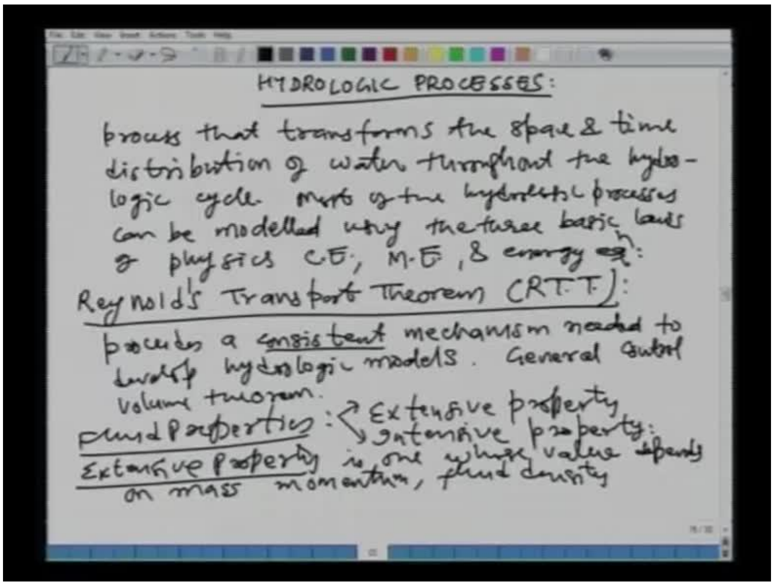
Advanced Hydrology
Prof. Dr. Ashu Jain
Department of Civil Engineering
Indian Institute of Technology, Kanpur

Lecture - 3

Hello friends, good morning and welcome to the 3rd lecture of this video course on Advanced Hydrology. Just looking back in the last class we looked at an example of lake water budget, then we looked at systems concept, we saw that a system consists of three components that is input, output, and a transfer function or an operator. Then we looked at couple of examples on the transfer function operators for a hydrology system model.

And then we moved on and looked at the classification of the hydrologic models, we said that hydrologic models can be classified from many different angles. Firstly, we said they can be either physical models or the abstract models, then we moved on to the basic dimensions then we classified the model into deterministic stochastic and then sub classification and so on. Then finally, we looked at some of the examples some of the combinations of hydrologic system models.

(Refer Slide Time: 01:26)



Today we will start with a new chapter or the new topic in this course and we will call it as hydrological processes, hydrological processes. What is a hydrologic process? It is nothing but a component of a hydrologic cycle for example, infiltration or evaporation or rainfall or runoff, a base flow or any of this component. Basically when we talk of a hydrologic process it is a system we are trying to model that system using the laws of physics.

So, if we are to define the hydrologic process, it is a process that transform the space and time distribution of water throughout the hydrologic cycle. That is to say hydrologic process is something which captures the variation with respect to space and time or determine, how the water or the fluid is distributed in space and time. For example, infiltration it will vary from place to place in a catchment, it may be due to the heterogeneity in the catchment, that is soil and the vegetation and so on.

All this factors may be different from one place to the other and also it will vary as a function of time. So, infiltration process is a hydrological process which we need to model and similarly, we can think of other processes. How do we model the next question that arises is how do we model or how do we understand or how do we study the mechanism of the hydrological process. Well as you all know there are three basic equation, which can be used to understand or model any hydrological process, which is continuity equation, momentum equation and energy equation right.

So, most of the hydrological processes can be modeled using the three basic laws of physics that is continuity equation, momentum equation and energy equation. Now, what we are going to do today is look at one theorem, which is the mother of all these equations. Any kind of fluid motion process can be modeled using this theorem and this is called Reynolds transport theorem and short we will refer to this in this course as RTT. As I said it is the mother of all the equation or at least in the fluid mechanics and in our case the fluid is water.

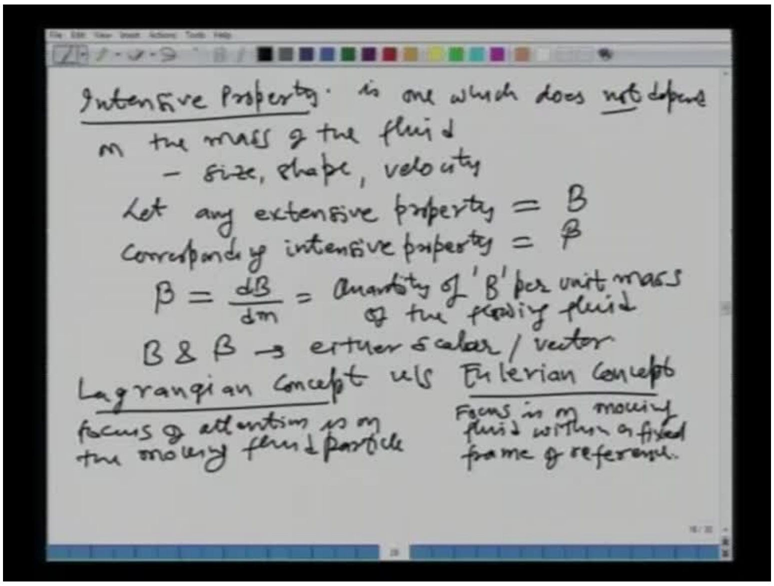
So, any kind of movement of water whether it is in the vapor phase or in a river or in the underground, it can be modeled using this Reynolds transport theorem. And we would see that all these three basic equation that is law of conservation of mass, law of conservation of momentum and law of conservation of energy all of them can be derived from this Reynolds transport theorem. So, first let us see what it is it basically provides a consistent mechanism, which is needed to develop hydrologic models the keyword here is consistent what do we mean by consistent.

Consistent means a process or a system or an equation which can be used in all the circumstances; the consistent the performer is somebody who can play well in all the conditions. So, this is the kind of equation we are looking at this is Reynolds transport theorem and it is also called a general control volume theorem; I hope that most of you would be aware of a control volume approach. However, I will go into the basics of this control volume approach, before we actually go in to the derivation of the Reynolds transport theorem.

So, before we go I would like to look at some basic definitions and also some concepts or mechanism that are used to analyze the fluid motion in this case what happen is. So, coming to the definitions the fluid properties can be classified or grouped into two different types of properties, I do not know if you heard about them, one of them we would say is extensive property and other one is called the intensive property. You know what is the difference between an extensive property and an intensive property?

Well an extensive property is one which depends upon the mass of the fluid and intensive property is something which is independent of the mass, that is the basic difference. So, looking at this extensive property are those or is one whose value depends on mass can you give me a example of an extensive property, a fluid property that depends on the mass can you think of one yes it is the momentum. So, momentum of a flowing fluid is an extensive property, because higher the mass, higher will be the momentum lower the mass lower will be the momentum.

(Refer Slide Time: 09:08)



Some other properties could be for extensive property fluid density, density itself depends upon the mass etcetera. And coming to the intensive property as we said is one which does not depend on the mass of the fluid, can you think of some examples of the intensive properties of fluid well it could be the physical properties such as the size, shape of the fluid when it is put into some container and velocity we can say is an intensive property and so on. There is a relationship between the extensive property and the intensive property.

So, let us look at it before we go into that, let us use some notations which we will be using in this course throughout. So, let any extensive property we will denote by capital B. So, what we are trying to do here basically is that we are defining the intensive and the extensive properties and we will write the Reynolds's transport theorem in terms of any extensive property. So, it is the most general form of theorem we will derive and then we for extensive property we can put either mass or momentum or any other property, so that we will have that particular law.

And let be corresponding intensive property of the fluid be denoted by beta. Now, the relationship between them is that the intensive property is nothing but d B over d n that is to say the intensive property is nothing but the extensive property per unit mass of the flowing fluid. So, if we write it, it is quantity of extensive property B per unit mass of the flowing fluid it is easy to see that or understand that, this both of these extensive and intensive property can be either scalar or vector.

For example, let me first write that both B and beta can be either scalar or vector; for example, momentum and velocity they are examples of a vector quantity and the fluid density is a scalar quantity. Now, we move to another definition of concept when we analyze the fluid motion any kind of fluid motion do you know what are the approaches that we use or in your 12th standard or in your earlier physics, I am sure you have looked at the solid mechanics or the mechanics of the motion of the solids. That approach which we have used is called the Lagrangian approach.

So, what we will do is we will look at different ways or concepts in which we can analyze the motion of the particles or the fluid particles. It can be done in two ways, one of them is the Lagrangian concept and other one is the Eulerian concept, I am not sure if you have heard about these concepts but before I get started, I will just like to give you a very simple example. So, that you understand what is the difference between the Lagrangian concept and the Eulerian concept, I am sure all of you would be familiar with cricket I am big fan of cricket myself.

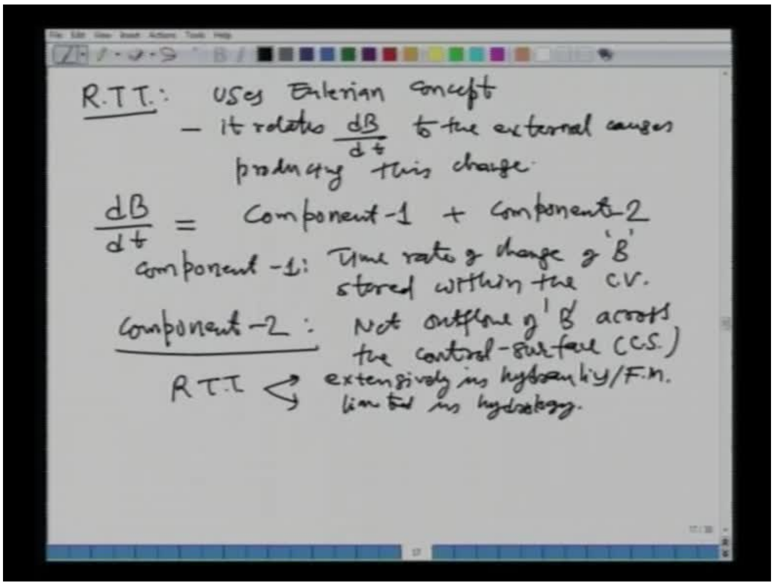
So, in a cricket match when a batsman hits a ball for a six the camera follows the ball. So, the camera when is working or following that particular ball or that ball is solid particle and we are focusing our attention on the movement of the ball all the way to wherever it goes that is called the Lagrangian concept.

So, what is a Lagrangian concept, we focus on the movement of the particles or a group of particles and we move along with them and then we try to analyze what is happening to their momentum, the velocity and so on, all the fluid property. So, it is something the analysis is moving along with the fluid particles. As oppose to eulerian concept in which our focus of attention is fixed in a particular frame for example, in the cricket match what happens, when there is a close run out we go for third umpire.

What happens there is the third umpire focuses his or her attention in a fixed frame and the batsman as sliding in is like a fluid particle moving in. So, we focus on a fixed frame and then fluid particles are flowing and then we try to analyze what is happening to the fluid motion. So, in simple terms if we want to define is when the focus of attention is on the moving fluid particle it is called a Lagrangian concept and when the focus is on moving fluid within a fixed frame of reference and that is when we analyze that is called the Eulerian views of motion or eulerian concept.

Why we are doing this theory is because the Reynolds transport theorem which we are going to derive uses the Eulerian view of motion. So, what we will do is we will use or fix our control volume, we will fix some volume in the space and then try to analyze what is happening into the fluid as it passes through that fixed space.

(Refer Slide Time: 16:21)



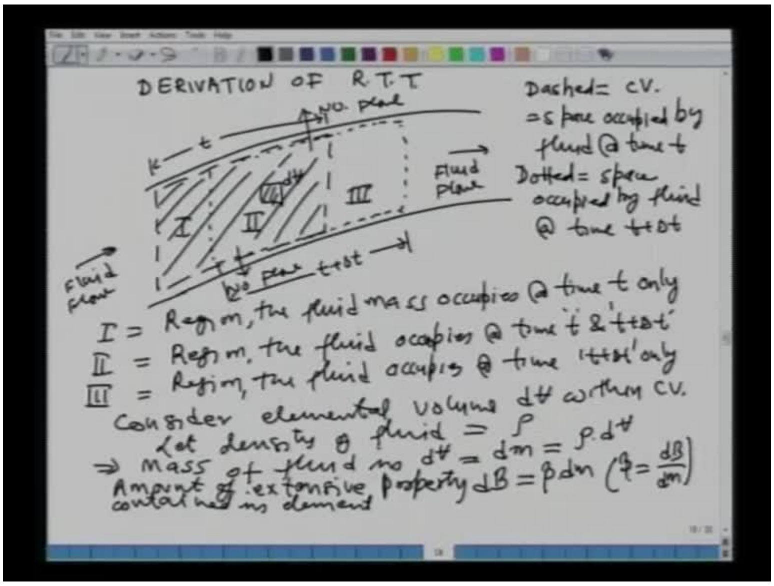
So, coming back to RTT with these concepts in mind, first thing I would like to say is it uses the Eulerian concept in analyzing the fluid motion and then what it does is it relates d B by d t to the external causes producing this change. So, what we are interested in, when we are looking at the Reynolds transport theorem is we take any extensive property and we look at the time rate change of this extensive property d B by d t. And look at why it is happening, what are the forces acting or what are the reasons, which are producing these changes.

So, what the Reynolds transport theorem gives us is this time rate of change of extensive property as a sum of two components and we will define them as component 1 and component 2. What is component 1? Component 1 is the nothing but the time rate of change of the extensive property stored within the control volume, because we are looking at the control volume approach. So, the component 1 gives us or describe the time rate of change of the extensive property stored within the control volume. Time rate of change of B stored within the control volume.

And what is component 2? Component 2 is nothing but what is called the net outflow of the extensive property flowing across the control surface alright, we will come into these terms, what is a control volume and what is a control surface in a minute. So, the component two in the Reynolds transport theorem is nothing but the net outflow or out flux of your extension property B across the control surface C S. Now, this Reynolds transport theorem this is the very basic overview I have given you before we actually move to the derivation.

This Reynolds transport theorem has been used extensively or widely in hydraulics and in fluid mechanics. However, it is you has been limited in hydrology, at this point of time I would like point out that this Reynolds transport theorem is a consistent mechanism to analyze the fluid motion alright. So, it is not only the civil engineers who used this Reynolds transport theorem but it can be used by any type of engineer or anybody who is you know, who has to deal with the motion of the fluid. It can be the fluid can be air or it could be any other kind of gases, it could be some chemicals or it could be water; so it is a very general theorem.

(Refer Slide Time: 20:26)



So, now we move to the actual derivation of Reynolds transport theorem for this what we are going to do is look at the or define our control volume. So, let me draw the figure first (No audio from 20:47 to 21:00), this is very important to understand this figure, I will go slow. So, that you can may be draw it on your notebook (No audio from 21:11 to 21:24) what I have done is I have drawn two different types of lines, one is the dashed which is this one, the dashed one, which we say is the control volume or it is the space occupied by the fluid at time t.

The arrows on the left and right show the direction of flow and what we are saying is that the dashed lines represent the control volume or nothing but the space which is occupied by the fluid at time at any time t. So, if you just take a snapshot if you take a picture at time t, we are focusing on such a control volume. And the dotted one which are the small dots is nothing but the space occupied by the fluid at time t plus delta t, just after very small time delta t the focus of attention is shifted to slightly to the right as the fluid is moving.

Now, what we do is we say that this is region I, this is region II and this is region III and what we will do is we will take a very small elemental volume we will call this d v and this is the direction of fluid flow. So, if we are looking at this figure just try to match in that this is a two dimensional figure we are looking at; however, there is a third dimension a perpendicular to the board or this your P C. So, try to imagine that there is a third dimension also and it is a it is like channel or it is a tube in which the flow is taking place.

One thing you want to understand is there is no flow taking place across this tube no flow in any of the directions. So, flow is moving from left to right only in one direction but we made a analyze this as a three dimensional flow within these boundaries. So, let us move on before I do that I would like to just actually control volume. So, this is the dashed area which is the control volume. Now, we define region one which is the first triangle, which is the first rectangle looking at it is the region the fluid mass occupies or the fluid occupies at time t.

Before I move on let me say that this is at time t and and this one is at time t plus delta t a dashed is at time t and the dotted one is at time t plus delta t. So, you see that the region one is the region the fluid occupies at time only t, the second one is the region the fluid occupies at time t and t plus delta t. I would like to draw your attention and you know look at this and try to understand this very carefully and there is third region, which is here on the right side it is region the fluid occupies, at what time.

(())

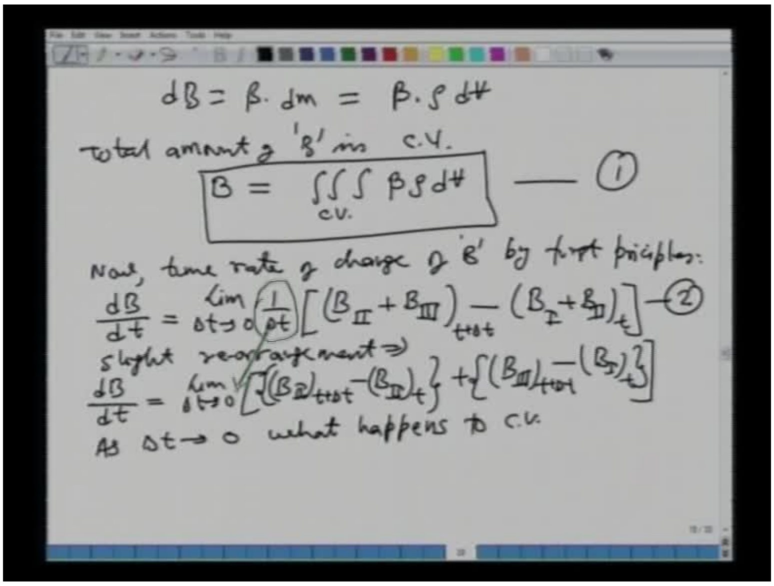
Good and t plus delta t only. So, we have a region 1 plus 2 which is at time t and you know some part of it is overlapping with the both t and delta t and region three is only at time t plus delta t. Now, what we do is we define try to calculate some of the properties within a small elemental volume. So, what we do is we consider an elemental volume d v within the control volume, what is d v. In this region two what we do is we take a very small elemental volume, it may be a very small cube depending upon the coordinate system we are using.

So, d v is a very small elemental volume, which we are going to use or focus our attention, net the density of the fluid which we are trying to analyze the rho. So, the density of the fluid is rho and it is volume is d v. So, can you tell me, what will be the mass of the fluid in the elemental volume d v and we will denote this as d m can you tell me what it will be mass is equal to density times volume good; d m will be nothing but density is rho times the elemental volume.

Now we want to find what is the amount of extensive property, contained in this elemental volume or on this element and I am going to call it d B can anybody tell me what will be the amount of extensive property stored in this small element. D b; good it will come from the definition or the relation between intensive and extensive properties. So, this d B will be nothing but beta times d n and it is basically coming from beta is equal to d B over d m which we are justifying beta is the intensive property which we said is the extensive property per unit mass. From this equation you see the d B will be beta times d m, let us move on what we just saw is the let me write it down first.

So, d b is nothing but beta times d m and what was d m which we had just written it is nothing but beta times d m verse rho d v. So, we have a small little expression about the amount of extensive property that is stored within a small element. Now, if I wanted to find out what is the amount of extensive property stored within the whole control volume. So, I have taken a small elemental volume I have found the extensive property in it, now I want to find out the same property in the whole control volume what do we do we integrate.

(Refer Slide Time: 29:17)



So, what we do is then the total amount we say, the total amount of the extensive property that is it B in the control volume and let say that will be capital B will be, I will be most general we will be integrating the whole control volume. So, if we have the if we are using the Cartesian coordinate system then you will be integrating with respect to x y and z and if you are in the polar system then r theta z or whatever coordinate system you are using. So, we will take beta rho d v, which was the d m and this is my I will call this first equation.

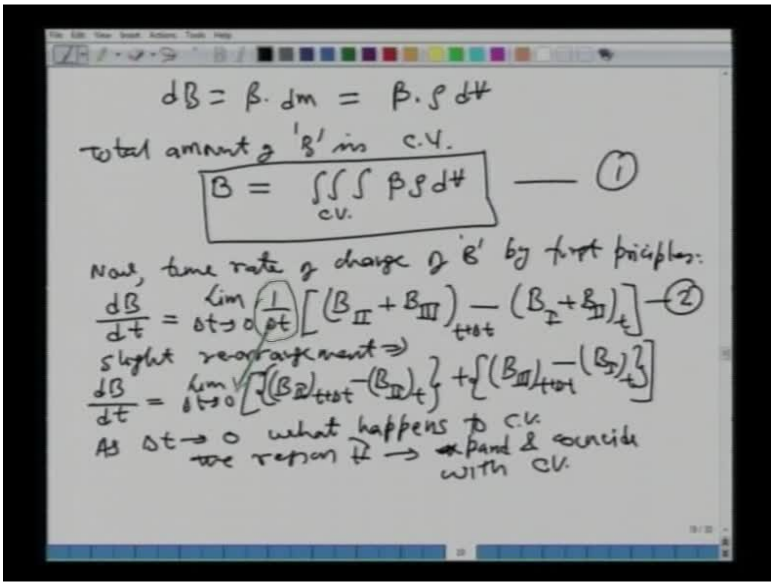
What we have till now is we have taken an elemental volume and we have tried to find out what will be the total amount of extensive property that is stored in this. So, we will leave this concept here and move further. So, when we are deriving the Reynolds transport theorem what we are trying to do is we are trying to find the time rate of change of extensive property. So, let us write the time rate of change of extensive property by first principles, what do we mean by first principles we go back to the calculus. So, we will write the expression for this time rate of change of extensive property which is d B over d t from our knowledge of calculus as delta t tends to 0 of 1 over delta t of your extensive property at time t plus delta t in our control volume.

And the same thing at time t, please look at this equation very carefully what we are saying is that the first derivative of any variable is nothing but the value of that variable at time t plus delta t minus the value at time t divided by time interval delta t that is all we have done here. So, what is the extensive property at time t plus delta t the B within region 2 and b within region 3 that is the first expression in equation 2 and minus B 1 and B 2 sum of the extensive property in regions 1 and 2 at time t.

Now what we do is we do slight rearrangement of this equation and that will lead me to d B over d t is equal to under the limits as delta t tends to 0 of 1 over delta t B 2 at t plus delta t minus B 2 at t. So, what I have done is I have combined the extensive property in region two which will be my control volume plus you will have B 3 at time t plus delta t minus B 1 at time t. Please understand this equation very carefully what we done is we have combined the extensive property in region II and regions I and III separately and the reason is that region II is my control volume. So, I will be working on the control volume and region I and region III are my inflow and outflow region. So, I will be working on the component one and component two of the Reynolds transport theorem separately.

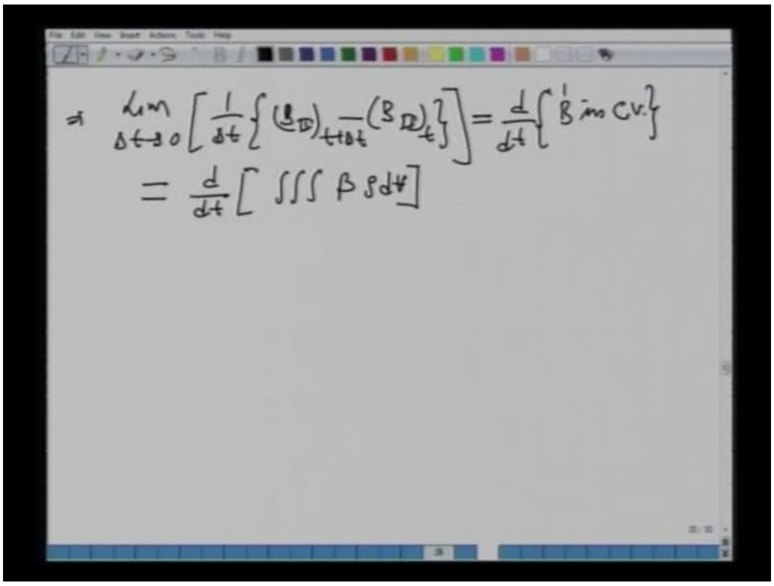
Now, as delta t tends to 0 what happens to the c v, I would like you to go back to your sketch which we have just drawn let me go back this figure region 2 is in the middle, what happens just try to think about it what happens to the region two as we reduce delta t, we have taken a picture at time t then we took a picture at time t plus delta t.

(Refer Slide Time: 35:19)



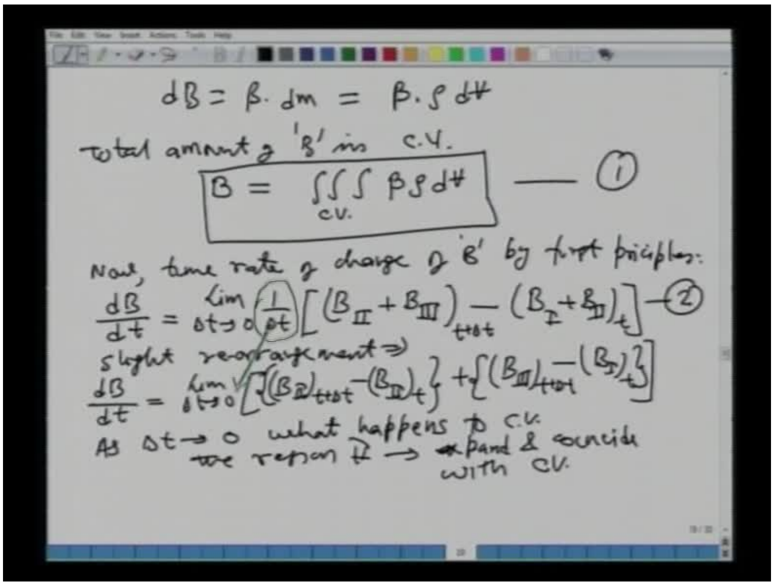
So, if we reduce that time interval delta t what will happen to region II this region II will keep expanding and it will coincide with the control volume, the region II will expand and coincide with the control volume.

(Refer Slide Time: 35:50)



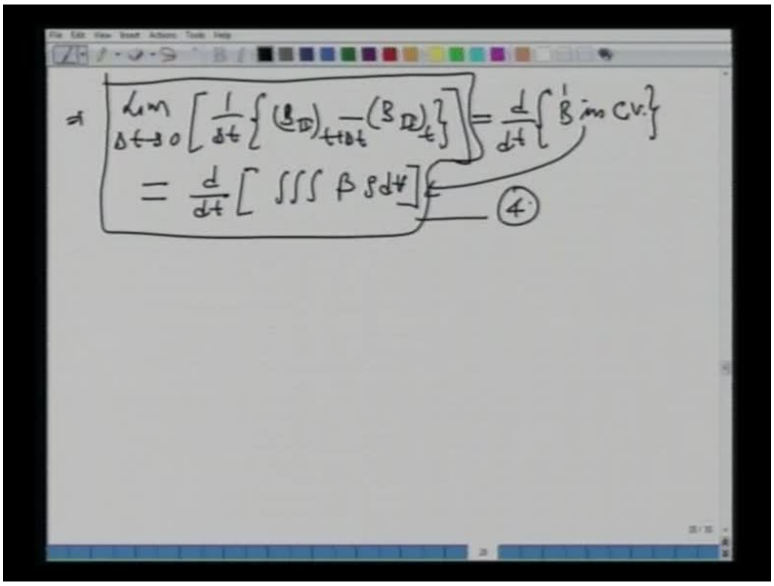
Now, let us move on and we would look at this equation under that scenario. So, what will happen is it will be the limit as delta t tends to 0 of 1 over delta t of b at 2 t plus delta t minus b at 2 at time t only this equation. So, I am looking at the first component in my equation two which involves only the region II under the limits this is nothing but my d of d t of your extensive property B in the c v, because under the limits your region II is approaching towards the control volume. This expression which we are looking at this will be then equal to d over d t of what is the b in c v, we have already derive this from equation one, this will be nothing but triple integral of beta rho d v.

(Refer Slide Time: 35:19)



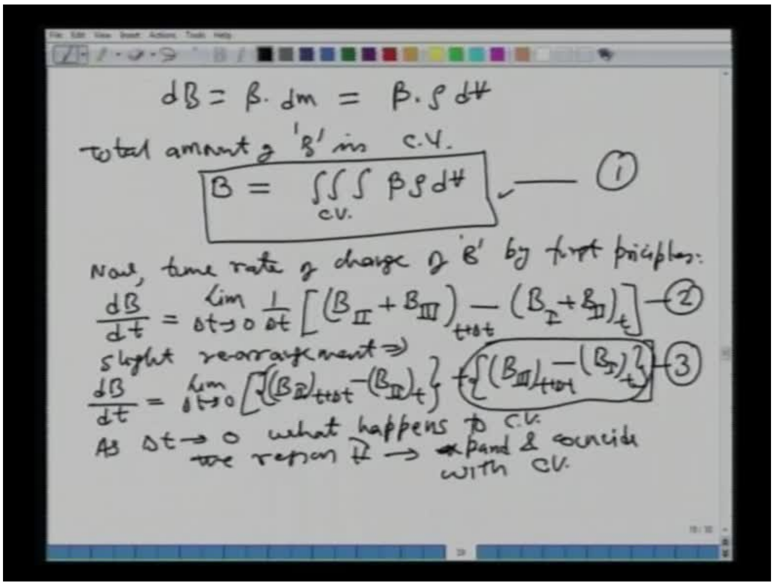
If you go back to your equation one this one you see that capital B is nothing but triple integral of beta rho d v.

(Refer Slide Time: 37:11)



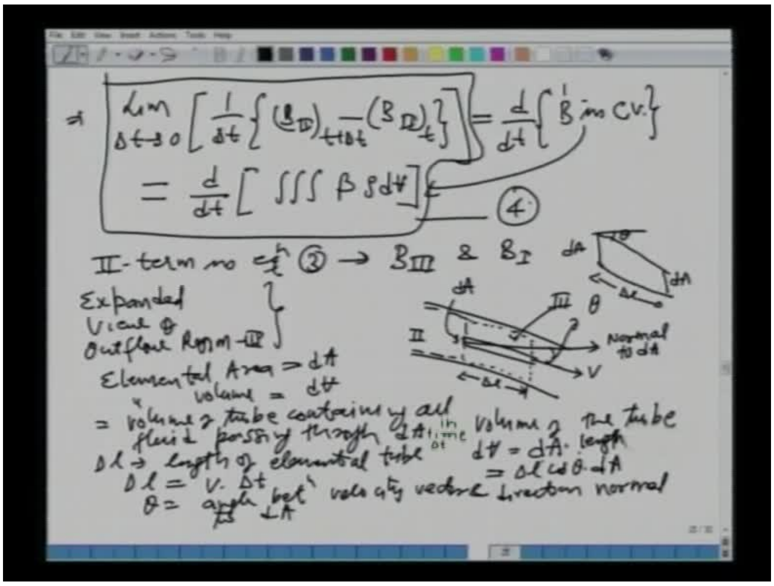
So, that is all we have done that this expression we have substituted here. So, what I have then got is that this expression is equal to this and I will call this equation as four. So, what we done is we have written down the expression for the extensive property within the region number 2.

(Refer Slide Time: 37:43)



So, if I go back to my equation number 2 you see that we collected the terms on B II let me first name this equation as. So, in this equation three it has two components, one is B II at t plus delta t minus B II at t which we have just derived as one expression then what is remaining is this whole expression. So, we need to work on the second part of this equation three and once we derive these expressions our time rate of change d v by d t we will have the equation.

(Refer Slide Time: 38:13)



In the second term it involves or the second term in equation three it involves B III and B I right, this B III and B I basically correspond to the outflow region and inflow region respectively. So, what we will do is we will look at the outflow region we will magnify the outflow region and see what is happening across the control surface close to the outflow region, then try to write the expression for that B III. So, we look at the expanded view of the outflow region. So, first I am going to draw the figure and you have first we have the dashed line and then after that we have the dotted line.

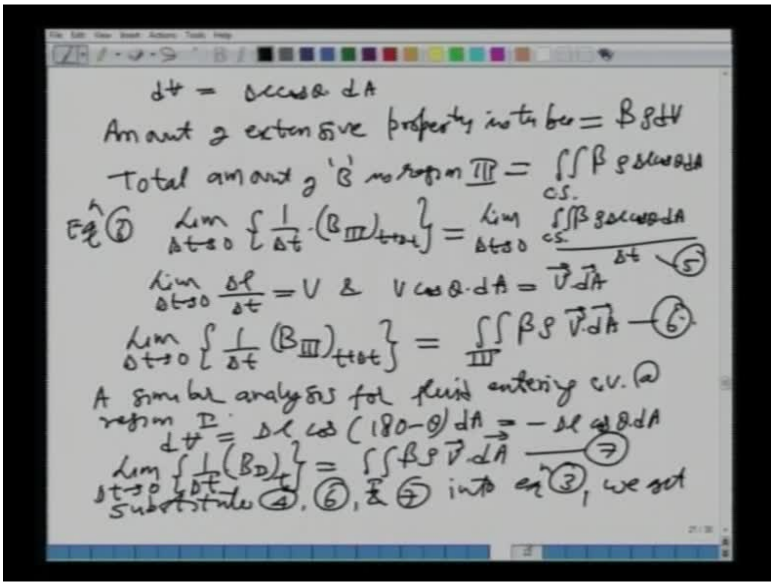
So, what we do is we take a very small elemental tube within this region III we see that this is your region II and this is your region III. The direction of this arrow is the velocity vector or the fluid flow motion, we take this area as some elemental area d a and we take this length of this tube as delta l, so we are into the region III. So, this is your region III and what we have done is we have taken a very small elemental tube within this region three which is the outflow region whose cross sectional area is d a and whose length is delta l and total volume is d v.

So, we have the we are considering the elemental area is equal to d a elemental volume of the tube is equal to d v, which is defined as nothing but the volume of the tube containing all the fluid passing through this area d A or the cross sectional area in time delta t d A. And we have said that delta l is the length of the elemental tube and what will that be equal to (( )) velocity times delta t. So, there is a fluid which is running at a velocity v in time delta t the distance it will cover will be nothing but the delta l; now what will be volume of the tube.

Let say d v before I move further let me define the angle theta which is the angle between the velocity vector and this is the vector which is normal to d a and this angle is theta. So, theta is the angle between the velocity vector and line, which is normal to the area d A. Now, if you want to find out the volume of the tube will be equal to what it will be equal to the area or the cross sectional area times the length this is your d A and this is your d A and the length slant length is delta l and this angle is theta.

So, how do we find the area of a parallelogram it is nothing but the length between the two parallel lines multiplied by that particular length which is parallel. So, this will be nothing but delta l cos theta will be the distance between two parallel lines and the area in that case is d A. So, this will be the volume of the tube and we have already define what is theta, theta is the angle between the velocity vector and the direction which is normal to the cross sectional area d A.

(Refer Slide Time: 44:11)



Now, we have found out the d v which is equal to delta l cos theta d A this is the volume of the elemental volume we have considered. And now we can write the amount of extensive property like we have done earlier, in the tube what will that be equal to like earlier beta rho d v. So, beta rho d v where d v is equal to this expression is nothing but the amount of extensive property within this small elemental volume, going by the same approach as we used earlier how can I find out the total amount of extensive property that is flowing across this control surface what do I do, so you can integrate.

So, the total amount of extensive property in your region three will be equal to what it will be double integral here over the control surface of your beta rho d v and d v is delta l cos theta d a. So, this is the expression in equation three or part of that. In fact, in equation three what we have is limit as delta t tends to 0 of 1 over delta t of B III at time t plus delta t. Time rate of change of extensive property in the outflow region and the expression for B we have just find out.

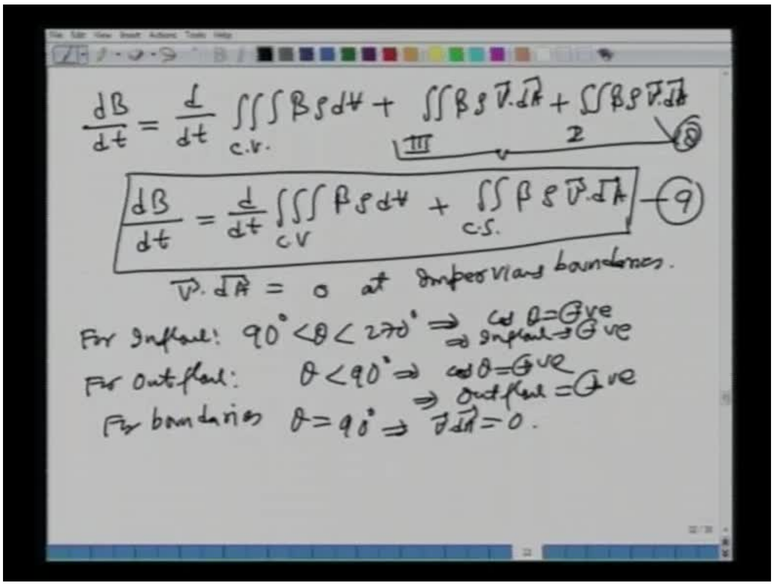
So, it will be limit as delta t tends to 0 of your double integral over the control surface of your beta rho delta l cos theta and d A and this whole expression divided by delta t. Let me say that this is my equation number five as I may need to refer this equation again now what is limit as delta t tends to 0 of your delta l over delta t it is nothing but the velocity. And noting that v cos theta d A is nothing but v dot d a using these two expressions in my equation number five, it can be written as limit as delta t tends to 0 of 1 over delta t of your extensive property in region III at time t plus delta t is nothing but double integral in your region III or control surface of your beta rho v dot d A.

It is important to come from equation number five with this equation we can see that very easily. So, what we done is then found out the expression for the extensive property that is flowing across the outflow region, which first component of the second part of this Reynolds's transport theorem.

Now, what we will do is we will go through the whole analysis and we should be able to say that we can carry out a similar analysis at the inflow region what we will do is a similar analysis for fluid entering the control volume at region I. We can carry out in which we again take that small elemental tube whose volume will be delta l cos of instead of theta it will be 180 minus theta because of the normal will be in the opposite direction. So, it will be 180 minus theta times d A which will be minus delta l cos theta d a.

So, that we will have under the limits delta t tends to 0 of 1 over delta t of your B at 1 time t is equal to double integral over region one of your beta rho v dot d A. So, v dot d A is the dot product which will take care of the negative sign which we have in the earlier equation. Now, what we do is we substitute before that let me name this equation as number 6, now what I do I just put or substitute the expressions in equation four and 6 and 7 into equation number 3, which is my basic equation what do we get.

(Refer Slide Time: 49:45)



What you will get is the d b by d t which is the left hand side of your equation three the first component we have derived was d over d t of triple integral over the whole control volume of beta rho d v, plus the double integral over region III of your beta rho v dot d A. And then plus you have the same thing in the inflow region one beta rho v dot d A. Now, what we can do is we can combine the last two expressions in to a single expression, why because these represent the extensive property that is flowing across the control surface, one is the inflow control surface other is the outflow control surface.

So, what we can do is we can combine this and then say it is a single term across the all control surface in the control volume, it is beta rho d v plus a single term c s represents the whole of the control surface and beta rho v dot d A. So, the earlier equation if it was eight then I will call this as 9. So, equation nine is our final Reynolds's transport theorem in it is final mathematical form, we treating that on the left hand side we have the time rate of change of extensive property due to the external forces and on the right hand side we have two components. Component one is the time rate of change of the extensive property that is stored within the control volume that is the first part d by d t triple integral term. And on the second on the right hand side the second component is nothing but the extensive property that is flowing across the control surface, inflow, outflow everything.

Now, this v dot d A term will be equal to 0 at impervious boundaries that is any boundary across which there is no flow taking place. So, that v dot d A term will be 0. So, just to tell you at the inflow region your theta will be between 90 degrees and 270 degrees always. So, that your cos theta will be negative; that means, your inflow is treated as negative all the time, for outflow your theta less than 90 degrees; that means, your cos of theta will be positive always, that is to say your outflow will be positive or it is treated as positive always or as I said earlier for the boundaries or impervious boundaries your theta is always 90 degrees; that means, your v dot d A term is 0. So, there is no flow of extensive property across the control volume.