Optimization Methods for Civil Engineering
Prof. Rajib Kumar Bhattacharjya
Department of Civil Engineering
Indian Institute of Technology, Guwahati

Lecture - 13
Constrained Optimization 1: Equality Constraints

Hello student. So, today, we will discuss Constraint Optimization problem. So, in the previous classes so we have discuss the unconstrained problems. So, we have discussed the necessary and sufficient condition for optimality of unconstrained optimization problem.

So, as we have discuss that the necessary condition for optimality of an unconstrained problem is basically you are taking the derivative and that the first derivative should be equal to 0 that is the necessary condition. And for the sufficient condition so, we are calculating the higher order derivatives.

In case of single variable, we have calculated the higher order derivative suppose second order, third order, fourth order, higher order non-zero derivative. So, depending upon odd or even so, if it is odd, then it in that case that is an inflection point, but if it is even so, we will see the value of the derivative, higher order derivative.

So, if it is positive so, then the solution whatever we have obtained that is a minima and if it is your negative so, in that case the solution is the point is your basically a maximum point.

For multivariable problems so, we are getting Hessian matrix that means, the second derivative will be an Hessian matrix, then we will basically evaluate whether the Hessian matrix is a positive definite matrix or negative definite matrix or neither positive nor negative.

If it is positive definite, then it is a minimum point and if it is a negative definite, then that is a maximum point. If it is neither positive nor negative so, in that case, the solution whatever we have got and that is neither a minima nor maxima.

So, we have already discuss these thing and now, today, we will discuss about Constrained Optimization problems. So, what is the sufficient necessary and sufficient condition for optimality of constrained optimization problem.

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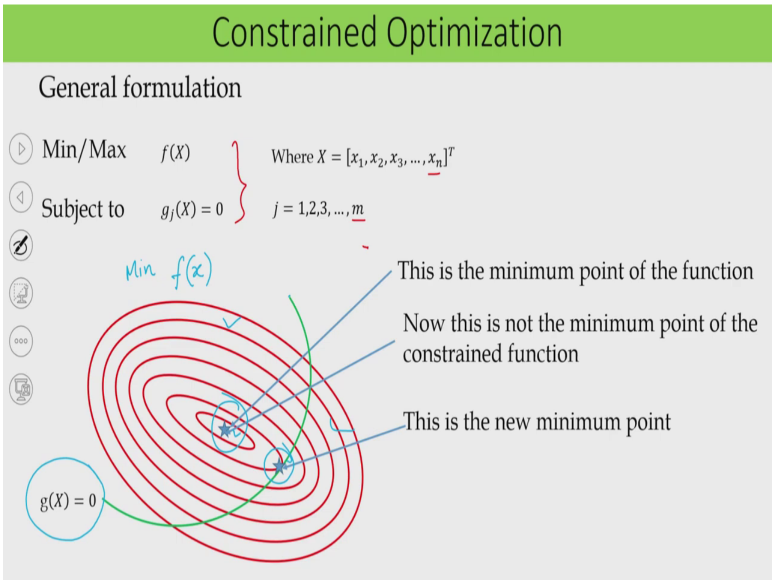

So, a general formulation of an optimization problem, constrained problem can be written like this. So, here the problem may be a maximization problem or minimization problem. So, I can write that it is a either minimization or maximization problem. So, minimization or maximization are f X. So, it is a multivariable problem so, therefore, we have total n variable here so, ok x 1, x 2, x 3 up to x n.

Then, the problem has two types of constraint here. So, the problem is subject to g j X equal to 0 where j is from 1 to m that means, total m numbers of equality constraints so, this is an equality constraint that g j X equal to 0 and you have inequality constraint that is h k X is less than equal to 0. So, this is an inequality constraint and suppose we have total n yeah so, we have total n number of inequality constraint. But today, we will only discuss the problem with equality constraint and inequality constraint, we will discuss in the next class.

So, today's lecture is basically on constrained problem with equality constraint. So, the problem with as I said that so, today, we will discuss this problem that problem with equality constraints so, either the problem is a minimization or maximization problem. So, the function is given and then subject to g j X equal to 0 that means, we have equality constraint and number of equality constraint are m and number of variable is your n ok.

So, now, let us suppose this is a function f x ok; suppose this is a function f x f and it is a two variable function so, we have so, only two variable function and this is the control lines and suppose this is a minimization problem so, we are minimizing these function ok. So, we are minimizing this function and, in that case, this is the minimum point.

So, this is the minimum point, and this is the solution of the unconstrained optimization problem. So, this is the minimum point of this particular function.

So, if I apply the sufficient and necessary condition so, I will get this solution and this is the minimum of this particular function. Now, if there is a constraint suppose g j; g X that is a constraint so, we have only one constraint I have considered. So, if there is a constraint g X, then what will happen this g X equal to 0 what does it mean that the solution should be on this particular g X line. So, the solution will be on this particular constraint, because this is the equality constraint.

So, therefore, whatever earlier solution we have that is not a your feasible solution and that is an that is basically in the infeasible region. So, any solution which is on this particular g X line that is the feasible solution and other solution whether it is in this side or on or in the other side, those solutions are not feasible because that will violate this constraint g X equal to; g X equal to 0. So, therefore, the solution must be on g X equal to 0 line.

So, now, that is not the constraint, that is not the solution of the your problem. So, maybe the new solution is somewhere here so, this is the new solution which will be your the constrained solution and that is somewhere here in this particular point.

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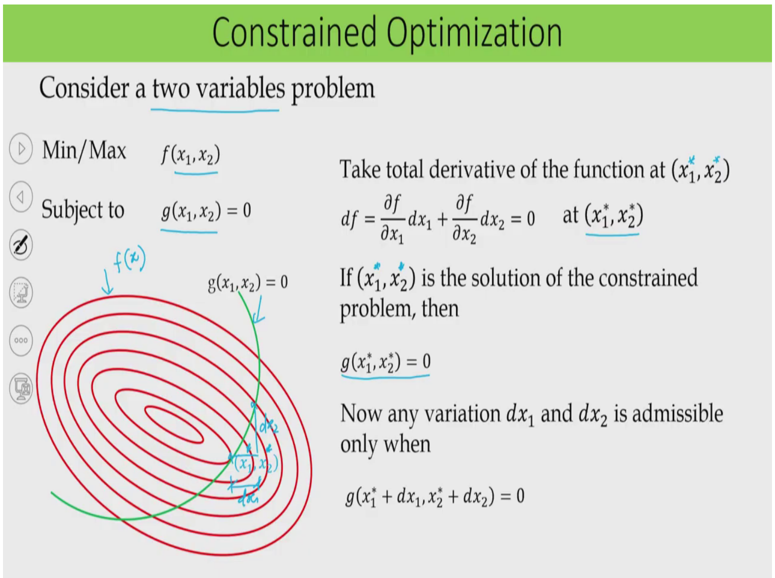

Now, let us see how we can find out the necessary and sufficient condition for optimality in case of constrained optimization problem with equality constraint. So, let us take the same example problem. So, only thing is that I am considering here two variable problems. So, just to introduce the concept so, we I have considered two variable that means, the function is f of x 1 and x 2 and subject to g x 1, x 2 equal to your 0. And this is the function, this is the function if I plot it, this is the function f x and this is basically the g equal to 0 line.

Now, suppose the solution is somewhere here. So, if I take the total derivative of this function at x 1, x 2 and basically, I would like to find out because I am trying to find out the necessary and sufficient condition. So, at optimal point that is your x 1 star and x 2 star at the total derivative should be equal to 0. So, this derivative df which is equal to del f by del x 1 dx plus del f by del x 2 dx 2 is equal to 0 at that your optimal point. So, x 1 star at and x 2 star, so this one, this will be equal to 0.

Now, if x 1 and x 2 is the solution of this constraint. So, if x 1 and x 2, this point is the solution of this constrained optimization problem so, in that case what happen? It has to satisfy the constraint g x 1, x 2 equal to 0 that means, if x 1 star and x 2 star is the solution of this particular problem, then this point should be on the constrained line. So, therefore, g x 1 and x 2 should be equal to your 0.

Suppose this is the point and this is the point that is your x 1 star and x 2 star ok so, x 1 star and x 2 star. Now, if we go for any variation of dx 1 and dx 2, then the variation is admissible only when that whatever new points you are getting. Suppose if I go ok; suppose if I go along dx 1 so, this is your dx 1 and this is your dx 2, then this variation will be admissible only when whatever new points you are getting.

So, this is the new points I am getting, at that point, the g should be equal to 0, g at that function value, that constant value at that particular point should be equal to your 0. So, therefore, I can write that g x 1 star plus dx 1 and x 2 star plus dx 2 at that point the g should be equal to 0.

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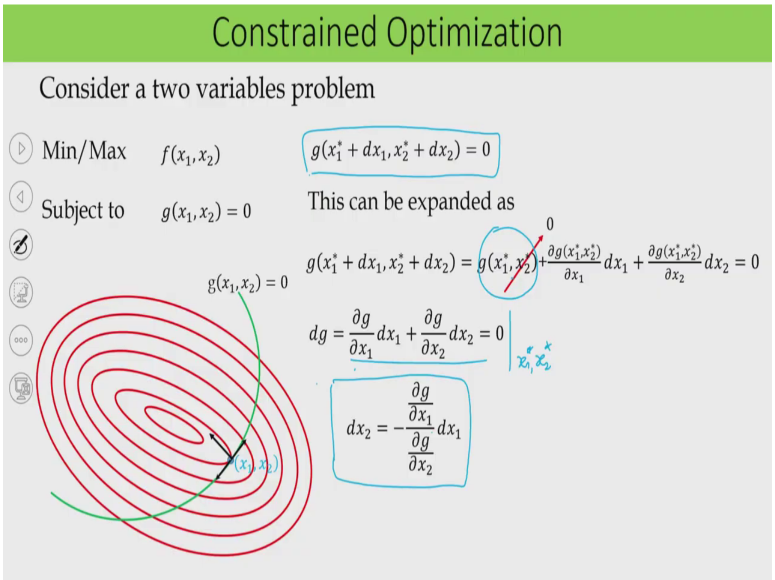

So, this must be equal to that g at x 1 star plus dx 1, x 2 star plus dx 2 should be equal to 0. So, now, I can expand these things using Taylor series, I can expand it. So, if I expand it, then I am getting that g x 1 star x 2 star, then del g by del x 1 dx 1 plus del g by del x 2 dx 2 and this must be equal to 0 as g x 1 and x 2 star on the g line. So, therefore, this must be this is equal to 0 and I am getting that these I am getting this part ok so, I am getting this part which is del g by del x 1 dx 1 plus del g by del x 2 dx 2 and this is equal to 0 and what is this?

This is nothing, but the total derivative of the constrained function at optimal point ok. So, at basically a x 1 star, x 2 star, so basically x 1 star and x 2 star ok so, x 2 star. So, from here, I can; I can write what is the value of dx 2. So, I can express these in terms of dx 2. So, dx 2 is equal to del g by del x 1 minus del g by del x 1 divided by del g by del x 2 dx 1. So, what is; what this equation is giving?

So, if I know if I suppose say that dx is basically if I want to change the value of x by dx 1, the corresponding dx 2 I can calculate using this equation and that only the new point will be an; the new point will be an your admissible point. So, if I look at suppose if I look at this point so, right now I am here so, I can go along this direction or along other direction, but I cannot go along these directions. Suppose I cannot go along.

Suppose this is the direction D so, I cannot go along the direction D because if I go along that direction, then whatever new point you are getting that will be in the infeasible region, that will not be in the constrained line. So, in that case, that solution is an infeasible region. So, I can go along that line. So, therefore, that new point can be calculated using this relation that is dx 2 equal to minus del g by del x 1 divided by del g by del x 2 dx 1 ok.

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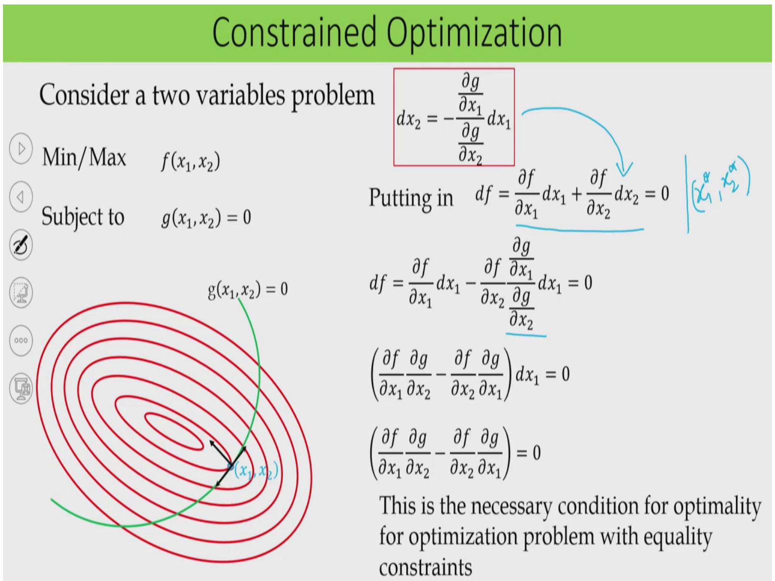

So, I am getting this relation and now, the optimally condition. So, this is the optimality condition, we got it from the necessary the from the necessary condition optimality that is the total derivative at the optimal point that is x 1 star and x 2 star should be equal to your 0. So, this is the total derivative of the objective function and if I replace dx 2 using this equation, then I am getting the total derivative which is equal to del f by del x 1 dx 1 minus del f by del x 2 into.

So, I am putting this part so, this part is coming from dx so, that is negative of del g by del x 1 divided by del g by del x 2 dx 1. And I can further simplify it and I will be getting it del f by del x 1 into del g by del x 2 minus del f by del x 2 into del g by del x 1 into dx 1 equal to 0 as d x 1 is not 0 so, this, this portion should be equal to 0. So, therefore, so, this is the necessary condition for optimality so, the necessary condition of optimality is del f by del x 1 into del g by del x 2 minus del f by del x 2 into del g by del x 1 is equal to 0.

So, this is the necessary condition for optimality for optimization problem with equality constraint ok. So, the necessary condition we have derived, but question is that now, it is actually not convenient to use this necessary condition ok. So, let us see that how we can use this in a little bit convenient way for finding the optimal solution of a constrained problem having equality constraint.

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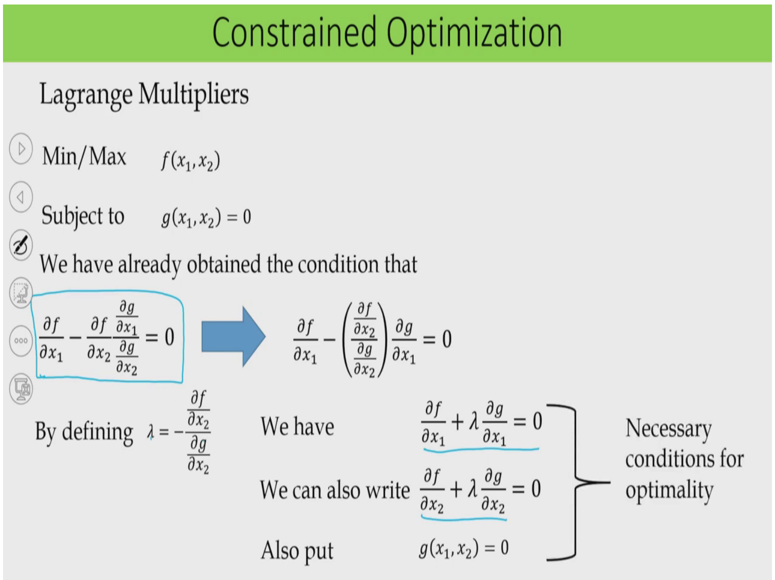

So, now, we will discuss the Lagrange multiplier method ok. Here, let us take the same problem with two variables that is minimization or maximization of x 1, x 2 subject to g x 1, x 2 equal to 0 and now, we have already obtained the condition for optimality in case of a constrained optimization problem having equality constraints. So, the condition is del f by del x 1 minus del f by del x 2 into del g by del x 1 divided by del g by del x 2 equal to 0. S

o, this is the necessary condition, this is the necessary condition for optimality; so, this is the necessary condition for optimality. Now, if I simplify, I can rearrange these terms and I can write the del f by del x 1 minus del f by del x 2 divided by del g by del x 2 into del g by del x 1 equal to 0 ok. So, now putting the lambda so, lambda is known as Lagrange multiplier, we will discuss about the significance of lambda later on. But if I put the lambda which is equal to minus del f by del x 2 and divided by del g by del x 2 ok.

So, what does it mean? It is basically showing that how if suppose del g by del x what is del g by del x? Del g by del x is basically, the change in the constrained function so ok, del g by the gradient of your constrained function how it is influencing? The gradient of the objective function with respect to a particular variable. So, I can write that this is lambda, and which is equal to minus del f by del x 2 divided by del g by del x 2. So, if I put it in this equation so, finally, I am getting del f by del x 1 plus lambda del g by del x 1 equal to 0 ok.

So, I am getting this equation and similarly, I can also write this equation in terms of your x 2 variable so, in that case, this is del f by del x 2 plus lambda del g by del x 2 equal to your 0. And basically, also put that this has to be satisfied because this is an equality constraint, so therefore, g x 1, x 2 must be equal to 0. So, I am also putting this thing and all together so, I am getting all the necessary conditions for optimality.

So, necessary condition for optimality I am getting del f by del x 1 plus lambda del g by del x 1 equal to 0, del f by del x 2 plus lambda del g by del x 2 equal to 0 and g x 1, x 2 equal to 0. So, if I solve these three equations now, there are three variables here that is one is your x 1, x 2 and lambda so, and I am having and I have three equation. So, therefore, I can solve these three equations and I can find out the value of x 1, x 2 and x 3 and whatever sorry x 1, x 2.

And lambda and whatever solution you are getting that will be the stationary point of this particular your function ok. So, I can find out that one.

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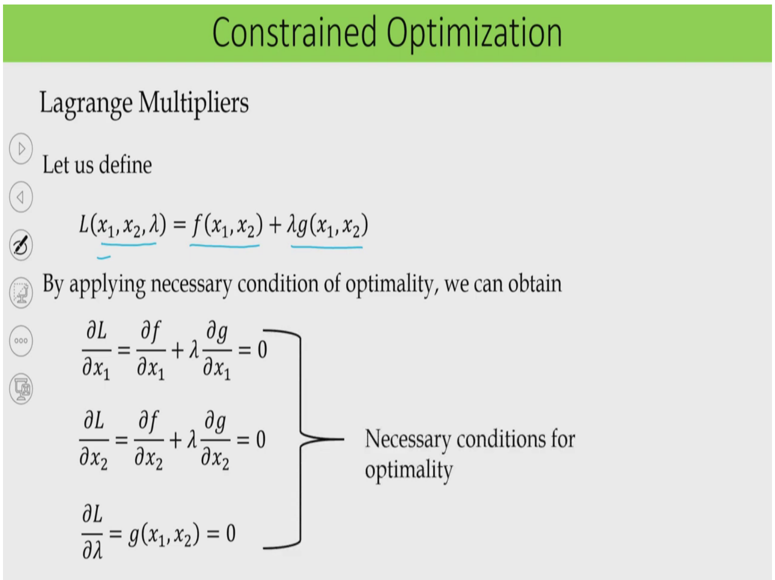

Now, I can also define a Lagrange function which is basically L. So, L is a function of now three variables that is x 1, x 2 and lambda and which is equal to f of x 1, x 2 plus lambda g x 1, x 2. So, this function is known as Lagrange function so, I can define this. Now, if I apply the necessary condition for optimality in on this particular function so, I can write it so, what is the necessary condition? That is the partial derivative with respect to your x 1, x 2 and lambda should be equal to 0.

So, if I do that so, I am getting del L by del x 1 equal to del f by del x 1 plus lambda, del g by del x 1 you please recall that one, the previous slide so, we also go we also got the same your equation so, but here, we are finding it from the Lagrange function. Similarly, I can also take the derivative with respect to x 2, I can write del L by del x 2 which is equal to del f by del x

2 plus lambda del g by del x 2 equal to 0. And if you take the derivative with respect to lambda so, in that case so, you are getting g x 1, x 2 equal to 0 ok.

So, we are getting all these three equations and these are the necessary condition for optimality in case of constrained optimization problem having equality constraint ok. So, I got it and now, if I solve these three equation so, I can find out what is the value of x 1 star, x 2 star and an your lambda star.

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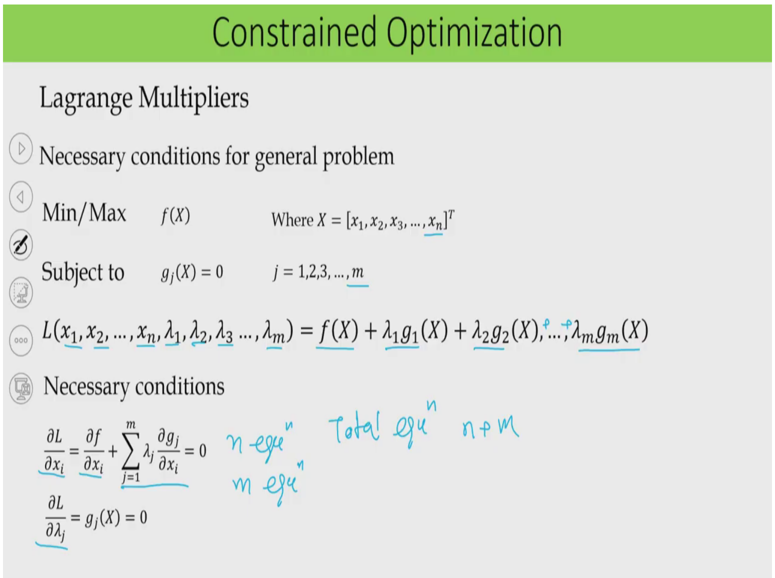

Now, let us see the Lagrange multiplier method for general problem. So, suppose I have n variable here. So, I have total n variable and we have total m constraints. So, m variable, m constraint, so in that case so, I can write the Lagrange function like this. So, now, Lagrange function is x 1, x 2, up to x n, then I have total m constraints so, therefore, lambda 1, lambda

2, lambda 3, up to lambda m and this is equal to f X, then lambda 1 g 1 plus lambda 2 g 2 then plus plus plus your lambda m g m ok so, you are getting this Lagrange function.

Now, you apply the necessary and sufficient condition on this Lagrange function, and you can find out the value of x 1, x 2 up to x n and you can also find out the value of lambda 1, lambda 2 up to lambda m. So, the necessary conditions are that is del L by del x i, the partial derivative with respect to the variable so, I have total n variable so, I will get n equation. So, del L by del x i which is equal to del f by del x i plus so, now, total constraints are m so, it is a summation of j equal to 1 to m lambda j del g j by del x i.

So, this is with respect to derivative with respect to x i and then, derivative with respect to lambda that is your del L by del lambda j which is equal to g j X equal to 0. So, how many equations we are getting? So, we are here, we are getting total n equation ok so, n equation we are getting and here, we are getting total m equation ok so, m equation we are getting. So, total equation so, therefore, the total equation we are getting that is n plus m ok so, n plus m so, this much equation we are getting ok.

So, now, you have to solve this equation to find out the value of x 1, x 2, up to x n and lambda 1, lambda 2, up to lambda m. So, you can solve this, and you can find out the solution and that is basically you are getting the stationary point.

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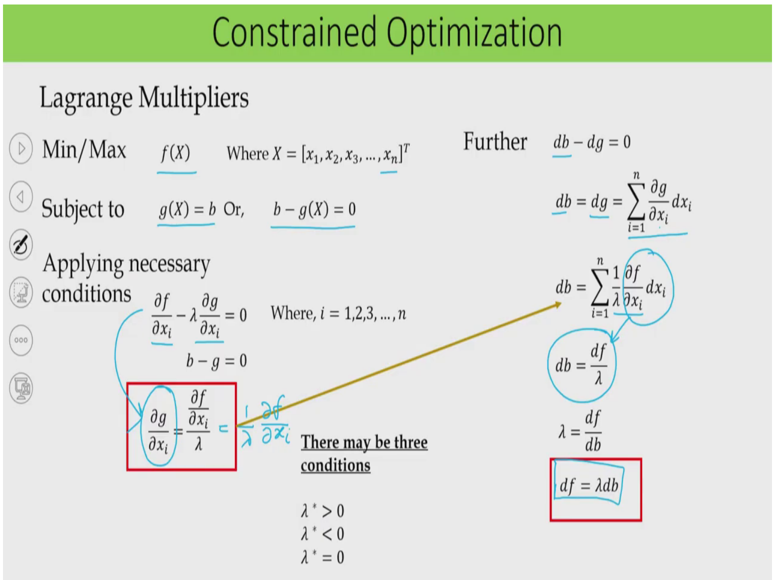

Now, let us see that we have an equality constraint. Now, what will happen if I change the equality constraint or basically, we can relax the constraint or we are tightening the constraint, then how basically it is influencing the optimal solution ok. So, that also we can evaluate or that we can find out. So, let us see here so, this is a minimization or maximization problem so, this is min, max f X and here, we have total n variable and subject to that g X equal to b ok, I can write these b minus g X equal to 0.

So, b is the right-hand side of the equality constraint that means, suppose g equal to some value ok so, I can write it b minus g X equal to 0. So, applying the necessary condition so, if I apply the necessary condition here, necessary condition is that the partial derivative with respect to the variable should be equal to 0.

If we apply the necessary condition, then I am getting del f by del x i minus so, minus lambda del g by del x i equal to 0. And here, i equal to 1, 2, 3, up to n and I am also getting that that b minus g is equal to 0 so, I am not writing the x here so, but this is g x so, b minus x equal to 0.

Now, from the first equation so, I can write that del g by del x i which is equal to del f by del x i by lambda so, I can from the first equation, I can write that one. So, del g by del x i which is equal to del f by del x i by lambda or I can also write it something like that this is 1 by lambda and this is del f by del x i. For a I can also write that if there is a small change in db, then and so, if I take the derivative of so, db minus dg equal to 0.

So, if I take the derivative of this constraint function and in that case so, db which is equal to the dg, dg is the total derivative of the constraint function, and which is equal to i equal to 1 to n del g by del x i dx i ok. So, this is basically I can write and if I write it so, what is the value of del g by del x i? So, del g by del x i already we have calculated; so, already we have calculated what is the del g by del x i and if I put it so, del g by del x i is 1 by lambda, this is 1 by lambda del f by del x i dx i.

So, now, what is this, what is this value? So, this is, this term is nothing, but the total derivative of the objective function. So, this is the total derivative of the objective function and therefore, I can write this term is equal to df. So, what finally, we are getting? So, we are we are getting finally this equation and this equation is telling that db equal to df by lambda or I can also write that lambda equal to df by db. So, lambda equal to df by db or df equal to db; lambda db; df equal to lambda db.

Now, there are total three condition. The lambda maybe greater than 0, lambda may be less than 0 and lambda is equal to 0. Now is a see if we look at this relation that is df equal to lambda db, what does it mean basically? So, lambda is telling that how sensitive this constraint is. So, if lambda is 0 that means, it does not have any impact; that it does not have any impact, if there is a small or if there is some sense in the constraint that is you are either relaxing that constraint or tightening that constraint that will not impact the objective function value.

But otherwise, if lambda is positive, if the suppose it is a minimization problem; it is a minimization problem and lambda is positive so, in that case, if you tighten the constraint; if you tighten the constraint, then what will happen? There will be a marginal improvement in the objective function. So, marginal improvement will be there in the objective function if you tighten the constraint that means, db is negative. So, db is negative, lambda is positive so, in that case what will happen? df is negative.

So, what will happen that whatever objective function value you have basically already obtained now, objective function value will reduce if you tighten the constraint. If lambda is positive and you are relaxing that constraint so, in that case what will happen? The objective function value will not improve basically, it will; it will increase. So, in case of minimization problem, if lambda is positive and you are tightening the constraint so, in that case the there will be a marginal improvement in the objective function value.

Now, consider the lambda is negative. So, lambda is negative so, in that case, if you are relaxing your constraint; if you are relaxing your constraint that means, del v is positive; del v is positive so, in that case what will happen? There will be a marginal improvement of your objective function value, but in that case, if you are tightening that one, lambda is negative and if you are tightening that means, del v is negative in that case so, what will happen? There will be a the function value will increase.

So, if we are relaxing in if we are tightening your constraint so, in that case the function value will increase, but if you are relaxing, then function value will reduce, there will be a marginal improvement. And the last condition; and the last condition that is that lambda is equal to 0 so, it does not matter actually whether you are relaxing, or you are tightening your constraint, it does not have any influence on the objective function value ok. So, that is basically the significance of lambda you are getting.

So, therefore, if lambda is very large, in that case, you have to be very careful in implementing the solution because if there is a small change in the constraint or if value so, then what will happen? There may be very large scenes in the objective function value. So, you have to be very careful if lambda value is large, but if lambda value is 0 or lambda value is very small that means, that constraint is not that significant so, it will not have mass influence on the objective function value.

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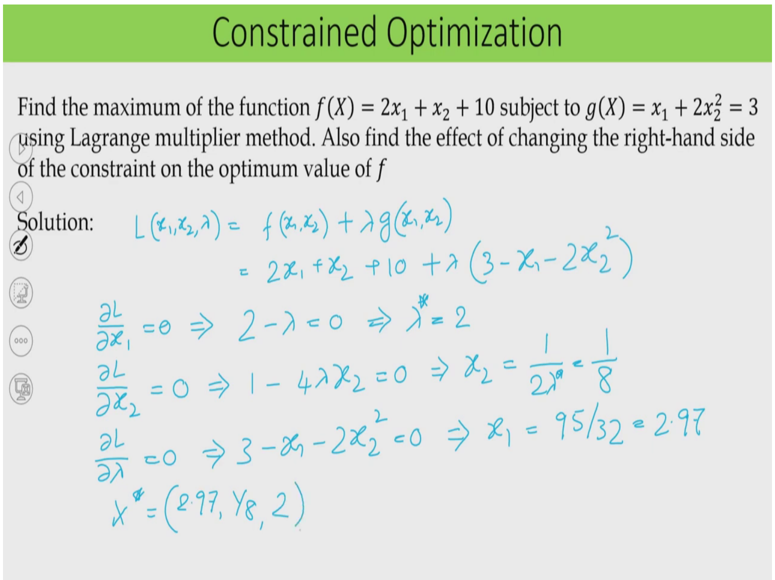

Now, let us solve one problem here. We have to find out the maximum of this particular function. Find a maximum of the function that is f equal to twice x 1 plus x 2 plus 10 subjected subject to there is a constraint that g equal to x 1 plus twice x 2 square equal to 3 using Lagrange multiplier method and also, we have to find out what is the effect of changing the right-hand side of the constraint on the optimum value of f ok.

So, let us solve this problem. So, what I will do basically? So, I will write the Lagrange function now. So, Lagrange function is x 1, x 2 and lambda and which is equal to f of x 1, x 2 plus lambda g x 1, x 2. So, I can write it like this that this is equal to twice x 1 plus x 2 plus

10 plus lambda 3 minus x 1 minus twice x 2 square. So, apply the necessary condition so, the necessary condition is that del L by del x 1 equal to 0. So, what I will get that 2 minus lambda equal to 0 and from here, I can find out the lambda equal to 2 or I can say lambda star equals 2.

The next necessary condition is del L by del x 2 equal to 0 and I can I will get 1 minus 4 lambda x 2 equal to 0 and from here, if I solve it for x 2, I will be getting 1 by twice lambda star and which is equal to 1 by 8. The third equation is del L by del lambda equal to 0. So, what I am getting that 3 minus x 1 minus twice x 2 square is equal to 0 and if I solve it for x 1 so, I will be getting x 1 equal to 95 by 32 which is equal to 2.97 ok. Whatever solution I am getting that is x star this is 2.97, then 1 by 8 and lambda I am getting your 2, so this is the solution I am getting.

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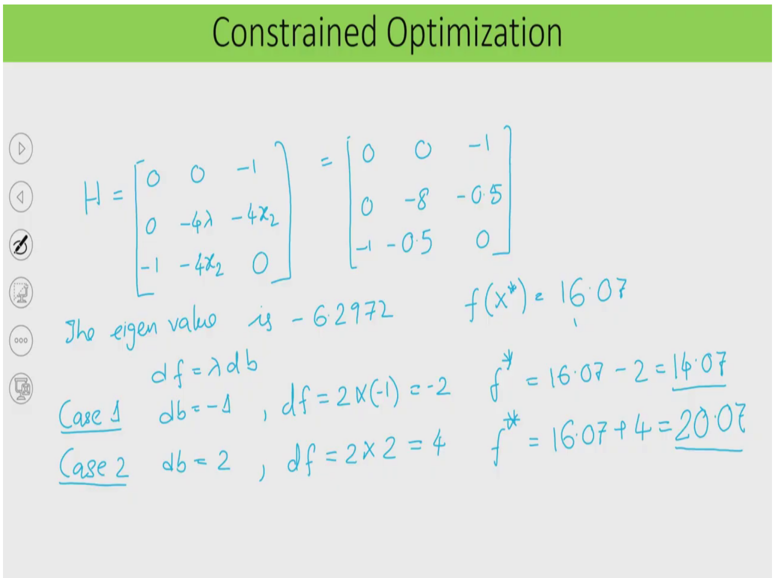

And now, apply the sufficient condition just to check whether this is a minima, maxima or a neither minima nor maxima. So, what I will do? I will calculate the Hessian matrix ok so, I will calculate the Hessian matrix. So, in this case, the Hessian matrix is coming 0, 0, minus 1, then 0, minus 4 lambda, minus 4x 2, minus 1, minus 4x 2, 0. So, if you put the Hessian matrix at x star so, then what I am getting? I am getting this 0, 0, minus 1, 0, minus 8, minus 0.5 and minus 1, minus 0.5 and 0.

The eigen values of this matrix is minus 6.2972. So, therefore, this is the negative definite matrix and whatever solution you are getting that is an maximum point. So, therefore, eigen value is your negative so, the optimal solution whatever we have got that is your maximum point. And if we calculate the function value; if we calculate the function value that is f X star that is equal to 16.07 ok. Now, also we have to check suppose if you sense the value of db ok value b, then what will happen?

So, there are two ways to do that. So, basic the next part of this question is that we have to evaluate that if there is a change in value of b, then what will happen? Now, what we can do basically for that? So, I can take a different value of b and I can solve this problem and just see whether how basically, it is changing the how objective function value will be same.

Now, or we can do what? We can apply the relation, already we have derived that is your df is equal to lambda db. Now, suppose case 1 ok that db is equal to minus 1, then what will happen that df is equal to 2 into minus 1 and which is equal to minus 2 and so, the new f star value ok. So, the new f star value will be that 16.07 minus 2. So, we will be getting 14.07 ok.

So, what does it mean that if you tighten that one so, in this case, the lambda is positive so, lambda is positive, problem is maximization problem, it is a maximum value so, what will happen? If we tighten that one so, if we tighten the constraint by one unit, then corresponding change in our objective function will be two unit ok. So, we are getting the new objective function value which is lower than that one and now, it is 14.07.

Now, there is another case if we consider case 2, that db is positive suppose 2 unit. So, in that case, what will happen that df which is equal to 2 into now, 2 so, that will be 4 ok. So, the new f star will be now 16.07 plus 4 so, that is 20.07. So, in this case, what is happening? As lambda is positive and if you relax the constraint ok so, in that case, the objective function value will improve ok. So, there will be marginal improvement in the objective function value and from 16.07 so, we are getting now 20.

So, therefore, if you (Refer Time: 38:49) it by 2 unit, then corresponding improvement in the objective function value will be 4 unit and you are getting 20.07 and that is the new objective function value and if you tighten it, then it will reduce and this is 14 point, we will get 14.07 ok. So, this is all about today. So, today, we have discussed the constrained optimization problem with equality constraint.

In the next class, we will discuss constrained optimization problem with inequality constraints. So, what is the necessary and sufficient conditions for optimality in case of the problem with inequality constraint. So, see you tomorrow.