Advanced Hydrology
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Lecture – 24

Good morning and welcome to the next class of this post graduate video course on Advanced Hydrology. In the last class, we started a new chapter on unit hydrograph. We looked at the basic definitions; the basic concepts of linear systems theory; the hydrologic system modal and then we looked at a general model proposed by (( )) swamy in 1971. Then we looked at the basic two principles of linear systems theory, which allow us to compute or make certain applications in complicated situations. These were the principles of super position and the principle of proportionality. Then we said that, we need to understand three basic functions and then we started looking at the first one, which is called the unit impulse response function.

We defined the unit impulse response function as a function or a response from a system, when it is subjected to a unit impulse input. What is a unit impulse input? A unit impulse input is nothing but an input of unit amount, which is applied to a system instantaneously. So, you have certain forcing function or certain force being applied to a natural system instantaneously. Alright? Momentarily and then it is taken of and then how the system would respond and the response is then called unit impulse response function. What was the first one? We said it is an example of a guitar string. In which, if we provide a certain force using our fingers to a guitar, then we get a response or the sound waves in the form of the impulse response function.

Then we said, another example was of shock absorbers of an automobile, in which we said that, if the system is the shock absorbers of a car, the force applied is, let us say passing of the car over a hump or over a speed breaker or a pot hole, then that acts as an impulse input to that system. What comes out by the response of the shock absorber is expressed as the impulse response function. We also said that this impulse response function also follows the principle of proportionality and the principle of super position, because this is also based on the linear systems theory. Alright? That is where we stopped. Today, we will start by looking at a third example of a catchment, which is also the you know, we can apply what is called this impulse response function.

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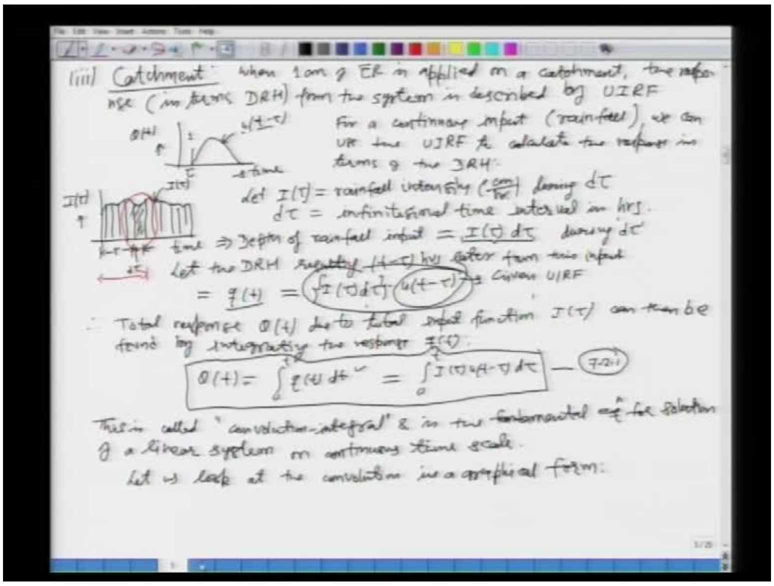

The third example which we will look at starting with today's lecture is of a catchment. What happens in a catchment when a unit amount of rainfall or 1 centimetre or 1 inch of rainfall is dumped on to the catchment instantaneously? What will come out of that is actually called the unit impulse response function. So, when 1 centimetre of effective rainfall is applied on a catchment, the response how in terms of your garage or the direct run of hydrograph, the response from the system and the system being your catchment is described by a unit impulse response function. How does it look like? We have seen this earlier, but we will look at it again today. So, this is your 1 unit of 1 centimetre of rainfall. What comes out is your u t minus tau. So, this is your time domain and this one is your q t, or I should say or u t t minus tau and the input is applied at time tau and we are calculating or we are expressing the response after sometime or delayed time is t minus tau. So, that is how this unit impulse response function is defined.

Now, this input which we have seen is instantaneous. The way this unit impulse response function is defined is that, this 1 centimetre of rainfall is applied instantaneously. However, in reality, rainfall does not occur instantaneously. Alright? It does take some time. Like for example, you may have certain amount of rainfall occurring in 1 hour, 2 hours, and 3 hours and so on. So, how do we determine or how do we find the response from a catchment? By using this impulse response functionor the unit impulse response function for a continuous output. So, for a continuous input I should say, which is your rainfall, we can use the unit impulse response function to calculate the response in terms of the d r h. Well, how do we do that?

Let us say that, this is your time domain and you have some continuous amount of rainfall like this. This we are saying is your intensity of rainfall represented by i tau. So, we say that, let your i tau be the rainfall intensity in, let us say centimetres per hour or some suitable units during d tau and this d tau is a very small time interval. Extremely small, so that, we can say that its duration can be neglected or infinitesimal or very small. So, i tau is the rainfall intensity during some very small interval d tau and what is d tau? It is nothing but infinitesimal time interval in hours. So, if you are looking at it like this, let us say you have this. So, this curve is what? It is nothing but the intensity as a function of time, tau.

Let us say that this is your time tau and this is your time d tau. So, we are looking at this strip of input, which is occurring almost instantaneously during interval d tau, which is infinitesimal or very small. Then can you tell me what will be the depth of rainfall input during this very small infinitesimal input d tau? What will be the depth of rainfall? You know the intensity and you know the very little duration, so what will be the depth of rainfall? It will be the intensity multiplied by d tau, the duration.

So, you can say that, this is going to be equal to i tau times d tau during your small interval d tau. Now, we have the instantaneous input of the magnitude i tau, d tau, right. Now, how can I find the response from the catchment in terms of d r h from this input i tau d tau, assuming, I already have the unit impulse response function.

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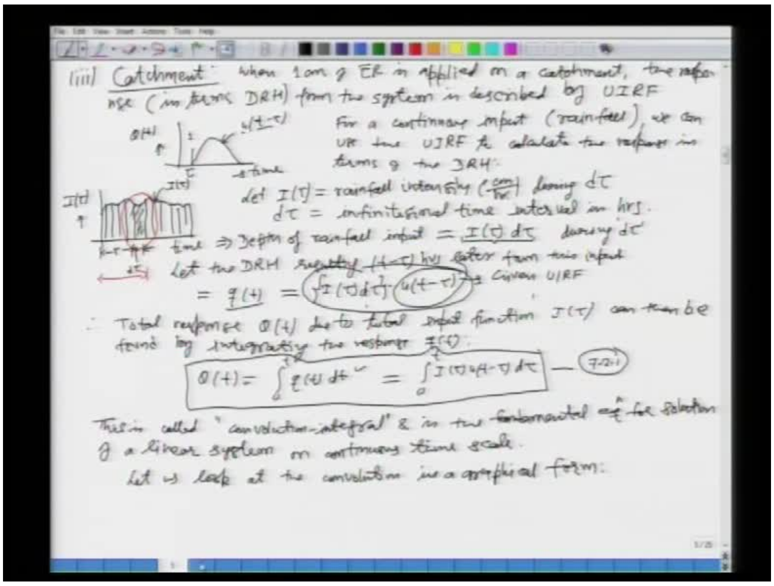

Now, we are looking at the using of the unit impulse response function to calculate the total response or the total d r h. So, this will be, let us say, let us define it first. Let the d r h resulting t minus tau hours later from this input. What input? This one, i tau d tau. Let us say that this is equal to small q of your t. What will that be? I can use the principal of proportionality. I know what is the ordinate of your response from the catchment due to this 1 centimetre of rainfall, alright. So, given your u t minus tau, I can multiply each ordinate by the amount of rainfall and the amount of rainfall is i tau d tau.

So, it will be equal to what? i tau d tau. So, this is the amount of rainfall multiplied by your u t minus tau. This let us say is given to us. This is your unit impulse response function and what is this q t? This q t is the response from this system due to this infinitesimal strip of your rainfall, which is expressed by this. Alright? So now, I have written the expression or calculated the response from this very small input i tau d tau, which occurs in d tau. How about or how can I now calculate the response from the total rainfall input? Now, this i tau is continuously occurring and I have calculated the output due to the small strip. So, what I can do actually is, I can divide this whole input into various strips, small small strips and then I can integrate the whole response. Alright? So, I will have very small small discreet intervals. Again, this is an assumption or simplification. So, I can divide this whole input to small small strips of inputs and then I can integrate, so that, I can calculate total response from this continuous input.

So, what does that mean? That the total response, let us call that as q t, due to the total input or the complete input for the whole domain, total input function i tau can then be found by integrating the response q t. So, mathematically you can write it as q t is equal to integral between 0 to t of what? Of your small q t d t. So, what we are doing is, you have this small small scripts between time limits 0 to t. Just before that rainfall. Alright? So, what we are doing is, we are integrating. So, let me go back here and try to demonstrate what we are doing is, you are integrating up to certain time. So, you have all this strips, this one, this one, this one, this one and so on. So, we are integrating over this whole time for the whole input and this you can continue for whole time duration. So, this is your total response. Alright?

Now, q t we have already written from this equation. This whole expression is your q t. So, if you put in here, what will you have is this. Integral 0 to t of your i tau u t minus tau and d tau. I am going to number this equation as 7.2.1, Let me put it in box also, because this is a very important equation, which is called the integral form of convolution equation. Extremely important result, which allows us to calculate response, d r h response from a catchment for any complex rainfall event. So, let us say that, this is called convolution integral and is the fundamental equation for solution of a linear system on a continuous time scale. It is called convolution integral. Alright? Integral because that integration process is involved, so your summing up in your certain response and convolution is just a term, which is used why because this function is giving you the convolution or you have the input being applied to a system. That input is being convoluted. It is being transformed to that system to create that output response.

So, this whole process of convolution involves lots of processes in the catchment. It may be translation, it may be attenuation and it may be many other storage, you know, e facts or characteristics. So, all this convolution process actually is taking care of the transformation of this effective rainfall into direct run of hydrograph. That is why it is a very important tool for us. Once we have developed this unit impulse response function, then we can find out the response from any continuous input or rainfall.

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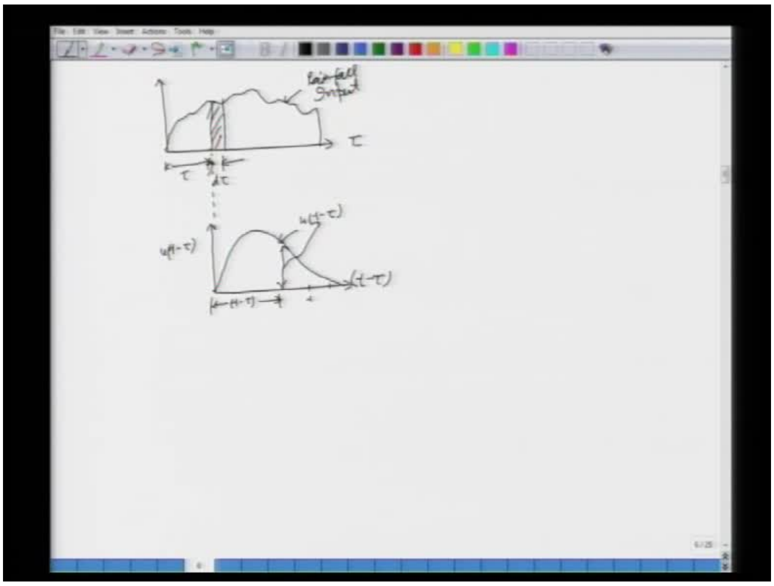

So, now what we are going to do is, we will look at this process of convolution in a graphical manner. That will probably clear, you know, this whole thing a little bit better. So, let us look at the convolution in a graphical form. So, this is your tau domain. We are talking about the input. This is your rainfall input as we have said earlier. This is tau. This is d tau and this is your rainfall input.

Now, when does the response from the system will start due to this input? Obviously, it will it will be this time. So, let us, then you say you have this response starting from here. This is your t minus tau domain. So, you have any general time t, but you are subtracting tau when the rainfall has actually occurred. So, this curve is what is your u of your t minus tau. Alright? Let us say that this is your unit impulse response function. So, what is this? This is u t minus tau. What does that mean? If you take the ordinate of this curve, what are the coordinates? Well, this is t minus tau is the x coordinate or the time domain and what is this? This is nothing but this. So, this curve is your unit impulse response function, which we know and what is the objective of the convolution integral? Well, the convolution integral uses this unit impulse response function and applies the series of these rainfalls. Small small strips and convolutes all of that through the catchment to give you a combined response as per the equation 7.2.1, which is this equation. So, we are looking at the expression of this equation 7.2.1 graphically. Alright? So, the response will actually start from time t is equal to 0, when this rainfall starts.

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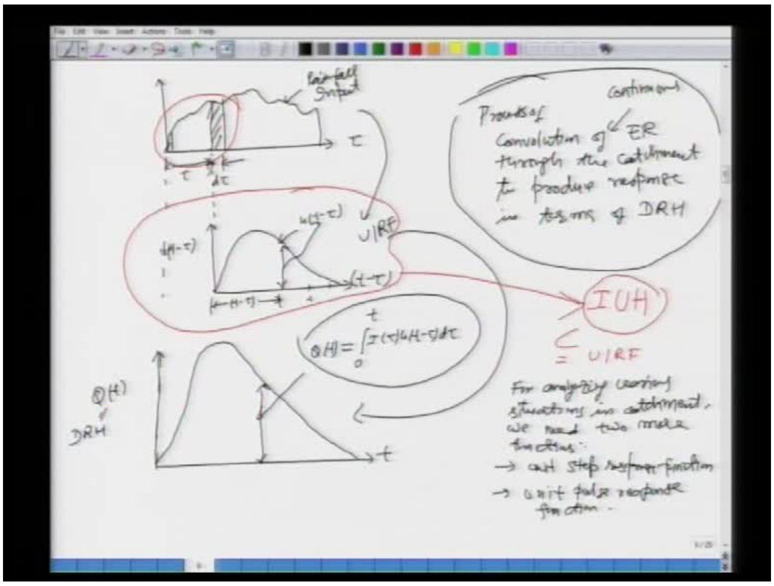

So, obviously you have a strip here and the response will start from here itself. So, this is your time domain t. What is this? This is your q t. q t is what? This is equal to your direct run of hydrograph, due to all of this input. What is that input? This whole input. Alright? So, I am just going to draw this d r h. Let us say it will be looking like this and if you take any point on this curve, what is this ordinate going to represent? This ordinate is obviously your q t, which is equal to what? Which is nothing but 0 to t of your equation 7.2.1, which is the convolution integral, i tau u t minus tau d tau. This is your convolution integral. So, what we have is, this is the input falling on the catchment and you are using the unit impulse response function, which we already have let us say and then the final output, then is given by this curve here.

So, this is called the process of convolution of your input, which is your effective rainfall, which is continuous. Let me say continuous effective rainfall through the catchment to produce response in terms of the d r h. This is an extremely important process. It is the basis of all of your rain fallen of models. Now, I am going to ask you a very simple question, which is related to what we have just seen and you have seen all these concepts in your earlier classes. It is just that we are looking at them in a most general form.

The question is, this unit impulse response function, let me try to show it to you, this whole thing, this whole guy here, what is this? Have you seen it earlier? What is this?

Physically it is representing what? It is the d r h from the catchment, when the catchment is subjected to 1 centimetre of rainfall occurring instantaneously. Not in any duration. That duration is extremely small. So, the response from the catchment, when it is subjected to unit input or 1 centimetre of rainfall occurring instantaneously is called what? It is nothing but instantaneous unit hydrograph. Have you studied i u h? I am sure some of you may have. So, this whole concept, which we have seen is nothing but the i u h, which is your equivalent to unit impulse response function. So, this is about the unit impulse response function and for analysing various situations in the catchment as far as the hydrograph analysis is concerned, we need to understand two more different types of functions. What are those? Well, first one is the unit step response function and the other one is the unit pulse response function.

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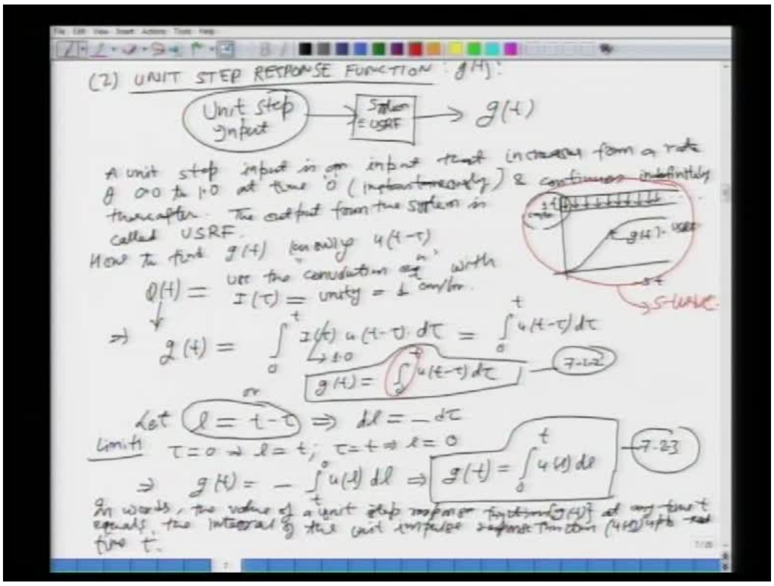

So, let us start looking at these two functions now. So, we have seen unit impulse response function. The next function we are going to look at is the unit step response function. The second one is unit step response function. Let me just give you the schematic. What is the unit step response function? Well, it is nothing but the response from what is called a unit step input. We have the unit step input, which is being applied to your system, which is a catchment we are going to look at and then this model acts as a reservoir and what you have is the unit step response function is coming out of it and this is denoted as g. So, in words, if you want to write it, let us first define what is this unit step input. A unit step input is what? It is an input that increases from a rate of 0 to

1.0. Alright? So, from 0 to 1, a rate, at time 0 or initially, in the beginning. What does that mean? Instantaneously and continues indefinitely or continuously thereafter.

So, what kind of input is this then? If you want to draw this, this is your time domain. So, this input is 0 at the beginning and then it quickly reaches some rate of 1 and then it continuous indefinitely. It is continued. So, this amount is 1 centimetre per hour or 1 or unit input, this whole thing and this is occurring continuously. Then the output from the system is then called your unit step response function. So, what is this unit step response function? It is the response from a catchment or output from the system, when that system is subjected to a very special type of input and what is this speciality of this system or this input? It is the rate going from 0 to 1 at the beginning at time t is equal to 0 and then continue indefinitely or continuously, infinitely. So, what will happen? What kind of curve do you see? You will have a curve going up and then coming down? No, because the input is not stopping here. Once the input stops, that is when the d r h response starts to come down. Here the input is not stopping, it is continuous. Therefore, your curve will be an or the response will be an ever rising curve.

So, the response would be something like this. Alright? This we say is called g t as a function of time, which is your unit step response function. Now, how can we find this gt or this unit step response function? Let us say you have the unit impulse response function. Alright? Now, the next question is, I am going to pose this to you. How to find g t, knowing or given u t minus tau? What is u t minus tau? We have just looked at. It is unit impulse response function and what is the unit impulse response function?

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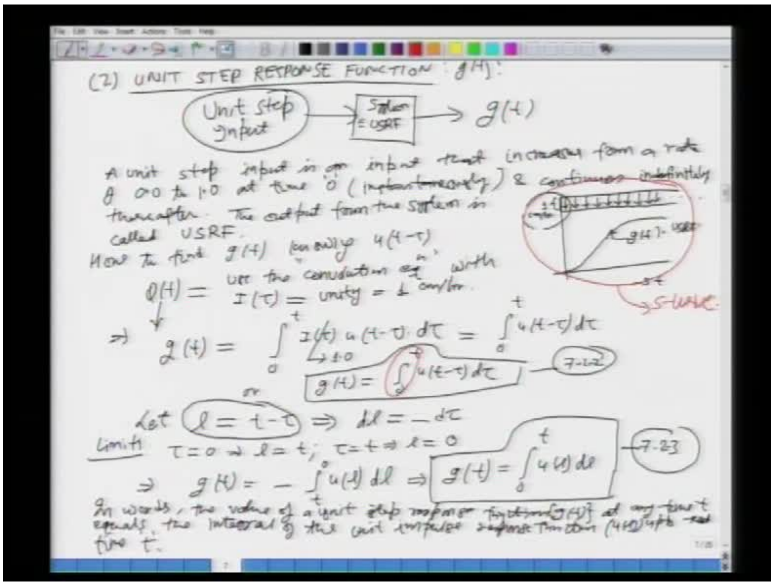

Well, you have unit amount of rainfall or unit amount of input occurring instantaneously and the response is u t minus tau. Now, what we have is an input, which is continuous of certain rate and what will be response from that, knowing u t minus tau. So, what we can say that, this q of t or the response from this input is equal to what? We can use the convolution equation or integral form of the convolution equation with what? Your i of tau is equal to what? What is i tau? It is the rate of rainfall. What is the rate of rainfall here? It is 1 or unity. Your i tau d tau is equal to 1, which is 1 centimetre per hour. Alright? So, you see that, to calculate the response or the unit step response from the catchment when it is subjected to a rate of 0 to 1 occurring indefinitely will be equal to what? You use the convolution equation and you put intensity is equal to 1.

So, what is this? That means, what is q? q is nothing but your g. This is equal to what? 0 to t. What was your convolution equation? Well, it was i tau u t minus tau d tau, right and what we are saying is that, this is equal to 1.0, right. So, what is this equal to then? This is nothing but 0 to t of your u t minus tau d tau or let me say that, this g of t is 0 to t u t minus tau d tau. So, this I am going to define a number as 7 to 2. Now, what I am going to do is slight mathematical manipulations because you have t and tau. Both of the time domains are appearing here in the integral. You have t and tau, both of them. So, what we will do is, we will get rid of these two and convert this equation into a single time domain, so that, we can integrate it properly. Alright? So, to do that, you say that, let some new parameter or variable l is equal to t minus tau. Am I right? So that, what is d l? d l will be equal to what? It will be minus d tau, because you are integrating with the spec to tau here. Alright? So, your d l is equal to minus of d tau.

Then, you need to change this limits of this integration. Alright? Limits of integration means what? These limits. So, if you work on them, then you have tau is equal to 0. This is with respect to tau and tau is equal to 0. The lower limit would mean what in terms of l? Well, this is 0 and l will be equal to what? T, is it not? For the second case, your tau is equal to t, which is the second limit. This one, all right. When tau is equal to t, you put t in this equation, means l is equal to what? l is equal to 0 from this equation, right. So, in this one, you put t is equal to 0 and you have l is equal to t, first in the lower limit and the upper limit, from this will be l is equal to 0. So that, once you put this whole thing into 7.2.2, then what we will have is, your g t is equal to d tau is negative. So, I will take the negative outside and integration limits are from t to 0 now. Then u of your t minus tau is what? It is l and d tau is minus d l. So, that is what I have got. That means, your g of t is equal to what? Minus you can take inside and change the limits of integration. 0 to t of what? u l t l. So, you see, we have done slight substitutions and some mathematical manipulations here to arrive at this equation, which is 7.2.3, I am going to number.

Now, what does this equation represent? g of t is equal to integral from the limit 0 to t of what? u l d l. What is u? u is nothing but your unit impulse response function. So, g is nothing but the cumulative of your u curve or unit impulse response function curve. Alright? So, mathematically you can say that or in words, the value of a unit step response function, what is that? g t, right. At any time t, equals what? Equals the integral of the unit impulse response function, which is u. That is, u l in this case. Up to where? Up to that time t. So, this g is a mass curve of u curve. Alright? Now, can anybody tell me, what is this curve? If you come here, this one. Have you seen this in your UG classes? I am sure you have. This is nothing but your s curve or a s hydrograph. So, the first one we have looked at is i u h and the second one is the s curve. Alright? We will look at the third one, which is called the unit pulse response function.

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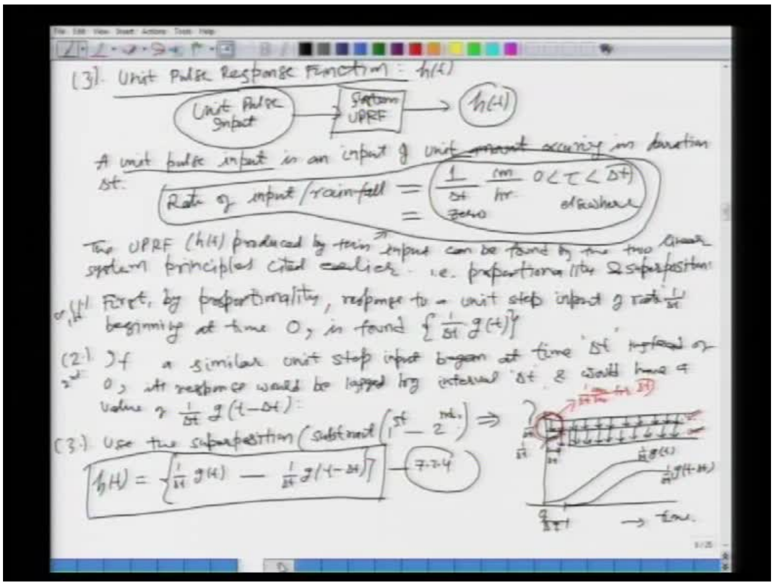

The third function we will look at is what is called unit pulse response function from a catchment. We are going to represent this by h of t. Again, looking at the schematic first, you have what is called the unit pulse input. What is the unit pulse response function? Well, it is the response from the system, which is catchment in the form of u p r f and that is your h of t. So, a unit pulse response function is what? It is nothing but the response from the system when the system is applied or is subjected to unit pulse input, right. What is the unit pulse input? Let us look at that. So, we have defined, you know, different types of inputs here and this is the third one. A unit pulse input is what? It is an input of unit amount, not the rate. Alright? Unit amount means, 1 centimetre of rainfall or 1 inch of rainfall or 1 millimetre of rainfall in this case and effective rainfall we are talking.

So, a unit pulse input is an input of unit amount occurring in duration delta t. We are saying duration delta t means what? It is occurring in certain finite duration delta t. It may be 1 hour, 2 hours, and 3 hours and so on. It is not infinitesimal. It is not extremely small. So, it is not instantaneous. Alright? It is certain duration of rainfall. So, this is your definition of the unit pulse input and then we want to find out, what is going to be h.

So now, we have the amount of rainfall is 1 centimetre and it is occurring in time interval delta t. So, what will be the rate of input or rate of rainfall? Can anybody tell me that? It will be what? 1 centimetre of rainfall occurring in how much duration? Delta t duration. Alright? This is, let us say your centimetre per hour. That is the rate or the intensity of the rainfall and it is occurring for what duration? Let us say tau is your input domain, input time domain. For how much time? delta t and 0. So, your rate of input is what? It is 1 over delta t centimetres per hour for time between 0 and delta t and what is the input elsewhere? Elsewhere it is 0, all right. So, this is your unit pulse input.

Now, we want to find out the response from this kind of input. So, you have a rate going from 0 to 1 by delta t and then continuing for certain time delta t and then becoming 0. Now, knowing the first two functions, that is your unit impulse response function u and the unit step response function g, how can we find this one function h, which is the unit pulse response function. So, what we will do is, we will take the earlier functions and then apply our linear system theory, proportionality, super position and so on and once we do that, we should be able to find out, what is this h. So, let us look at that first.

The unit pulse response function h, produced by this input, which input? This one, which we have just defined, can be found or described by the two linear system principals cited earlier. That is, proportionality and super position. So, what I am going to do is, I am just give you step by step procedure for that first and then we will see graphically how we can deduce this h or the unit pulse response function. So, the first step is, by proportionality, response to a unit step input of rate 1 by delta t, what is the rate? Well, you just add, it is 1 by delta t, beginning at time 0 or at the beginning is what is found, which will be what? Nothing but 1 over delta t multiplied by what? The step response function g of t. So, we are using the principal of proportionality. You know the unit step response function g, let us say and that is given to us. That is, the g of t is what? It is response from the function due to what? Due to an intensity of 1 centimetre per hour. Alright? But, it is a linear system. So, we can get the response from the system when the intensity is not 1 but x and that x is 1 by t here.

So, all we do is, we multiply the g by 1 by delta t, which is the intensity of rainfall. For how much? For certain time. Only delta t. So, this is what we do. That is the first step. Then what do we do? We say that, if a similar unit step input began at time delta t, which is the duration and instead of 0, its response would be what? It would be lagged by interval delta t. Is it not? Because, the input has started delta t time later or delta t hours later, so the response will also start or it will lag by delta t and would have a value of what? 1 by delta t multiplied by what? g lagged by delta t.

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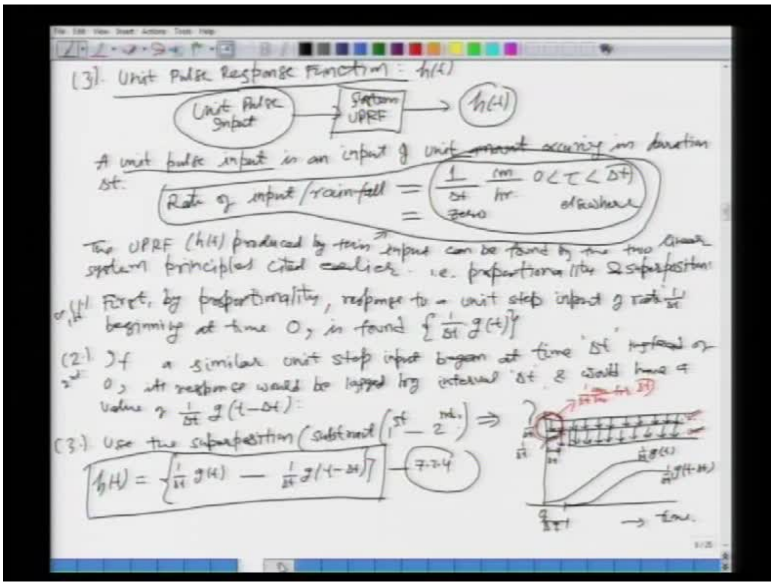

Then, what you do is, the third step, then we use the super position. How we subtract the first or carry out this subtraction operation first minus second. Alright? That is, this is the first one and this is the second one. So, there are two responses. One is g t 1 by delta t g t another is one by delta t g t minus delta t. So, if you subtract or carry out this operation, that will give you what? That should be your h of t. Let us look at this graphically. What we are saying is, that is, this is your time domain. Alright? Then what we said is, we are using g f t at time and this is time is equal to 0 and what is this? This is 1 over delta t and this is occurring indefinitely and so on. The response from this is, first one is 1 by delta t g f t. Now, what I will do is, I am not plotting the response here. Then you take a time interval. How much? delta t and let us assume that another rainfall starts at this time indefinitely. I am plotting it just below this, but it is at the same scale. Just keep that in mind. Alright?

This is also, this intensity is also 1 over delta t. But, this is occurring what? delta t later. So, what you do is, this is 1 by delta t g of your t minus delta t. That is the input and then there is a response from this in terms of this s curve and other one is this. So, let us say this is the first one and this is the second one. g f your t minus delta t and then there is 1 upon delta t of course. Now, what you want to find out when we are talking about this unit pulse response function. What is the unit pulse response function? Well, it is the input, which is going from 0 to 1 or 0 to 1 by delta t actually ahead is. I would like to draw your attention here, this one. It goes from 0 to 1 by delta t and it occurs for duration delta t. Alright? So, you come down here, so you want to find out what will be the response from the system due to only this input. Right? What is this? This is your nothing but 1 by delta t centimetres per hour, occurring for what? For a duration delta t. So, what we have done is, we have 1 by delta t occurring indefinitely at time t is equal to 0 and the second one occurring indefinitely starting delta t hours later. So, if you subtract the response from the first one and the second one, so if I did, this one minus that one. That will be what? That will be the response due to which input? This input. Do you see that? Alright?

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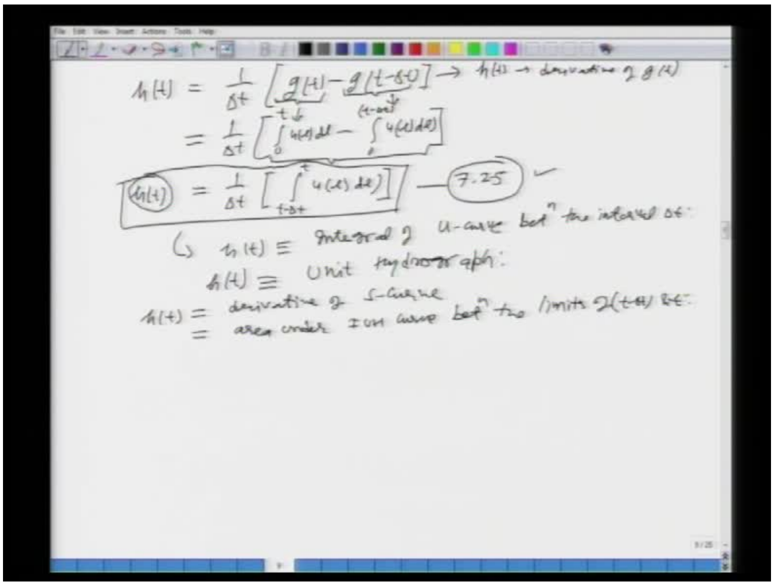

We are using the principal of super position. The first one is response due to this and second one will be the response due to this. So, you subtract this one from this. What will you get? You will get the response due to this guy here, which is of intensity 1 by delta t and it is occurring for duration delta t. So, this you can say will be equal to what? This I can say is equal to h of t. So, this is your unit pulse response function and I am going to say that, this is 7.2.4.

Moving on, then you can define your h of t is, you can take 1 by delta t outside and then you have g t minus g t minus delta t. This you can represent in terms of u. We have already found out what is g t. g t is nothing but it is integral between 0 to t of what? u l d l. We have just said that. Similarly, what will be this guy here? It will be minus 0 to what? 0 to t minus delta t. Is that right? Whatever is in the brackets here, so that will be the limits of integration of your u l d l. What is this? Using your knowledge of integral calculus, you can say that this is 1 over delta t of what? You have integral. You have 1 integral going from 0 to t and other going from 0 to t minus delta t. So, essentially what you have is. t minus delta t to t of your u l d l. Let us say, this is 7.2.5.

So, we have to write this equation for the unit pulse response function. Alright? We have used the two basic principles of a linear systems theory, which is proportionality and superposition, both of them. In the first one, if you look at this equation, what is this telling us physically? Well, what it is saying is that, your h is nothing but what? The derivative of the g curve. So, what is the h curve? h curve is nothing but the derivative of the or the slope of the g curve. So, you have the g curve or the unit step response function curve. To differentiate that with respect to time, each time step, what you will get? The curve you get, which describes the slope of g curve is nothing but your h. Alright?

Similarly, h is also equal to this equation 7.2.5. What is this telling us physically? Your h is equal to what? It is nothing but integral of your u curve, which is your unit impulse response function between the interval. What is that interval? delta t. So, you see that this h is nothing but your unit hydrograph. So, if you remember what is this? It is nothing but your unit hydrograph. The way we have defined this h or the unit pulse response function, it is due to the unit pulse. What is that? 1 centimetre of rainfall occurring in duration delta t. What is the unit hydrograph? 1 centimetre of rainfall occurring in duration d. Alright? So, it is the similar concept. So, that is your unit hydrograph and unit hydrograph is equal to what? Well, it is nothing but the derivative of your s curve. What is the s curve? s curve is the summation of the unit hydrograph for certain duration.

But theoretically, this unit hydrograph is nothing but your integral between the limits of t minus delta t to t of your i h. So, what we have seen is that, then h of t is equal to the derivative of s curve, that is number 1 and this is also equal to the area under your i h curve between the limits of t minus delta t and t. So, basically what we have done today is, we have looked at these three very important functions, unit impulse response function, unit step response functionand unit pulse response function. These are the three basic concepts which are your unit, instantaneous unit hydrograph, the s curve and the unit hydrograph. They are all related to each other, the way we have described or we have derived all these equations.

In the next class, what we are going to do is, we will look at what is called the discreet convolution equation and we will look at the unit hydrograph. The way we have seen them earlier, the basic definition and look at some examples. So, I would like to stop here today and then we will continue further.