Optimization Methods for Civil Engineering
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Lecture - 15
Constrained Optimization

Hello student. So, today we will continue our discussion on Constrained Optimization methods. We have already discuss the constrained optimization problem with equality constraint, then we have discuss constrained optimization problem with inequality constraint and today we will discuss transformation method ok.

So, already we have discuss two types of problem, one is constrained problem with equality constraint. So, we have derived the Lagrange function and then we have defined what is necessary and sufficient condition. And then we have discuss constrained optimization problem with inequality constraint and there we have derived the Kuhn Tucker condition. So, we discuss about Kuhn Tucker conditions and these conditions are necessary condition, but not sufficient condition ok.

But for convex problem Kuhn Tucker conditions are the necessary as well as sufficient condition. So, if a particular solution is satisfying the Kuhn Tucker conditions then that is a global optimal solution in case of convex optimization problem. So, today we will discuss transformation method for handling equality and inequality constraint.

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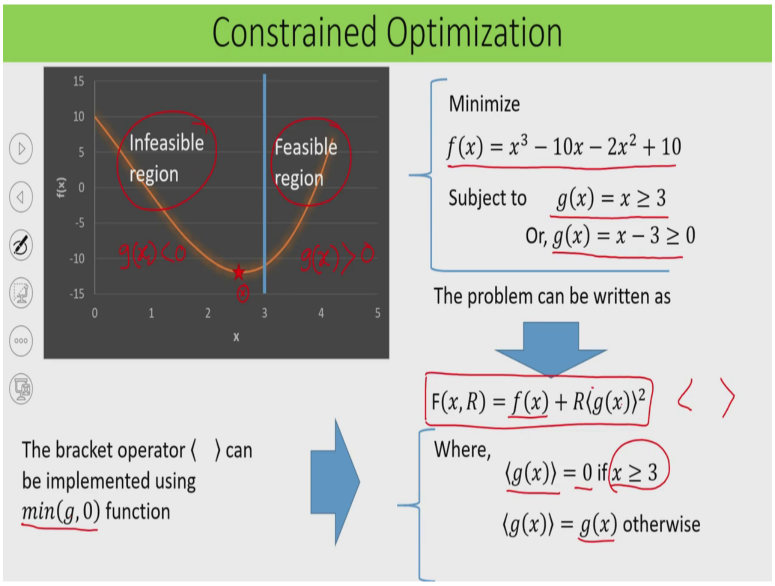

Let us start our discussion with an example problem. So, first I will show an example problem and then I will give you the transformation methods. So, basically we will discuss an example problem initially. The problem is this that it is a single variable function and we it is a and basically we are trying to minimize this function and we are trying to find out the solution for find out the minimum your function value.

Now, I can solve this problem. So, this is a very simple problem. So, I can take the derivative with respect to x and equating to 0 will give you the stationary points ok. So, I can find it and let us plot this function. So, if I plot it the solution is somewhere here ok, the solution is somewhere here. So, this is the solution of this particular problem. Now this is as I said this is a very simple problem.

So, I have just taken to explain the method here and now this is the your solution of this problem that minimum of this particular function. But let us see there is a constrained what is this constrained? The constrained g x which is x greater than 3 ok; that means, the solution should be greater than equal to 3. So, therefore, the earlier solution is somewhere at 2.7 or something like that. So, this solution is not valid actually because the solution should be greater than 3.

So, if I plot this line x equal to 3, so; that means, this line is dividing in the solution space in two region. One is feasible region where the constrained well that constrained function will be satisfied. So, we call it feasible region here, this is the feasible region that x should be greater than 3 and x less than 3 ok. So, that is your infeasible region.

So, we are getting two regions. So, one is the feasible region where constrained function is satisfied and or on the other hand in the infeasible region constrained function is not satisfied in this case. So, therefore, any solution in the infeasible region is not acceptable ok the solution should be in the feasible region.

But if you look at the unconstrained solution so, that is actually less than x less than 3. So, therefore, that is not an acceptable solution for us. So, that solution is not acceptable and we have to get a different solution which will satisfy the constrained function that will be in the feasible region.

So, I can write this thing. So, g x which is equal to x minus 3 greater than 0. So, I can write that one now let us try to solve this problem. So, what we do basically? The idea is that you convert this problem to a unconstrained problems. So, idea is that the this is a constrained optimization problem, it has the objective function and it has constraint and so, I would like to or basic idea is that you just convert this problem to a unconstrained optimization problem ok.

So, this is the conversion here. So, you are. So, we are writing this function. So, here this is the unconstrained function that is capital F x, R. So, R is a new variable I will explain that what is R and which is equal to the original objective function f x plus R. So, we have used bracket operator here, I will also explain what is this bracket operator. So, this is your bracket operator and then I am taking square of that one. So, I would like to take just positive value ok.

So, that is why I am just I will take square of that one. So, I will explain one by one. So, what we are doing here? What this bracket operator is doing that, if g x. So, I will explain the trans of this particular function.

So, here g x equal to 0. So, this bracket operator will give you 0 if x is greater than 3 what does it mean? That means, if the solution is on the right hand side in the feasible region this bracket operator will give you your 0; that means, what we can say that if the solution is somewhere in the feasible region then the there is no violation there is no constrained violation ok.

So, therefore, I can say the constrained violation is 0. So, therefore, if x is greater than 3, the constraint violation is 0 and this bracket operator will give you 0 there is no constraint violation. Now, the bracket operator will give you g x ok. So, g x otherwise; that means, if it is in the infeasible region the bracket operator will give you your g x value ok.

So, that is the function of this bracket operator. So, I can implement this bracket operator using the mean function. So, the bracket operator can be implemented using mean functions. So, I can use mean. So, if it is a infeasible region that g will be less than 0. Here g x is less than 0. So, therefore, I will take minimum of that; that means, on the right hand side g x is positive. So, here g x is positive ok. So, in this side it is greater than 0.

So, if it is positive then this mean function will give you 0 and if it is negative then mean function will give you the value of that g basically. So, I this is the purpose of this mean function. So, what does it mean? What we are doing here that we are trying to or we are with this actually transformation, we are penalizing the solution in the infeasible region because I do not want that my optimal solution is in the infeasible regions. Just to avoid the infeasible region. So, I am putting penalty ok penalty to that solutions ok.

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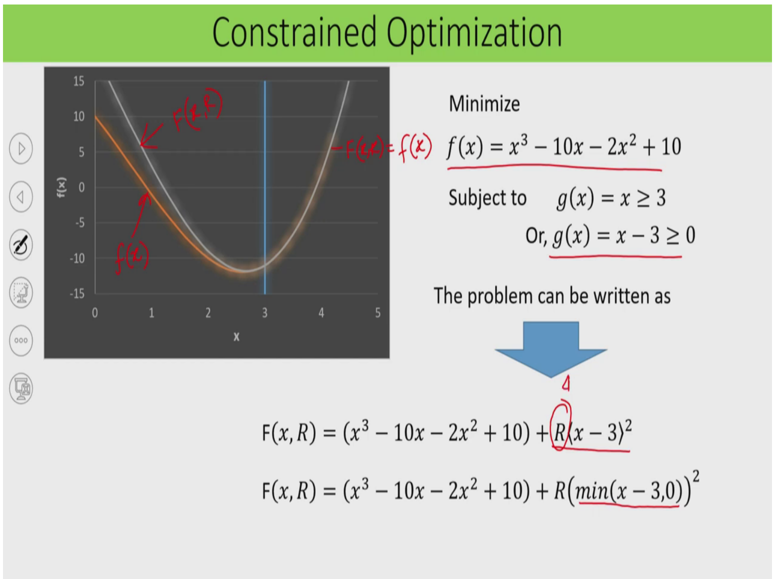

So, this bracket operator is giving penalty and basically now if I. So, if I plot it. So, what I am getting here? So, here in this case this is a again this is the; this is the function and I am writing this is g x is equal to x minus 3 and then I am just putting this bracket operator and I am multiplying with R I will explain what is R basically. So, R is nothing, but this is a penalty parameter ok. So, I will explain what is the significant of R basically.

Now, basically, so, this is I am using the bracket operator here and then this as I said that I can implement this bracket operator using mean function; that means, if it is a infeasible region that x minus 3 will be negative. So, mean function will give you negative. If it is on the feasible region then mean function will give you 0 and I am just squaring it up just to have a positive value and then I am adding to this particular function.

So, if I plot it that this is the this is your small f x ok. So, this is your small f x and this is basically the transform function and this is capital F x, R basically. So, you are getting this one. Now basically I will try to find out the minima of f R. So, there is no penalty on the feasible region. So, this is here this is f x R and is equal to your f x basically in this side.

But here the small f x and capital F x are is different ok. So, now, this is for a particular value of R ok. So, maybe I can start the R with 1 and then I can increase it.

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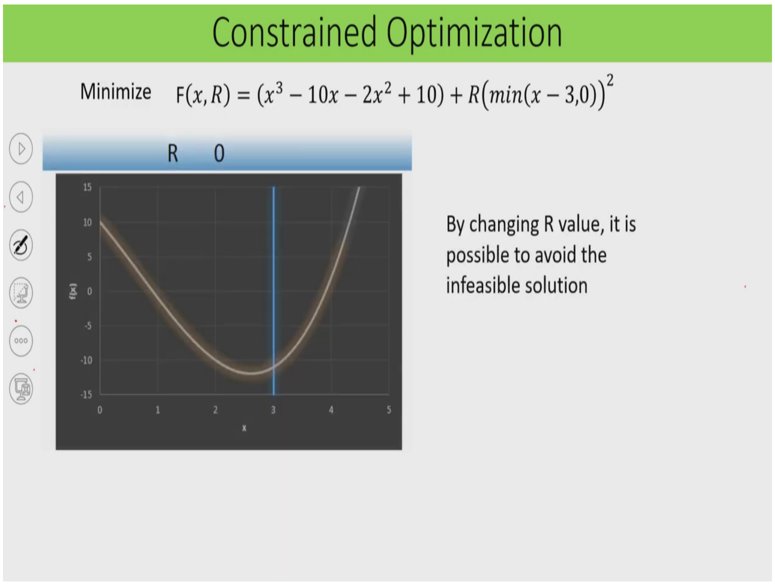

Let us see the effect of R in finding the constrained minimum solution of this problem.

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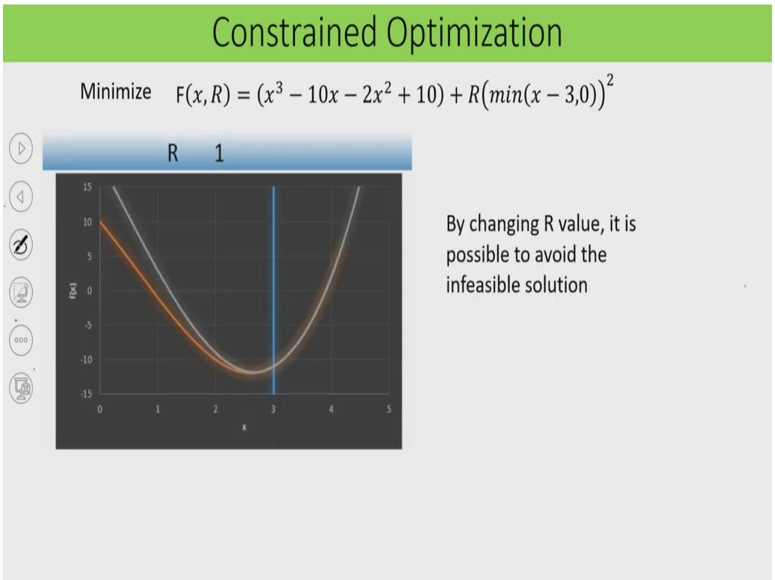

So, will change the R. So, this is the solution for R equal to 0.

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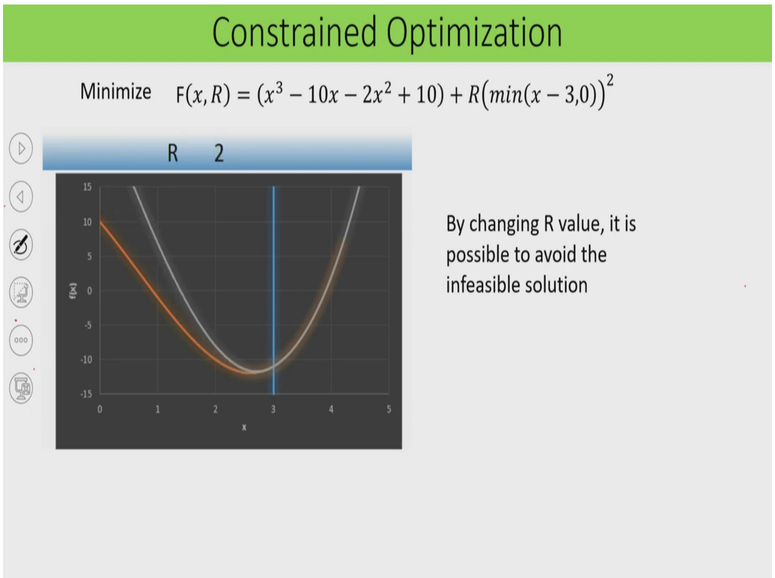

So, we will be getting somewhere here.

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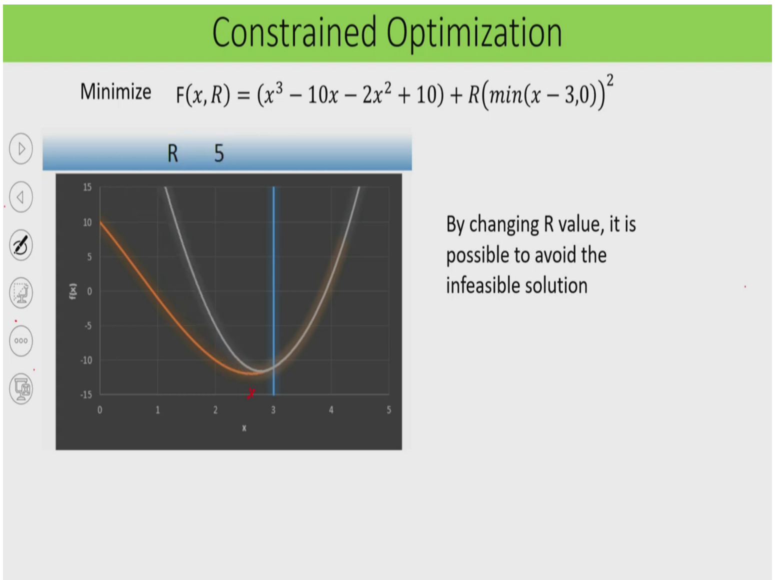

And then this is the solution for R equal to 2 then 5 something like that.

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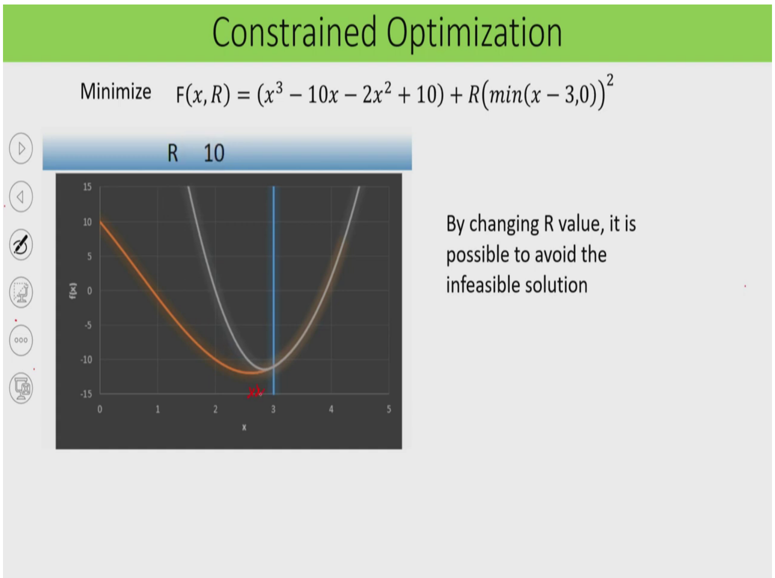

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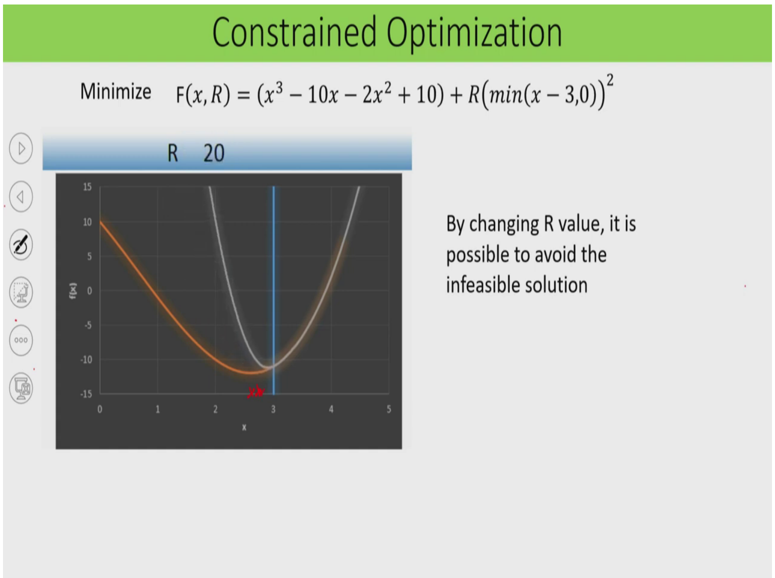

So, you just see once you are increasing the value of R.

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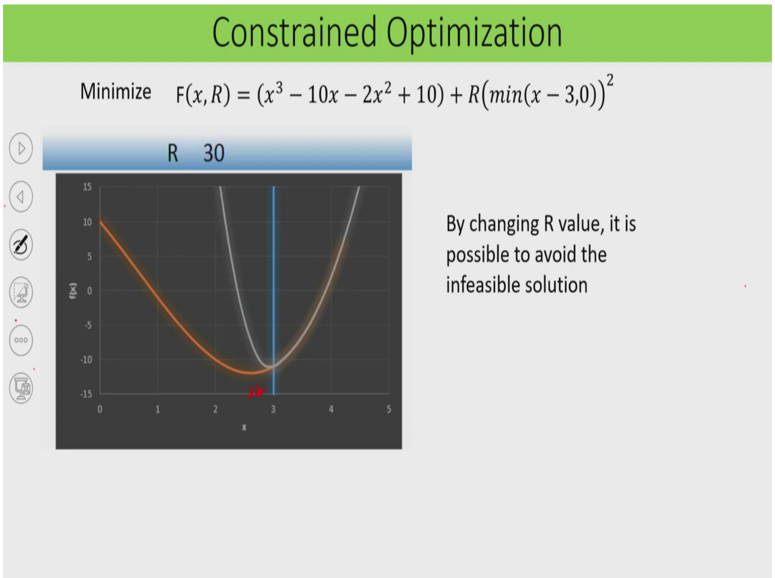

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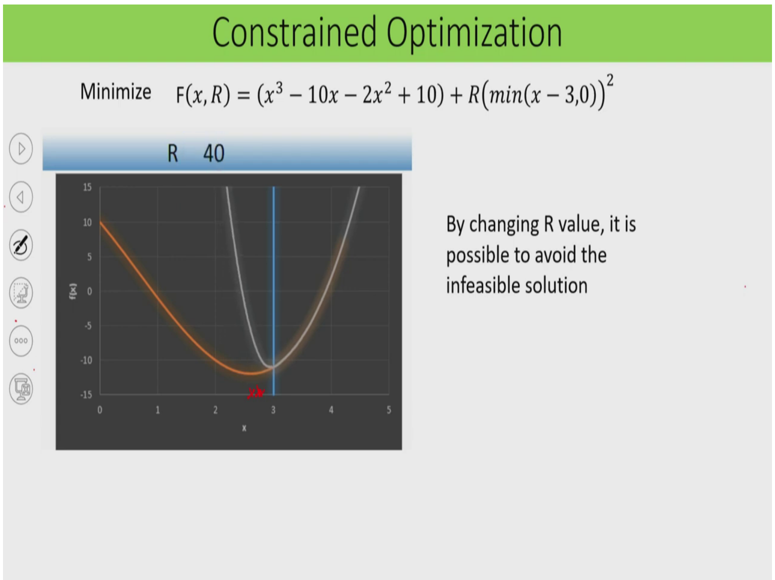

Then your solution is basically moving from infeasible side to the feasible constrained boundary.

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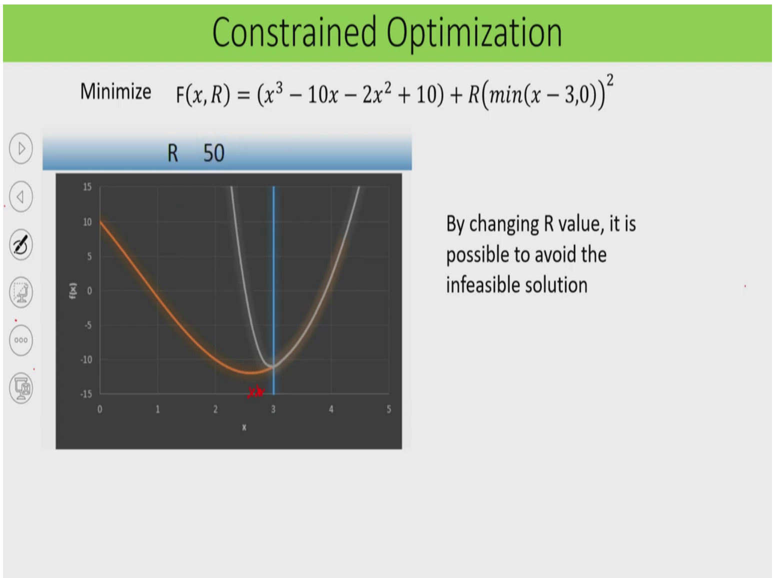

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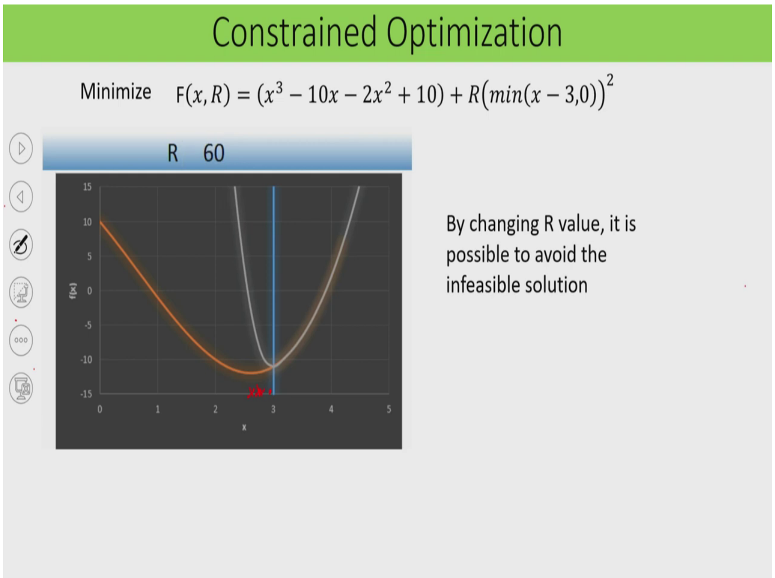

So, at R equal to 50, you are getting solutions somewhere here.

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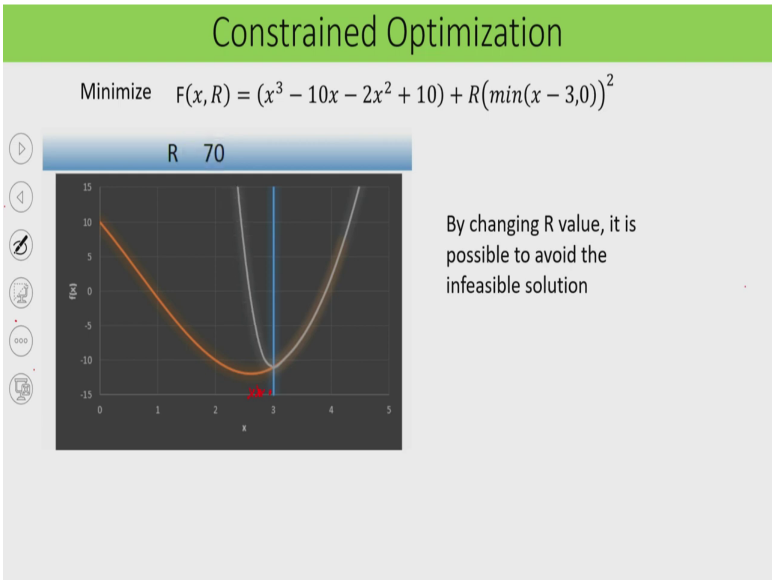

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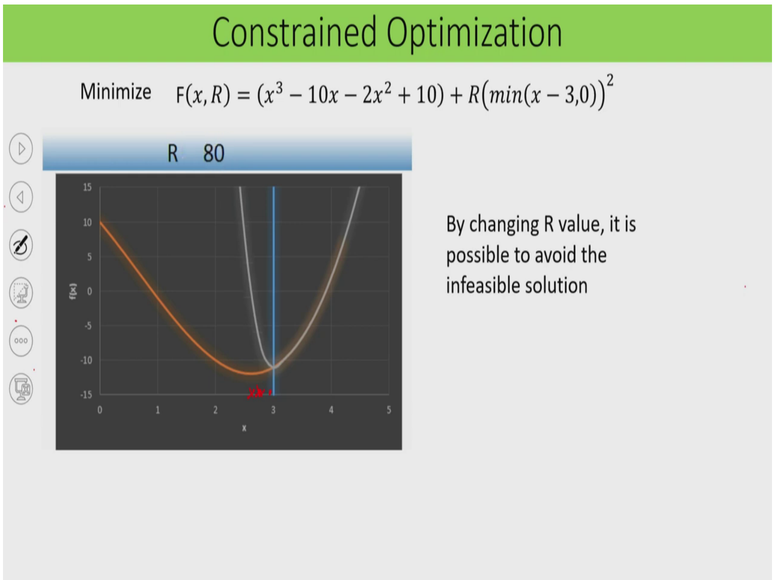

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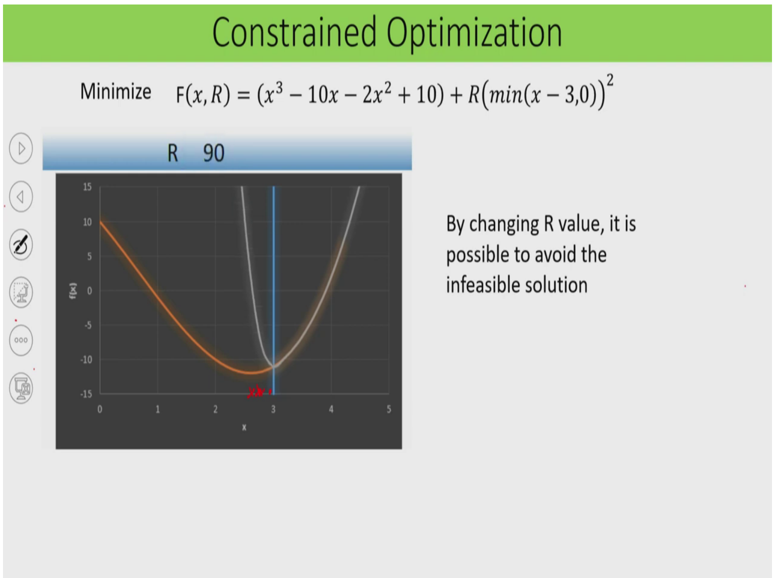

And for large value of R so, you will get the exact constrained optimum solution of these problem.

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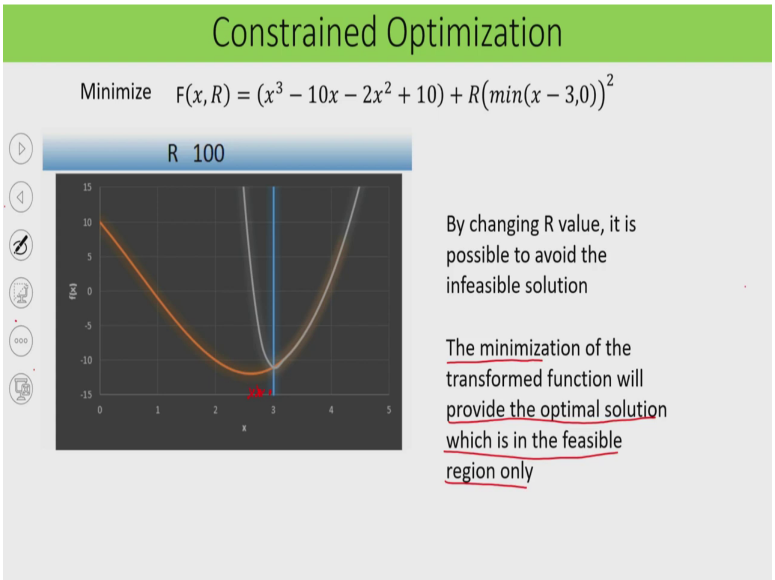

So, therefore, in this method. So, what you have to do? You have to use a very high value of R large value of R ok. But question is that; question is that, we cannot use a very large value initially. So, if you are using very large value initially, the gradient will be very steep and your gradient based algorithm will not work. So, in that case.

So, therefore, in this case what you have to do? We have to start with a smaller value of R and then we can increase the increase then we can increase the value of R in a step manner and for large value of R you will be getting the constrained minimum solution of this problem.

Now, another issue is here. So, if you look at the solution procedure. So, what we are doing? We are penalizing the solution in the infeasible region ok. So, we are not penalizing the solution and in feasible region, we are penalizing the solution in the infeasible region.

So, therefore, this method is known as exterior penalty method ok exterior penalty method. So, here we are penalizing the solution in the infeasible region and in this case of Forbes method. So, you have start with a smaller value of R may be 1, may be point 0.5 or may be 0.1 or something like that and then depending upon what type of problem you have, what is the constraint value you have.

And then you can increase the R value in a step manner and finally, you will get a solution where the further increase in R value will not give you the better solution. So, in that case you can stop your iteration. The minimum of the transformed function will provide optimal solution which is the which is in the feasible region. So, we are basically solving this transformed function and finally, for large value of R it will give you the feasible solution ok. So, it will give you the feasible solution.

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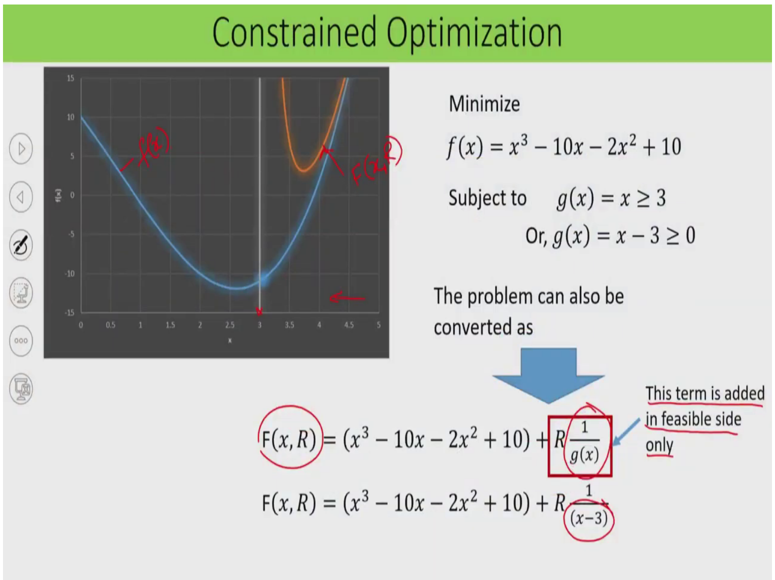

Now, let us look at another way of doing this thing. So, what we can do basically, in place of penalizing the solution in the infeasible region we can also penalize the solution in the feasible region ok. So, let us see this procedure. So, what I can do basically, in place of this your bracket operator I can use 1 by g x. So, what is 1 by g x? So, if you look at suppose if I look at this function ok.

So, 1 by g x. So, in this case 1 by g x means x minus 3 ok. So, what we are doing here? We are penalizing the solution in the feasible region. So, these terms is added in feasible side only. So, what is happening here? So, once you are away from optima. So, optimal solution is this, optimal solution is this. So, once you are away from optima so, in that case this value will be large then 1 by x minus 3 will be very small.

So, basically you are not penalizing the solution which is away from optima, but which is near to the optima. So, you are penalizing those solution near to optima what will happen? x minus 3 will be very small and 1 by x minus 3 will be very large. So, in that case you are penalizing the solution when you are approaching the optimal solution, but at x equal to 3 that 1 by 1 by g x will be infinity.

So, at x equal to 3, 1 by g x will be infinity and therefore, once you are starting your iteration from the feasible side and you are going towards the boundary. So, at boundary this capital F x, R value will be infinity. So, therefore, you will not be able to cross that boundary ok. So, this is acting as a barrier in this case. So, here I have shown that blue line is the original small f x function ok and then this orange line is your capital F x, R ok.

So, this is capital F x, R now if you look at. So, in this case what we are doing? We are starting with a large value of R and then we are decreasing the value of R. I will explain you in the next slide and you will see what is the effect of R in this particular procedure. Another issue is that. So, in this particular method we are penalizing the solution of the feasible region and therefore, this method is also known as interior penalty method.

So, earlier method whatever we have discussed that is an extra penalty method we have penalized the solution and the infeasible side and in this side we are we have penalized the solution in the feasible side. So, therefore, this is interior penalty method and the earlier one was exterior penalty method.

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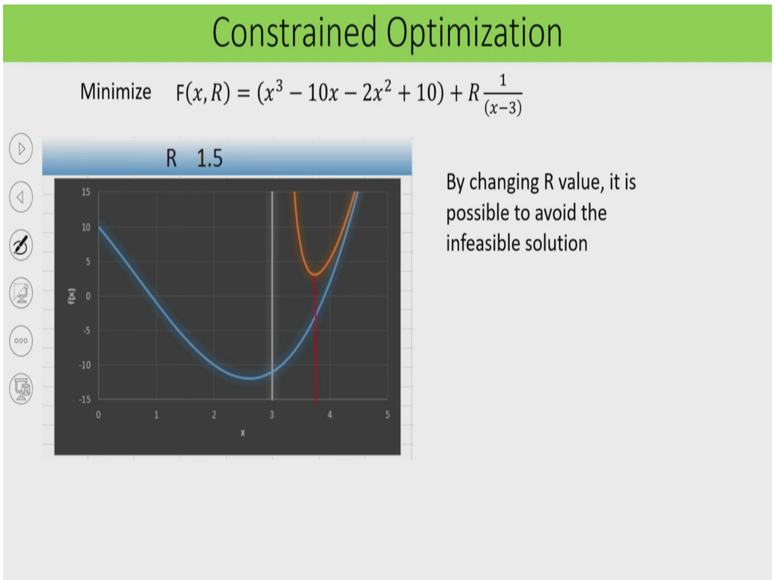

So, now let us see what is the effect of R here in this case. So, if you look at this. So, this is for R equal to 1.5. So, you are getting solution somewhere here.

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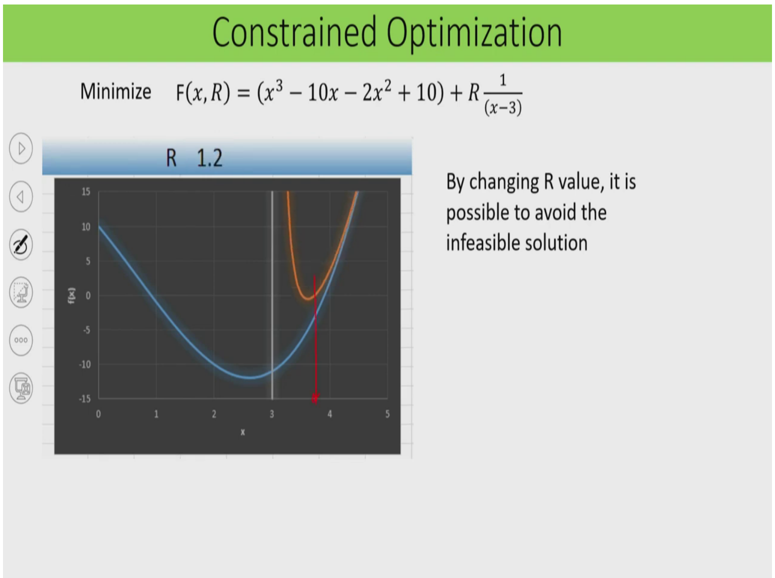

So, this is the solution somewhere here.

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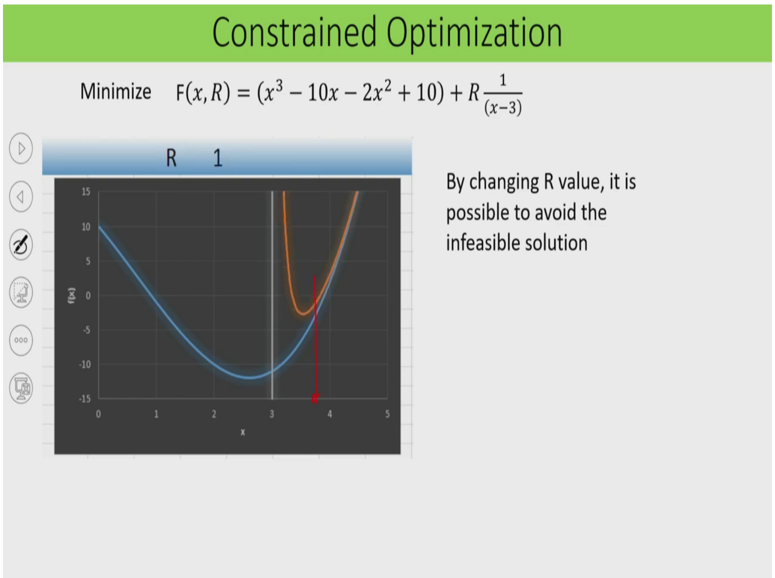

You are getting for R equal to 1.5, then 1.2 you just see the solution is moving suppose 1.2 you are getting here.

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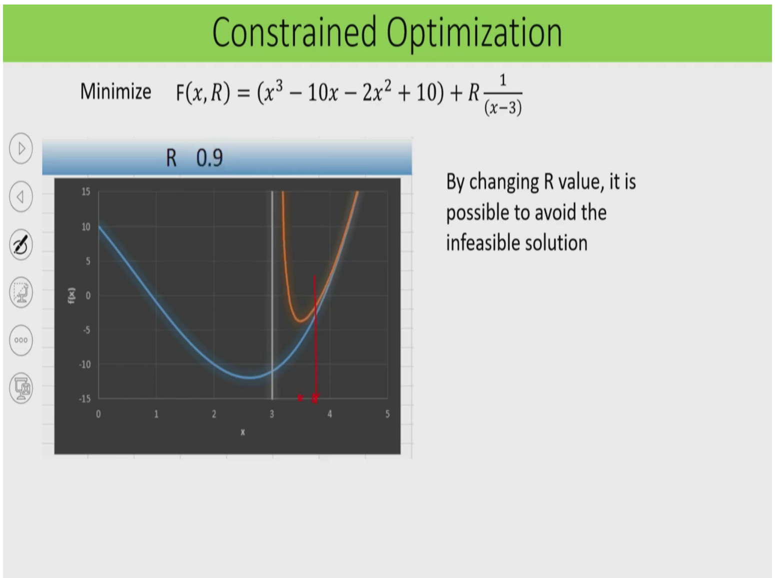

Then one you are getting here then this is for 0.9.

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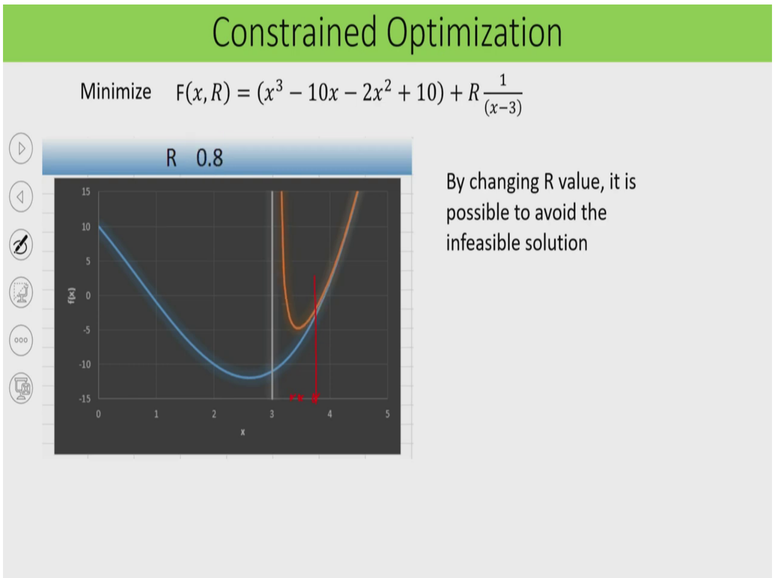

And for smaller value of R so, you will get the exact constrained optimal solution of this problem.

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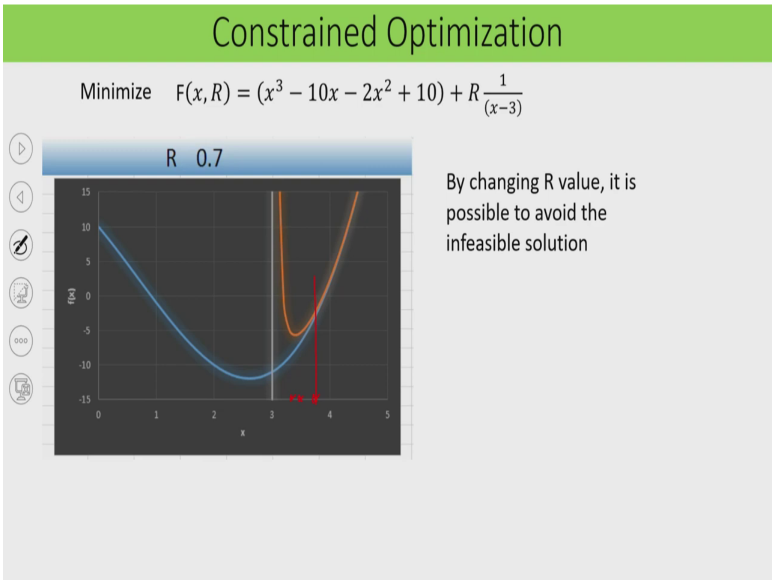

So, this is for 0.8 and then this is for 0.7, this is for 0.6 ok.

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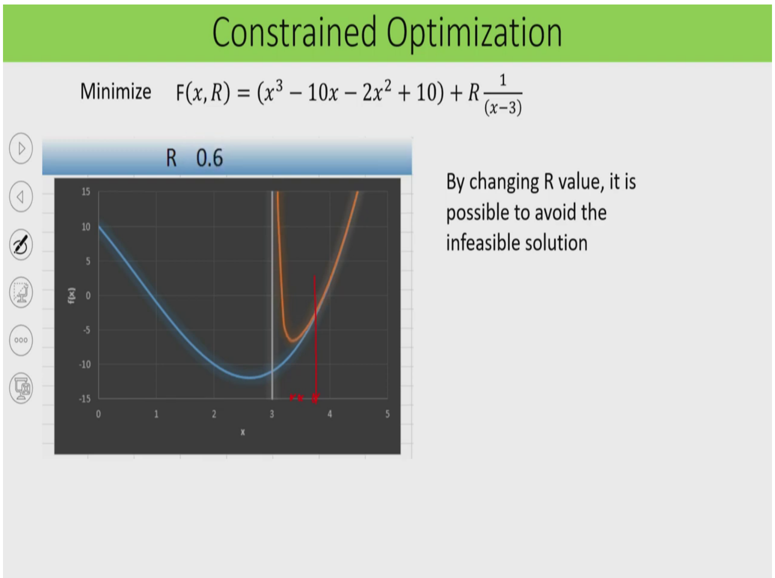

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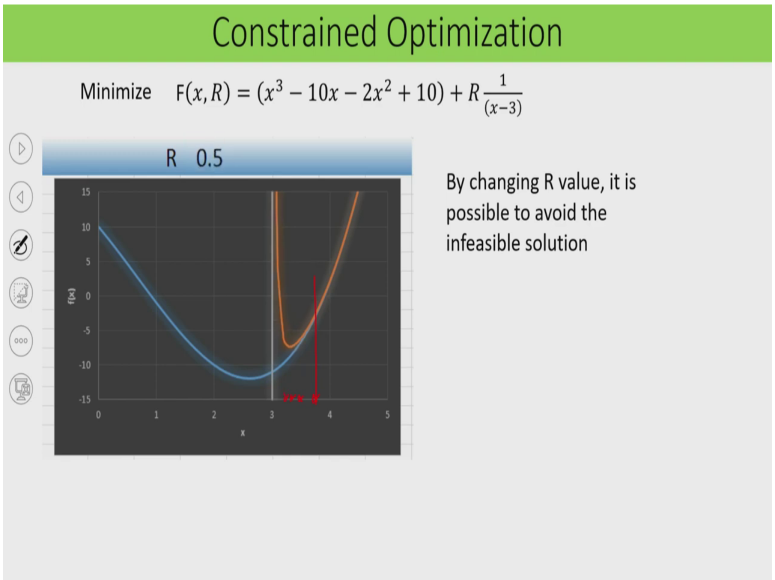

So, you are getting somewhere here, this is for 0.5 ok, this is for 0.4, this is for 0.3, this is for 0.2.

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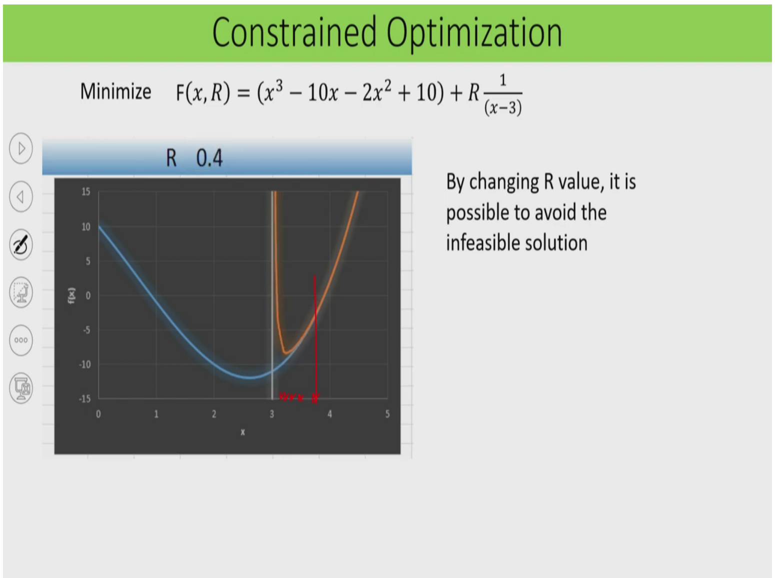

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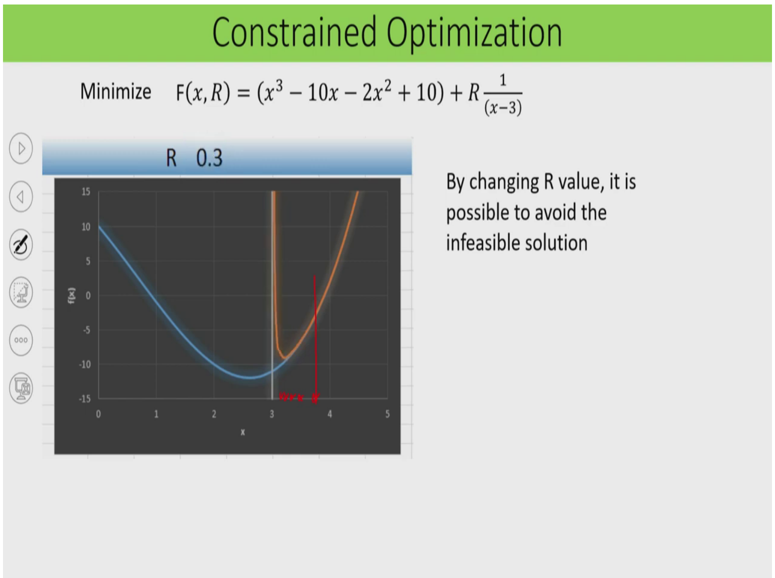

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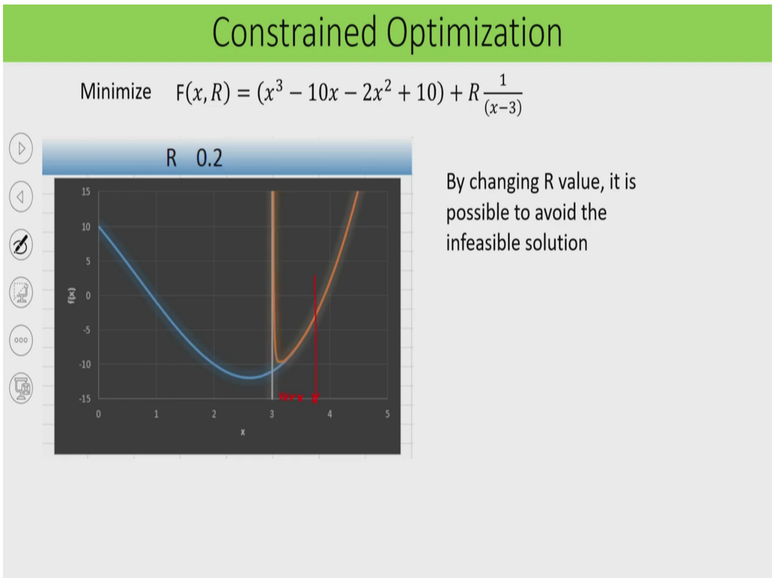

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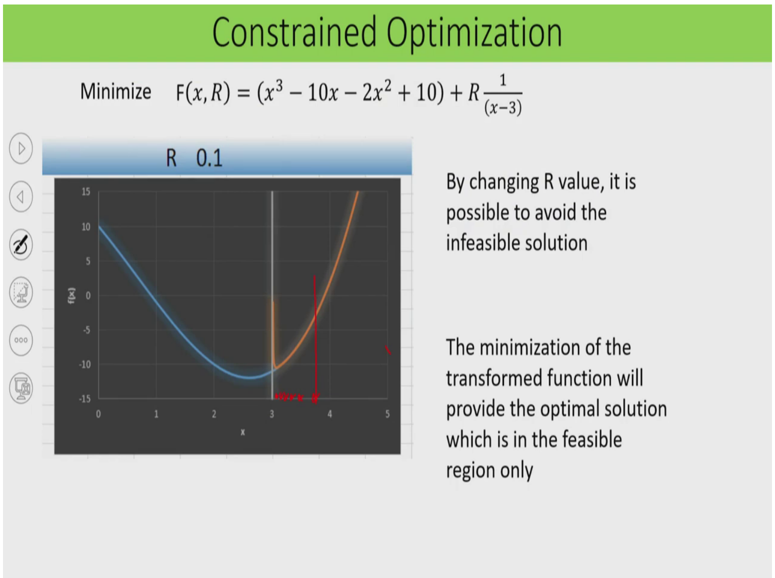

So, you are approaching the actual optimal solution at 0.1. So, you are getting near to that optimal solution may be I can further reduce and I can get the exact optimal solution. So, in this case what is happening that, you are starting with a little bit larger value of R and then you are reducing it for the value near to 0. So, you will get the optimal solution of this constrained minimization problem.

So, as I said. So, here what we are doing? We are we have penalized the solution in the feasible region and therefore, this method is known as interior penalty method. Now if we compare both the methods. So, whatever we have discuss earlier the bracket operator we have used and this particular method.

So, here the difference is in the bracket operator. So, initial that initial solution should not be a feasible solution. So, you can start from anywhere or you can take a initial solution and you can find out the optimal solution, but in this case what you have to do? You have to start your iteration from the feasible region ok.

So, initial solution must be a feasible solution; that means, from the feasible region you have to start. So, therefore, suppose this is a simple problem. So, I can find out where whether my initial solution is feasible or not, but when you are solving a problem with multiple constraint, then it is quite difficult to find out an initial solution.

So, therefore, you should have a different algorithm for finding the initial solution in order to apply this particular method. So, I can have an algorithm. So, there are some algorithm available, this algorithm can be used just to get a initial feasible solution and once you are getting an initial feasible solution you can apply the this procedure for finding the constraint optimal solution ok.

So, this is the I have discuss what is the your impact of R or in finding the optimal solution in case of exterior penalty method. So, you are starting with smaller value of R and then you are increasing and in this case you are starting with a little bit larger value of R and then you are reducing it you are reducing the value of R.

Now, again as I have explained that you cannot start initially with a big value it is a larger value of R in case of exterior penalty method because the gradient will be very very steep and your gradient based algorithm will not work. Similarly, for interior penalty method you cannot your start your means iteration with a very small value of R ok.

So, you have to use a very little bit large value of R otherwise again this method gradient based algorithm will not work ok and then with step only you have to reduce either you have to increase the value of R or you have to reduce the value of R.

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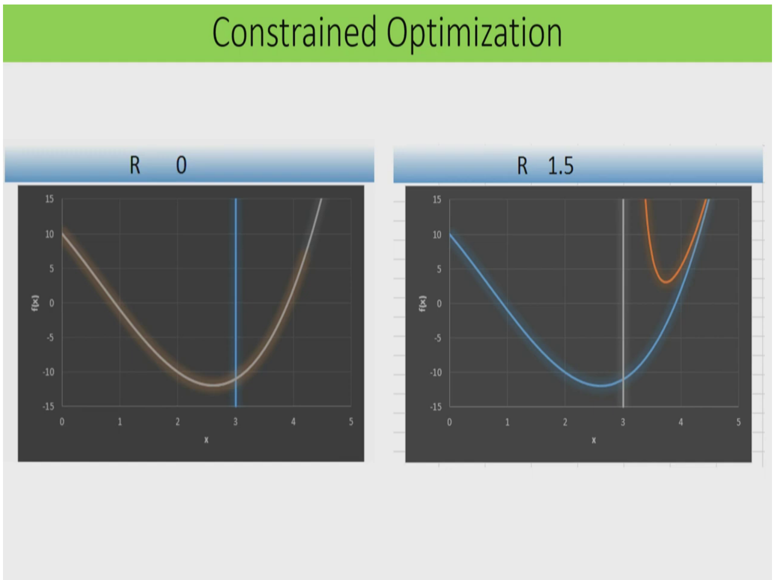

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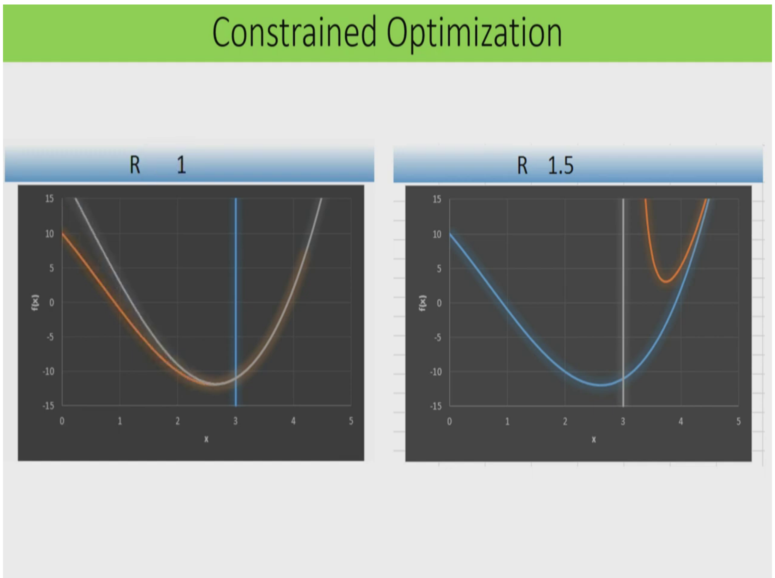

So, now I have just shown you that how this is for the interior penalty method.

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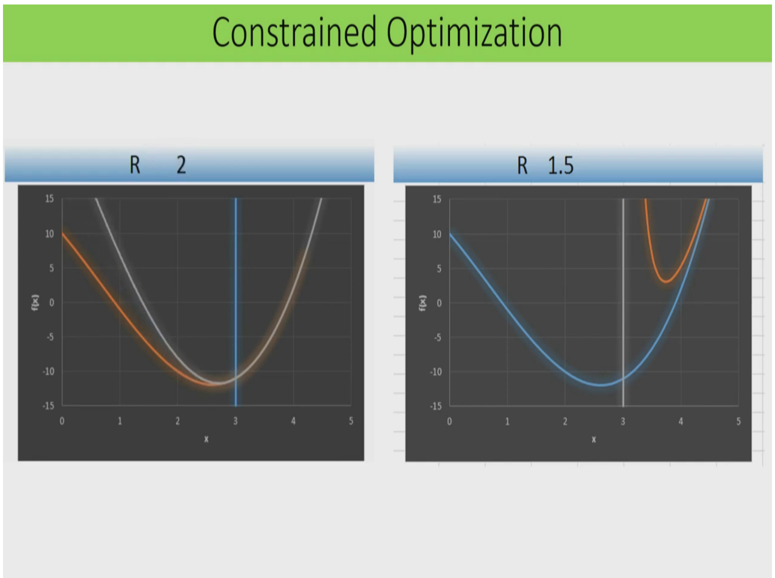

That the left hand side figure is for exterior penalty method.

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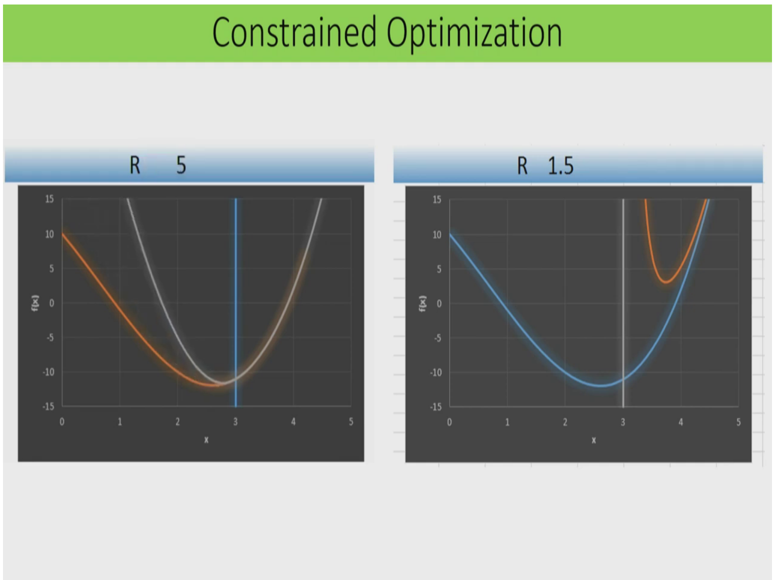

And you can see the affect of R.

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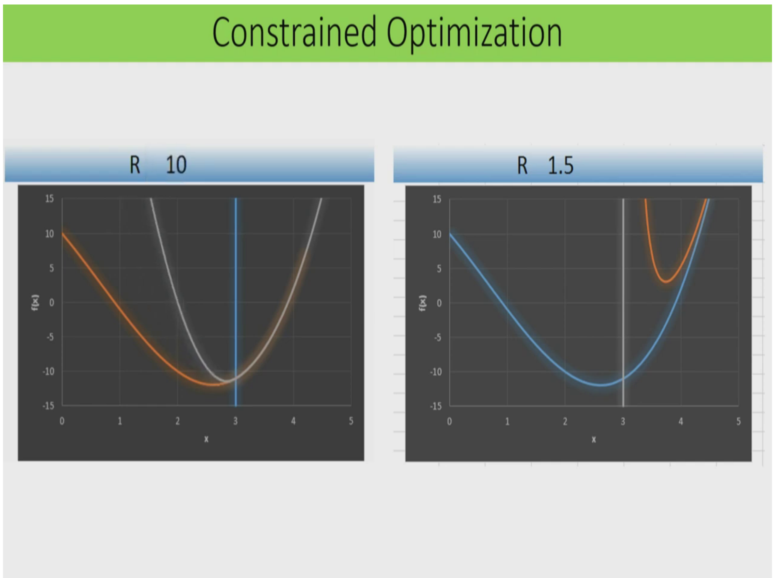

So, once you are increasing the value of R.

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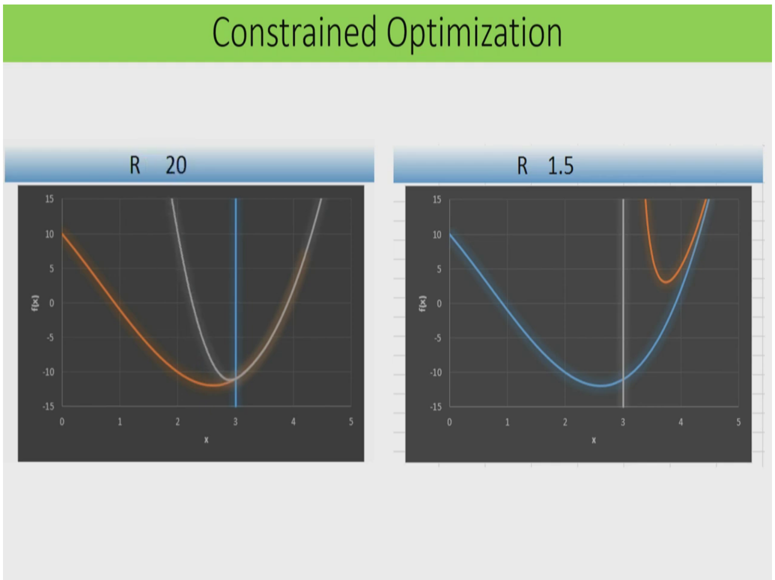

So, you are moving towards the actual constrained solution of these particular problem.

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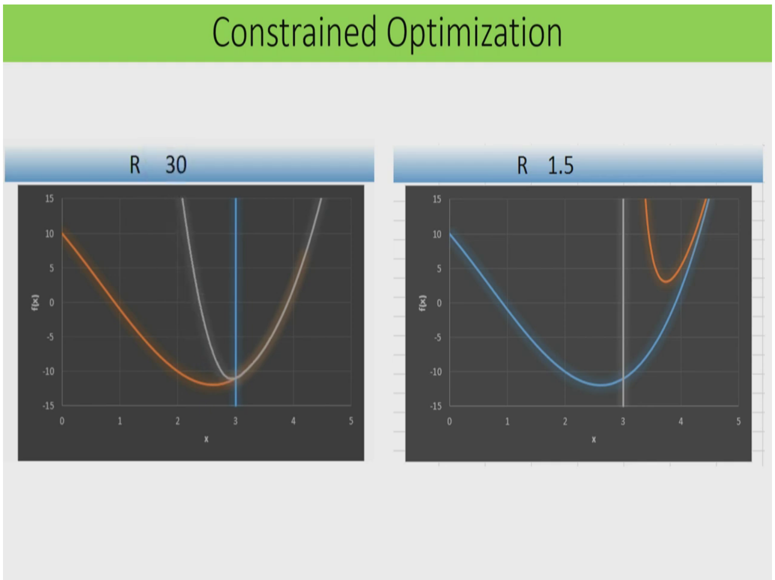

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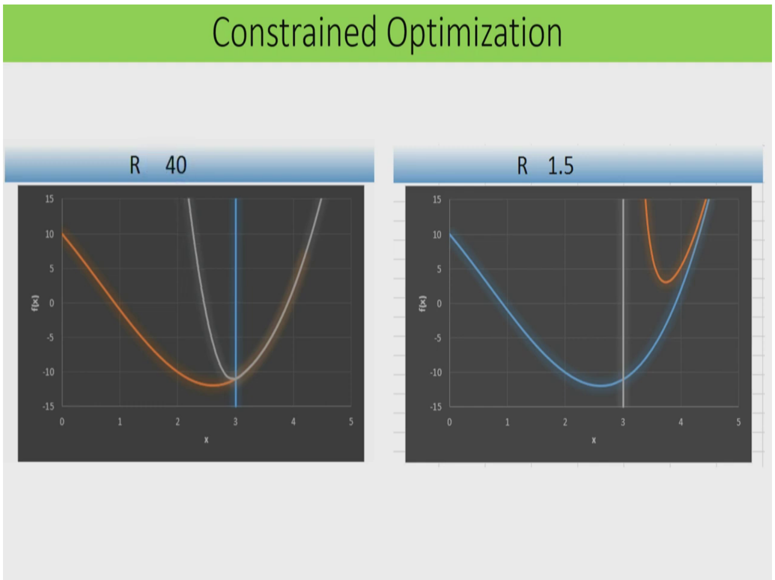

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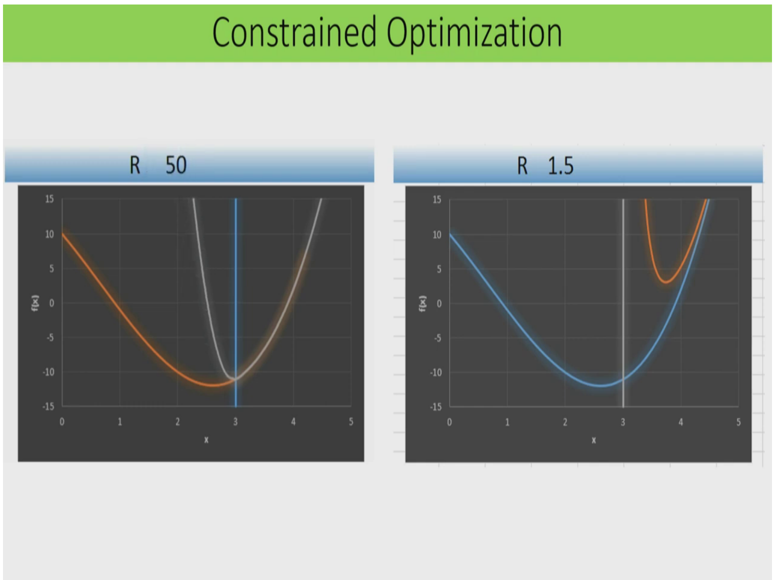

So, this is again. So, this is we and in this case we have penalized the solution in the infeasible region.

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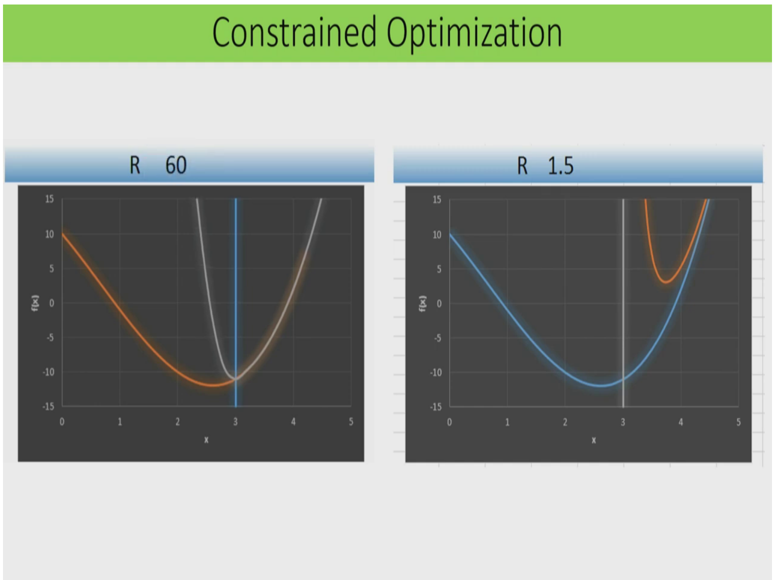

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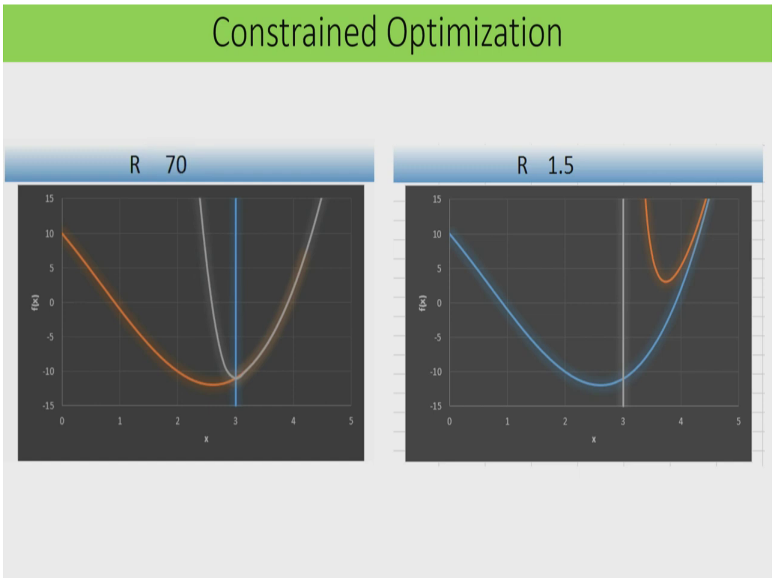

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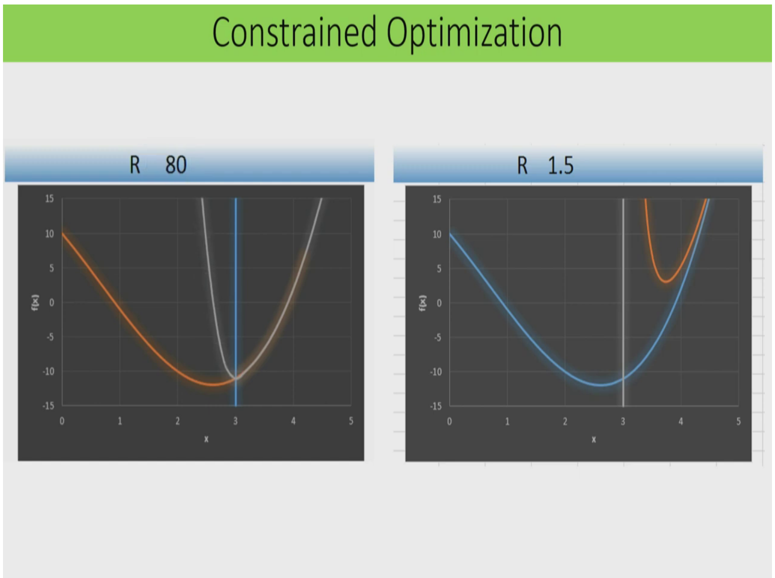

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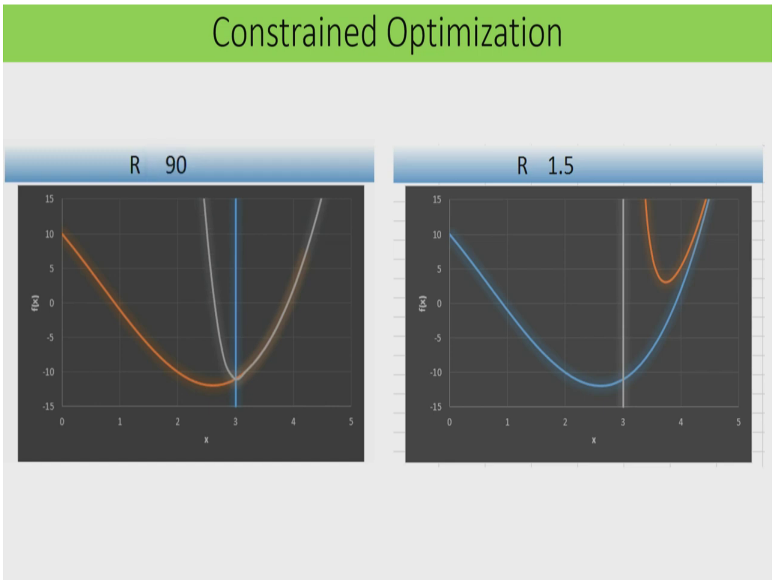

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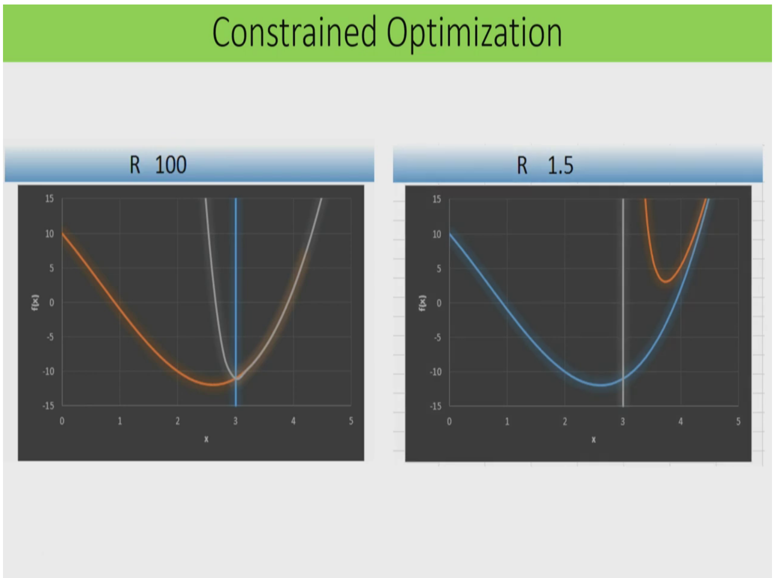

In case of interior penalty method.

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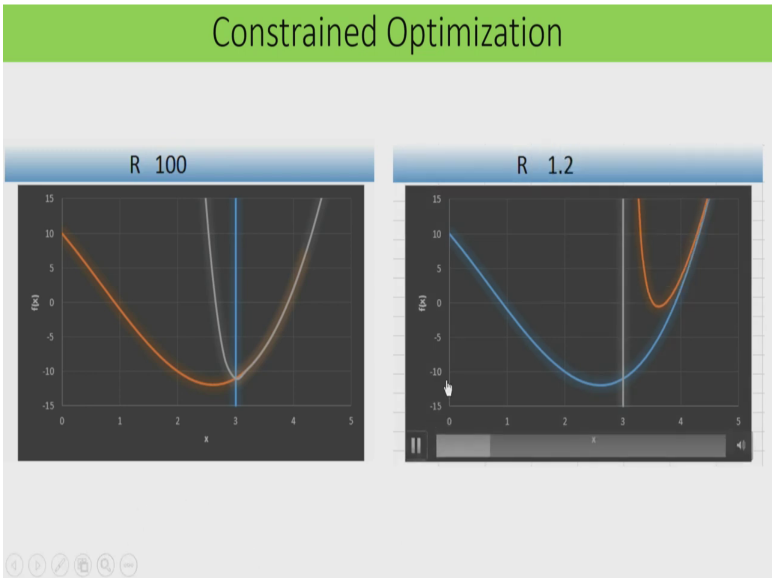

So, in case of interior penalty method.

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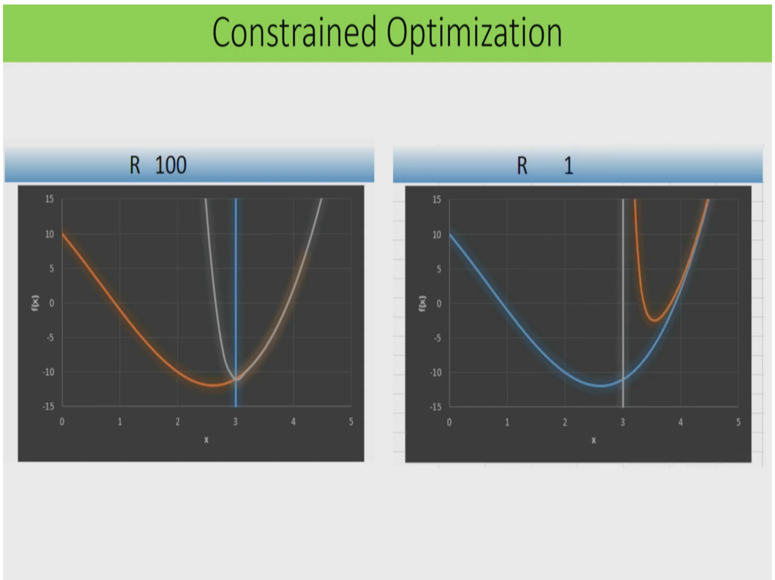

So, you are starting with a relatively higher value of R.

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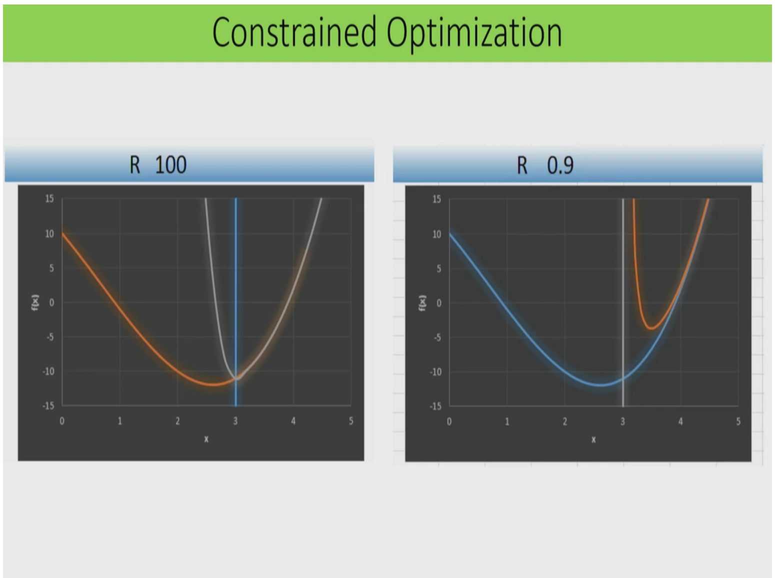

And then you are reducing the value of R and when R is near to 0 for smaller value of R near to 0.

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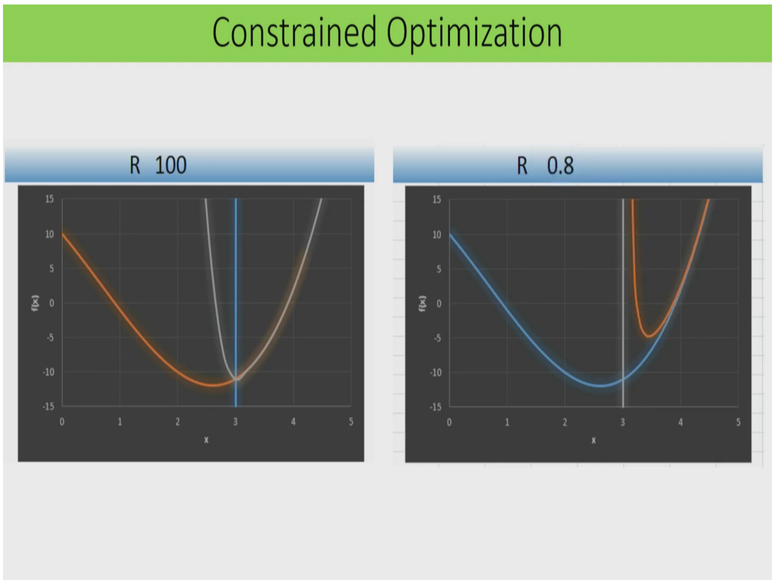

So, you will get the exact constrained optimal solution of this problem ok.

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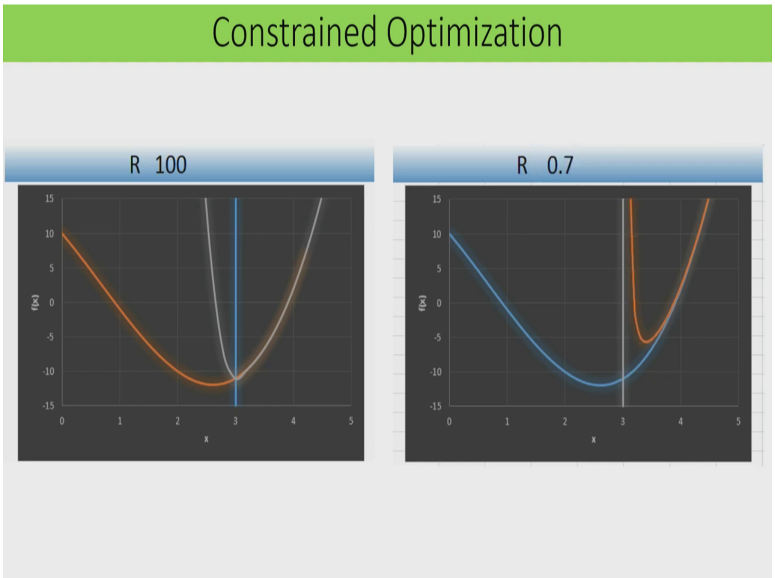

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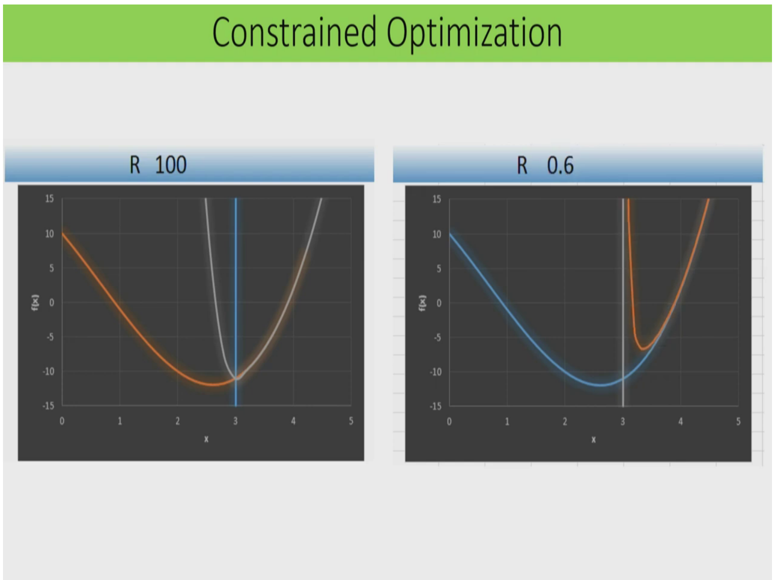

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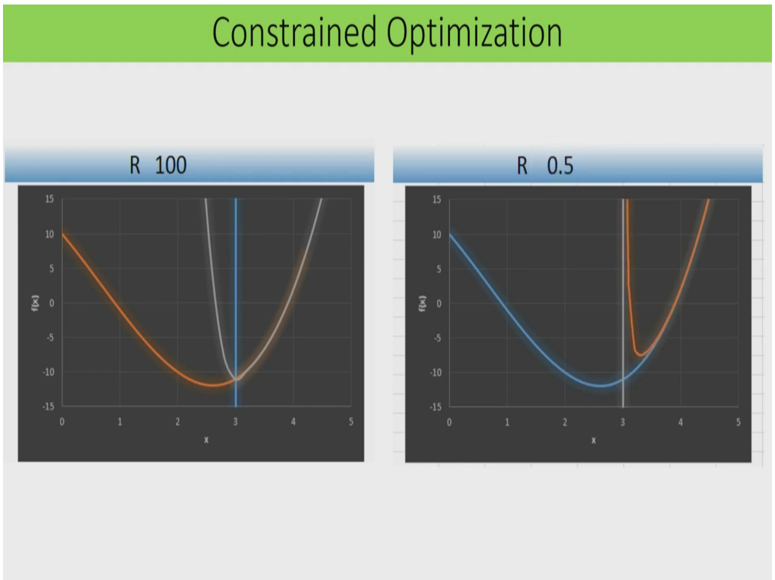

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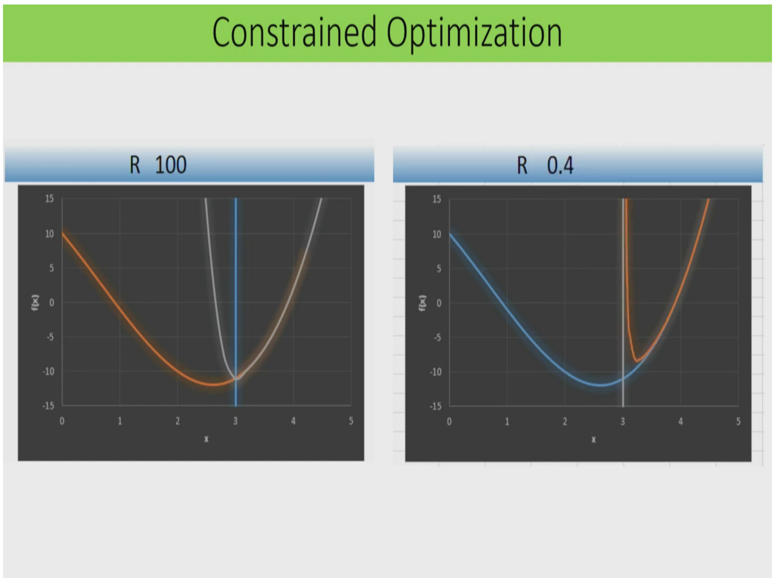

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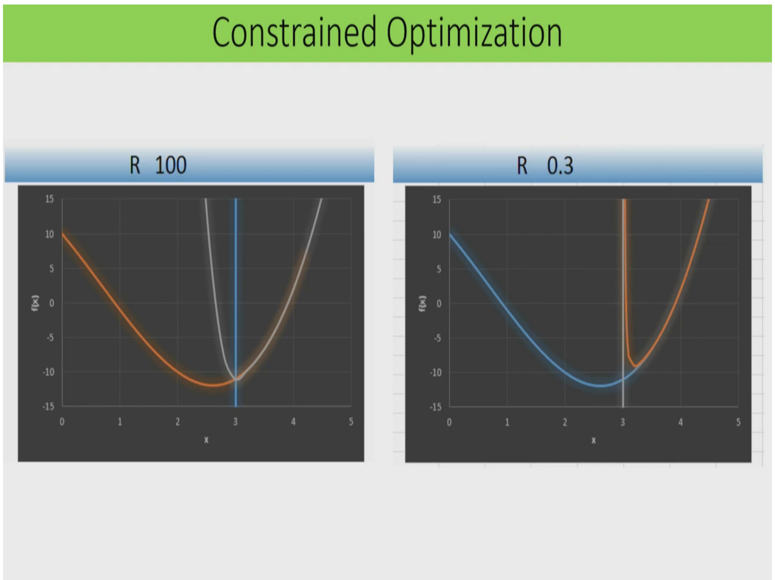

So, you are getting for R equal to; R equal to 0.1.

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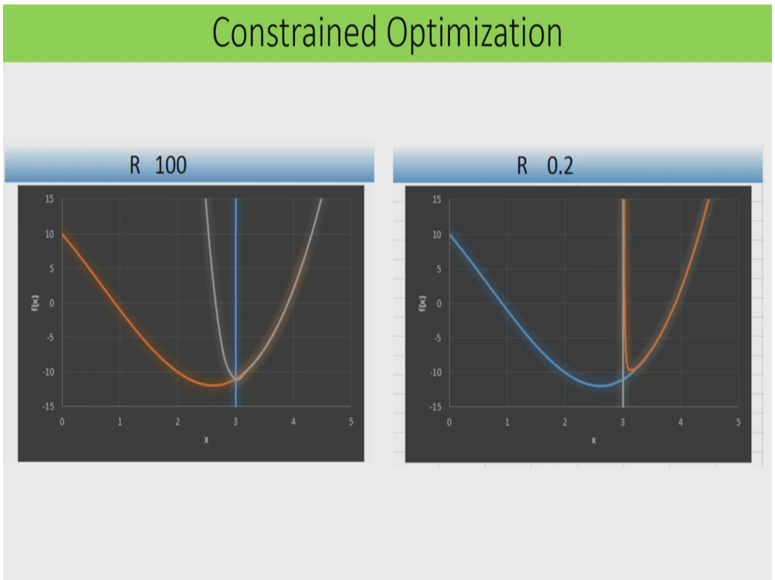

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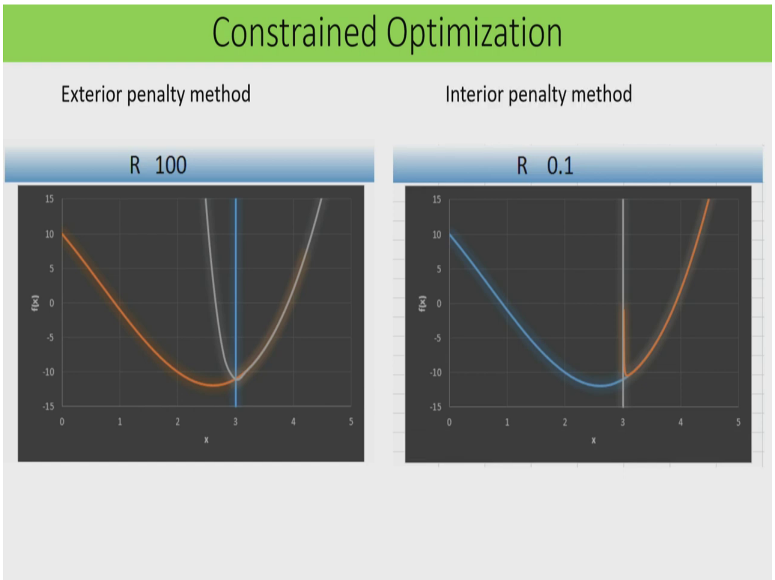

So, you will get the or 0.1 or may be 0.001. So, you will get the exact optimal solution of this problem. So, this is exterior penalty method and then that is interior penalty method.

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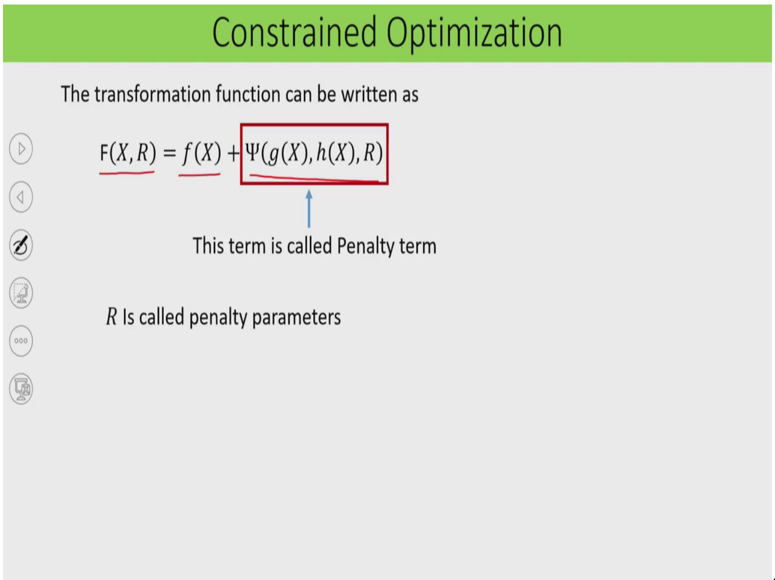

So, the transformation function can be written as F X, R which is equal to f X this is the objective function plus the penalty term. So, penalty term is a function of g X h X. So, if you have inequality constrained that is your g X and h X is equality constraint and R is the penalty parameter ok.

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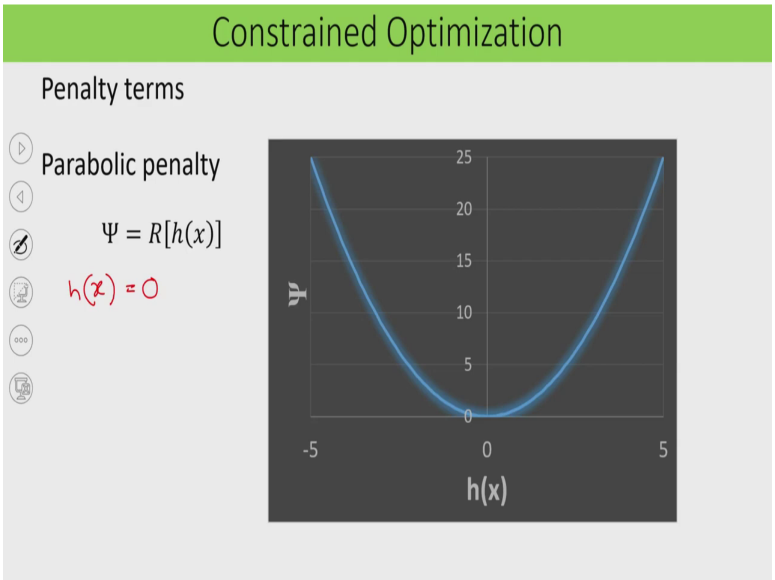

So, this term is called penalty term and R is called penalty parameters. Apart from the function two function we have discussed, we can also use some other function. So, one is parabolic function. So, this function is use in case of h x; that means the equality constraint ok h x equal to 0.

So, now in this case so, we are using parabolic function and this function will give you the violation basically. So, in this case also we are penalizing the infeasible solution and therefore, this method is exterior penalty method.

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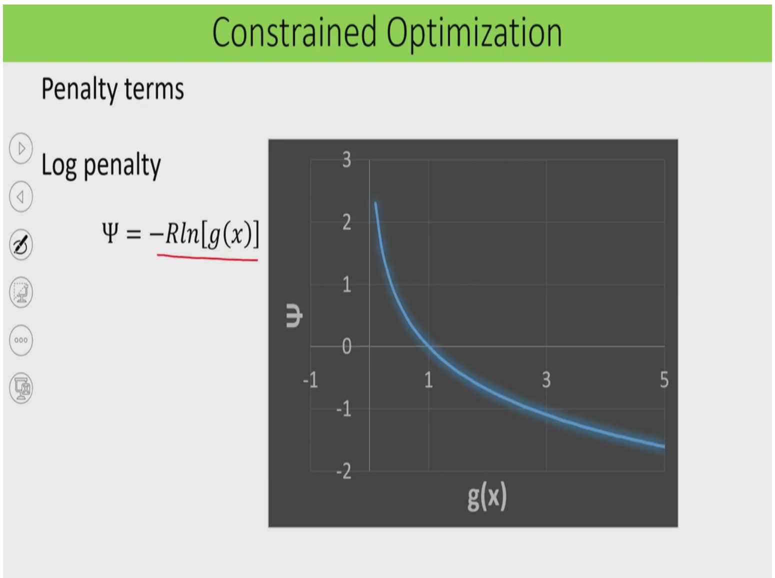

So, I can use this parabolic function and similarly I can use log penalty function. So, here this is acting just like a barrier. Once you are moving towards the constrained boundary this value will be very large ok the penalty term value will be very large and this will act as a barrier ok.

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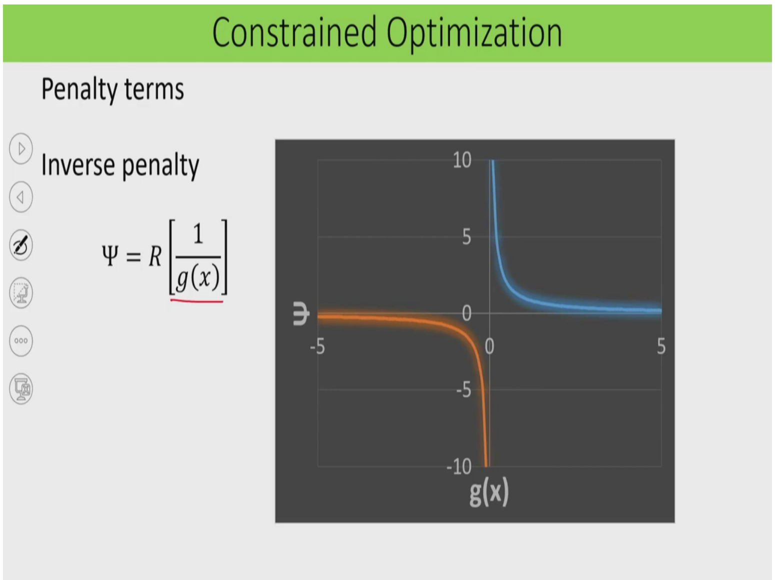

So, I can also use lock penalty term and inverse penalty already I have discuss. So, here also we are penalizing the solution in the feasible side ok. So, this is also acting as a barrier and here this is your interior penalty method.

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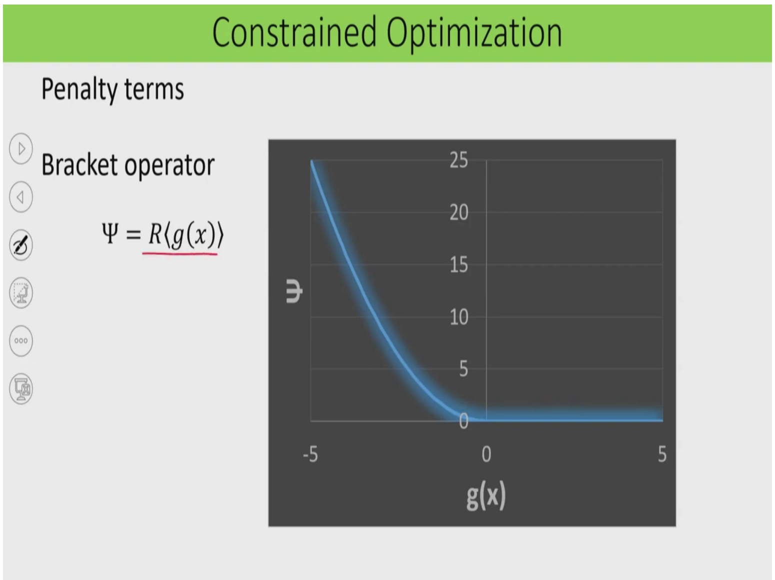

And similarly bracket operator is already explained. So, we are penalizing here we are penalizing the infeasible solution and therefore, this is exterior penalty method.

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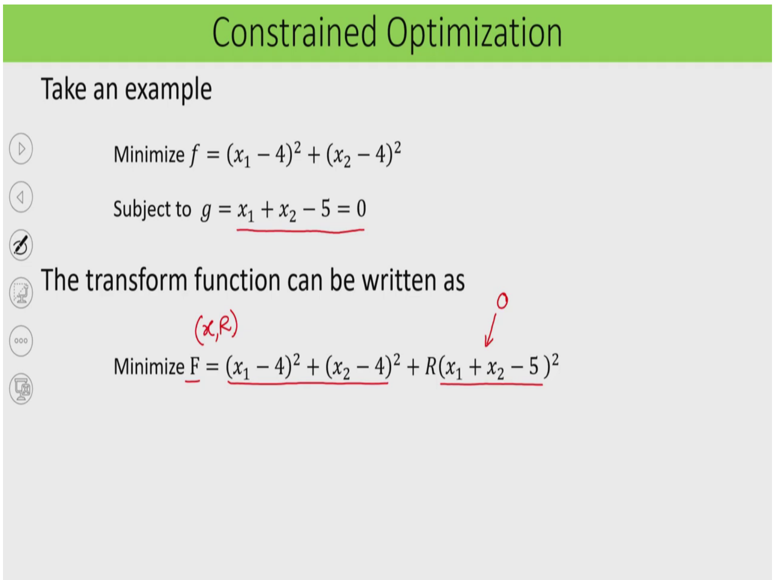

Now, I will discuss one example problem. So, let us take this example problem that is minimize f equal to x 1 minus 4 whole square plus x 2 minus 4 whole square and subject to g equal to x 1 plus x 2 minus 5 equal to 0. So, here this is a equality type problem the constraint is equality type. So, I will be using parabolic function. So, I have used I have written this function that capital F which is a function of x and R. So, this is a function of x and R. So, this is the objective function and this is your constraint.

So, parabolic function I have used R into this is x 1 plus x 2 minus 5 whole square this is a equality type constraint. So, these value should be equal to 0. So, this part is equal to 0 and if it is 0 if this part is 0; that means, there is no penalty or otherwise if it is not 0; that means, you are putting penalty to those solution. So, I can minimize this function using any algorithm.

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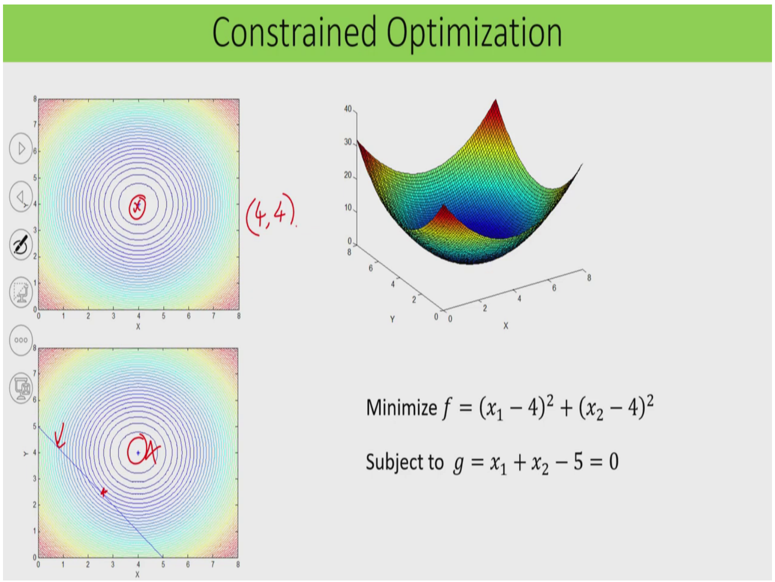

So, I have shown the solution here. So, this is the; this is the solution 4 and 4. So, this is the solution 4 and 4; 4 and 4. So, this is the solution of the objective function and constrained optimization problem. So, here this is the constrained. So, this is the constrained and therefore, the solution should be on this particular line.

So, constrained solution will be somewhere here and this solution is in the infeasible size and therefore, that is not acceptable ok. So, that is not acceptable the solution should be on this particular line, this is an equality type constraint. So, let us start the iteration and as I said; as I said. So, what method we have used? We have used the exterior penalty method. So, therefore, we have to start our algorithm or start our iteration from the smaller value of R and then we are increasing.

So, we cannot start our algorithm with a very large value of R and then decreasing. So, that is in this case you have to start your iteration with a smaller value of R and then you increase it.

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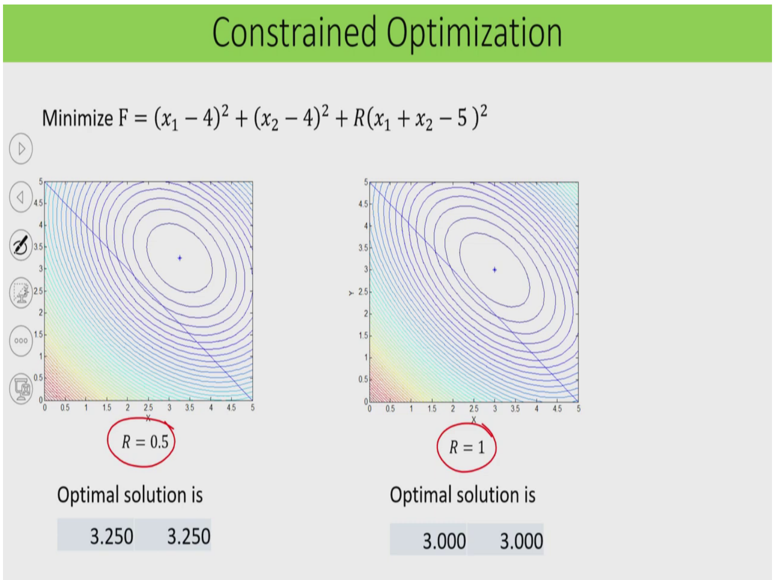

So, this is for R equal to 0.5. So, I have started the algorithm with R equal to 0.5 and then solution is 3.250 and 3.250 and so. So, somewhere here we are getting that one 3.25 and 3.25 and this is for R equal to 1 and this solution is 3 and 3.

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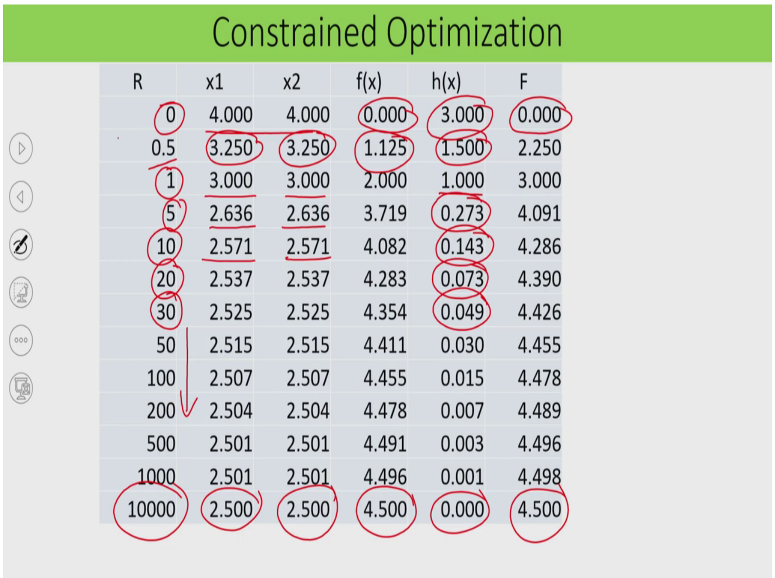

R	x1	x2	f(x)	h(x)	F
0	4.000	4.000	0.000	3.000	0.000
0.5	3.250	3.250	1.125	1.500	2.250
1	3.000	3.000	2.000	1.000	3.000
5	2.636	2.636	3.719	0.273	4.091
10	2.571	2.571	4.082	0.143	4.286
20	2.537	2.537	4.283	0.073	4.390
30	2.525	2.525	4.354	0.049	4.426
50	2.515	2.515	4.411	0.030	4.455
100	2.507	2.507	4.455	0.015	4.478
200	2.504	2.504	4.478	0.007	4.489
500	2.501	2.501	4.491	0.003	4.496
1000	2.501	2.501	4.496	0.001	4.498
10000	2.500	2.500	4.500	0.000	4.500

So, let us see. So, if I put R equal to 0; that means, I am not putting the constrained violation. So, therefore, whatever solution you are getting that is an R constrained that is the solution of the objective function ok without constrained and that solution is 4 and 4 and this is the function value is 0 and constrained violation is 3 and capital F is 0.

So, what is your idea? You have to reduce the constrained violation and basically the constrained violation should be 0. So, now, you increase your R. So, R equal to 0.5 and this is the solution 3.25 and 3.25 and f x is 1.125 and constrained violation is 1.5. So, then increase the R value to 1 and then you are getting solution 3 and 3 constrained violation is 1. For R equal to 5 you are getting 2.636 2.636 the constrained violation is 0.273.

Similarly, for R equal to 10 you are getting solution 2.571, 2.571 and the violation is 0.143. For 20 you are getting the violation is 0.073, then I am finding for 30 this is 0.049 and then if you continue your iteration for 10,000. So, R equal to 10,000 and constrained violation is 0 almost 0.

And the solution you are getting is 2.5 and 2.5 and objective function value is 4.5 and capital F is also 4.5 ok. So, here what we have done? We have started our iteration for a smaller value of R and then we are increasing it and at almost R equal to 10,000. So, we are getting the solution 2.5 and 2.5 and that is the constrained solution of this problem violation is 0 and objective function value is 4.5. So, I have shown the plot here.

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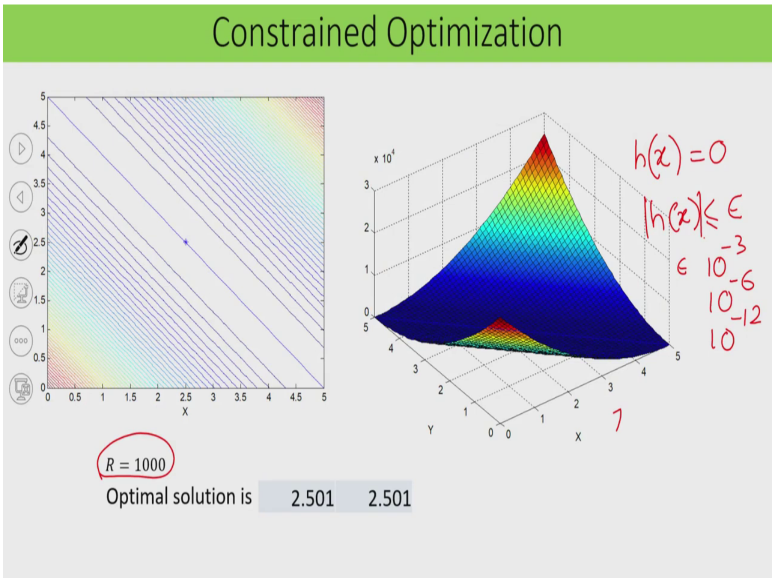

So, you can see this is for R equal to 1000 and this is the solution somewhere here 2.5 and 2.5 and this is the surface plot I have shown, but if you start with start or if you start your iteration with R equal to 1000 you may not get the solution ok. Because as I said so, for larger value of R the gradient will be very steep and your gradient based algorithm will not work. So, therefore, you have to start with a smaller value of R and then you increase it in case of exterior penalty method ok.

So, this is all about the constrained optimization problem. So, we have means we have discuss the Lagrange multiplier method that is used for solving constrained optimization problem with equality constrained then we have discuss Kuhn Tucker condition and then we have discuss transformation method.

These transformation method can be used for any types of constraint whether it is a equality type constraint or whether it is a inequality type constraint. So, you can apply this constraint method. So, as I have shown here. So, this problem was a constrained optimization problem with equality type constraint, but I can solve any problem equality or inequality type problem.

But what we should do basically, if you have a equality constraint it is very difficult because your solution has to be on that particular constraint. So, sometime you may not get the solution like that there may be little bit of violation. So, therefore, may time what we do, the equality constraint we convert it to an inequality constraint with some admissible error. So, what we can do basically? Suppose if you have a equality constraint something like that h x is equal to 0 ok.

So, what I can do basically that, h x I can write that not 0 h x is less than absolute value of h x is less than some epsilon. So, I can define that one. So, I am converting the equality constraint to an inequality constraint and basically. So, this absolute value we can define suppose this is 10 to the power minus 3 or 10 to the power minus 6 or 10 to the power minus 12.

So, depending upon the problem requirement. So, I can take the that value of this epsilon. So, suppose you want to have a very precise solution. So, you can increase this value. So, depending upon the permissible error or error you can permit. So, you can define this epsilon value, but the idea is that you are converting your equality type constraint to an inequality type constraint.

So, that also generally we do. So, what we do basically? So, we covert the equality type constraint to an equality inequality type constraint and then basically you we solve the problem. So, let us stop here. So, this is all about the constrained optimization problem.

Thank you.