Optimization Methods for Civil Engineering
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Lecture - 08
Region Elimination Methods

Hello student. Welcome back to the course on Optimization Methods for Civil Engineering. So, in today's class, we will discuss Region Elimination Method. It is a method for finding the optimal solution of a single variable function; the function has to be an unimodal function.

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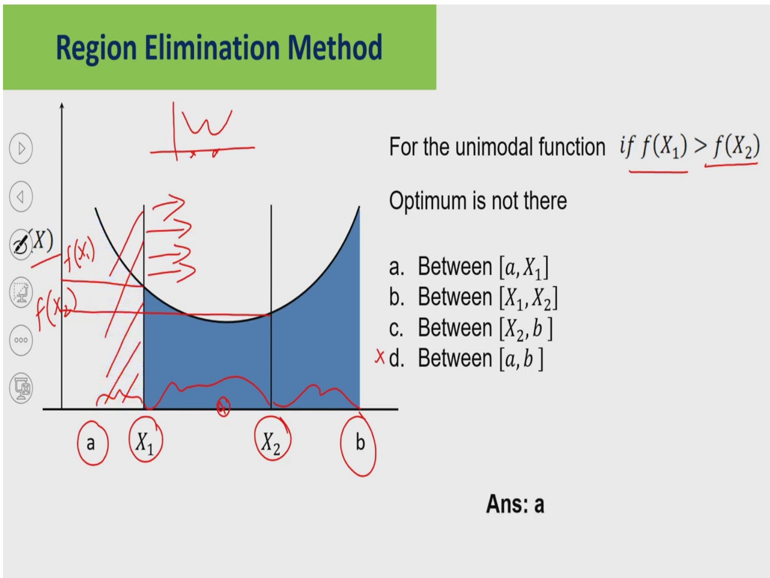

Let us discuss the method. Before describing the method, let us take an unimodal function f X. This is a unimodal function and the function is continuous between a and b and so that means, there is 1 optimal solution. So, this is an unimodal function. So, I can also give you an example of bimodal function. So, there are 2 modes and that means, there are 2 optima here. But in this case, there is only 1 optimal solution somewhere here.

So, this is an unimodal function and now, for this particular function f X, if I take two points X 1 and X 2 and for the unimodal function, if f X 1 is greater than f X 2, optima is not there. So, these are the options. Optima is not there between a X ok. So, that means, this particular region a X; then X 1 X 2 ok, so this is another region and this is another region ok. And optima is not there between a b.

So, there are four options. Optima is not there between a X of X 1, optima is not there between X 1 and X 2, optima is not there between X 2 and b and optima is not there between a and b. So, this is a unimodal function. So, therefore, this option is not correct ok.

So, this option is not correct. The reason minimum solution of this particular function, this is a unimodal function. Now, question is that which one is true; whether a is true, b is true or c is true? Now, what it is telling that this is an unimodal function and f of X 1, this particular point this is f of you can say this is X 1 and this point is f of X 2 ok; this is f of X 2.

So, now as f of X 1 is greater than f of X 2; that means, optima is not there in this particular region. So, optima is somewhere in this side; is not it? So, somewhere in that side. So, therefore, the solution is here a, that means, optima is not there optima is not there between a and X 1. I hope this is clear.

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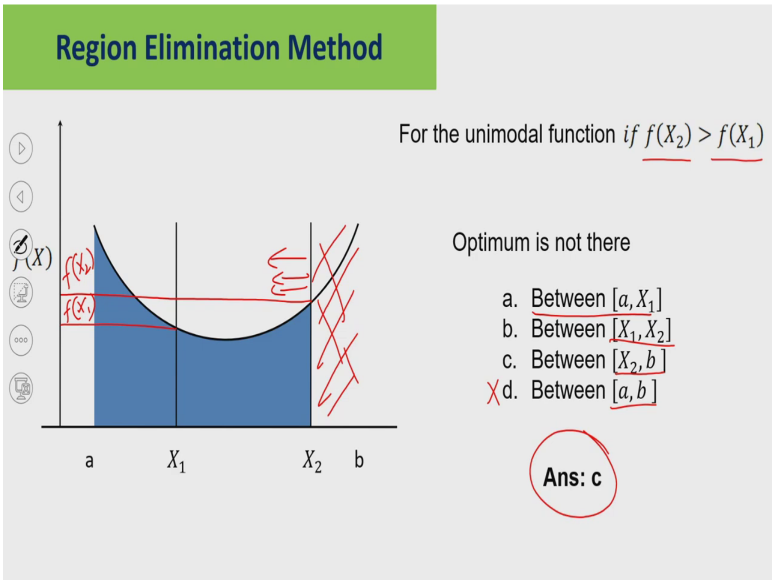

Now, if I ask the second question, then again for the same function for the unimodal function, now f of X 2 is greater than f of X 1; that means, this is my f of this is f of X 2 and this is f of X 1 ok. Now, in this case, f of X 2 is greater than f of X 1. So, in that case, optima is in this particular side; not in this side ok.

So, therefore, if I give you the option that optima is not there between a X 1, between X 1 and X 2, then X 2 and b and a and b, anyway this is not correct. Then, out of these three now the solution is c; that means, optima is not there between X 2 and b. So, in this region.

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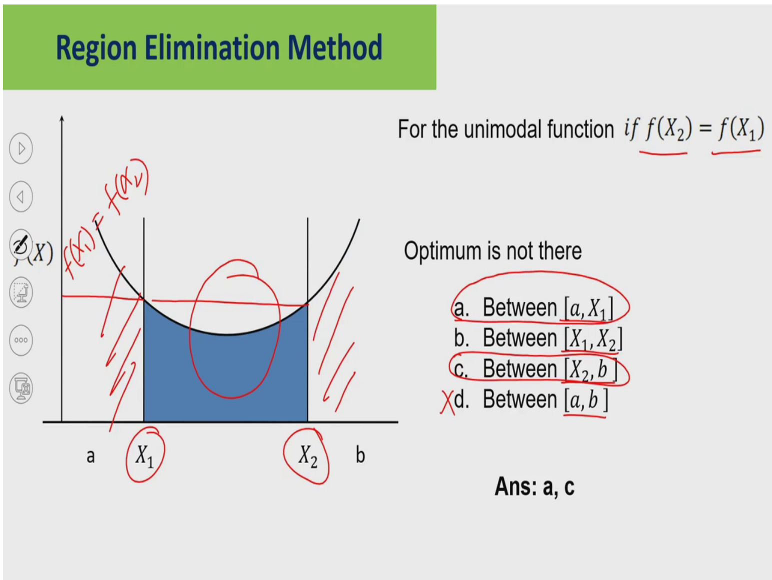

Now, let us take the third question. So, for the same unimodal function. So, if I take another two points X 1 and X 2. Now, in this case f of X 2 is equal to f of X 1. So, in that case, which one is true? Optima is not there between a X 1, X 1 X 2, X 2 b and a b. So, this is not correct.

Now, in this case, what will happen because this whatever value you are getting and this is the same f of X 1, f of X 1 is equal to f of X 2 ok. So, therefore, optima is not there in this particular region and optima is not there in this particular region. So, optima is somewhere here. So, therefore, the correct answer is a and c. So, this is your correct answer and c is your correct answer.

Now, you just see by looking at by comparing this function value at two points, what I can do? I can eliminate some of the region ok. So, in the first case that if I take these two points and if I know that f of X 1 is greater than f of X 2. So, what I can do basically? So, I can eliminate the region between a and X 1. So, by comparing the function value at two points, I can eliminate some of the region.

Similarly, for the second one by comparing these two function values at X 1 and X 2; in this case X 2, f of X 2 is greater than f of X 1. So, I can eliminate this particular region. Similarly, if this condition is true, that means, f of X 2 equal to f of X 1, so in that case, I can eliminate both the region.

So, this rules we can apply and we can eliminate the region, where optima is not there ok. So, therefore, whatever method we will discuss, so these methods are known as Region Elimination Method. Basically, we are trying to eliminate the region, where optima is not there ok; so, where optima is not there.

So, what we are trying to do here? We would like to eliminate the region where optima is not there. So, therefore, this method is called Region Elimination Method. So, basically we are trying to eliminate the region, where optima is not there. So, with some iteration, so what will happen?

The area will be eliminated and finally, you will get a very small region, where optima is there and you if your precision is a, c then you can say that the particular point is your the optimal solution ok. So, we can declare that particular solution as an optimal solution. So, let us discuss the methods. So, we will apply basically this region elimination technique.

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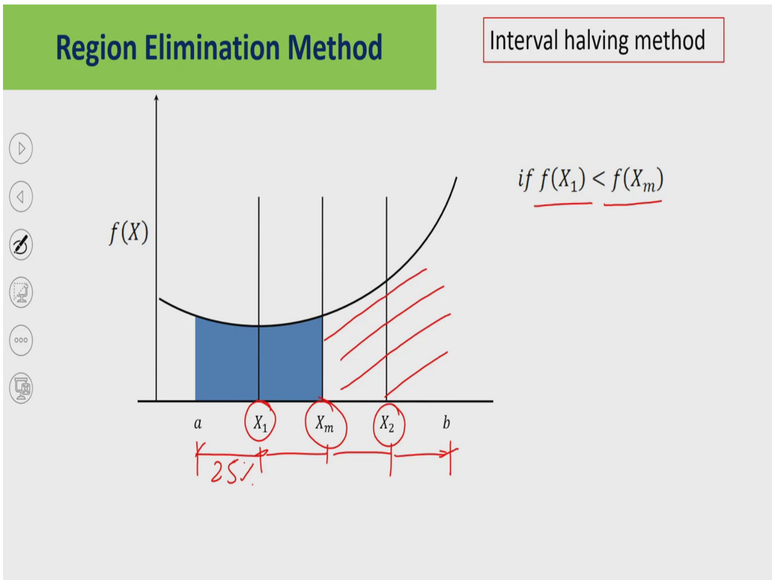

Now, the first method is Interval halving method ok. So, what you are doing in this particular method? So, we are taking three points. So, one is at the mid, that is X m. So, what is X m? X m is equal to a plus b by 2 and X 1 is between a and X m and you can say that X 2 is between X m and b.

So, exactly at the middle of X m and b and X 1 is exactly at the middle of a and X m. So, basically what you are doing? You are dividing this region in four equal divisions ok. So, that means, this is your 25 percent ok, 25 percent and this is also 25 percent; this is also 25 percent and this is also 25 percent.

So, what you are doing here? You are dividing the this length between a and b in four equal division. So, each division will have 25 percent. Now, what I will do? So, we will basically apply the region elimination that rules basically. So, if f of X 1 is less than f of X m, so what does it mean? That optima is not there in this particular region because f of X m is greater than f of X 1.

So, therefore, you can eliminate this particular any region. So, how much you are eliminating? You are eliminating 50 percent of the area ok.

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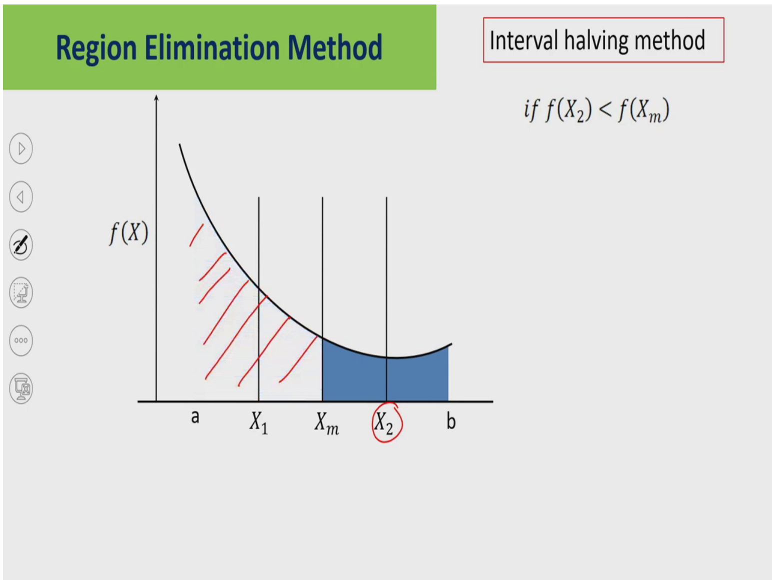

For this particular condition, if f of X 2 f of X 2 is less than f of X m, so in that case, what will happen? Optima is not there in this particular region. So, you can eliminate 50 percent of the area. Is not it? So, this area you can eliminate.

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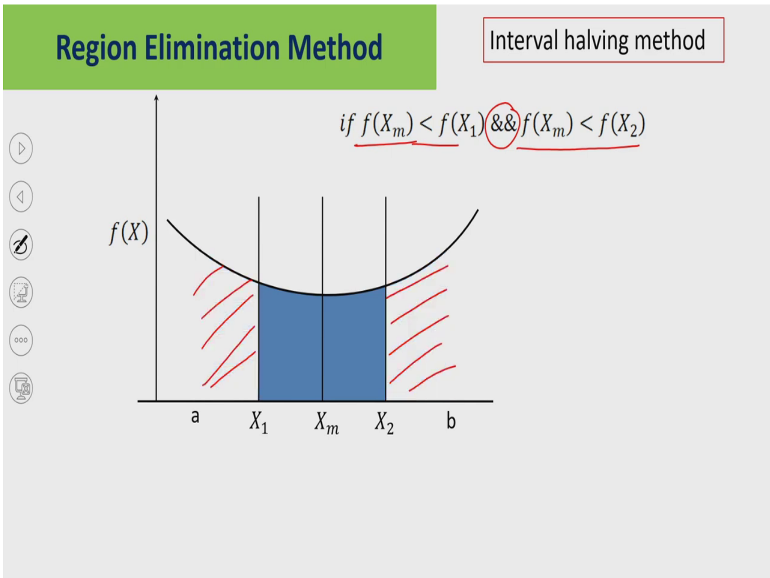

Now, if this condition is true; that means, f of X m is less than f of X 1 ok, so f of X m is less than f of X 1 and at the same time and f of X m is also less than f of X 2 ok. So, in that case what you can do? So, optima is not there in this particular area ok. So, in this particular area, so I can eliminate these areas.

So, therefore, what we are doing? We are checking three conditions here that the first one is f of X 1 is less than f of X m, then you are eliminating 50 percent of the region. Then, if this is true f of X 2 is less than f of X m, you are eliminating this 50 percent and if f of X m is less than f of X 1 and f of X m is also less than f of X 2, then you can eliminate this 50 percent of the area. So, out of these three, so one condition will be true and you can eliminate 50 percent of the area.

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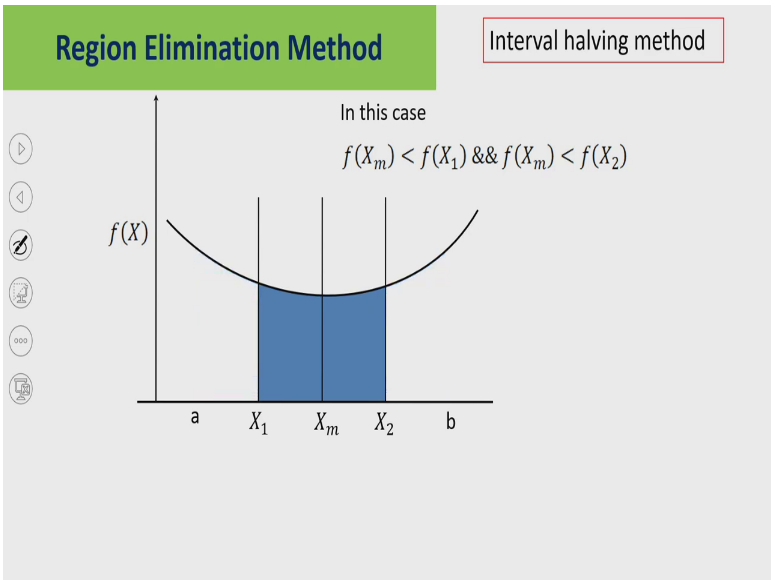

So, now what is the method? So, if I in the first iteration, so I have divided in four equal division and now, I will check suppose in this case, this is true f of X m is less than f of X 1 and f of X m is also less than f of X 2, so that means, this area will be eliminated.

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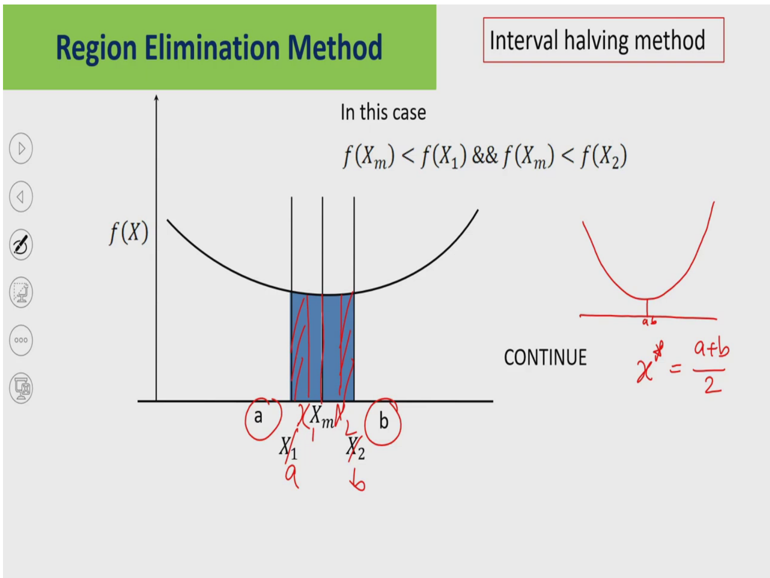

So, then what will happen? Now, this is the new a and b. So, again you will divide in four equal divisions and again, you will say suppose in this case also this is true that f of X m is less than f of X 1 and f of X m is also less than f of X 2, so therefore, this area will be eliminated.

So, now this will be a and this will be b and I can take another two point ok. So, this is your X 1 and this is X 2 and I can eliminate some of the region again. I can check the conditions and I can eliminate some of the region and if I continue then finally, I will have a very narrow area ok. So, very narrow area that finally, I may get something like this, something like this.

So, I will have a and b is something like that and the difference or the distance between a and b is basically we can call it precision. If the distance between a and b is reasonably smaller value, so in that case what you can do?

You can terminate your iteration and you can say that X star that is the optimal value is a plus b by 2 ok. So, a plus b by 2 and that is your optimal solution. So, what we are trying to do basically that we are trying to eliminate the region in every iteration and finally, we will get a very narrow strip.

And if suppose if I say that it is reasonably very small may be 10 to the power minus 3 depending upon what precision you need, 10 to the power minus 6, 10 to the power minus 12 or something like that. So, if it is suppose acceptable, then in that case what we can say that optimal solution is a plus b by 2 ok. So, this way, so in this case, we are eliminating the region, where optima is not there.

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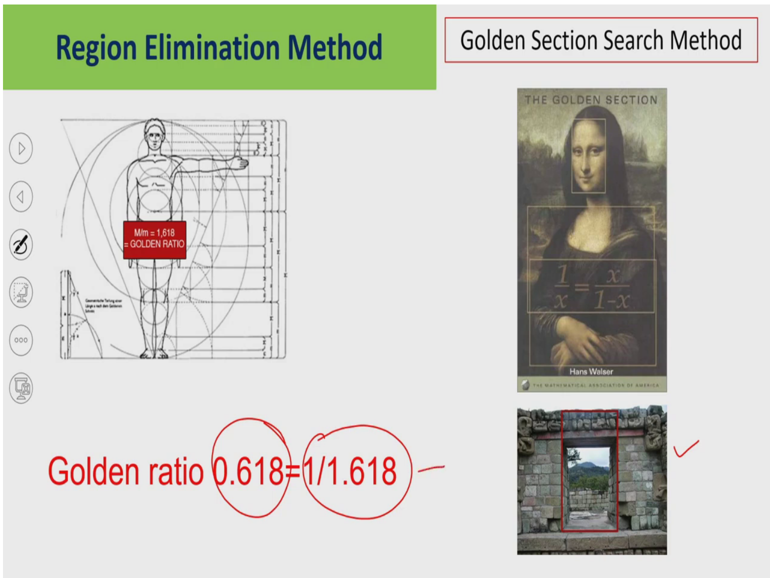

Now, let us discuss another method and this method is Golden Section Search method. So, there is a golden number or we can say golden ratio that is 0.618 and or 1 by 1.618. So, if you look at measure, so you can see that this ratio is somehow maintained.

So, even if suppose if I take the dimension of our different body parts, so you can see that somehow this ratio 0.618 or 1 by 1.618 is maintained ok. So, even when this Monalisa was constructed, so you can see that many facial dimensions are maintaining this particular ratio.

So, if you look at the whole structure, so you can also observe that these ratios are maintained and therefore, this ratio or this number is known as a golden number and basically in measure many dimensions are following this particular number ok. So, in optimization method also, so we can apply this number and we can eliminate the region.

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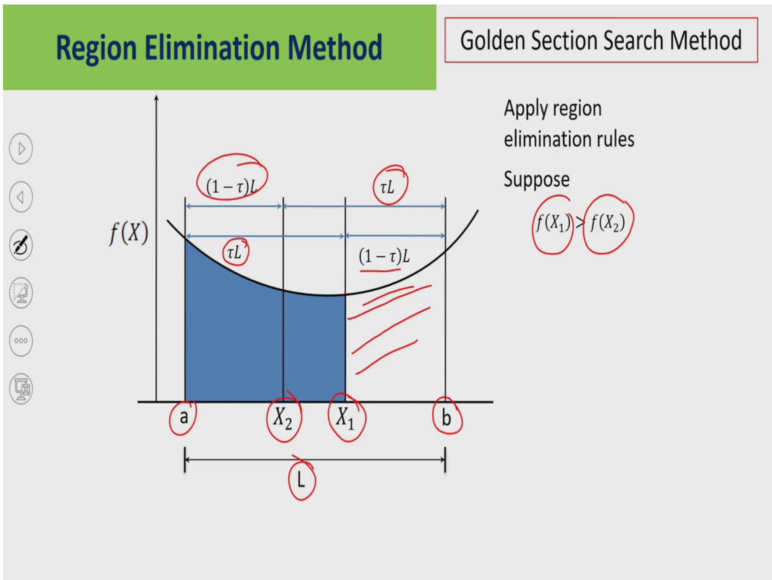

So, let us discuss the method. In this case, what you are doing? So, again f X is a unimodal function. So, this is an unimodal function; it has only 1 optimal, 1 optima ok or you can say in this case 1 minimum solution and the function is continuous between a and b. Now, the length is here the distance between a and b is L ok.

Now, if I say that this distance, so if I take a point X 1 which is at tau L distance, distance between a and X 1 is tau L and the distance between X 1 and b is 1 minus tau L ok. So, this is basically. So, I will calculate the value of tau later on, but you just assume that tau is a the tau value is between 0 and 1.

Now, if I take another point X 2 which is tau L from b and 1 minus tau L from a ok. So, that means, distance between X 2 and b is tau L and distance between a and X 2 is 1 minus tau L ok. Now, apply the region elimination rule. Now, suppose f of X 1 is greater than f of X 2, so this is f of this is f of X 1 and this is greater than f of X 2.

So, in that case, what will happen? The optima is not there in this particular region. So, therefore, I can eliminate this region. So, how much I am eliminating? I am eliminating 1 minus tau into L region ok. So, I am yet to calculate what is the value of tau, but what I am doing here? I am eliminating 1 minus tau L region.

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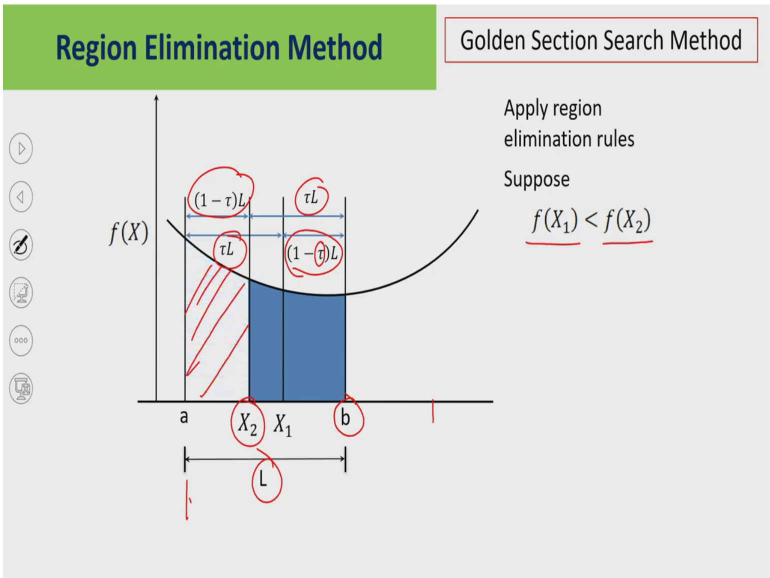

Now, this will be my b now ok. So, earlier b was somewhere here and it was X 1, now this is my b and the new length ok that is distance between a and b length is L now. Now, if I take that this is tau L ok, so tau L and this is 1 minus tau L. So, now, if for a particular value of tau, then this will be your tau L for the new L and this will be 1 minus tau L for this new L.

So, now I will take another point and this point is again tau L from b and 1 minus tau L from your a. Now, again, I am I will apply the region elimination rules and suppose in this case f of X 1 is less than f of X 2. So, that means, f of X 2 is greater than f of X 1; that means, this area or this region can be eliminated ok.

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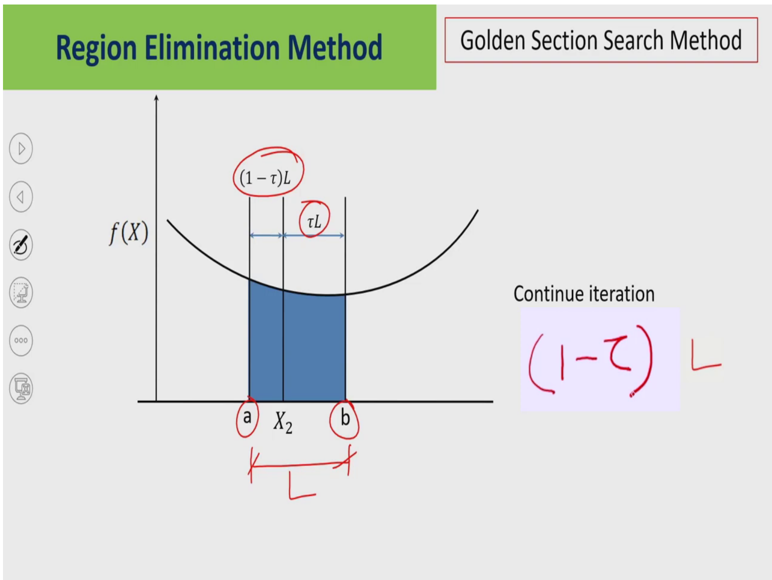

So, I am getting now new a b. So, now, this is new a b and this will be your new L. Now, if you look at that you can see that this value will be again tau L for the new L and this value will be equal to 1 minus tau L for the new L.

So, now this iteration will continue. So, we can continue this iteration and you can see that every iteration that we are eliminating 1 minus tau into L region and how many function evaluation you are doing here? At the first iteration, I am doing two function evaluations that is at X 1 and X 2.

Now, after the first iteration that every iteration, I am only doing one function evaluation because the earlier point will be either X 1 or X 2; that means, earlier point will be either at tau L or 1 minus tau L. So, therefore, the one point I am getting from the earlier iteration, so I have to take a new point. So, therefore, every iteration you are eliminating 1 minus tau percentage of the region and you are doing one function evaluation. Now, look at the interval halving methods.

So, in the first iteration, you are doing three function evaluation. So, that means, at X 1, X 2 and X m and after that, from second iteration itself how many function evaluation we are doing? We are doing two function evaluation in case of interval halving method and how much region I am eliminating? I am eliminating 50 percent of the area in each iteration.

So, each iteration, I am eliminating 50 percent of the area and at the same time, how many function evaluation I need? I need two function evaluation in each iteration. Therefore, the area eliminated per function evaluation is 25 percent in case of interval halving method.

So, if I compare two algorithm that is this golden section search method and interval halving method, in this case every iteration after first iteration, you are eliminating 1 minus tau percentage of the area ok. So, now, I have to calculate the value of tau if 1 minus tau value is more than 25 percent, so in that case, this particular golden section search method will be more efficient than the interval halving method. So, let us see how what is the value of tau.

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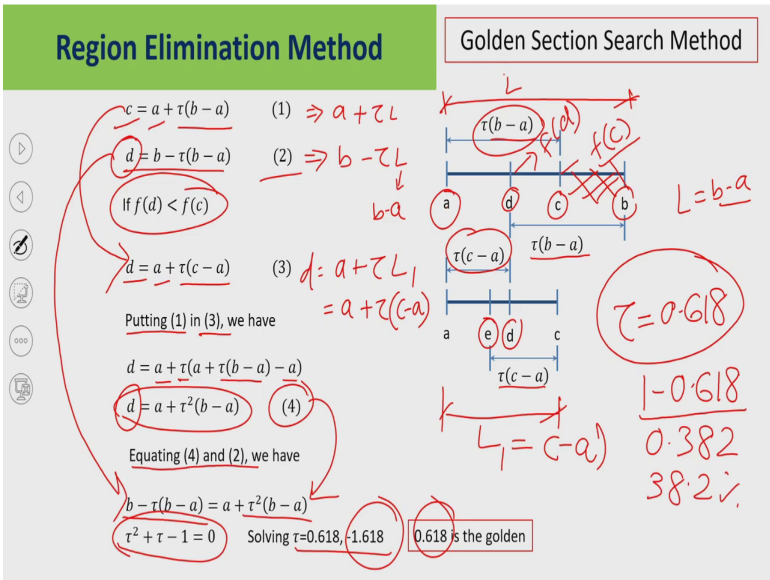

So, let me explain this part. Now, in the figure this a and b and the length between a and b is your L; a and b is your L. Now, I have taken two points here that is c and d. Now, the length L is nothing but b minus a. So, L is b minus a ok. So, this is your L. Now, the c point the distance between a and c is tau b minus a and that is d b is tau b minus a.

So, therefore, I can write what is c? So, c equal to a plus tau b minus a. Is not it? So, this is c equal to a plus tau b minus a and similarly, I can write what is d? d is b minus tau b minus a or you can say I can write it, this is b minus tau L and this is and this is a plus tau l. So, L is nothing but b minus a ok.

So, I am getting this one. Now, if suppose this condition is true that is f d is less than f c; that means, f c is greater than f d. So, this is f c and if I calculate f d, so f c is greater than f d; that means, this particular area will be eliminated ok. So, therefore, once you are eliminating. So, you are getting this new L ok.

Now, what is this new L? L is nothing but now c minus a. So, this is c. Now, the earlier point will be, so earlier point that d will be now tau c minus a ok. So, this is tau c minus a and I will take another point e and that is tau c minus a from the right hand side. So, from c. So, now, if this condition is true, now what is the value of d now?

So, d is a plus tau c minus a. So, d equal to a plus tau this is you can say suppose if I say L 1. So, this is tau L 1 and which is equal to a plus tau c minus a. Is not it? So, if I put 1 in 3, so that means, I am putting 1 in 3, so what I am getting? This is a plus tau. Now, what is c? c is a plus tau b minus a minus a and if I simplify that one, so I will be getting this equation that is d equal to a plus tau square b minus a. So, I am getting 4.

Now, you just see this is d and this is also d; that means, if I equate 4 and 2; so, this is 4 and 2, if I equate, so what I am getting? The 2 is b minus tau b minus a ok, so that is your equation 2 and this is your equation 4 which is equal to a plus tau square b minus a and if I simplify this one, so finally, I am getting this quadratic equation that is tau square plus tau minus 1 equal to 0.

And if I solve this equation for tau, what is the value of tau? Tau is 0.618 and minus 1.618. So, now, the tau value cannot be negative ok. So, that ratio cannot be negative. So, therefore, the possible value is 0.618 and you just see this is nothing but the golden ratio ok.

So, therefore, if you are using tau is equal to 0.618, then every iteration, how much area you can eliminate? You can eliminate 1 minus 0.618. So, we can eliminate 1 minus 0.618 and which is equal to 0.382 ok. So, that means, every iteration, I can eliminate 38.2 percent of the area. So, now, if I consider this golden section method, so every iteration, I am eliminating 38.2 percent of the region ok.

So, that much region, I am eliminating in every iteration and how many function evaluation we are doing? We are doing only one function evaluation in every iteration except the first iteration; that means, after first iteration, we are doing one function evaluation and we are eliminating 38.2 percent of the region. So, if we compare with the interval halving method, in the case of interval halving method, we are eliminating 50 percent of the area in every iteration and every iteration, we are doing two function evaluations ok.

So, therefore, the area eliminated per function evaluation is 25 percent and in this case, this is 38.2 percent. So, if you compare these two methods, in that case the golden section search method is more efficient.

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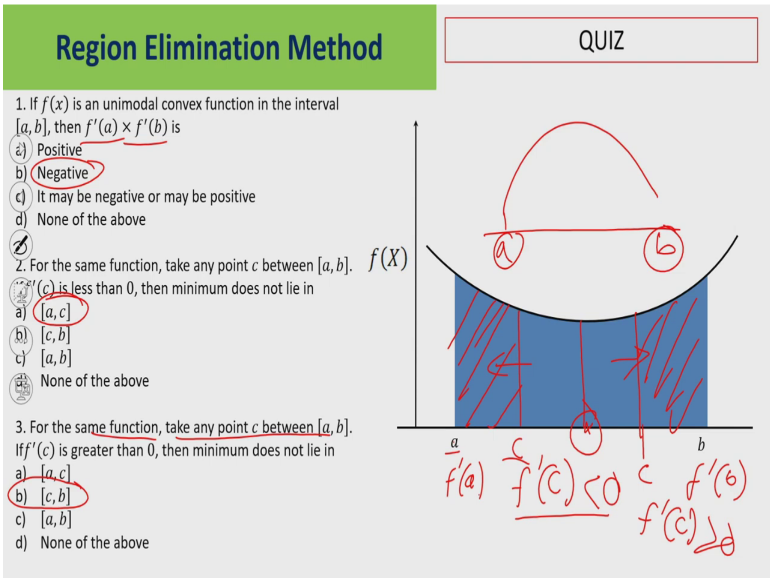

Now, let us have another quiz. The question is if f x is an unimodal convex function in the interval a and b, then the derivative at a into derivative at b is ok the options are is positive. So, I am just asking about the sign; whether it is positive, whether it is negative; it may be positive, it may be negative or none of the above ok.

So, what you are doing here? You are taking derivative at a; that means, I am calculating derivative at a and derivative at b. Now, if I multiply these two derivative ok; derivative at a and derivative at b, so what will be the sign of whether it will be a positive value, whether it is a negative value or it may be positive it may be negative none of the above?.

So, in that case, if it is an unimodal function, so for this function only because there is an optima somewhere here. So, therefore, derivative on this side suppose if I can say the slope in this side will be negative ok and in this side will be positive. So, therefore, if there is an optimum solution, then the derivative that the sign of f dash a into f dash b should be negative because one will be negative, one will be positive. So, therefore, the product will be negative. So, answer is negative ok.

Now, what I can do basically? So, that means, if it is an unimodal function, there is a minima between up between a and b. So, in that case that f dash a into f dash b should be should be negative or otherwise, what I can say that for a unimodal function, if this product is negative, then there must be an minimum between a and b.

So, I can say that if the product of these two derivative is negative, so in that case, I can say that there is a minima between a and b or there is a maxima between a and b. This is true for this type of function also. So, in that case, the derivative at a suppose this is a and this is b.

So, if I take derivative at a and derivative at b and if I multiply these two derivative then the that the sign will be negative in case of maximum also. That means, if I say that if the product is negative, so in that case, there is a optimum solution between a and b ok. So, this is your first question.

Then, second question is for the same function that is for this particular function, take any point c between a and b and derivative at c is less than 0. So, somewhere here, I am taking a point c and derivative at c is less than 0 ok. Then, minima does not lie between a c ok a c, then c b, then e b and then, none of the above.

So, in this case what will happen? The derivative at c is less than 0. So, in that case, the optima is not there in this particular region. So, you can say you can eliminate that particular region. So, therefore, the solution here or answer here is a c ok. Similarly, the third question is for the same function take any point c between a and b and derivative at c is greater than 0; that means, positive ok.

So, now, c is somewhere here and derivative at c is greater than 0. So, in that case, the minima is not there between a c, then c b, then a b and none of the above. So, in that case because now this is on the other side, so minima is not there in this particular region. So, what you can do? So, you can eliminate that region and here answer is c b ok.

So, you can the minima is not there in that particular region and you can eliminate this particular region. So, what we can do? Now, instead of looking at the function the way we are doing interval halving method and golden section search method, so we can also look at the derivative value ok. So, if derivative is positive, so we can eliminate the region on the right hand side of that particular point and if derivative is negative, so we can eliminate the left hand side of that particular point.

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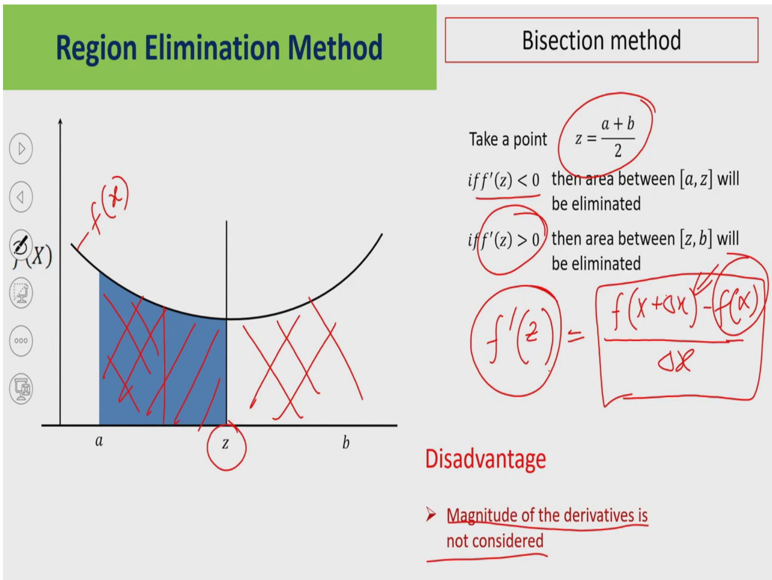

So, let us apply this method. This method is known as Bisection method. So, what we are doing? Then again, f x is a function unimodal function; that means, there is only one minimum and the function is continuous between a and b and now, what we are doing that we are taking a point z and z is the midpoint between a and b and z equal to a plus b by 2 ok. So, we are taking z and now, we are calculating the derivative at z ok. So, f dash z derivative as z.

If the derivative is negative ok, less than 0; then, this area will be eliminated. Is not it? So, optima is not there in this region. So, I can eliminate this particular region. Now, if the derivative is positive ok; so, derivative is positive, then I can eliminate with the other region, this region can be eliminate.

Now, what we are doing? Suppose, if I say this is my first iteration and the second iteration, I will take another z and I will see basically or I will check whether derivative is positive or negative and that way I can eliminate the other region.

Now, question is that every iteration, we are eliminating 50 percent of the area because we are taking z at the midpoint. So, 50 percent of the region we are eliminating here. So, every iteration and every iteration, we have to calculate the value of z ok.

Now, if I compare the efficiency of this particular method with the other method, suppose interval halving method or golden section search method, what is the computational complexity here that here we have to calculate the derivative of the function at that particular point.

Now, question is that how to calculate this derivative? So, derivative suppose if I calculate numerically at this particular your point, so what I can do basically? That I have to find out f x plus delta x minus f x divided by delta x, numerically if I calculate the derivative. So, that means, to calculate derivative, I have to do two function evaluation; I have to calculate two function evaluation at x plus delta x or f x. So, I have to I have to do means I have to calculate two function evaluation. Is not it?

So, if I calculate the derivative numerically, then I need two function evaluation, two function evaluations I need ok. So, therefore, every iteration you are eliminating 50 percent of the area and there are two function evaluation.

So, the power function evaluation I am eliminating 25 percent of the area. So, you can say this is similar to interval halving method in terms of computational complexity. Is not it? So, every function evaluation, I am eliminating 25 percent of the region.

Now, let us see what is the disadvantage here. So, disadvantage here is the magnitude of the derivative is not considered. So, we did not consider; we have considered only the sign of the derivative ok, so whatever the value, the sign of the derivative, we have considered; whether it is positive or the negative value, but we are not considering the magnitude of the derivative.

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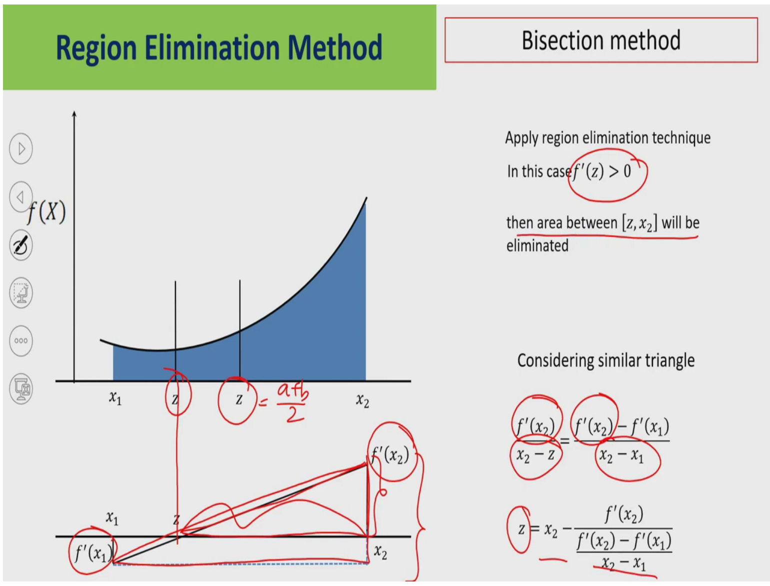

So, we can modify this one. Now, let us see that this is the derivative at x 2 and this is the derivative value magnitude value at x 1 ok. Now, this is the earlier z, this is z equal to a plus b by 2 and now, I would like to take the z value somewhere here ok. So, where the sign has changed. Now, if I consider this similar triangle ok, so this is one triangle that is this is one triangle and another triangle is the larger triangle ok. So, this is another triangle.

So, if I consider these two similar triangles, so I can write that f dash x 2 divided by x 2 minus z. So, I need to calculate what is the value of z here. So, this is from the first triangle that is f 2 dash and this is your x 2 minus z.

Now, from the other triangle, so I can calculate what is this value. So, this value is derivative at x 2 f dash X 2 minus f dash x 1 and the distance is x 2 minus x 1. So, if I simplify that one, so I will get the value of z. So, z equal to x 2 minus f dash x 2 divided by f dash x 2 minus f dash x 1 divided by x 2 minus X 1.

So, I can calculate the value of z. So, now, if I take this is my z value and if I apply the region elimination technique, so in that case, the derivative is positive ok. So, what we can do? So, in that case, the area between z x 2 can be eliminated ok. So, can be eliminated.

So, I can eliminate this region. So, in this case, we what we have done? Apart from the sign of the derivative, we have also considered the magnitude of the derivative and we can eliminate some more region ok. So, and this is this will be certainly more efficient than the earlier method.