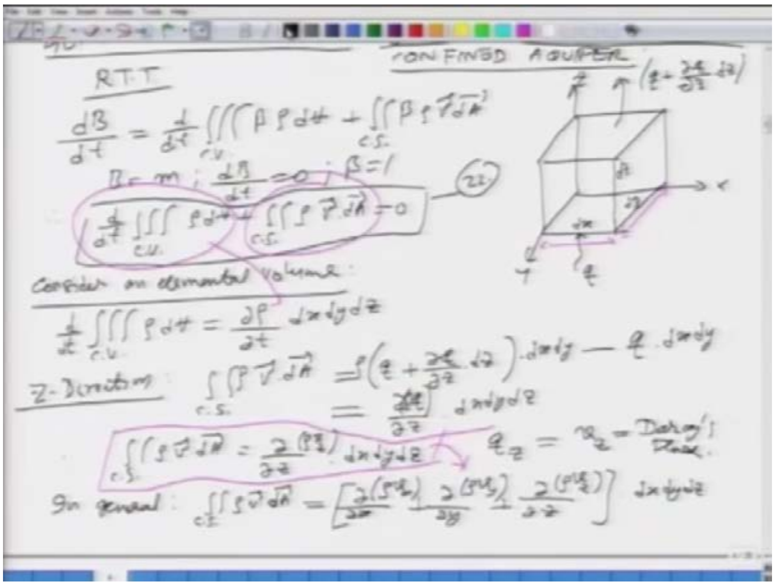

So, let us get started with the general flow equations. And first we will look at saturated steady ground water flow in confined aquifer. And then we will look at unsteady and then move to the unconfined case. So, what we are going to use is we will go back to our Reynolds transport theorem this as I said this is going to hound you throughout this course and many other courses on the fluid mechanics. What is the Reynolds transport theorem? I am going to write it again, you should remember by heart d B by d t is what d over d t triple integral beta rho d v over the whole control volume plus the extensive property flowing across the control surface. The flux beta rho and v dot d A B is equal to the mass of the moving water d B over d t, the left hand side would be 0 by the law of conservation of mass and your beta is 1.

Therefore, your equation is d over d t of triple integral over the control surface of your rho d v plus double integral over the control surface of beta is 1 so rho v dot d a is equal to 0. Now, what we will do is we will derive this governing differential equation by using this we are working actually on the continuity equation. So, what we will do finally is the governing differential equation will be the combination of the law of conservation of mass and the law of conservation of momentum. So like we have been doing we will write the continuity equation we will consider a control volume in the aquifer. We will take a cube and we will see how much is the accumulation term and how much is the flux across the mass plus flux across the control surfaces in different directions? And then we will combine all that thing and derive the expression.

So, let us do that quickly consider an elemental volume which is shown. So, let us see you have this cube where this is your x direction this is your y direction and z is of course, your vertical direction. So, this is your d x the dimensions of the cube you are considering is d x d y and the vertical one is d z. This is d z let us say that some volumetric flux q is entering in the vertical direction z is positive upward. So, we are considering the flow in that direction what will come out is by Taylor series expansion you will have q plus change of q with respect to z in a distance of d z. This d z actually should be inside it will be q plus del q del z d z.

So, what we will do is we will look at these expression this equation let me call this 22. For that small elemental volume what will be d over d t of your triple integral of your rho d v. This will be nothing but del rho over del t right times what, what is the total volume? Total volume is nothing but d x d y and d z that is the first part this one; this expression now what about the second part? So, what we will do is we will write the mass flux flowing across the z direction or the control surface in the, our z direction. First we will do the z direction and we are writing the second term in your Reynolds transport theorem which is rho v dot d A. So, I am not writing rho. Now, let us look at what will be the v dot d A term what is mass flux which is flowing across the control surfaces in the z direction, this is q going up and this quantity coming out. So, it is going to be nothing but you have q plus del q del z over a distance of d z this is flowing across what area this is flowing across this area d x times d y right the q.

So, this is multiplied by d x d y so this is the total flux flowing across the bottom phase minus then you have inflow which is q times d x and d y is the area. So, what will that be then this is nothing but del of your q over del z times d x d y and d z. How about if I put A rho in front. So, you will have rho embedded in the end then you will have rho inside of this. So that you will have rho v dot d A will be equal to del over your del z of your rho v x times d x d y d z. So, what we have done here is this q x, this q z this is in fact, d z.

So, what we are saying is that your q z is nothing but your v z this is the Darcy's flux in your z direction. So in general then we can say the total mass flux which is flowing across all the surfaces will be we can write a similar expression in all the 3 directions. So, you will have del over del x of your rho v x plus del over y of rho v y plus del over del z of what rho v z times, what times d x d y and d z? So, what I have done basically is that I have written down the second term in your continuity equation for z direction like this; this one. And this I have generalized for all the 3 directions this all we have done if you simplify this equation.

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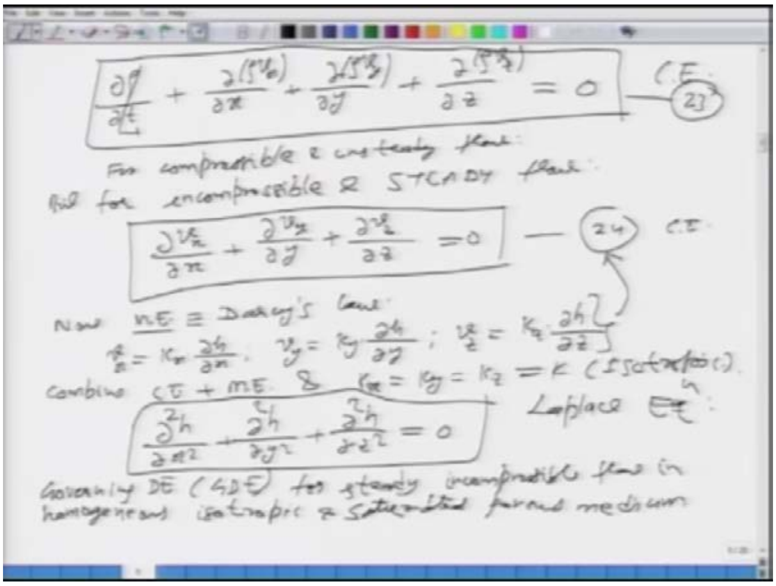

What you will have is this del rho over del t plus del o del over del x let us say of your rho v x plus del over del y of your rho v y plus del del z of your rho v z, this is equal to 0, this is your continuity equation. So, this is actually a general equation for compressible flow in which actually we have put the rho inside the differential and the integral. So, we have try to be general here so this is for compressible and unsteady flow, but if we are considering incompressible flow in which the density changes are not important or we ignore them incompressible and steady what will be the continuity equation? Obviously, this term will drop out and the rho will come out of this differential and rho will also cancel out. So, this is the continuity equation which actually you know already del v x del x plus del v y del y plus del v z del z is equal to 0. And I am going to number this equation as 24 and this was my 23. So, this is your continuity equation for which type of flow in compressible and steady flow.

Now, we write the momentum equation what is the momentum equation for the ground water flow as we have said it is nothing but the Darcy's law. And what the Darcy's law say well it is v x is equal to K x times del h del x v y is equal to K y of del h del y and v z is equal to K z times del h del x assuming we know the values of K x K y and K z. If we do not then we say or we assume that the aquifer is isotropic which we actually we will do. So, what you do is you combine the continuity equation and the momentum equation. That is to say you put all these expressions into this and once you simplify all that. And then you say another thing actually assume is K x is equal to K is equal to K z. So, you are assuming what isotropic aquifer let us say this is equal to K. So, if you did that what you are going to get is this del 2 h del x square plus del 2 h del y square plus del 2 h del z square is equal to 0 is it familiar what is this equation called? This is called Laplace equation it is very famous very popular very important equation for solving many ground water flow situations.

This is the governing differential equation or your G D E for what? For steady incompressible flow in homogeneous isotropic and saturated porous medium, and how we have derived it by combining the continuity and momentum equation? So, we saw that the governing differential equation for steady flow which is incompressible in the ground water in the confined aquifer actually comes out to be what is called the Laplace equation? Now, what we will do is and that is for the steady case. Now, what we will do is we will look at the derivation or the governing differential equation. For the unsteady case in the confined aquifer we will actually not go through the derivation. But what I would like to do is I will give you the final expression, because derivation will take a lot of time. And we do not have the number of lectures in which we can compute this.

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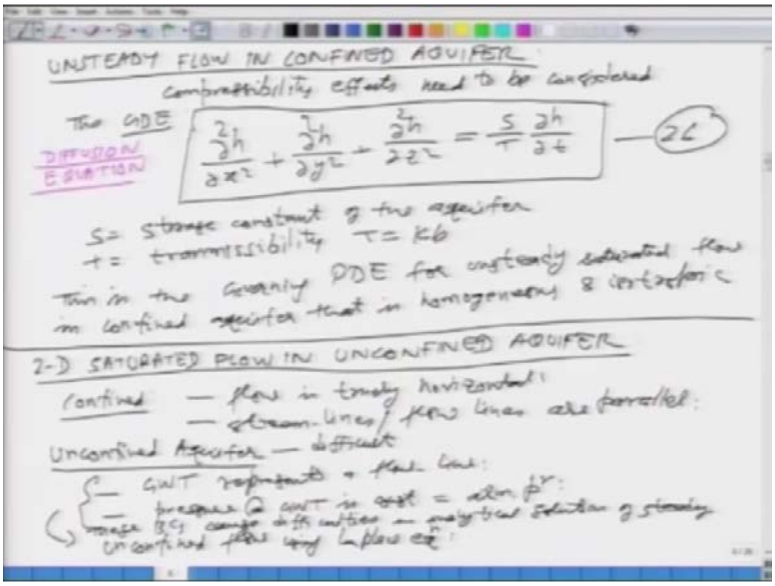

So, let us look at the unsteady flow in confined aquifer and this only thing I will say is that the compressibility effects become important, remember you had a del rho by del t term which we had derived earlier. So, the del rho by del t term cannot be neglected. So, we have to take care of that so what are the density changes in a confined aquifer? In a unsteady case will become important and they have to be considered. So, for compressibility effects need to be considered. How we do that actually I will not go through this. And what we will get is the governing differential equation which will come out will have this form; del 2 h over del x squared plus del 2 h del y squared plus del 2 h over del z square.

So, it is something similar to what we had found earlier that the expression on the left hand side. However, it is not 0 if it is 0 it is a Laplace equation, but what we have is S by t del h del t so h is changing as a function of time and also as a function of space that is with respect to x y and z where what is S? S is the storage constant of the aquifer and T is the transmissibility as we had seen yesterday transmissibility T is equal to K b; b is the thickness of your aquifer. So, what is this? So, this equation actually and let me number this as 26. So I maybe skip a few equations here, because I have not done the derivation, but let us not worry about that. This is the governing partial differential equation for unsteady saturated flow in confined aquifer that is homogeneous; again these assumptions are still valid and isotropic.

Now, this equation is actually called a diffusion equation, it is very famously known as diffusion equation. So, somebody asks you in a confined aquifer what is the governing differential equation for a steady case? It is Laplace equation. And what is the governing differential equation for unsteady case? It is the diffusion equation in the Laplace case the left hand side is equal to 0, but for the unsteady case you have the del h del t term S bt t del h del by del t. So, these are the governing differential equations and we will come back to this little later in this course where we will take some simplified cases and try to solve some ground water flow situations. But now we will move on to the unconfined aquifer case. So, let me move on and what we will look at is a two dimensional saturated flow in unconfined aquifer, the ground water flow equations which we are just derived were applicable for through confined aquifer, everything which we have seen above is what the confine in the confined aquifer.

The things are actually very simple in the sense that that the flow is truly horizontal in a confined aquifer the flow is taking place which is in the horizontal direction whether you are taking pumping water out or whether it is between 2 water bodies the flow will always be horizontal. As a result of that the stream lines or the flow lines are parallel to each other that is why we can apply those equations easily and calculate the flow and velocities and all those things. However, in the case of unconfined aquifer things are slightly complicated in case of the unconfined aquifer things are difficult what do we mean by that well the ground water table represents a flow line right or a stream line. And also the pressure at the ground water table is constant which is equal to the atmospheric pressure. So, what we have is that, because the ground water table is it maybe sloping which maybe very steeps in a certain a situations.

So that the flow lines will not always be parallel to each other and also the pressure is constant along that in a confined aquifer it may have been a different case the pressure between 2 distances would be different. So, here applying the boundary conditions becomes a very difficult task, because your boundary is undulating it has a constant pressure and the things are not parallel. So, this causes these boundary conditions cause difficulties in analytical solution or solving them analytically of steady. We are looking at steady case only actually unconfined flow, how using the Laplace equation which we have earlier derived which is applicable only for the confined case.

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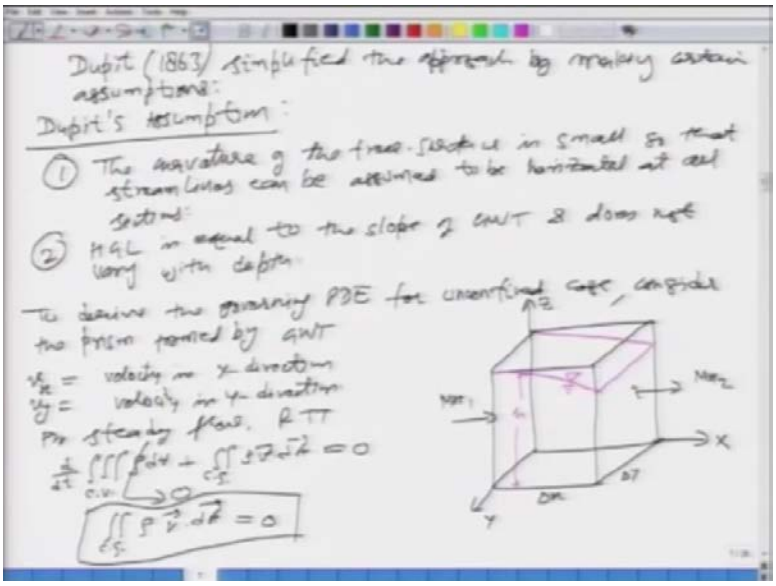

How do we get around this problem? Well a very famous scientist name Dupit in 1863, simplified the approach or presented a simplified approach how by making a certain assumptions which are actually named after him as the Dupit's assumptions. I am sure some of you may have seen this Dupit's assumptions in your earlier classes. So that is the next thing we are going to look at what is called the Dupit's assumption which will allow us to carry out our analytical solution formation of your governing differential equations under these Dupit's assumptions in the case of the unconfined flow. So, if we look here the first one of them is the curvature of the free surface that is your ground water table is small. So that the stream lines can be so that the stream lines can be assumed to be what horizontal at all sections horizontal at a locations or sections.

So, this is one of the basic assumptions what we are saying is that the ground water table or the pre surface or the steepness is so small that we can take the flow line or the stream lines to be horizontal or parallel to each other. So, you see that this Dupit's assumption or the unconfined case analysis which we will do will be applicable only in areas or only in locations where these assumptions are not violated or if they are violated they are violated very slightly. So, that is what we have to keep in mind that everything where which we are going to do will be applicable under certain you know limitations or restricted areas.

So, this is the first assumption, what is the second one? It says that the hydraulic grade line is equal to is equal to the slope of the ground water table and does not vary with depth. It is a simple one in which we say that the hydraulic grade line this equal to the slope of the ground water table. Now to derive the governing differential equation or partial differential equation for the unconfined case, what we will do is we consider what the prism which is formed by the ground water table. So, what we will do is let me try to explain this to you what are we talking about here? So, we will take the control volume approach like we had done earlier similar to earlier. And then you have this is your ground water table and as far as the direction is concern this is your x; this is your z; this is y; this is again your delta x delta y and delta z of course or height actually delta z is not important here. So, what I am going to define here is this height is h from the horizontal.

The other thing we do is we assumed that the mass fluxes let us say in x direction, the mass flux entering is M x 1 and what is coming out is M x 2 and similarly, we assume the things in the other directions. So, if we define certain things here now, v x is the velocity in x direction, and v y is the velocity in y direction, we are looking at two dimensional flow only. So for steady flow, flow if you write your Reynolds transport theorem your first term actually will drop out d over d t if you write the continuity equation beta is 1 rho d v plus this of your beta is 1 rho v dot d a is 0. And for steady case I am going to say that this is. So, basically you have the continuity equation as rho v dot d A is equal to 0. So, all we need to do is we need to write the expression for the rho v dot d A term. So, let us try to do that.

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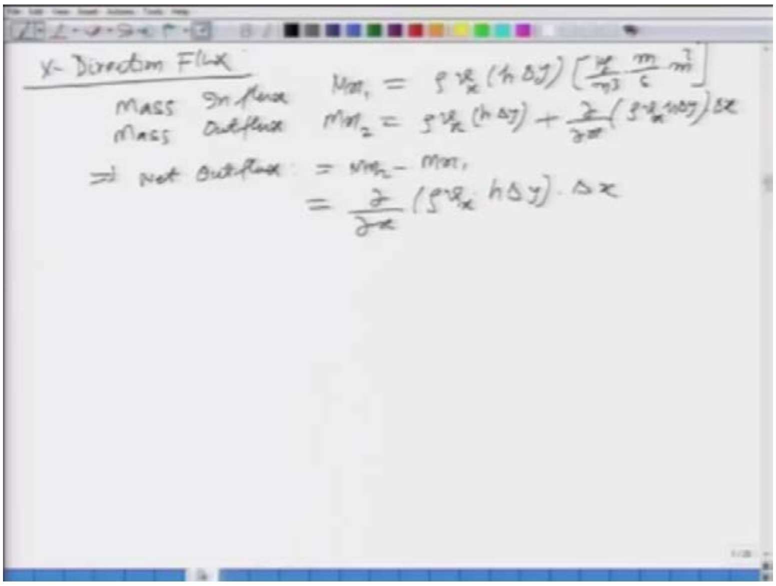

First we will write the x direction fluxes. What is the unit flux which is M x 1 it is rho times v x times across h of delta y. If you look at the units it is going to be flow is kg per meter cubed times v x is the velocity it is in meters per second and h and delta y are your length units. So, it is square meters, so what is the overall unit you say see that the meter cube will cancel it is kg per second. So, M x 1 is equal to what it is the mass in flux which is entering in the x direction we had shown here this guy here this is what is entering. So, we are not using the total delta z here we are using the height h which maybe actually varying. So this is rho v x h time delta y is the phase the area of the phase across which it is happening. Similarly, you write the mass flux out from the other phase which is M x 2 which is nothing but use the Taylor expression.

So, you have rho v x times h times delta y plus the same thing del del x of your rho v x h delta y across delta x. So, what will be the net out flux? That is the one of things which we have to find out which is nothing but M x 2 minus M x 1. And that is what is going to be equal to del by del x of your rho v x times h delta y times delta x. So, we see that what we have done is we have taken the control volume in a unconfined aquifer in which the water surface or the ground water table is sloping. So, I am running out of time today. In fact, so all we have done is we have written down the continuity equation and in that continuity equation we have written the mass flux in the x direction. We are considering a two dimensional case. And we will come back and look at the mass flux in the y direction. And we will write it similarly, we will combine these two we will combine it with the momentum equation. And the final equation we will come out as the governing partial differential equation for the unsteady case, sorry the unconfined aquifer case. So I would like to stop here and come back to this tomorrow.

Thank you.