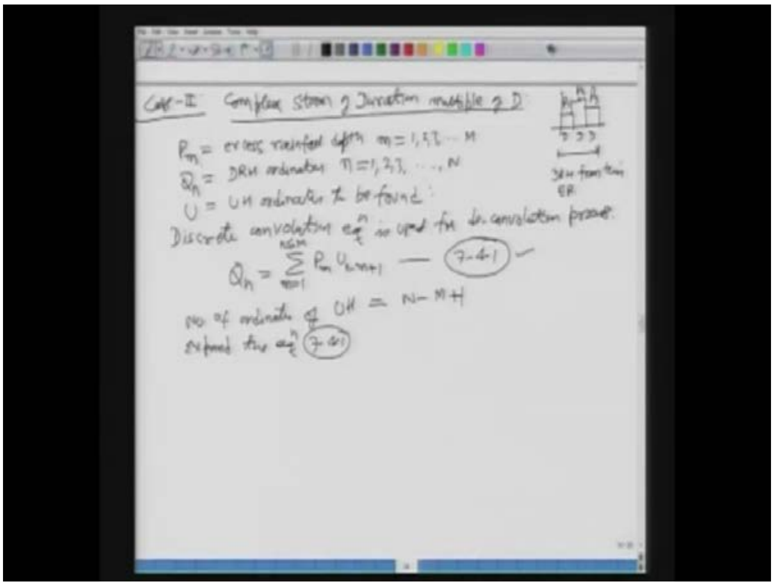
Advanced hydrology
Prof. Dr. Ashu Jain
Department of Civil Engineering
Indian Institute of Technology, Kanpur

Lecture – 26

Good morning and welcome back to this post graduate video course on Advanced Hydrology. In the last class we looked at the discrete convolution equation, definition of unit hydrograph, then the assumptions and limitations that are involved with the theory of unit hydrograph. Then we looked at the calculation or derivation of a unit hydrograph and we look at the case 1 which was for the simple storm or an isolated single storm.

Then we started looking at the derivation of a unit hydrograph given a complex storm or the rainfall and flow data corresponding to a rainfall event in which there is more than 1 rainfall impulse of duration d. And then we said that we can use the discrete convolution equation in the reverse direction, which is called the d convolution process to calculate the unit hydrograph.

(Refer Slide Time: 01:17)

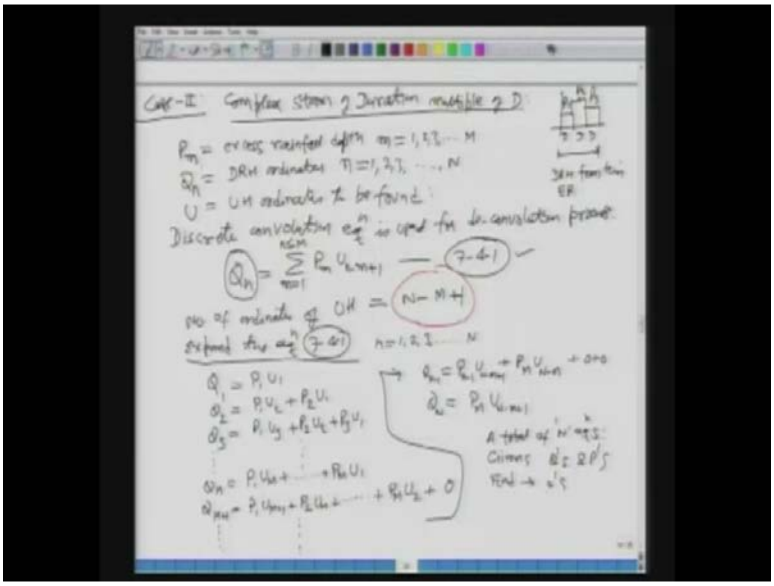


So, we will come back to that problem and start looking at this derivation or unit hydrograph in a complex storm. So, you see here we are a case 2, which is the complex storm of durations of multiples of d. Let us say this is the effective rainfall hyetograph in which you have more than 1 effective rainfall impulse of duration d. And then we define various terms, then we defined these various terms, for example this P m is the access rainfall depth that with m going from 1 to capital M, Q n was the direct run-off hydrograph ordinates, small n going from 1, 2, 3 all the way up to capital N.

And then let us say u is the unit hydrograph ordinate which we need to find out. Then we said that, we are going to use this discreet convolution equation, which is given by equation number 7.4.1 and then we said that the number of unit hydrograph ordinates; which we need to find are this, that is N minus M plus 1. So, we will get started with this the step is to expand this equation 7.4.1. If we did that and we will do that for what n values going from 1, 2, 3 all the way to, what? Capital N.

So, we will have n number of equations. So we will right this Q n or small n varying from 1 to capital N, so we have seen this earlier in the last chapter actually, when we look that this time area diagram. So what I will do is I will not spend too much time on that and I will write these equations directly and then go to the 2 methods, which we can use for the derivation of the U H.

(Refer Slide Time: 03:16)



So, let us look at this now, so you will have Q 1 is equal to P 1 U 1; Q 2 is P 1 U 2 plus P 2 U 1; Q 3 will be equal to what P 1 U 3 plus P 2 U 2 plus P 3 U 1.

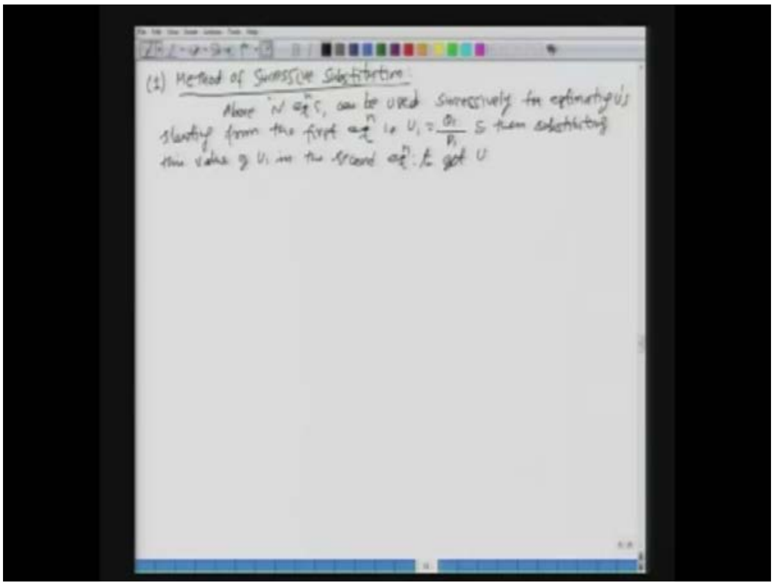
And then, we keep on doing this and we go to let us say Q capital M will be what P 1 U capital M plus, you can keep on doing that all the way up to P M and U 1. And then, the next 1 would be, what Q M plus 1 would be P 1 U M plus 1 plus P 2 U M and then all the way to your P M of U 2; and the next quantity would be your P M plus 1which is actually not there so that will be 0 and then so on.

So, what I am going to do this from here is I will go here ok. And then write the equation for let us say Q n minus 1. I am giving you all equations, so that you can expand this for any value of N and M. It will be P M minus 1, U N minus M plus 1 plus P M U N minus M and then rest of the terms will be 0. Because P M plus 1 is not defined or P M plus 1 does not exist, because there is only M plus 1 sorry there is only M number of rainfall impulses.

And in the last ordinate of your direct runoff hydrograph, it will be P M U N minus M plus 1, this going to be only one expression in the summation. So this way, we see that we can expand this discreet convolution equation to get us N different equations. So, you have N equations. What are the unknowns in this equation? The unknowns are the unit hydrograph ordinates. This Q's are known to us, that is the direct run-off hydrograph ordinates. We have this data that is given to us. We also know the effective rainfall data and our objective right now is to find out, the U's through a de-convolution process. So, you have N equations or let me write it. This is a system of equations, in which A total of N equations are there, givens are what your Q's and P's and we have to find or determine the U's.

So, let us move on and then see that, you are going to say that this can be done by or it can be done in many ways in fact that is determination of the unit hydrograph ordinates; given the rainfall and direct runoff hydrograph data. But we are going to look at 2 methods and the first 1 of them is called the method of successive substitution.

(Refer Slide Time: 07:20)



So, let us look at that method of Successive Substitution. The above capital N number of equations, which are given to us. Equations can be used successively, what do we mean by successively? One after the other, for estimating the U's starting from for example, the first equation. What is the first equation?

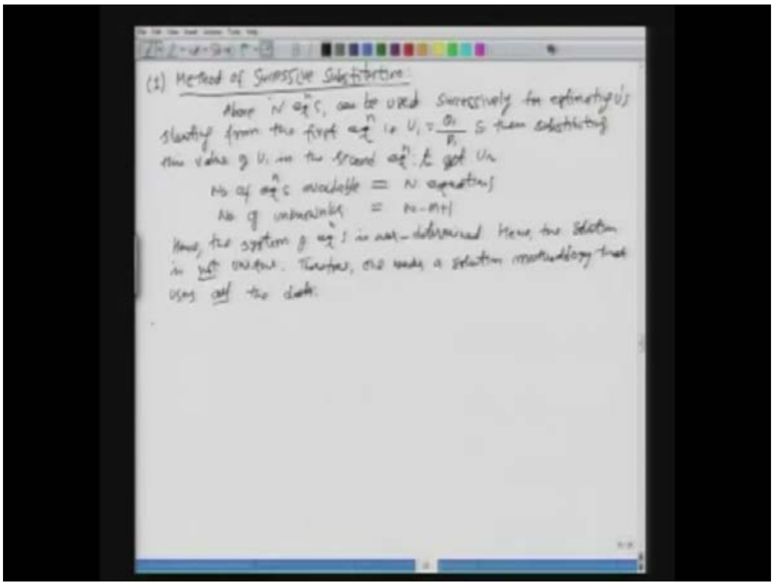
If you go back, the first equation is this one, this one, which says Q 1 is equal to P 1 U 1? What are the known's, what are the unknowns in this? Well Q 1 and P 1 is known, only thing unknown is U 1. So, can we calculate the value of U 1, that is the first unit hydrograph ordinates? Sure, that is very simple U 1 is going to be equal to what? Q 1 over P 1. So, that is very simple and then so you have the value of U 1. What do we do then? Substituting this value of U 1 in the second equation, to get, what U 2? Let us go back to look at second equation, that is this one, the one I am underlining with the pink colour. Q 2 is known, P 1 is known, U 2 is what we want to find out. P 2 is the rainfall which is know and U 1 is something we just calculated.

So, only thing unknown in this equation is what? U 2, which we can find easily from this equation. And then what we do is we keep on marching ahead using these equations. Next equation we can calculate U 3, next equation we can calculate U 4 and so on up and until we can calculate or we are able to calculate all the ordinates of the unit hydrograph.

So, this is what is called the method of successive substitution, in which you just start from the top; calculate one ordinate, calculate the next ordinate and so on of your unit hydrograph.

So, this is the very simple method. However, there are problems with this. Can anybody think of some problems? Well it says that you start from the top. What if somebody starts from the bottom; for example you look at this equation. In this equation only thing unknown is, what? This Guy here the last ordinate of your unit hydrograph, U N minus M plus 1. Q is known, P is known; so what we can do is actually, we can go from the bottom go up use N minus M plus 1 number of equations and then come up with their different set of unit hydrograph ordinates or a different solution. So you start from the top you will have one unit hydrograph you star from the bottom you will have another unit hydrograph and somebody else may pick up any number of N minus M plus 1 equations from the middle .So, you will get another set of solution. So, you do not get a unique solution using this method. Why is that?

(Refer Slide Time: 11:56)



Well let us look at this, the number of equations available are how many? As we just said there are N equations. How many number of unknowns are there? What are the number of unit hydrograph ordinates? Well that is N minus M plus 1. That means you have more number of equations than, the unknowns. So, you have a system of, linear system of equations which is, what is, called over determined. What is an over determined system?

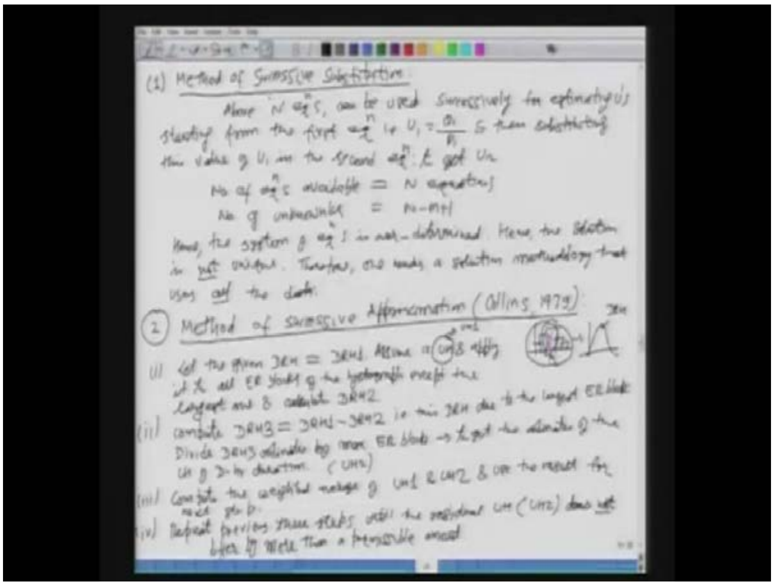
When you have more number of equations than, unknown. If you have number of unknowns and then you have number of equations same, then it is very easy to solve; you will get a unit solution. But, when the number of equations are more your system is over determined. And then depending upon which you know N minus M plus 1 equation you would use, you will get are different solution. So, there is no uniqueness to the solution.

So, the system of equations is over determined. The solution is not unique, as a result of that you will get different solution. Therefore, one needs a solution methodology, that uses all the data or all the equations.

What is happening is actually, we are using less number of data. Total number of data that is available to us, in terms of D R H ordinates is capital N. N number of ordinates are there, but we are using only part of that. That is why you are getting a wrong solution and which is not unique. Now, what we will do is, will look at a second method which is called a method of successive approximation.

This is again it is a modified version, in which you will carry out the alternative procedure and this is a method which uses all the data or all the ordinates. It is an interesting one, so what are do is. First we will look at the step by step procedure and then we see if there is any problem.

(Refer Slide Time: 14:53)



The second method is called, the method of successive approximation and this is proposed by Collins in 1939. So, remember what we have is this, you have this complex storm data and then from there you have a single D R H. And our objective is to find the unit hydrograph ordinates.

So, let us look at the step by step procedure of this. First thing you do is, you select the given D R H be is equal to you will say D R H 1. Will do a series of computations in which will redefined your D R H. So, let us say the D R H that is given is we have redefined that as D R H 1 the first one. Then we assume a unit hydrograph. What do we mean by a unit hydrograph? We assume some ordinates of the unit hydrograph. It can be arbitrary, it can be the result of the previous method, we can use the method of successive substitution and get the initial gas or the initial ordinates.

So, let us say, we we assume some unit hydrograph and apply it, means the unit hydrograph to all axes rainfall blocks of the hyetograph, accept the largest one and calculate the runoff response and let us call that D R H 2.

So, what we are doing is, we are saying that the given D R H is D R H 1. And then we calculate the direct runoff hydrograph response from the catchment from a effective rainfall hyetograph, the whole accept the largest one. So, if you look here in this one, which is the largest one, this one. So, what you do is you calculate the D R H from all the effective rainfall this one, this one, this one and this one accept this one. You calculate the D R H and let us called that D R H 2. The next step you compute D R H 3 as D R H 1 minus D R H 2. So, what we have done is, we have subtracted; so subtracted means basically what we are trying to do is, we are using the principle of super position here or you are doing the process D R H 1 minus D R H 2.

So, D R H 2 is what we have found out. So, you have subtracting that from the original D R H to calculate D R H 3. What will that be? This is, which is your D R H 3 is, what is the D R H due to the largest E R block. Is this clear, if you look at this figure here the original D R H is due to the whole input. The second one is due to all theses effective rainfalls except the largest one. So, if you subtract the 2 responses from the method of super position, what you will get is the result due to this largest one. That is what you are saying, then what do you do, then you divide the D R H 3 which you have just obtained ordinates by what, by the max E R block that is this one.

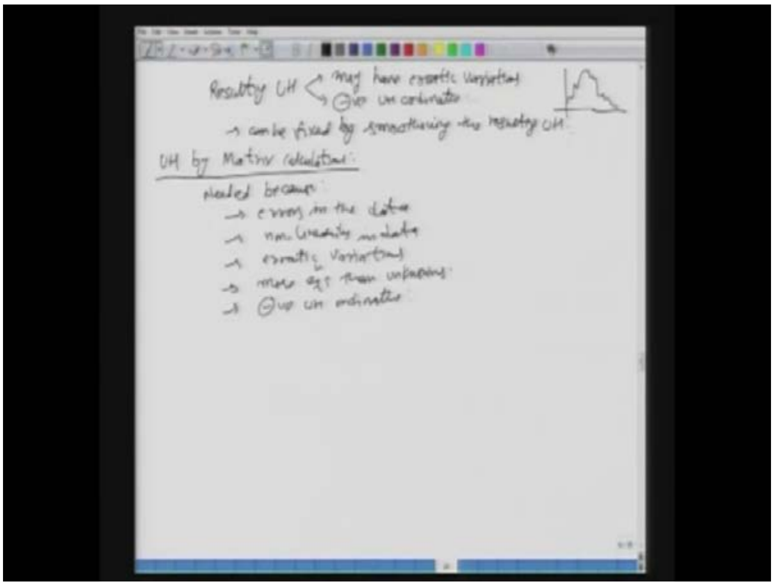
If you do that, what will it give you; you have a D R H resulting from certain amount of rainfall largest effective rainfall block. So, you divide the D R H by that amount of rainfall, what should you get? Well unit hydrograph for that duration, isn't it? by the principle of proportionality.

So, you divide the D R H 3 ordinates by the maximum E R block to get the ordinates of the U H of D hour duration. And that is resulting from the largest E R block. Let us call that some U H 2 and initially we had assume some U H this one and let us say that this is U H 1. Then in the next step what you do is, you compute the weighted average or average of what, we assumed U H, U H 1 and the residual U H which we have just found out in the previous step let us say U H 2. So, what we do is, we take some kind of average either the weighted average or the simple average or in some form we find out the combined effect of these 2 unit hydrographs and use the result for next step.

What is the next step? Number 4 is repeat the previous 3 steps. So, what all you do is repeat this previous 3 steps until the residual U H. What is the residual U H? Well it is H 2, which you are refining at each alteration, does not differ, does not differ, by more than by more than a permissible amount. So, what we are saying basically is that, we keep on iterating. We assume some certain unit hydrograph we calculate the D R H due to the largest E R block, divide by the largest E R to get the revise estimate of U H, do some weighted average, do the whole process again and keep on doing it until some level of acceptable error is reached in terms of the unit hydrograph ordinates or the total volume or whatever criteria.

So, that way we can find, the final unit hydrograph which uses all the data. So, this method is much more accurate than the first one. However these 2 methods or this method even has some problems. So, what are those we are going to look at next?

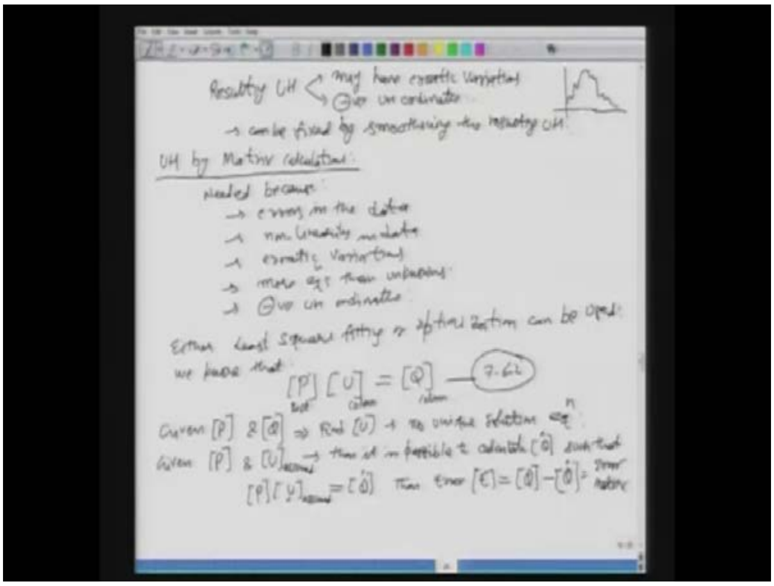
(Refer Slide Time: 24:23)



The resulting U H may have some problems for example, may have erratic variations. What do we mean by that? Erratic variations means, the unit hydrograph may look like this. Rather than a smooth curve, it maybe you know undulations or you know something like that or erratic variation. The second one, which is actually not acceptable is, you may get negative U H ordinates. It is possible to get negative U H ordinates which is actually not acceptable, due to certain numerical and other problems. Some of these can be fixed by smoothening the resulting unit hydrograph.

What we are going to do next is, we look at couple of more methods which are more sophisticated than these two. And which are more robust and take care of some of this problem. So, the next thing we are going to look at is, what is called the unit hydrograph by matrix calculations. Why do we need this? Well it is needed because of one of these problems. There may be errors in the data, there is nonlinearity in the data, there is a erratic variations, there is a more equations than unknowns that uniqueness problem. The first one at least and the last one is the negative U H ordinates. So, these are some of the problems in the determination of the unit hydrograph or a complex storm. For a single storm, we have seen the procedure which is very straight forward. So, what we do is we look at the couple of methods. First one is, what is called the linear regression and which will take care of some of these problems not all of them. And the final one is method by optimization technique and in this case we will look at only the linear optimization method, and which will take care of actually all of these problems.

(Refer Slide Time: 27:41)



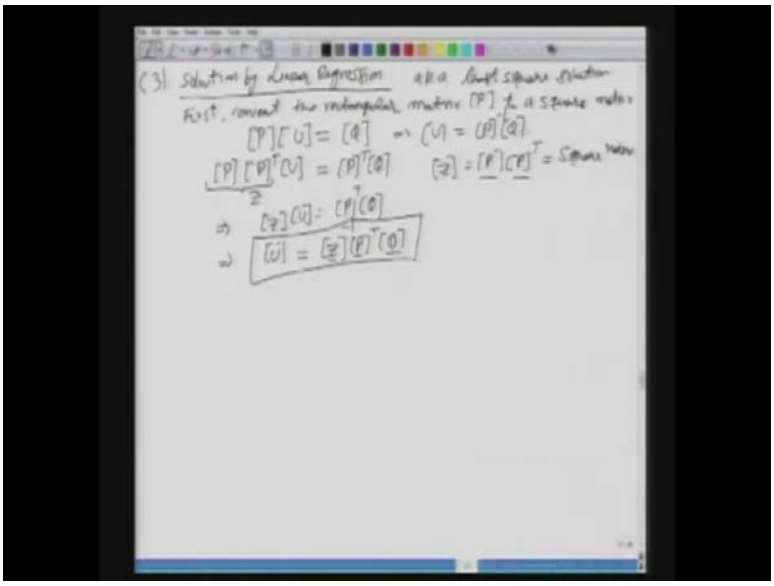
So, let us look at the first one or we say that either the least square fitting or optimization, either of these two can be used. And both of these methods actually are classified for you know matrix calculations, because it will involve the matrix algebra and then we can do the inverse inversion you know in of a matrix inverse of a matrix and you know, we can calculate the things in a nice manner and compact manner. So, let us look at the matrix form of these equations and how we can formulate these problems.

So, we know that, the discreet convolution equation in a compact form can be written like this. Do you agree? Let me first number this equation as 7.6.2. What is this equation saying? This equation is saying that, the matrix P multiplied by matrix U, U is a column matrix. This is just the column matrix of unknowns, P is a rectangular matrix and Q is also a column matrix. So, let me say that, this is column matrix, this is also a column matrix and this is a rectangular matrix. So, if you go that to your discreet convolution equation.

These one’s, all of the set. Can you write the set of equations in a matrix form. So, if you look at the right hand side, this is all the equations in which P’s are known and U’s are unknowns. So, you can this is basically the matrix operations, you can have P times U is equal to right hand side which is the left hand side here Q 1, Q 2, Q 3, Q 4 etc. All of these are known to us. So, we can write this discreet convolution equation like this. Now the problem is that, given what are the known’s P and Q that is known to us right. If that is the case we want to find out U. There are problems of no unit solution etc, if we did that, we have seen that. However, what do we do is, if we say that, the given P and U, let us say some assumed U which was one of the second methods actually we just looked at. Some assumed or initial value, then what; then it is possible to calculate. What, I am going to say Q hat or estimated value of the D R H ordinates, such that your P matrix times your U matrix which is assumed, this is your assumed value should be equal to what, Q hat. So, it will not be equal to exactly the U or sorry, exactly the Q or the D R H ordinates, which is known. But, it will be some other values Q hat and then if you find that difference between Q hat and Q that will be the error. So, what we are doing is that initially, we start with some assumed value of U, calculate the computed value of Q, calculate the error by finding out the difference and then we are try to minimise that error. So it kind of becomes and optimization problem.

So, then we defined the error matrix your E as what, as your true values of Q minus the computed values of Q, this is your error matrix. And then what is the perfect solution? Well the perfect solution is obtained, when your error matrix has all the 0's. So, obviously it will be very difficult to achieve that but we try to minimise the entries in your error matrix.

(Refer Slide Time: 32:49)

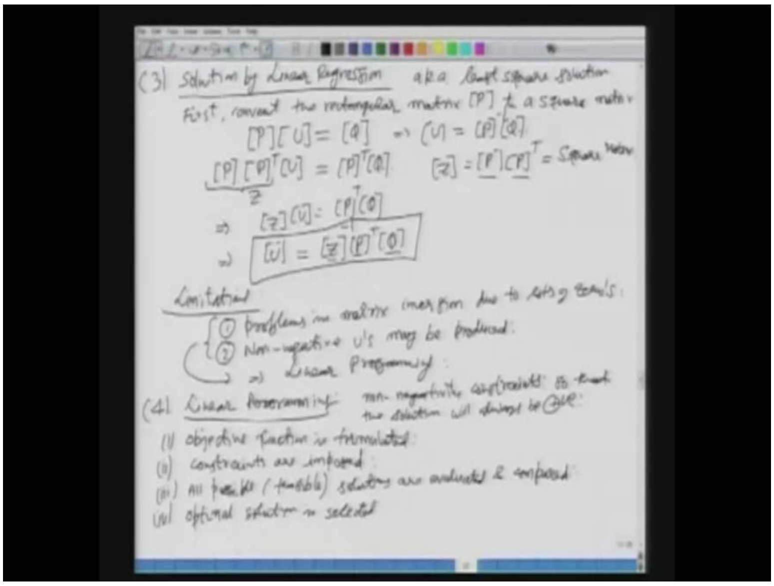


So, let us look at the least square method first, which is also called the, and I am going to numbers this as the number 3. Because we have already seen the 2 methods will say Solution by Linear Regression. Actually it is also known as least square solution. As you may know that, what we do in the linear regression or the regression analysis, it basically uses the least square method to calculate the co-efficient of the regression co-efficient. So, either you say it is a linear regression or the least square method. So, what we actually do in this particular method is, we first convert the rectangular matrix of rainfalls which is P to a square matrix.

So, what we are doing actually is let us say you have this equation P times U is equal to your Q, is it not? That is what we have. So, basically if you see that would give you U is equal to what, your P inverse time's Q. So, you can directly do that but P is a rectangular matrix. So, it is not possible to invert a rectangular matrix, so let us first converted into a square matrix. How do you do that? Well multiplied that matrix by the transpose of it. So, if you did that, will have P multiplied by the transpose of symmetric times U is equal to what, transpose of your P matrix times your Q. So, I multiplying by transpose on the both sides and let us say that this we call as Z matrix,, were your Z is equal to what, P times P transpose, which is a square matrix right.

So, then that would mean you have your Z times U is equal to P transpose of your Q that would mean your U is equal to what, you invert the Z matrix then you already have this thing here. So, this is your solution in terms of U. So, given the values of P and Q and the Z which is defined like this, you can find out the U matrix, which is your unit hydrograph ordinates or a column matrix.

(Refer Slide Time: 36:28)



Now, this method also has a certain limitations, this method of least square or linear regression, whatever we have looked at what are the limitations? There maybe, number one problems in matrix inversion itself, finding the z matrix itself can be problematic why, because there may be lots of 0's in it. And not all matrixes are invertible. So, there may be some problems related to that. Also you have or you may have non-negative U's may be produced. It is still possible to have solution of U, such that you will have negative values, negative unit hydrograph ordinates which are again not correct and not acceptable. What do we then we resort to the next method which is called the linear programming and that is what we are going to look at next.

So, the next method we will look at is called the linear programming, this is an optimization technique or solution by linear programming and the major advantage of this method is that it allows us to use non-negativity constants. So, that the solution which will obtain in terms of the unit hydrograph ordinates or the decision variables will always be positive. So, in this course actually, I will not have a time to go through the details or the you know the inter cases of a optimization method itself. I will expect that you understand what is a linear programming or the basic concepts of optimization. What will do is, I will look at this step by step to see their which we use in this linear programming then, we will formulate the optimization problem. And will I will say that the simplex or any other method which is available for the solution of linear programming can be used.

So, will not go too much in to the detail, will just look at the formulation of the problem. So, that we are able to get the positive values of the unit hydrograph using this linear programming method. So, what are the steps, if you look here; in any optimization problem or in the linear programming problem in this case. What we do is, we need an objective function and objective function is formulated. What is an objective function in an optimization problem, well an objective function can be the cost function which we are trying to minimise or it can be a benefit function which we are trying to maximise.

So, there is a function which will involve the values of the decision variables and the decision variables shares are what, it is the unit hydrograph ordinates. So, you will formulate some kind of function which you will try to minimise or maximise. So, in this case we will formulate the objective function at the error. So, you have the actual values of Q you have the computed values of Q and the difference of those will be your error matrix, which we have already seen. So, the error function will be the objective function we will minimise. So, this is a first step in which, we formulate an objective function and which in our case will be the error function.

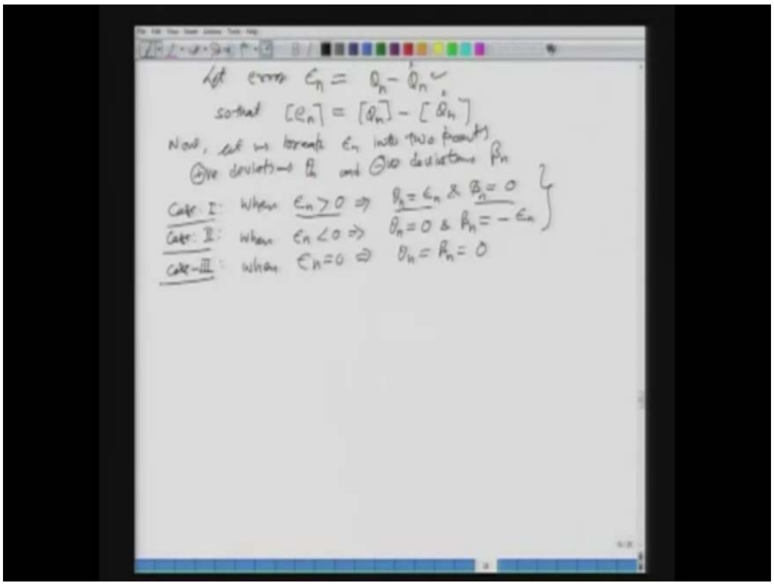
Then, what you do in the linear programming is, we impose the constraints or we say the constraints are imposed. What do we mean by that? When we say that the constraints are imposed, the objective function consists of some formula, let us say some equation which involves the decision variables right U 1, U 2, U 3, U 4 etc or for any other linear programming problem it may be X 1, X 2, X 3, X 4 and so on.

Now, when the linear programming solution methodology will search in the visible region or in the search space, we want to put certain constant. So, that the solution which we obtained satisfies certain restrictions or certain constraints. We should not look for the possible solution in the whole domain, because that solution may not be acceptable because it does not follow the certain basic principles or the laws of nature.

In this case the unit hydrograph should be such or the unit hydrograph ordinates, it should be such that, they obey the discreet convolution equation. So, we will look at the constrains which we will impose in this particular problem, as I said we will formulate the problem. So, basically the constraints are the one in which we put the limitations on the values of the decision variables. So, that the obey those laws or those equation or those constraints.

So, that is the constraints, the third step is all possible which is called feasible in your optimization terminology, solutions are evaluated and compared. So, we are doing, you are looking at all the visible solutions in the linear programming, we are normally on the boundaries. So, we look at all the possible or feasible solutions on the boundaries in the case of linear programming and we compare various solutions that are available and then we select the best. So, the last step is optimal solution is selected or chosen, what is the optimal solution well the one which gives you the 0 error.

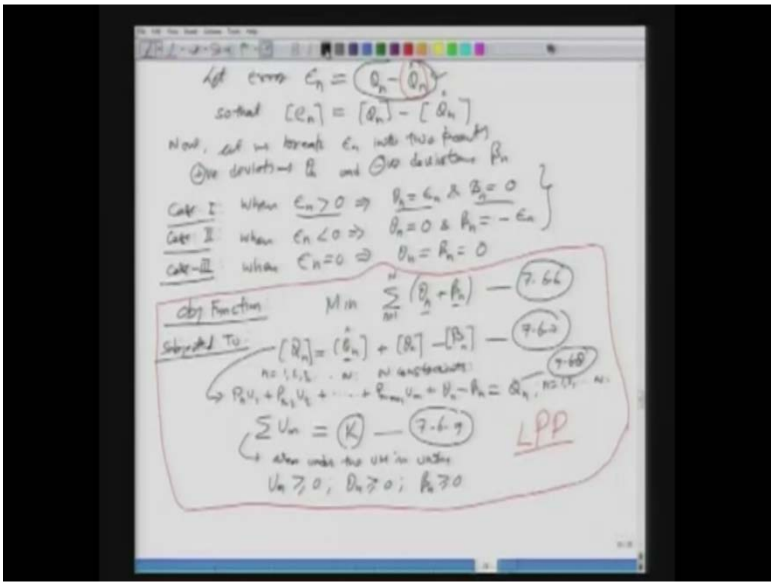
(Refer Slide Time: 43:48)



So, let us formulate this problem let the error E for n equation n be defined as your Q n, which is the actual value of your D R H ordinate minus Q hat n. So, that your error matrix, is equal to what, is nothing but this matrix minus Q n hat matrix. Now, let us break this epsilon. This one, into 2 parts that is positive deviations, which we are calling theta and negative deviations, which we will call beta. Now there can be 3 possible cases case one: when your epsilon n is greater than 0, that would mean what, that would mean your theta n is equal to is epsilon because epsilon is positive and we are saying theta is positive this one. So, we said eta is equal to theta and what will be beta for this case, well it is the negative component which is 0. So, if your epsilon is positive then you say that epsilon is your theta and the beta component is 0. We are doing all these so that we are able to impose the non-negativity constraints. Then you have case 2: when you have epsilon n less than 0 that is, when the error is negative then what, then you say your positive component is what obviously, we say that it is 0 and then you say beta n is equal to what, you want that to be positive so you say negative of a n. So, all you are doing is you are just doing some mathematical manipulations, so that your epsilon you are converting into positive error all the time. If you dispositive you are calling it theta if it is negative you are taking one more negative and you are calling that beta.

So, we do this and then the third case is when there epsilon is 0, then what well then theta n is 0 beta n is 0 and we are having a perfect time or the perfect case. So, with this notations what I will do is I will give you the formulation of the linear program, in which will have the objective function and we will have the constraints. And then we can solve that problem using any mathematical software or the simplex method or some other method which are available in a met lab or you know mathematical or any other mathematical software.

(Refer Slide Time: 47:51)



What is the objective function? We say that we are going to minimise this, sum which is theta n by positive errors plus beta n corresponding to the negative errors and then n is what, n is an index going from 1 to what capital N. This corresponds to all the D R H ordinates remember that and this I will say 7 6 6. Subjected to certain constraints, what are those constraints? Well well before i go to that, what are those thetas and betas, these are the errors. What are these errors, these are coming from this equation on the top, if you see here, where is the U’s in this, well U is coming from this guy here which is the estimated value of Q n.

So, the constraints we obtained or we we impose will be in terms of the discreet convolution equation. The first one is we say your Q n is equal to Q n hat plus theta n minus beta n. This is your column matrix, this is also column matrix and these are all your matrix. So, this is your one constraint which is 7 6 7. This is nothing but your what is Q n, this equation is written for n is equal to what 1, 2, 3 all the way to n. So, how many constraints are there, there are capital N constraints corresponding to each and every equation. So, what you have is, then actually let me write one of the equations. So, you will have P n U 1 plus P n minus 1 U 2 plus P n minus M plus 1 of your U M plus theta n minus beta n, corresponding to that is equal to what is your is your your Q n.

So, this and this is exactly the same. So, we have written it down in the equation form that is in the matrix form, where n goes from 1 2 all the way to capital N. So, what is this constraint doing is that, it is ensuring that the unit hydrograph ordinates follow the discreet convolution process. So, the unit hydrograph ordinates which we will get will be such that they will satisfy all these equations. So, we are enforcing these use to follow the discreet convolution equations all n of them. So, that we are using all the data so it will give you an optimal solution.

So, that is one constraint, the second constraint we impose is that this unit hydrograph ordinates which we find should be such that, the area under the unit hydrograph should be equivalent to 1 centimetre of effective rainfall or the direct runoff depth should be equal to what unity. So, we ensure that and the next constraint, then we say that the summation of U m's, should be equal to K. I am not going to go in to the detail of this too much and then say that this is 7 6 8, no this is 7 6 9 in fact; this equation was 7 6 8 and this is 7 6 9. So, what is this equation is doing is, this is ensuring that the area under the U H is unity. What is this K by the way then? This K is nothing but the after you convert the units you know the area of the catchment in the square kilometre and the you know depth of the rainfall and so on. So, this is just taking care of that unit conversion, so that this is equal to some constraint, corresponding to area under the unit hydrograph equal to 1 or 1 centimetre on depth. Then that is it basically.

So, and other constraints are always there, so basically you will say that all the U m's are greater than equal to 0. All the theta n's are greater than equal to 0 these are automatically imposed, in any linear programming problem and then you have betas are also 0. So, this is your complete linear programming problem, linear programming problem L P P. We have formulated this problem and as I said we can solve it using any of the method. There is some discussion given in the book at the end of this formulation of the problem, which discusses some of the limitations of this linear programming method. This is also not you know perfect no solution methodology is perfect.

So, I will request you to kindly go through it yourself which is given at the end of this particular topic in your Venti Chau's book. So, I think I would like to stop at this point of time because I am at a good stopping point and in the next lecture, we will take up the new topic of synthetic unit hydrograph. What is a synthetic unit hydrograph and how, we can actually calculate it or generate it artificially generate it and this is useful in the catchments where there is no data in terms of a flow or rainfall or anything, so will stop here.

Thank you.