Advanced Hydrology
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Lecture – 23

Good morning and welcome to this post graduate video course on advanced hydrology. In the last class, we completed the chapter 5 on surface water and in that in that class we looked at the concept of travel time, then we looked at what is the time area diagram, we looked at one example. An example, we solve in which there is a rectangular paired area as the catchment and we had to find out the isochrones the inter isochronal areas are basically we have to develop the time area curve. We did that then I gave you a home work example in which you were supposed to find out the direct run of hydrograph response from that time area diagram, which we develop for the example problem.

I hope you are able to do that. If there are any questions you can contact me. Then we looked at a certain geo-morphological parameters using the topographic data. In the absence of any gauge in the catchment we can use such geo-morphological parameters to compare to catchments to get some idea about the behaviour of the catchment, how it is going to respond when the catchment is subjected to rain fall event. With that we ended that chapter and what we are going to do today is will start chapter seven on unit hydrograph.

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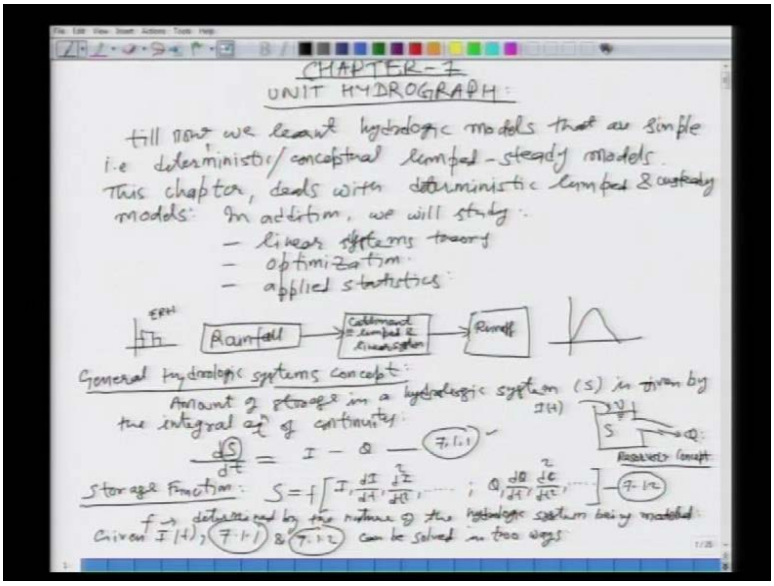

What we will do in this chapter is we will look at a lot of things, we will look at the definition of the unit hydrographs, we will look at the definition of various response functions which are needed to understand this.

Then we will look at how we can actually derive the unit hydrograph using the rain fall and float data at certain river in catchment. And then how we can opt a use the optervasation techniques to develop a unit hydrographs for a multiple storm case. Upon till now we have been looking at the lumped model or the deterministic model under the stradistate conditions and the unit hydrograph is something which gives us the response from the catchment under unsteady conditions or it gives you the time distribution of your direct run of response with respect to rain fall or subjected to rain fall. Then you have the different time distribution of the rain fall. So, what we are saying is that till now we discussed or we learnt hydrologic models that are simple are most simple in nature, that is we looked at the deterministic or I can say conceptual lumped and steady models.

Now, this chapter we will deal with or deals with deterministic lumped and what and unsteady models. For example, in the last chapter we looked at the derivation of the expressions of flow depth and velocity from over length flow, and in the channel flow remember they were for the steady state condition that was the inherent assumption. When the sufficient time has elapsed what is going to happen to the flow depth and velocity in the over length flow and in the channel flow, those kind of analysis is useful for the worst case scenario when you have continuous input of rain fall. Let say what will be the worst case scenario under steady state conditions, but what happens initially as the time elapses from the beginning that we would not be able to do based on the expressions which we derived earlier.

So, the unit hydrograph is a very important a theory which was proposed in 1932 by Sherman. And this is something which is still used today in it is earlier form or the original form. So, it is important to understand this unit hydrograph what it represents, how it can be derived and how it can be used in the practice. So, in this chapter this is what we are going to look at and in addition we will study or we will need to look at what is called a linear systems theory. On which actually it is a based on linear systems theory. We will also use our knowledge of optimization I hope that a you would know the basic concepts of optimization which will be assumed and then also, some statistics some knowledge of statistics actually will be utilize to carry out some of the things.

So, if you look at the linear system concept in the first chapter, we had looked at the systems concept then let us look back or review what is systems concept will reference to this particular chapter on unit hydrograph, and then we will go to the a linear systems concept. So, we had seen earlier that you have some input which is rainfall. In a systems concept that goes into your physical system which is the catchment in this case which we say is equivalent to a lumped, and linear system in the case of unit hydrograph and then we output or the response of the catchment is what we get is the runoff. And how does this look like the runoff may look like this, and the rainfall you may have in this form and this is basically your effective rainfall.

So, this is the systems concept and then if you look at what is call the general hydrologic systems concept that is what we are looking at. We have seen that the rainfall that falls on a catchment, where does it go? Well it goes into many or different storage components in the catchment. So, we have seen in the last chapter that these storage components are what it could be interception, depression, subsurface, surface, ground water and Soil moisture so on. And each of these different storages has different characteristics. So, the storage function is actually is extremely complex you know in nature. So, what we do is we say that the amount of storage in a hydrologic system or in a catchment like the one I just drawn above is given by the integral continuity equation.

No matter how complex the storage is it will satisfy or it will follow the integral equation of continuity. We have seen that earlier, because the fluid motion follows the law of conservation of mass. So, we can always apply that and then we say I am going to write this equation again we have seen it many times in this course I minus Q and this time we will number it as 7.1.1.

So, this is a key element in this equation when we are doing the rainfall runoff modelling or when we are trying to develop a unit hydrograph or some kind of catchment behaviour. As I said this S is consists of many components they are interrelated with each other. The inputs and outputs may also be interrelated with each other. These relationships are complex these are non-linear they are dynamic in nature. So, over all we are looking at an extremely complex thing. Mathematically this storage function can be expressed as follows.

The storage functions the key in the rainfall runoff modelling is to be able to capture this complexity, the non-linearity and the dynamism in your in your natural process which actually comes from this storage function. The storage function in general can be expressed like this it is a function of what it will be a function of your inputs and various derivatives dI over dt d2I dt square and so on. We can go all the way to the n-th derivative and then also the output Q dQ dt d2Q dt square and so on. And let me say that this is your 7 1 2. So, basically what we saying is that we are saying that this catchment behaves like a reserve wire in which you have some input going in which may vary as a function of time it may have certain storage. This is the storage which is express by this equitation 7.1.2 which is quite complex as I said and then you may have something coming out of it Q.

It is a very generic very simple you know a slimily I am trying to draw here this is like a reservoir for the catchment. So, this is the schematic that shows the simulation of the catchment using a reservoir concept. So, what we are saying is that the catchment behaves like a reservoir in which there is some input there is some storage and what comes out is the out flow Q. And this storage is going to be the function of input as well as output and there lots of derivatives.

Now when we are developing a model it will depend upon the accuracy desired how many derivatives of which variable we are going to consider in our modelling. So, this is the general concept where this function f is determined by the nature of the hydrologic system being model.

So, now you have the input available to us the rainfall that is falling on the catchment I then I have equation 7.1.1 which is nothing but the continuity equation that is a 7.1.1 here and then I also have these storage function. Let us say I have assumed that it is a linear reserve wire. For a linear reservoir we have seen what is the relationship between the storage function and the other things it is S is equal to K times Q. We say it for the linear reservoir the storage is function of only we outflow and linear, but we can make the things more complicated.

So, given the nature of your input equations 7.1.1 and 7.1.2 can be solved. So, it is a mathematical problem. Now, once we have said that your catchment is going to look like this as a function of this storage function can be solved in two ways what are those two ways well.

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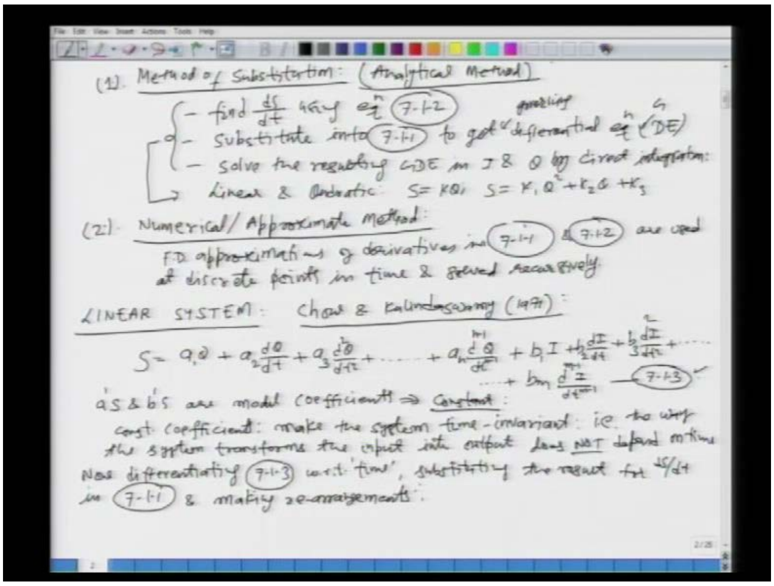

The first one is what is call the method of substitution or the direct method, method of substitution. We also call this analytical method or direct method what do we do in this well we find the function ds by dt we have already defined our function as from equation 7.1.2. So, we can always differentiate that and find out the first derivative. So, we find out ds over dt using we assumed function given by equation 7.1.2 and then substitute into 7 1 1 which is your continuity equation to get the differential equation I am going to use this as DE, and then we solve the resulting or I should say to get the governing differential equation or your governing differential equation here solve the resulting governing resulting equation along with your boundary and initial conditions n I and Q by direct integration or direct solution.

So, this is called the direct method or the analytical method or the method of substitution. If you recall we have seen a couple of examples of this using the continuity equation and different forms of the storage function. If you recall in this course we have seen linear and quadratic. So, in first case it was S is equal to KQ we have found in governing differential equation the form of the transfer function operator we have already done that. And also for the quadratic in which we said S is was equal to K1 Q square plus K2 Q plus K3. I think that was the form which is a quadratic polynomial. So, we have seen all that. So, that was as far as the first method is concern which is the direct method all right will not go too much into the detail of this. Then we have the second one which is called the numerical or approximate solution or approximate methods. I am sure all of you know that there are certain situations in which the equations become extremely complex, and we are not able to solve the system of equations or the differential equation we have analytically.

Then what do we do? Well we take help of numerical methods or the approximate methods, what are those? The finite difference methods, finite element methods and there are you know newer methods coming you know every day, but we will not going to the details of that what we do in the approximate methods is that this space domain, and the time domain what we do is we discretize these domains we use certain elements or at each notes we try to satisfy the governing differential equation. And then we keep on integrating or differentiating the basic equations, and then we try to find out the values of the variables at each note after applying the boundary conditions.

So, in the numerical methods what we do is the finite difference approximations of the derivatives in 7 1 1 and 7 1 2 are used. So, instead of direct values of this derivative we are using we are approximations at certain discreet steps points. I can say it could be able with respect to time a space in time and then solved recursively. So, most of this methods are iterative in nature when we say recursively means what we would apply this equations we will get started with some initial solution, and then we will have to alliterate or successively keep on doing the alliterations. And then we solve it recursively until certain level of acceptable errors is reached.

So, this is as far as these two methods are concerned. The next thing we are going to look at is the concept of what is called a linear system, and in this linear system what we are going to look at is the formulation which is given by Chau and Kolinda Swamy in 1971. What they have done is they taken a very general concept or general equation for the storage function. And using that general storage function which is very general you know we can take the values of the certain parameters in that and solve any kind of problem. So, using this general storage function concept under the linear assumption or under the assumption of the linear system story they have tried to derive the form of what is called the transform function operator. So, we would look at that.

What is that well with this what they have said is S is equal to or is given by this equation a1Q plus a2 the first derivative dQ dt plus a3 d2Q over d2 square and so on. All the way up to an dn minus 1 Q over dt square plus this is as per as the dependence of S on a Q is concerned and then we have the similar expressions for dependence of S on the inputs. So, you will have use another co-efficient that is b is you will have b1I plus b2 of your dI dt plus b3 d2I d2 d t squared plus all the way to let us say bm I am using another a subscript here and then you have m minus 1 I our dt what m minus 1. This should be an over dt n minus 1, let me call this equation 7 1 3. So, this is the general expression.

Where let us define all these parameters or the variables involved in this what are the a’s and rather I should say a’s and the b’s are what are the model constants or coefficients which have to be determined after calibration. Now these constant coefficients are we are saying constant. In this model these are constant what do we mean by constant coefficients a model with constant coefficients is one which is not adapted. As a function of time or when more data is available the values of these parameters or the coefficients will remain constant. So, we say constant coefficient it is an important assumption to understand what does it do? These coefficients make the system what is called time in variant. What is time in variant? I am sure when we have studied the unit hydrographs in your earlier classes you have seen that the basic assumptions with the unit hydrograph theory is that the unit hydrograph theory is what it is linear and it is time in variant.

You take up any book on you know on hydrology you look up at the unit hydrograph chapter these are the basic assumptions. So, what do you mean by the time in variant assumption well what it means is that the coefficients of your model are constant. So, they do not depend upon the time. So, if you apply this model once you have calibrated the model or once you have developed the unit hydrograph you apply that model to the data and those data may be in January, it may be in June, it may be in a December. You will get the same response. So, the coefficient does not depend upon the time or they are time in variant that is what we mean by that. That is the way system transforms the way system transforms the input into output in our case the input is rainfall and the output is direct runoff does not depend on time.

So, this is as far as the definitions of these parameters are coefficients are concerned and then you have s is the storage Q is the out flow and I is the input. Now as per this linear system or the Chau and Kolandi Swamys theory what do we do, well we differentiate in order to develop this a model differentiating equations 7.1.3 with respect to time of course, and then substituting the result for ds over d t in our original continuity equation and making slight rearrangement. If you did that what we are going to get is this equation, which you should be able to verify very easily

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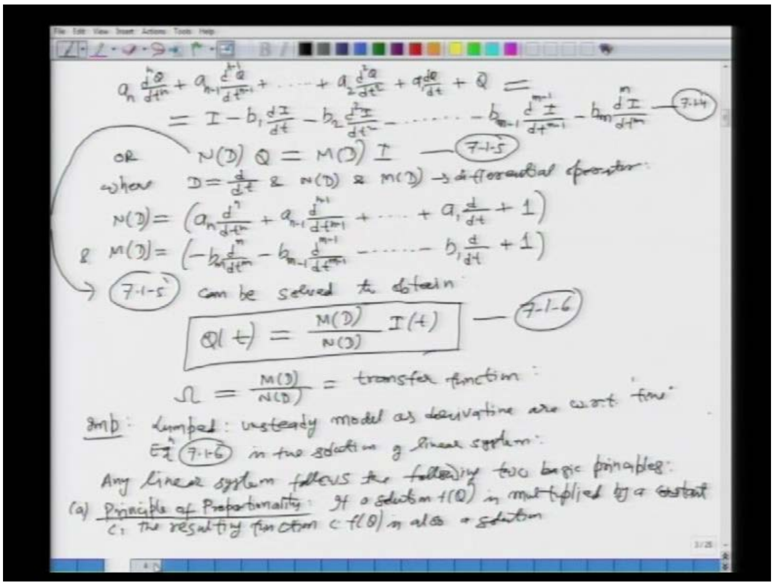

a n dnQ by dtn plus an minus 1 dn minus 1 Q dt n minus 1 plus all the way to your a2 d2Q dt2 plus a1 of your a1 of dQ our dt plus you will have Q coming from the continuity equation all of that is equal to what or I am let me write it here is equal to I minus b1 of your dI by dt minus b2 of your d2I dt square minus all the way to minus of your bm minus 1 dm minus 1 of your I over dt m minus 1 and then minus last term was going to be bm of your dm I over dtn. So, m-th derivative the last one and let me number this whole equation as 7.1.4 or in a compact form I can write this equation as some function N of the operator D applied on Q that is the left hand side if you see this equation.

This is equal to your some function of M of your differential operator D applied on I and let me say that then compact format is 7 1 5, where I have assumed certain things here where D is a differential operator which we have defined earlier as d over dt and what is N (D) and M (D) well these are differential operators which have very complex form and this form can be derived from this equation or let me write what is your N of D. If you think about it well it is going to be an of your dn over dtn this is applied on Q. So, I am not writing the Q here plus an minus 1 dn minus 1 dtn minus 1 plus all the way to a1 d over dt. So, you see that anything else well there is a Q there. So, there will be a 1 coming out.

So, all of this actually you will be applying to Q and on the input you have M what is that in that case you have minus bm the last term corresponding to the m-th derivative dtm minus bm minus 1 dm minus over dt m minus 1. And all the way up to b1 d over dt anything else yes there will be plus 1 also here corresponding to I which comes on the continuity equation. So, with this the 7.1.5 which is this equation can be solved or written as to obtain what well Q as a function of time is equal to what from this equation Q will be nothing but this differential operator M(D) over N(D) whatever that form is. So, it will depend upon what is the value of N and what is the value of M? What values of these the order of these derivatives, we are taking that will depend upon the complexity or the dynamism or the non-linearity you are trying to model in a catchment.

So, depending upon that form of this M(D) and N(D) will be dictated. So, you have that and this is multiplied by your input function. So, this is a curry general linear system models which was developed by Chau and Kolinda Swamy and let me number this as one 7.1.6. Where let me define this transfer function operator remember we had seen just earlier in our chapter one and chapter two for the couple of cases and this is the most general one M(D) over N(D) which is the transfer function. Now one thing one important just note I would like to make here is this is a lumped model lumped means what we have seen in the earlier in the classification of the models is that what is the difference between a distributed and lumped model? Well in a lumped model you are not accounting for the variations of your variables with respect to space you are accounting the variations of the variable with respect to time only.

So, we are saying that this is a lumped model, but this is unsteady. Lumped and unsteady model as the derivatives which are taken are with respect to what time only in this case we have taken and then equation 7.1.6 is the solution or represents the solution of a linear system. Now we have been talking about this linear system. What do we mean by a linear system? or why do we need to make this assumption? We know that the rainfall runoff process is extremely complex and it is non-linear, but modelling or developing a non-linear model is very difficult. So, what we will do is we will at the couple of basic properties of a linear system the assumption of linearity allows us to do certain operations on our solutions. And using those operations on the solutions we can solve many complex non-linear problems with reasonable accuracy. So, that is what we are going to look at next.

So, what we will do is we will look at the two basic principles of a linear system. So, any linear system follows the following two basic what are those well I am sure you have seen this earlier. So, let me look at the first one which is called the principle of proportionality. What is a principle of proportionality in a linear system? I am sure you know, if you have a solution f all right the principle of proportionality states that if you multiply that solution by a constant any constant, then that constant multiplied by that function or the solution will also be a solution to the position problem. So, if you have a unit hydrograph or if you have a solution you multiply that solution by 2.3 or 5.4 or any number then that is also going to represent the solution.

So, if you want to write it down what do we mean by that if a solution f of Q is multiplied by a constant multiplied by a constant c. Let us say c is some constant the resulting function c f of Q is also a solution it is as simple as that.

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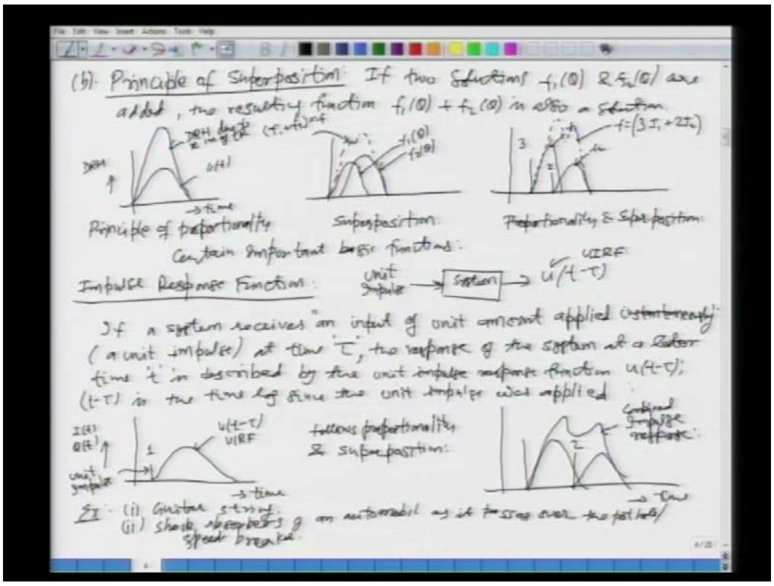

The second one or the second basic principle is what we call the principle of super position. Any solution which is based on the linear system theory also follows the principles of super position. This also I am sure you have seen earlier. What is the principle of the super position? Well let say you have two solutions we have found two different solutions f1 and f2 to certain problem, then what you can do is you can super impose those two solutions then that is if you add those two those two or subtract those two. So, you will do some kind of super position process.

So, 1 plus f2 will also represent this solution to the problem or to the same problem. So, if you want to write it down if the two solutions let say f1(Q) and f2(Q) are added I am going to say here or super post the resulting function f1(Q) plus f2(Q) is also a solution.

Now, what I am going to do is I am going to look at this two concept or this two basic principles graphically what do we mean by that? I am sure you have used these two principles in your undergraduate hydrology course in your unit hydrograph chapter earlier, but it is important to understand a various computations or various operations which we have done using the unit of hydrograph why we are able to do those. So, if we look at the principle of lecture proportionality first. What does it do? Is it allows us to find out the direct runoff hydrograph all right from certain amount of rainfall what is a unit hydrograph. I am going to look at the definition of the unit hydrograph very quickly here because I want to demonstrate what is a proportionality principle?

A unit hydrograph is nothing but the direct runoff hydrographs response from a catchment due to what due to unit rainfall unit effective rainfall it could be one centimetre or it could be one inch or it could be one millimetre. So, from one centimetre rainfall if it is subjected uniformly with respect to time and space on the catchment right what will be the direct runoff hydrograph that is call your u or the unit hydrograph.

So, this is your time and this is your a DRH response. So, this curve we are looking at is the direct runoff hydrograph response due to one centimetre of effective rainfall applied uniformly over space and time on the catchment. Now if we want to find out what will be the direct runoff hydrograph let say due to x centimetre of rainfall or 2.3 centimetres of the rainfall then what do we do? Well we use the principles of proportionality and then we say that all of these ordinates of the unit hydrographs we can multiplied by x without making any changes with respect to time. So, the proportionality would allow us to use or get the direct runoff hydrograph like this.

These two curves are similar the way I have drawn they may not look very similar, but you understand what I am trying to demonstrate here. So, this is the DRH due to x centimetre of effective rainfall. So, you have a solution due to some amount of rainfall which is 1 centimetre. So, another solution which is the DRH due to the x centimetre of the rainfall also represents a solution of a system. The system is same, but the input is different. So, you can apply that different input on the same basic solution. So, that is why the principle of proportionality allows us to find out the direct runoff hydrograph from complex term not the complex term, but the increased value of x. Now this is as far as the proportionality is concerned this is principle of proportionality.

Now next thing we are going to look at is the super position. What is super position? As we have said the two functions f1 and f2 you just add them up you get the combined response or some resulting function which is called f1 plus f2 we you can say that is f3 or z or something that also represents the solution of the system to a different problem. So, let us say that you have one is this is your f1(Q), you have another solution which is let us say this f 2Q .

So, what we are looking at is we have two different direct runoff hydrograph responses on the catchment due to two different input impulses these may be let us say two unit hydrograph 1 centimetre of rainfall occurs some time it produces a DRH, 1 centimetre of rainfall occurs after some time that also produces the same response. Now how do I find the combined response? well I just add the corresponding ordinates at each time. So, the principle of super position will allow me to add the corresponding coordinates at the same time and get the combined response. So, what you will then have is the combined response it may look like this.

So, this is your f1 plus f2 let us say this some f due to some rainfall occurring at some time plus another rainfall impulse occurring at little later. So, you are able to get the super position. Now you can have a situation in the real life where you need to apply the principle of super position as well as the principle of proportionality together we can do that. How do we do that? Well let us say let me first write the application of proportionality and super position together. This was f1 plus f2 due to 1 centimetre of rainfall, but we may have a situation where this is 3 centimetre of rainfall and this produces some response after some time you have 2 centimetre of rainfall occurring and that produces another response.

So, now you have f1 and f2. What is f1? Well f1 is the direct runoff hydrograph due to 3 centimetres of rainfall. So, f1 has actually accounted for the principle of proportionality.

So, there is a unit hydrograph in the catchment and then you apply that with 3 centimetre of rainfall impulse that is what represent it f1 on this graph.

Similarly, due to the 2 centimetres of the rainfall or some input which has a magnitude of twice or two that will result into to DRH response f2. Then what we can do is we can get a combined response that will start from here. So, this is again your f is equal to maybe I can say 3 times I1 plus 2 times I2 which is applied on your f where I1 is the response due to the unit hydrograph at that particular time and I2 is the response to these second impulse. So, this way we see that we can use or we can apply the principle of proportionality and super position to solve very simple or sometimes you know complex problems. Now, what I would like to do is when we apply all these linear systems theory or the unit hydrograph or you know solve a complicated problem, then we will encounter or we will need to understand some 3 or 4 basic different types of functions.

So, the next thing we are going to look at is some or certain important or basic functions which will need to define. So, the first one of them is what we call an impulse response function. What is an impulse response function? Well graphically I am going to represent this as if you apply a unit impulse to any system. So, this is your system what you are doing? You are applying a unit impulse input to this system and then what comes out is what is called u t minus tau this is called unit impulse response function.

So, you have a system you apply a unit impulse input to the system. The response of the system is given by a function which is called the unit impulse response function. So, let us first look at what is this unit impulse function it is important to understand that. So, let me then say that if a system receives an input of unit amount applied instantaneously. Let me put that in quotes n input of unit amount applied instantaneously what is this type of input is actually called unit impulse input

So, if somebody asks you what is the unit impulse input? Well it is an input of unit amount occurring instantaneously. So, you dump one centimetre of rainfall instantaneously on a catchment or you apply a force of unit amount to a system, what will be the response that is called the unit impulse response function. So, let me complete this in the brackets I am going to say our unit impulse for this at time tau. So, this unit impulse unit is applied at certain time tau. Then the response of the system at a later time‘t’ is described by unit impulse response function.

And this we are going to denote as u t minus tau why we are saying t minus tau because t minus tau is the time lag. Since the unit impulse input was applied. So, if you look at this concept graphically, this is your time domain and then we say that you have some on the same graph let us say you are representing I as well as the output Q.

So, what we are saying is that if we apply a unit impulse what is this? This is your unit impulse. What is going to be the response of the system is going to be described by this function. Let us say the response is described by this function which I have just drawn this is then called u t minus tau. So, and this is your unit impulse response function, this unit impulse response function follows proportionality and super position both.

So, that you can do this, we have time here let us say a three impulse is occurring here it is this an impulse of two is occurring here the response is this you can combine these two will look like this. So, this is your combined response like we have done earlier.

Now let us look at some interesting examples of this impulse response function all right for example, first one is what is call the guitar string. We are looking at some examples other than the hydrology. What happens when you touch the strings of the guitar let us say you are applying a unit impulse to the system which is called guitar these response or the sound that comes out will be described by certain sound waves which is called the unit impulse response function. So, this concept is use not only in hydrology, but in many other physical and natural systems.

So, one example is the guitar string, another example is what we called as the shock absorbers of an automobile as it passes over the some pot hole or speed breaker. Let us say you are sitting in a car you are going suddenly you go over a hump speed breaker or a pot hole what happens to the shock absorbers of that automobile or that car, let us say some impulse due to the movement of the vehicle on the your speed breaker what happens is some impulse or some force is applied on the shock absorber. So, they will have some response after sometime it will like down.

So, that kind of response is also described by what is called the impulse response function which will be different in nature of-course. So, what I am going to do is we have looked at these two examples which are not in hydrology. In the next class we would look at some examples or at least one example of this unit impulse response function in hydrology. And then we will look at some more functions I will stop at this point, because I am running out of time and in the next class we will look at some other functions.

Thank you.