**Optimization Methods for Civil Engineering**
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**Module - 01**
**Lecture - 02**
**Classical Optimization**

Hello, student. Welcome to the course on Optimization Methods for Civil Engineering. So, today, we will continue our discussion from the part we have discussed in the last class. So, in the last class, I introduce you what is optimization, then also we discussed few example problems – basically we discussed how we can formulate an optimization model and also we have discuss what is optimization.

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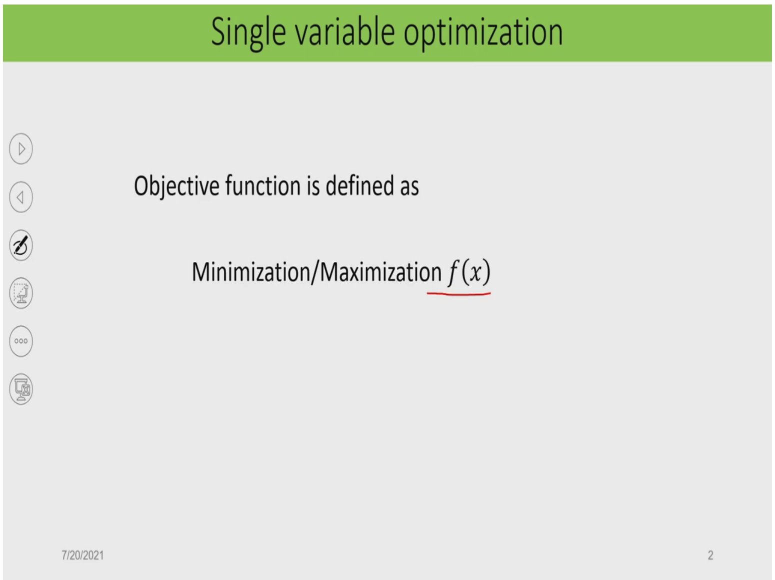

Let us consider a single variable optimization functions. So, we are not considering any constraint here or in that case we are calling it unconstraint optimization problem. Now, so, we have only the objective function. So, we have the objective function f x. Now, this objective function can be a minimization function or a maximization function.

So, here we do not have any constraint. So, therefore, this is an unconstrained optimization problem and we are only considering one variable that is single variable and therefore, this is single variable optimization problem.

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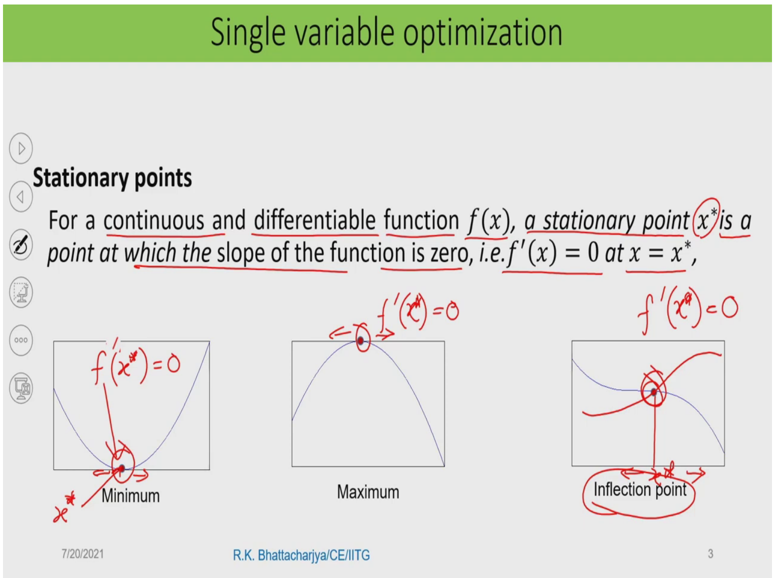


Let us discuss what is stationary points. For a continuous and differentiable function f x, so, here f x is a continuous and differentiable function. A stationary point x star; so, x star, so, when the point is stationary then we will denote it by x star. So, that means, x star means it is a stationary point a stationary point x star is a point at which the slope of the function is zero, ok. That means the first derivative is 0 at x equal to x star.

So, otherwise what I can say that if first derivative is 0; that means, slope of the function is 0 at x star. So, in that case x star is a stationary point. I hope this is clear. So, what is stationary point? The first derivative should be 0; that means, slope of the function at that particular point is 0 so, in that case x star is a stationary point.

Now, let us consider this particular function. So, here we have one point. So, this is a stationary point; that means, the first derivative is 0 here. So, at this particular point the first derivative; so, this is a stationary point. So, I will write x star and this is equal to 0, but this is also a minimum point. Why minimum? That if I go if I, so, I am at a x star right now. So, this is the point x star. This is the point x star.

So, I am at x star. So, if I go slightly right hand side or left hand side my function value will increase; that means, this is the minimum point of this particular function and if you go slightly on the right hand side or left hand side, so, if you go on this direction or in this direction the function value will increase. So, therefore, this particular point; therefore, this particular point is a minimum point.

Now, let us consider this particular function. So, here we have a stationary point. So, why it is a stationary point? Because the first derivative at this particular point is also 0, but this is a maximum point. Why maximum? Because if you go slightly right hand side or left hand side in this case the function value will decrease.

So, therefore, that is the maximum of this particular function and if you go slightly on the right hand side or left hand side the function value will decrease. So, therefore, that is a maximum point, but this is a stationary point. Why stationary? Because the first derivative is 0 or you can say the slope of the function is 0 at that particular point.

Now, let us consider this particular function here. Now, this is the point here. So, again this is your x star and at this point the first derivative is 0 first derivative is 0. Therefore, this particular point is also a stationary point; slope of the function is 0 at this particular point. So, therefore, this is also a stationary point.

But, this is neither maximum point nor minimum point because if you go in this case if you go right hand side the function value will decrease and if you go left hand side the function value will increase. So, therefore, this particular point is neither a minimum point nor a maximum point. But, this is an inflection point and we call it inflection, but this is also a stationary point.

Now, from this example problems, so, we can conclude that a stationary point maybe a maximum point, maybe a minimum point or maybe an inflection point. So, your function may be like this also. So, in that case also this particular point may be an inflection point where derivative is 0.

So, therefore, the stationary point does not mean that the particular point will be a maximum point or the particular point will be a minimum point, but it will say that this particular point is a candidate for either minimum point or maximum point or an inflection point. I hope this is clear to you so, what is stationary points.

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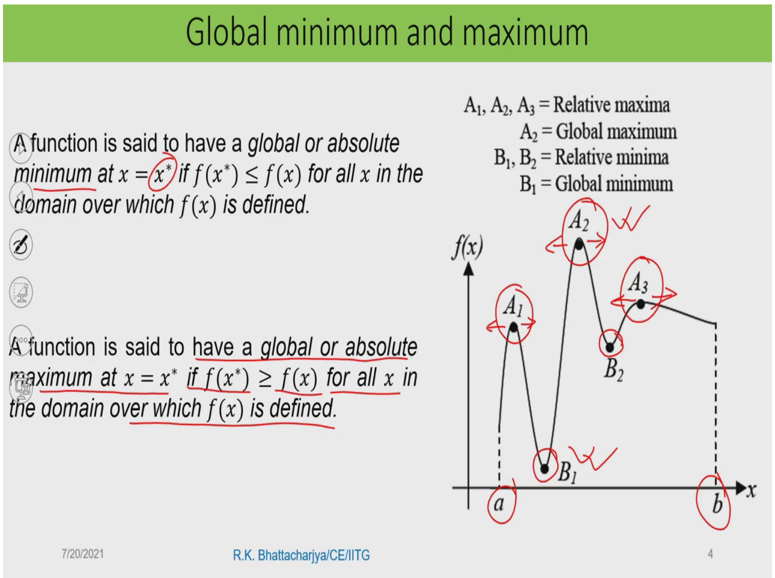


Now, in this particular curve, so, I have shown you a function. So, if you look at this particular function A 1 is a point and this is a maximum point then A 2 is also a point this is also a maximum point and A 3 is also a maximum point, ok or you can say these are all stationary points. Why stationary? Because the first derivative is 0 at A 1, A 2 and A 3. So, first derivative is 0, the slope of the function at this point is 0.

Similarly, if you look at the first derivative is also 0 at B 1 and B 2. So, therefore, A 1, A 2, A 3 then B 1, B 2 all these points are stationary points because the first derivative is 0. Now, if you look at the point A 1, A 2 and A 3. So, here what is happening if in case of a one if you go slightly right hand side or left hand side the function value will decrease. So, therefore, this is a maximum point.

Similarly, A 2 also if you look at if you go on the right hand side or if you go on the left hand side the function value will decrease. So, therefore, this is also a maximum point. Now, if you go to A 3 here also if you go on the right hand side or left hand side the function value will decrease. So, therefore, A 1, A 2, A 3 these three points are maximum point.

Now, one of them; one of them; that means, I either A 1, A 2 and A 3 so, one of them is basically the best solution within the search space. So, in that case what I will say the best solution will be the global maximum solution. So, in this case A 2 is the best solution out of A 1, A 2 and A 3. So, therefore, A 2 is your global maximum. I hope this is clear.

So, let us see the definition of what is global minimum or what is global maximum. So, a function is said to have a global or absolute minimum at x equal to x star. So, x star is a stationary point. If the function value at x star is less than f of x for all x in the domain over which f of x is defined.

So, here the f of x is defined between a and b and within that domain that A 2 is the best solution and you can say that is the global maximum solution or if I say in case of minimum so, B 1 and B 2 are minimum point here and B 1 is the best solution and therefore, you can say that B 1 is the global minimum.

So, similarly I can define global maximum point and global minimum. So, global maximum is a function is said to have global or absolute maximum at x equal to x star if the function value at x star is greater than f of x for all x in the domain over which the f over which f x is defined.

So, if I look at this definition what is global minimum and global maximum global minimum and global maximum. So, if I look at this definition and this particular function so, A 2 is the best solution; that means, the best maximum solution. So, I can say that A 2 is global maximum. Similarly, B 1 is the best solution in terms of minimum point. So, therefore, I can say B 1 is your global minimum.

Now, A 1, A 2, A 3 all are relative maxima or you can say these are local optimal solution. So, A 1, A 2, A 3 are relative or local maxima and similarly, B 1, B 2 are also relative or local minima, but A 2 one is the global maximum solution and B 1 is the global minimum solution. I hope the difference between local solution and global solution is clear to you.

So, what is what we call it global solution that is the best solution over which the function is defined in terms of either minimization problem or maximization problem. So, in this particular function B 1 and A 2 are the best solution. So, B 1 is the global minima and A 2 is the global maxima and all other solutions are either local maximum solution or local minimum solutions.

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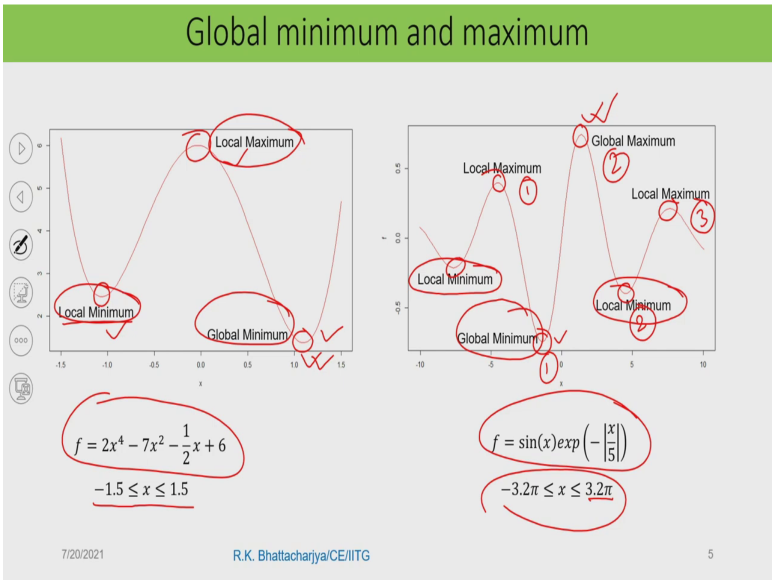


Now, I have shown here two different function. The first function is f equal to twice x to the power 4 minus 7x square minus half x plus 6 and if I plot this particular function between minus 1.5 and 1.5, ok. So, within minus 1.5 and 1.5, so, you can see that this particular function has one stationary point somewhere here, another stationary point somewhere here and another stationary point somewhere here.

So, this particular function has two minimum points. So, two minimum; so, this is one minima, this is another minima and it has one maximum point. So, if I say that if I compare these two minimum solutions then one is the best solution. So, this is the best solution of this particular function.

So, therefore, this is global minimum and other one is local minimum. Similarly, we have only one maximum solution and therefore, you can say this is the local maximum point or you can say the because we have only one maximum. So, therefore, that is the global maximum point of this particular function or.

So, now, if I look at this particular function the second function the function is f equal to sin x exp within bracket minus absolute value of x by 5. And, we are defining this particular function between minus 3.2 pi and plus 3.2 pi. So, between minus 3.2 pi and plus 3.2 pi.

So, you can see there are several stationary points. What is stationary point? The first derivative is 0. So, this is one stationary point, this is another stationary point, this is another stationary point, this is another stationary point, this is another stationary point and this is also a stationary point. Why stationary? Because the first derivative is 0. The slope of the function at these points will be 0. So, therefore, they are stationary point.

Let us see whether there is a there are minimum point or maximum point. You can see this particular function within this range that is range is minus 3.2 pi and plus 3.2 pi. So, we have two minimum solution. So, one is here. So, this is the 1st minimum solution and this is 2nd minimum solution and one of them is the best solution. So, I can say that this is your global minimum and other one is local minimum solution.

So, we have another one. So, another minimum, so, somewhere here, so, we have total three minimum solutions in this particular function and out of these three, one is the best solution and that is the global minimum solution of this particular function. And, if I look at the maximum point, so, we have three maximum point. So, this is number 1, this is number 2 and this is number 3. So, you can say all of them are relative maximum.

And, one of them is the best solution and the best solution here is this one and you can say that is the global maximum solution of this particular function. So, we have two local maximum points and one global maximum. So, therefore, this is your global maximum point of this function and other two are local maximum point.

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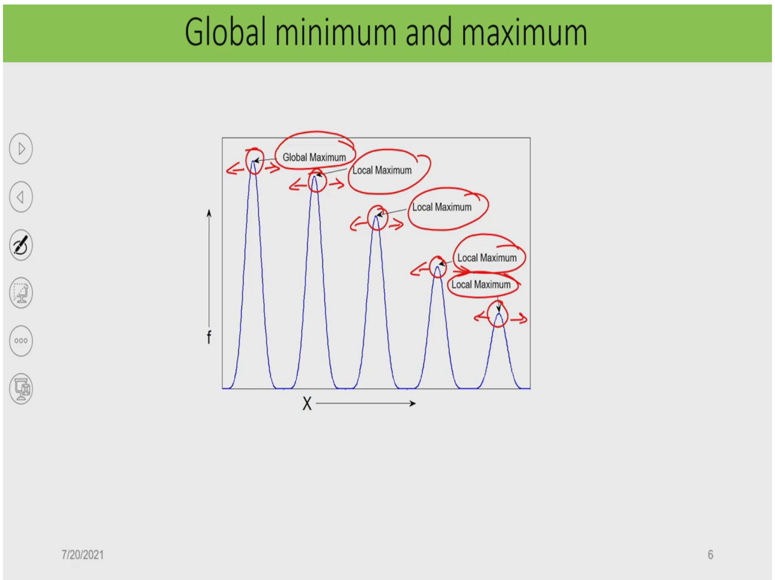


Now, let us see this particular function. So, you can see that this particular function has 1 2 3 4 5 stationary points. So, why stationary points? Because the first derivative at this point is 0. So, first derivative is 0. So, therefore, these points are stationary point and these points are maximum point because if you go slightly right hand inside or left hand side the function value will decrease. So, therefore, this is a maximum point.

Similarly, here also the function value will decrease if you go on the right hand side or left hand side similarly here also similarly here also and similarly here also. So, therefore, all these five points are stationary point and they are also maximum point. So, you can say these are the maxima of this particular function.

Now, if I compare these five solution, so, one of them is the best solution and you can say this is the global maximum point and other solutions are local or relative maximum points, ok. So, other solutions are local solution or you can say local optimal solution, local maximum solution or local maximum point and one of them is the best solution and I can say that is the global maximum of this particular function.

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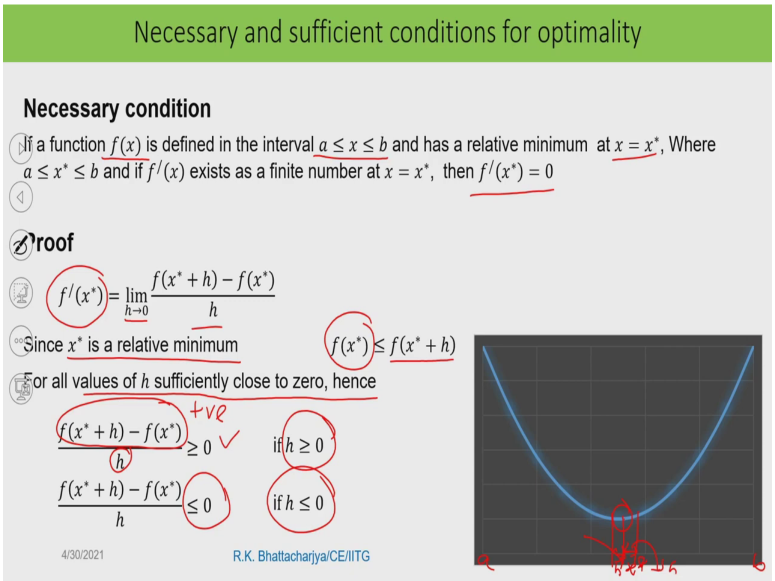

Now, let us discuss what is the necessary condition for optimality. Suppose, a function is given to you then how you can actually tell whether whatever solution you are getting or a particular point is a minima, maxima or an inflection point or whether it is a stationary point, ok. So, we call it necessary condition for optimality. So, we will discuss here what is necessary and sufficient conditions for optimality.

So, in this case so, I will take a function like this. So, this is a convex function I will also discuss what is convex later on, but right now so, in this particular function we have only one optimal solution or you can say one stationary point, ok. So, now what is necessary condition? A function f x is defined a function f x is defined in the interval ab; that means, between a and suppose this is b within ab and has a relative minima at x equal to x star, ok.

So, this point is your x star ok. So, x equal to x star where that x star is between a and b and if the first derivative exists as a finite number at x equal to x star then the first derivative is equal to 0, ok. So, if this condition is satisfied so, I will say that the x equal x star is a relative minima, ok.

So, now, what is the necessary condition a function f x is defined in the interval a x b and has a relative minimum at x equal to x star, where x star is between a and b and if first derivative exists as a finite number at x equal to x star, then the first derivative should be 0 ok if it is a relative minimum.

Now, let us prove it. So, how we can prove it? That I can write that derivative at x star which is equal to limit h tends to 0. So, f of x star plus h minus f of x star divided by h, ok. So, I have taken this particular point this particular point which is x star and you take another point at a distance of h. Suppose this distance is h, maybe I can also take another point which is at a distance h, ok. So, this distance is also h.

So, now I can write or I can take the derivative. So, derivative is equal to limit h tends to 0 f of x star plus h minus f of x star divided by h. Now, since x star is a relative minima. So, as already I said or already in the definition itself we said that x star is a relative minimum. So, this is the point x star and x star is a relative minima.

If it is a relative minima, so, therefore, f of x star should be less than f of x star plus h. So, what does it mean that whatever function value here at f of x star plus h that function value is more than f of x star, if x star is a relative minimum for all values of h sufficiently close to 0 ok h tends to 0, then what will happen? Because this particular portion ok this particular portion is always positive is not it? Because x star is a relative minimum so, therefore, f of x star plus h minus f of x star is always positive, ok.

So, therefore, the sign of this one f of x star plus h minus f of x star divided by h will depend on whether h is positive or negative. Now, if h is positive; that means, you are approaching x star from the right hand side. So, in that case this value will be greater than 0, and if h is negative if h is negative; that means, you are approaching from the left hand side. So, in that case this value will be less than 0 ok.

So, I hope this is clear because f of x star plus h minus f of x star is always positive. So, the sign of this term will depend on whether h is positive or negative. h positive means you are approaching x star from the right hand side and h will be negative if you are approaching x star from the left hand side, ok.

So, therefore, depending upon whether you are approaching x star from right hand side or left hand side this value will be either positive or negative; positive means greater than 0 or negative means less than 0, ok.

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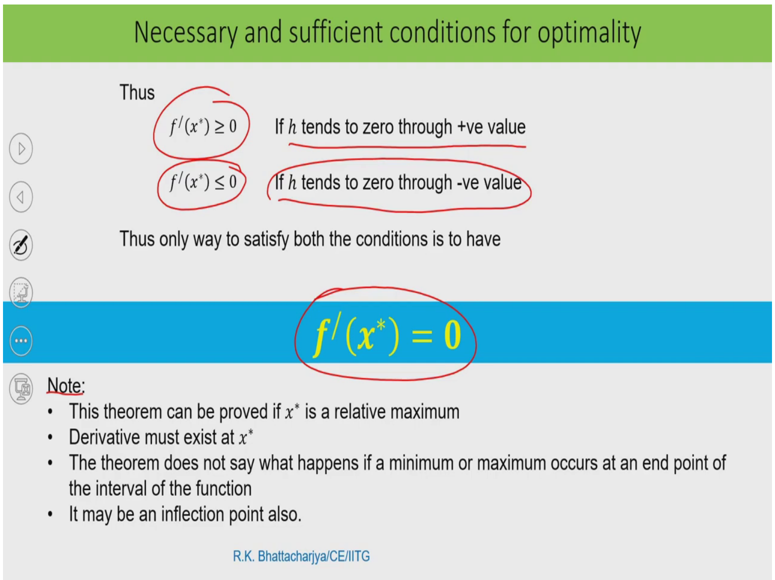


So, thus the derivative the thus the derivative will be greater than 0, if h tends to 0 through positive value; that means, if you are approaching x star from right hand side. So, in that case derivative will be positive and if you are approaching from the left hand side; that means, for h tends to 0 through negative value so, in that case derivative will be negative, ok. So, I hope this is clear.

So, what I am telling here, so, if you are approaching from the right hand side the derivative will be positive and if you are approaching from the left hand side the derivative will be negative, ok. So, therefore, derivative will be positive based on how you are approaching to x star. Thus only way to satisfy both the condition; that means, at that particular x star these two conditions to be satisfied so, therefore, the only way to satisfy these conditions is to have the derivative is equal to 0, ok.

So, therefore, this is the necessary condition for optimality; that means, first derivative should be equal to 0. So, let us see the node here. So, this theorem can be proof for if x star is relative maxima. So, we have prove it for x star is relative minimum, but I can also prove it for x star equal to relative maximum.

So, same way I can prove it the derivative must exist at x star; that means, the function must be differentiable at x star, otherwise you cannot apply this necessary condition. The theorem does not say what happen if a minimum or maximum occurs at the end point of the interval of the function, ok. So, if the minima and maxima your occurs at the end point; that means, at a or b so, this theorem we cannot actually this theorem is not telling anything.

And, apart from that so, as I said that it can also be a inflection point. So, neither maxima nor minima. So, therefore, this is not the sufficient condition for optimality. So, it is a necessary condition that once the first derivative is 0, the particular point may be a maximum point maybe a minimum point or it may be an inflection point also. So, but it is a necessary condition for optimality.

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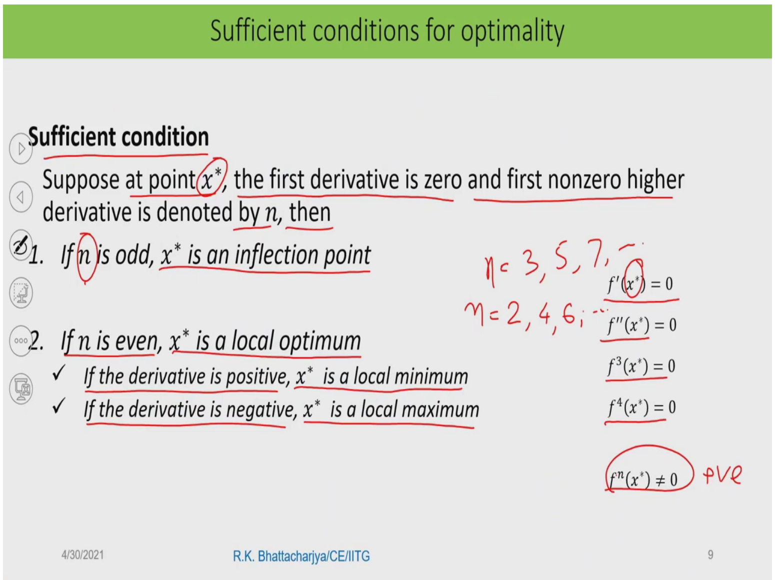


Now, let us see what is the sufficient condition for optimality. So, if you apply the necessary condition then the point whatever point you are getting where derivative is zero; that means, slope is zero so, that point maybe a maximum point may be a minimum point or it may be an inflection point also.

So, therefore, I cannot tell whether that is a maximum or minimum or inflection point, but I know that this is either maximum or minimum or an inflection point. So, therefore, I need a sufficient condition for optimality. Now, let us see what is sufficient condition for optimality, ok.

So, sufficient condition let us see the definition suppose at point x star ok this is an stationary point already. x star is a stationary point where first derivative is zero, suppose at point x star the first derivative is zero ok because this is a stationary point and first nonzero higher derivative is defined by n then if n is odd x star is an inflection point. If n is even, x star is a local optimum and then if the derivative is positive then x star is a local minimum and if the derivative is negative, then x star is a local maximum.

So, here I am defining. So, it is a stationary point. So, x star is a stationary point. So, therefore, first derivative is 0. Suppose, the second derivative is also 0, third derivative is also 0, fourth derivative is also 0 and n derivative is not 0. So, this is the n derivative of this particular function which is not 0.

Now, what I will look? I will look at this value of n. So, now, if n is odd ok so, n is odd; that means, that is 3, 5, 7 something like that. So, it is odd. So, then x star is an inflection point; that means, whatever your x star, x star is already a stationary point, but this stationary point will be an inflection point if n is odd. I hope this is clear.

So, if n is odd so, n is n value if it is 3, then 5, then 7 something like that so, in that case that x star is an inflection point. Now, if n is even so, n is even means the n equal to; n equal to 2, then 4, then 6 something like that ok. So, in that case that is a local optimum. So, what is local optimum? The local optimum means either it may be a maximum point or it may be a minimum point.

So, now then we look at the derivative value if the derivative is positive; that means, this is a nonzero derivative and this derivative is positive ok. So, positive so, positive means then x star will be a local minimum, clear? So, now, if the derivative is negative ok derivative is negative. So, in that case x star is a local maxima.

So, let us remember that one. So, now, what I will do? First I will look at the value of n; if n is odd, then x star is a inflection point. Now, if n is even ok so, n is even; that means, 2, 4, 6 something like that then what I will do? I will look at the function value ok. So, if function value is positive then x star is a minimum point if function value is negative then x star is a maximum point. So, by going through this I can actually tell whether x star is a inflection point, it is a maximum point or it is a minimum point.

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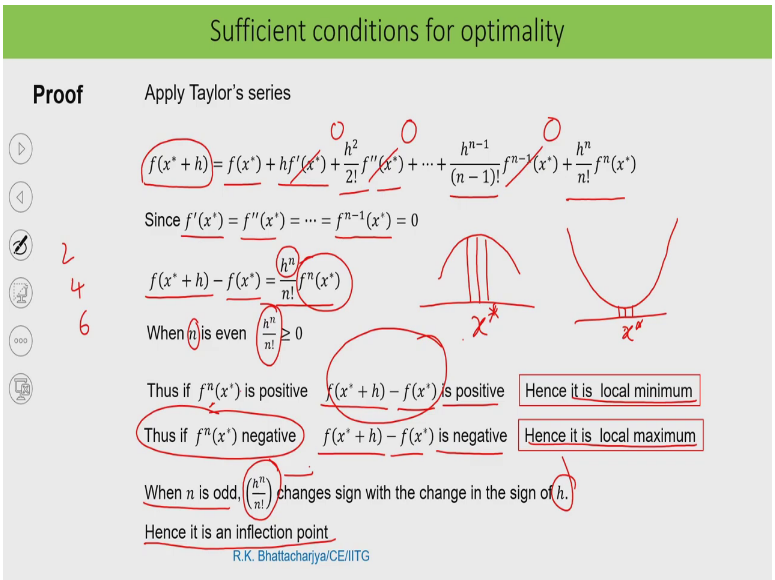


Now, let us see the proof ok, how we can prove that one. So, proof is that suppose f of x star plus h so, if I apply the Taylor series, ok. So, I can write f of x star plus h which is equal to f of x star plus h f dash x star plus h square by factorial 2, second derivative x star and then plus. So, I can go up to this is h n minus 1 divide by factorial n minus 1 f to the power n minus 1 x star and this is h to the power n by factorial n f to the power n x star that is the nth derivative, ok.

Now, since as per the definition the first derivative is 0, because that is stationary point suppose second derivative is 0, third derivative is 0 up to n minus 1 derivative is 0. So, here the first derivative is 0. So, therefore, this will be 0; this will second derivative is 0 and n minus 1 derivative is also 0. So, therefore, I can write that f of x star plus h f of x star plus h minus f of x star which is equal to h to the power n divide by factorial n that n derivative at x star.

Now, if n is even ok so, if n is even; that means, again I said that 2, 4 ok 6 something like that if it is even then what will happen this value will be always positive because n if it is an even number n is even number then entire value will be positive. So, therefore, if the n derivative is positive ok so, what does it mean? If this is positive; that means, the right hand side is positive. So, in that case what will happen? f of x star plus h minus f of x star is positive, ok.

So, what does it mean? That if this is a minimum point and this is your x star ok and you are taking either h on this side or this side; that means, f of x star plus h minus f of x star is positive. So, when it will be positive? If x star is a minimum ok. If x star is a minimum so, in that case this value will be positive. So, therefore, I can say if this value is positive then x star is a minimum. Hence it is a minimum, ok.

Now, if the derivative the n derivative is negative, ok. So, in that case f of x star plus h minus f of x is negative. So, when it will be negative? In case of maxima. So, I can say that this is the case. So, this is your x star this is your x star and if you take any value either left hand side or right hand side so, in that case what will happen? That f of x star plus h minus f of x star is negative in case of maximum, ok.

So, therefore, if it is negative, then I can say that x star is a local maximum ok. So, what we are discussing here if n is even, so, in that case I will look at the derivative value. If derivative value is positive, then f of x star plus h minus f of x star is positive, then I can say that this is a minimum point and if derivative value is negative, then f of x star plus h minus f of x star is negative then I can say that is a maximum point.

Now, if n is odd ok so, in that case what will happen? This value will basically changes sign whether you are approaching from left hand side or you are approaching from the right hand side. So, this value will changes sign with the change in the sign of h; that means, if h is positive; that means, you are approaching either from left hand side or from right hand side so, what will happen? H will be either positive or negative. So, based on whether h is positive or negative the sign of this value will change.

So, therefore, in this case what will happen? If you are approaching from one direction this will be positive and if you are approaching from the other direction then it will be negative. So, in that case that is a condition for inflection point. So, therefore, we can say if n is odd so, then the particular point is an inflection point, ok. So, if n is odd so, in that case because if you are approaching from one side, then it will be positive if you are approaching from the other side that will be negative.

So, therefore, this is a condition for inflection point. So, in case of inflection point as you have seen the function value is increasing in one side and decreasing on the other side or it may be other way also; that means, increasing one side and decreasing on the other side. So, therefore, there is a condition for inflection point. So, I can say that if n is odd. So, in that case x star is an inflection point.

So, what we have discussed? Let me summarize. So, what I will do? I will take the first higher order nonzero derivative value, ok. So, suppose the ordarity n, now you look at n if n is odd then what you will do? If n is odd then x star is an inflection point directly you say that is an inflection point that is neither maxima nor minima.

If x star is even, ok so, in that case you will look at the function value, ok. So, what is the derivative value. So, if x star is even. So, you look at the derivative value. Now, if the derivative value is positive, then x star is a minimum and if the derivative value is negative then x star will be maximum ok.

So, that way, I can tell a particular point, a stationary point whether that is a maximum point whether that is a minimum point or whether that is an inflection point ok. So, I can tell looking at or going through these sufficient conditions.

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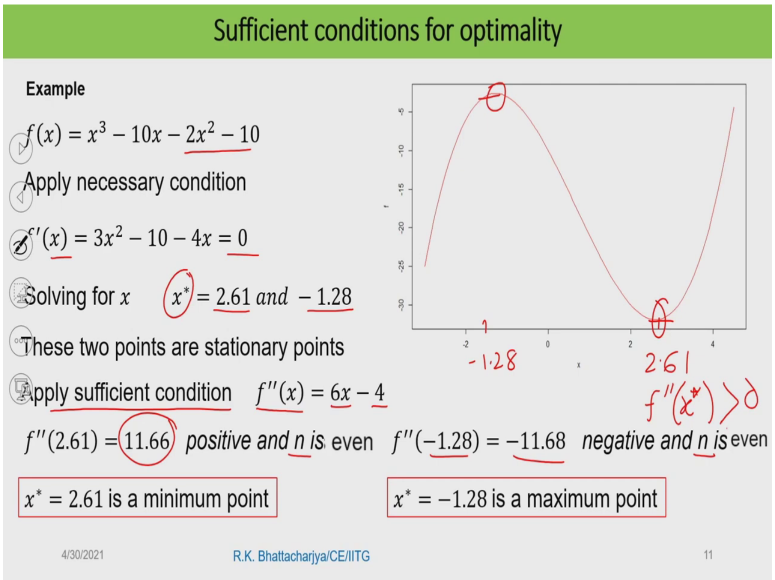


Now, let us see this example problem. So, I would like to solve this problem and find out the stationary point and also I would like to find out whether it is stationary points or a maximum point, minimum point or an inflection point. So, what I will do? Suppose for this particular function f of x equal to x cube minus 10x minus 2x square minus 10.

For this function what I will do? First, I will apply the necessary condition. What is necessary condition? The first derivative is 0. So, let me take the first derivative. So, first derivative is 3x square minus 10 minus 4x equal to 0 and if I solve this particular equation. So, I will get that x star is 2.61 and minus 1.28. So, that means, I am getting two points – one point is somewhere here and that is minus 1.28 and I am also getting another point that is 2.61, ok.

So, looking at this particular curves so, I can say that this point is a maximum and this point is minimum, but let me apply the sufficient condition here ok. So, these two points are the stationary points. Why stationary? Because the first derivative is 0 at this two points, ok. So, derivative is 0 here and derivative is 0 here. So, therefore, I can say that these two points are stationary points.

Now, let me check whether these points are a maximum point or a minimum ok or an inflection point. So, to do that, what I have to do? I have to apply sufficient condition. So, what is sufficient condition? I will take the second derivative first. So, now, second derivative I am getting 6x minus 4, ok.

So, if I calculate the value of second derivative at 2.61. So, I am getting 11.66. So, now, in this case the n value is n value is 2; that means, even number n is even number. This is second derivative, n is even number. So, n is even number in that case what will happen? That will be either maximum or minimum point.

Now, let us look at the derivative value. Now, derivative value is 11.66 and which is positive and therefore, that x star equal to 2.61 is a minimum ok minimum point that is 2.61, here the second derivative value is; second derivative value is positive, ok. So, therefore, is positive. So, an n is even because the second derivative. So, therefore, x star equal to 2.6 is a minimum point.

Now, let us look at the other point that is x star equal to minus 1.28. So, here the derivative value is minus 11.68; that means, negative and n is even and the derivative value is negative. So, therefore, this is a maximum point, ok.

So, that way by applying necessary condition and sufficient condition so, I can find out the stationary points all the stationary points and also I can tell that a particular stationary point is a minimum or maximum point or an inflection point. So, what I have to do? I have to apply the sufficient condition.

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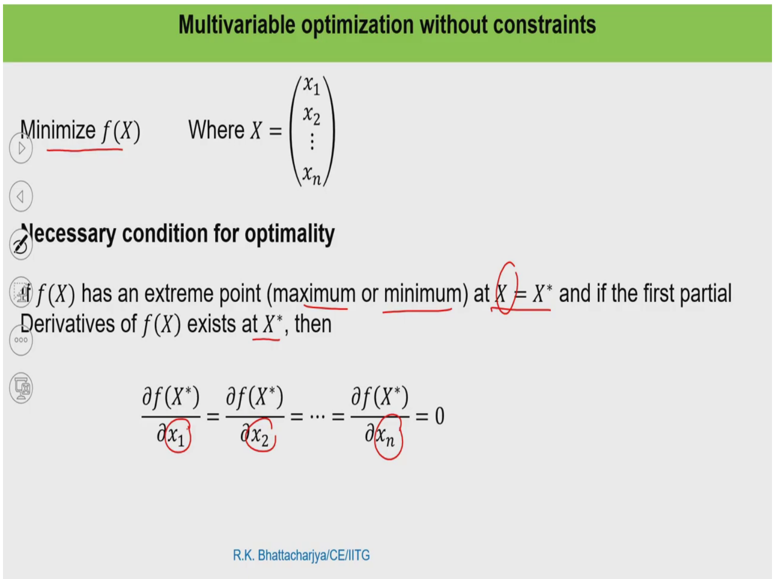


So, far we have discussed the necessary and sufficient conditions for optimality in case of single variable optimization problem. So, we have only discussed the single variable optimization problem, but what is the necessary and sufficient conditions for optimality in case of multivariable optimization problem? So, let us discuss that one.

Let us define a multivariable function; that means, I would like to find the minimum of these particular function f X. So, where X equal to x 1, x 2, x 3. So, its a it has n variables. So, necessary condition for optimality is if f X has an extreme points that is maximum or minimum at X equal to X star and in this case X is a vector, ok and if the first partial derivative of f X is exist at X star then the partial derivative of f with respect to x 1, x 2, x n should be equal to 0.

That means partial derivative with respect to the variable suppose we have two variables then in that case del f by del x 1 del f by del x 2 should be equal to 0. So, we have n variable so, in that case del f by del x 1, del f by del x 2 then del f by del x n up to del f by del x n should be equal to 0. So, that is the necessary condition. By applying that, so, I can find out the stationary points; that means, whatever X star I am getting so, that X star may be a maximum point or minimum point or an inflection point.

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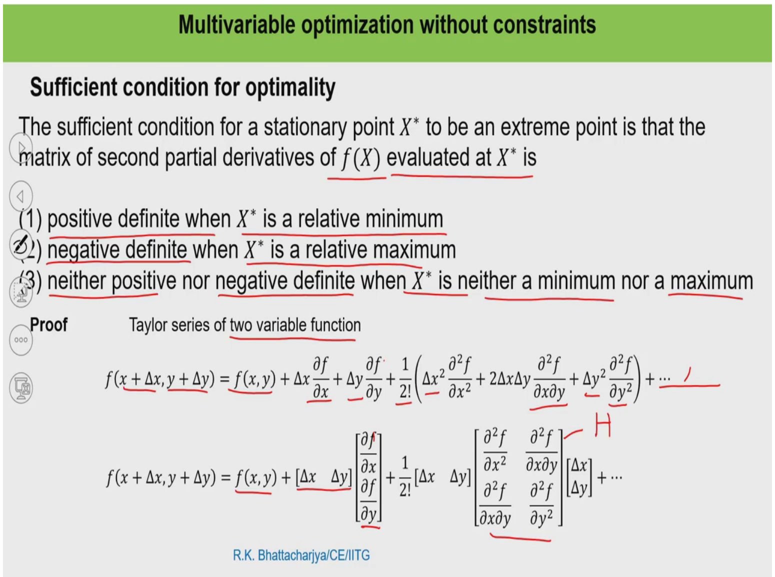


So, we have to apply the sufficient condition for optimality. The sufficient condition for stationary point X star to be an extreme point extreme; point means either it is a maximum point or minimum point or an inflection point to be an extreme point is that the matrix of second partial derivative of f X evaluated at X star is positive definite, ok. So, when X star is relative minima.

So, whatever the matrix I am getting so, the matrix of second partial derivative of f X evaluated at X if it is positive definite then X star is a relative maxima. If it is negative definite then X star is a relative maxima; neither positive nor negative definite matrix then X star is neither minima nor maxima then in that case that is a saddle point, ok.

So, in this case, what we have to do? We have to take the second derivative second partial derivative of f X. So, I will get a matrix. Now, what I will do? I will evaluate that matrix at X equal to X star if the matrix is positive definite, then X star is a minimum point; if the matrix is negative definite the point is a maximum point; if the matrix is neither positive nor negative. So, in that case the that X star is neither maxima nor minima it is a saddle point.

Now, let us prove that one. So, we can apply the Taylor series. So, in this case I have considered two variable function. So, I can write f of x plus delta x, y plus delta y. So, I can write it f of x, y plus delta x del f by del x plus delta y del f by delta y, then 1 by factorial 2 within bracket del x square then del square f by del x square plus twice delta x delta y del square f by del x del y plus delta y square del square f by delta y square. So, I am not considering the higher order derivative, ok.

So, if I write in matrix form so, I can write it something like that. So, f of x, y then del x del y then del f by del x then del f by del y. So, I can write and I am getting this particular matrix and this matrix is known as Hessian matrix, ok.

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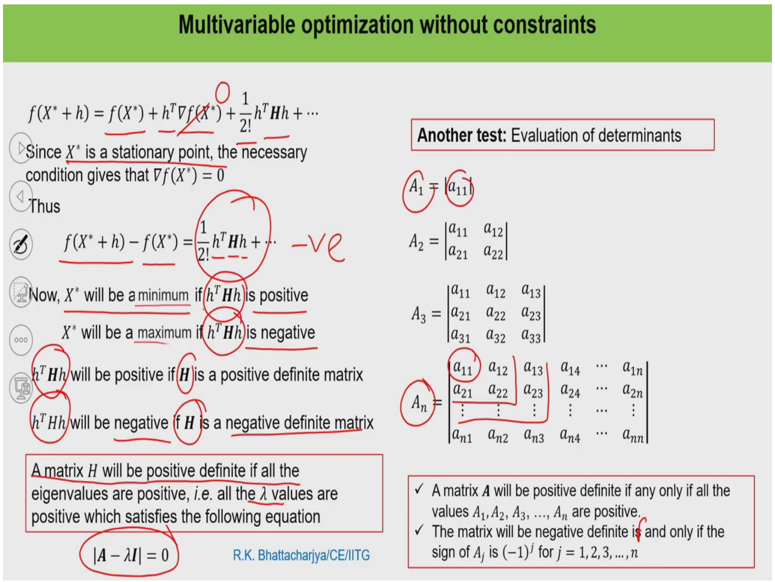


So, now, I can write it something like that in matrix form that this is f of X star then h transpose then del f at X star and 1 by factorial 2 h T Hh ok. So, in that case if I write it something like that then X star is a stationary point; X star is a stationary point. So, therefore, this term will be equal to 0, ok. So, in that case what I can write that f of X star plus h minus f of X star which is equal to 1 by factorial 2 h T Hh ok.

Now, depending upon what is this value, whether this value is positive or negative? So, if it is positive, then what will happen? If it is positive then f of X star plus h minus f of X star is positive in that case X star will be minimum. And, if this is negative, ok so, if this is negative so, in that case X star will be maximum.

So, now, question is that whether this right hand side is positive or negative, ok. So, now, X star will be minimum if this term h transpose Hh is positive and X will be maximum if h transpose Hh will be negative ok. So, now this term h transpose Hh will be positive if H is a positive definite matrix, ok. So, I can check actually whether it is a positive definite matrix or negative definite matrix and if this term will be negative if H is a negative definite matrix.

So, what you have to do basically? You have to look at this particular Hessian matrix. If the Hessian matrix is positive definite so, in that case that will be a minimum point and if Hessian matrix is negative definite matrix so, in that case that is a maximum point. If the Hessian matrix is neither positive nor negative in that case the X star will be neither maximum nor minimum point.

So, a matrix H will be positive definite matrix; that means, how we can check whether the Hessian matrix will be a positive definite matrix or not. So, a matrix H will be positive definite if all the eigenvalues are positive. So, what I can do? I can calculate all the eigenvalues of these particular matrix. If all the eigenvalues are positive, so, in that case I can say that H is a positive definite matrix ok that is all lambda values are positive which satisfies the following equation, ok.

So, all the lambda values will be positive which satisfy the following equation. So, what I can do? I can calculate the eigenvalues if eigenvalues are positive. So, in that case I can say this particular Hessian matrix is a positive definite matrix or otherwise there is another test we can evaluate the determinant ok.

So, what I can do? Suppose this is my Hessian matrix. So, A 1 is the first value; that means, this one; A 2 is the second order then the third order something like that. So, if I calculate A 1, A 2, A 3 up to A n. Now, a matrix A will be positive definite if and only if all the values that is A 1, A 2, A 3 up to A n are positive.

So, if A 1, A 2, A 3 and A n are positive so, in that case I can say that A is a positive definite matrix. The matrix will be negative definite if and only if the sign of A j is minus 1 to the power j for j equal to 1, 2, 3 up to n, ok. So, if the value of A 1, A 2, A 3, ok so, the sign of the A 1, A 2, A 3 follows this rule that is minus 1 to the power j, j means suppose j is equal to

1, 2, 3; that means, A 1 should be negative, A 2 should be positive, A 3 is negative, A 4 is positive something like that. So, in that case I can say that is a negative definite matrix.

So, I can apply any one of these 2. So, either I can check the eigenvalue see if eigenvalues are positive. So, in that case I can say it is a positive definite matrix or I can also calculate A 1, A 2, A 3 up to A n and look at the sign of A 1, A 2, A 3, A n. So, if it is positive I can say it is positive definite matrix; if it is follow minus 1 to the power j so, in that case I can say it is a negative definite matrix.

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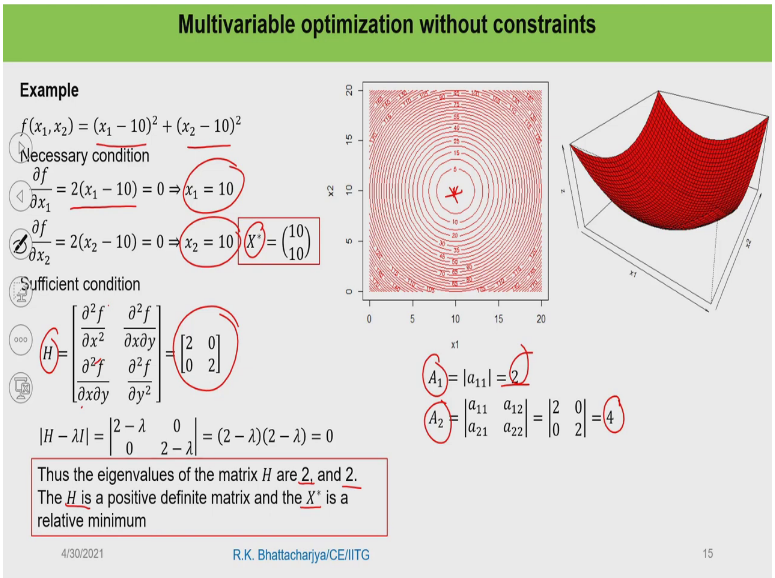


So, let us see an example problem. So, I would like to solve a simple example problem that f of x 1 and x 2. So, two variable problem and that is x 1 minus 10 whole square plus x 2 minus 10 whole square. So, solution of this particular problem is known to me. The solution is 10 and 10. So, I have plot it. So, this is the solution and this is the X star value. So, X star value is 10, 10. I have also shown the surface plot of this particular function.

So, let us apply the necessary condition. So, what necessary condition will give? Necessary condition will give me the stationary points, ok. So, necessary conditions are del f by del x 1; that means, the partial derivative with respect to x 1 and with respect to x 2 should be equal to 0.

So, if I take the derivative of this particular function then with respect to x I will get twice x 1 minus 10 equal to 0 and if I solve it then I will be getting x 1 equal to 10 and similarly if I take the derivative with respect to x 2, so, in that case I am getting x 2 equal to 10. So, therefore, X star X star is the stationary point that is 10 and 10 ok. So, this is the solution of these particular problem that is 10 and 10.

Now, question is that I know in this case this is a minimum point, but let us see by applying the sufficient condition for optimality. So, in this case the Hessian matrix should be a positive definite matrix, ok. So, let us apply the sufficient condition. The Hessian matrix is del square f by del x square del square f by del x del y. So, in this case x 1 and x 2 so, I have written del x del y and similarly, this is del square f by del x del y and del square f by del y square.

So, if I do that if I take the derivative of this one. So, finally, I am getting 2 0 0 2. So, this is the Hessian matrix. Now, I have to check whether this Hessian matrix is positive definite or negative definite. So, what I have to do basically. So, I have to check whether this Hessian matrix that is 2 0 0 2 is a positive definite matrix or a negative definite matrix or neither positive nor negative. So, in this case I know this is a minimum point. So, therefore, the matrix should be a positive definite matrix, ok.

So, let us see I can calculate the eigenvalues. So, if I calculate the eigenvalue so, eigenvalues of these Hessian matrix 2 and 2; that means, both are positive. So, therefore, the H the Hessian matrix is a positive definite matrix and X star is a relative minimum ok. So, I have calculated here the eigenvalues. So, eigenvalues are 2 positive 2 and positive 2. So, therefore, this matrix is a positive definite matrix and therefore, X star is a relative minimum.

Similarly, I can also calculate what is A 1. So, A 1 equal to 2 and in this case A 2 equal to 4 and both are positive. So, if both are positive A 1, A 2 are positive so, in that case the matrix is a positive definite matrix and therefore, the Hessian matrix is a positive definite matrix. So, therefore, I can say that X star is a relative minima.

So, I have also shown you the contour map here. So, you can see from the contour map also this is a minimum point. So, just I have shown you using the Hessian matrix also the Hessian matrix is also a positive definite matrix and therefore, X star is a relative minimum ok.

So, that is all for today. So, we have discussed the necessary condition for optimality in case of single variable optimization function and we have also discussed the necessary and sufficient conditions for optimality in case of multi-variable problem.

Thank you.