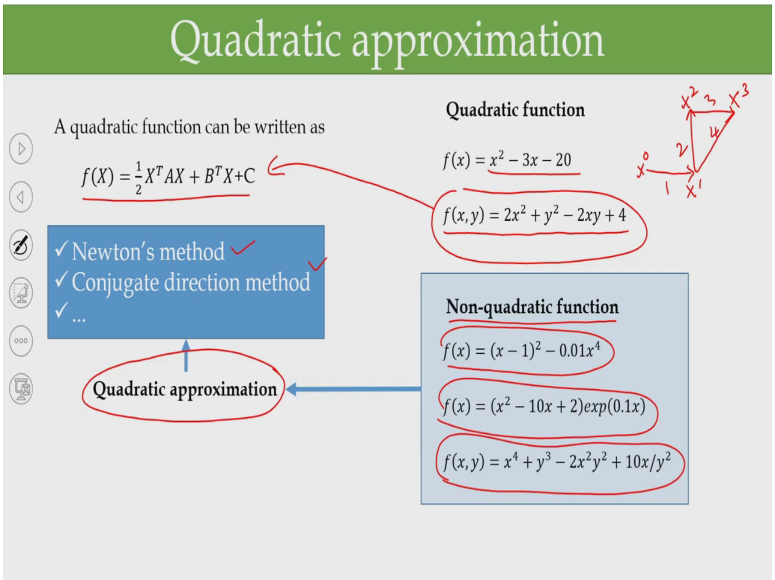
Optimization Methods for Civil Engineering
Dr. Rajib Kumar Bhattacharjya
Department of Civil Engineering
Indian Institute of Technology, Guwahati

Lecture - 12
Quadratic approximation

Hello student, welcome back to the course on Optimization Methods for Civil Engineering. So, today, we will discuss Quadratic approximation.

(Refer Slide Time: 00:45)



Now, as you know that a quadratic function can be written in this form. So, I can write a quadratic function so, in this form. Now, these are I think you know that this is a quadratic function of single variable, this is a quadratic function of two-variable, and I can write this quadratic function so, in this form.

Now, if we have a quadratic function so, as we have discussed. So, we can apply Newton's method. So, as you know that in a single iteration of Newton's method so, you will get the optimal solution of the problem, if your function is a quadratic function. Similarly, I can also apply conjugate direction method.

So, as you know, it has convergent proof that after so, as you know that what we are doing in case of conjugate direction method so, if we start from a single point, then we are going along x direction, then we are going along y direction and we are going along x direction suppose this is x naught, this is x 1, then this is x 2 and this is x 3, then optimal solution is in this particular direction that is x 3 minus x 1.

So, therefore, if your function is quadratic and if you are applying conjugate direction method so, you need 1, 2, 3 and 4 iteration that means, after 4 iterations, you will get your solution and if you are applying Newton's method so, you are getting your solution in one iteration. So, therefore, so, if you know that your function is a quadratic function so, then you apply this method so, there are some other methods also.

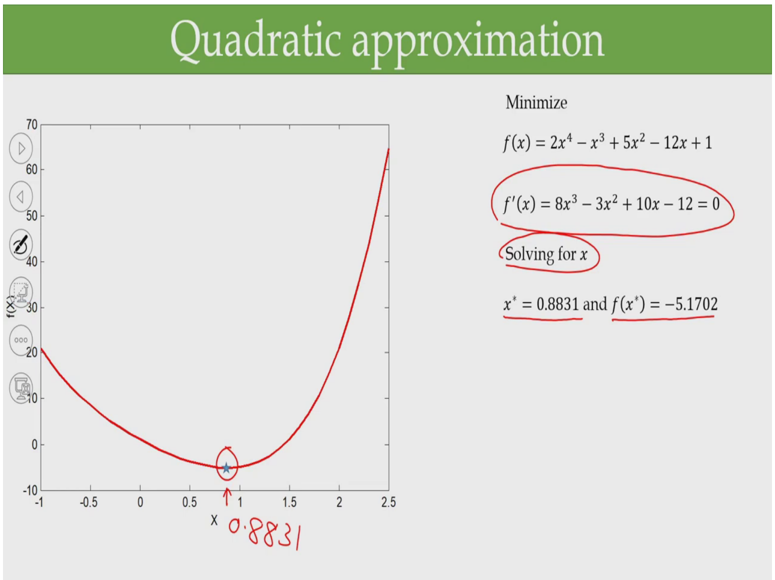
Now, if you have non-quadratic function, suppose I have shown you some functions so, these are non-quadratic function. So, this is one non-quadratic function, this is one non-quadratic function, and this is one non-quadratic functions are there.

Now, what I will do here? So, I can apply this method, then you need some more iterations so, if you are applying Newton's method so, you will not get your solution in one iteration and similarly, if you are applying conjugate direction methods so, you will not get your solution in four iterations. So, you need more iteration.

So, this is one way, you can directly apply this methods. However, what I can do, I can approximate this function as a quadratic function ok. So, I can approximate that one. So, this is an iterative based process so, what I will do? I will approximate this function as a quadratic function and then, I will apply either Newton's method or this one and then, I will get optimal solution of the approximated function, but that is not the actual solution.

Then, what I will do? I will make another approximation and that way with some iteration so, I will get the solution of this non-quadratic function ok. So, what I am doing? I will approximate the function as a quadratic function, then I will apply this method and then again, I will approximate it again, I will solve it. So, with some iterations, I will get the solution of this problem. So, that I am explaining with some example problems.

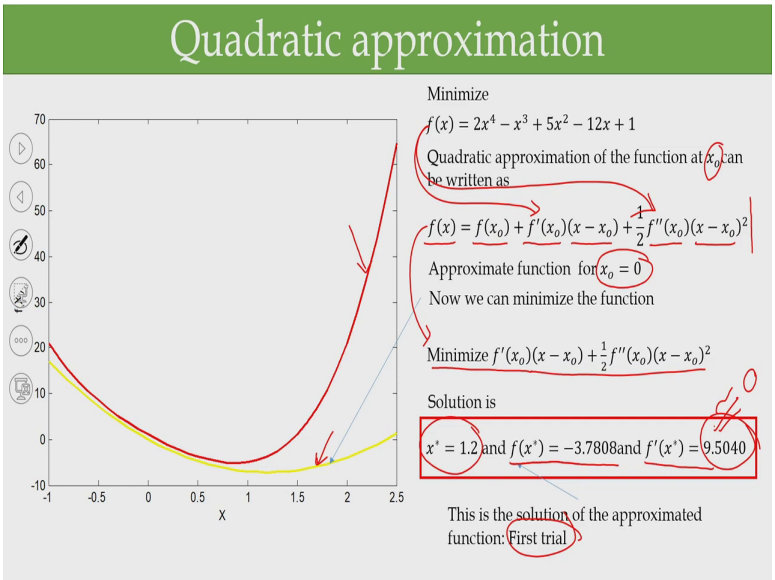
(Refer Slide Time: 04:37)



Now, let us see this particular function. Now, this is a non-quadratic function, and I can solve this problem. So, suppose if I apply the necessary condition so, this is the necessary condition that means, the first derivative should be 0 and if I solve it for x now, if I solve it for x, then I am getting one solution that is x star equal to 0.8831 and the function value at optimal point is minus 5.1702. So, this is the solution of these particular problem.

So, here, the solution is somewhere here and that is 0. 8831. So, this is the solution, this is 0.8831 and the function value is minus 5.1702. So, this is a non-quadratic problem, and we have one solution between minus 1 and 2.5 ok and that solution is 0.8831. Now, let us see how we can solve this problem using quadratic approximation method.

(Refer Slide Time: 05:59)



So, now, what I will do? I will take a starting point here x naught and I can approximate this function as a quadratic function using Taylor series. So, what I will do? I will use the truncated Taylor series that means, up to the second derivative. So, if you are considering up to the second derivative, then the function will be a quadratic function. So, I will approximate that one at x naught.

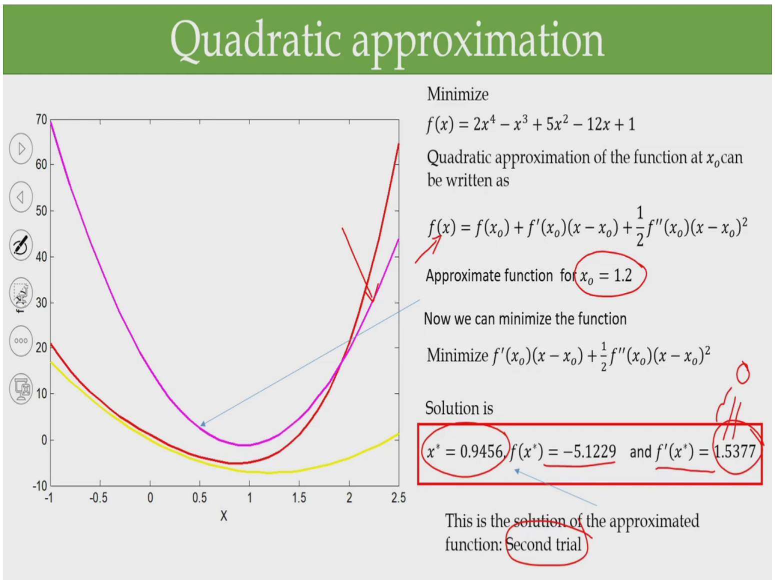
So, therefore, I can write f x which is equal to f x naught plus first derivative at x naught into x minus x naught 1 by factorial 2 second derivative at x naught x minus x naught whole square so, this is the Taylor series. So, we are considering up to the second derivative, then the function will be a quadratic function.

Now, if I consider x naught equal to 0; x naught equal to 0 so, what I am doing here? So, I am putting the first derivative of this function here and similarly, I am also putting the second derivative of this function at this particular equation that means, in Taylor series and I know what is x naught, x naught value I will put and then, I am whatever equation I am getting that is the quadratic approximation of the original function and here, I have shown you this yellow colour line that is the quadratic approximation of the original function and red line is basically the actual function.

So, this is the quadratic approximated function. So, I will minimize this one. So, now, if I solve it so, I am getting the solution that is x star equal to 1.2, but that is not the actual solution of the function. So, this is the solution of the approximated function is not it and the function value is minus 3.780 and derivative value is at the original function and that is 9.50. So, that means, this 1.2 is not the actual solution because if it is an actual solution, this derivative value will be this particular value will be or it will be near to this 0 ok, but this is the solution of the approximated function.

So, you can say that this is the trial one, first trial ok. So, this is the first trial, and I am getting a solution of 1.2 that means, what I am doing here? So, I have started from x naught so, x naught is 0 so, I am getting another solution that is x 1 you can say and that is 1.2. Now, next time what I will do? I will approximate this function, the original function at 1.2 ok so, I will get a different approximated function.

(Refer Slide Time: 09:15)

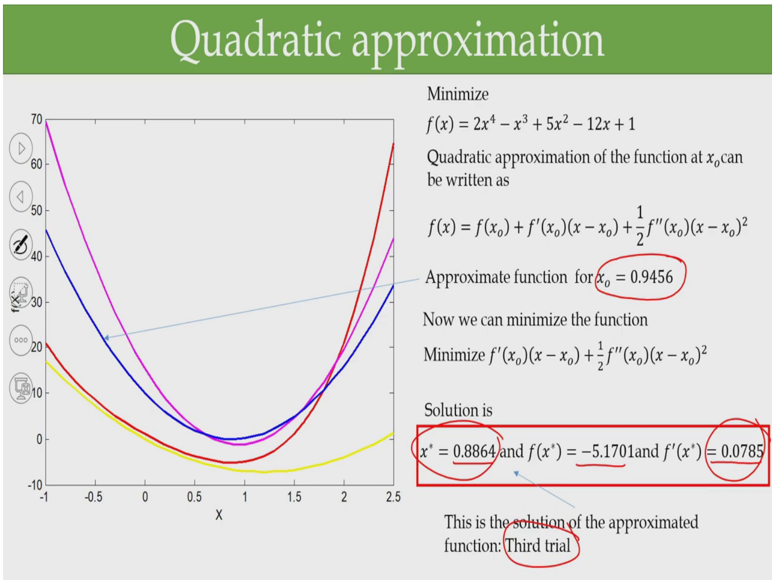


So, what I will do? Now, I will approximate this particular function at 1.2 ok. So, if I put this value in this particular equation so, then I will get the approximated function and here, now I have shown you the purple color, this one so, this is the new approximated function, and we have approximated at x naught equal to 1.2. So, earlier approximation was at 0 and now, the new approximation is at 1.2 and so, if I solve it, then I am getting this solution x naught equal to 0.9456 and function value is minus 5.1229.

And now, you just see derivative value has reduced. So, earlier derivative was something around you can see 9.5 so, now, whatever you are getting, you are getting around 1.5377 so, derivative value has reduced, but this is not the actual solution because this derivative should be; this derivative should be equal to near to 0. So, you can say this is the second trial. So, I am getting another solution that solution is 0.9456.

So, what we are doing? We have started from 0 and from 0, the first solution is 1.2, you can say first trial. Now, we have approximated at 1.2 and we are getting a new solution that is 0.9456. So, now, what we will do? So, we will not stop our iteration because the derivative value is not 0. So, this derivative is at the original function and this derivative value is not 0. So, therefore, we have to continue our iteration though series at the second trial. So, let us see the third trial.

(Refer Slide Time: 11:30)



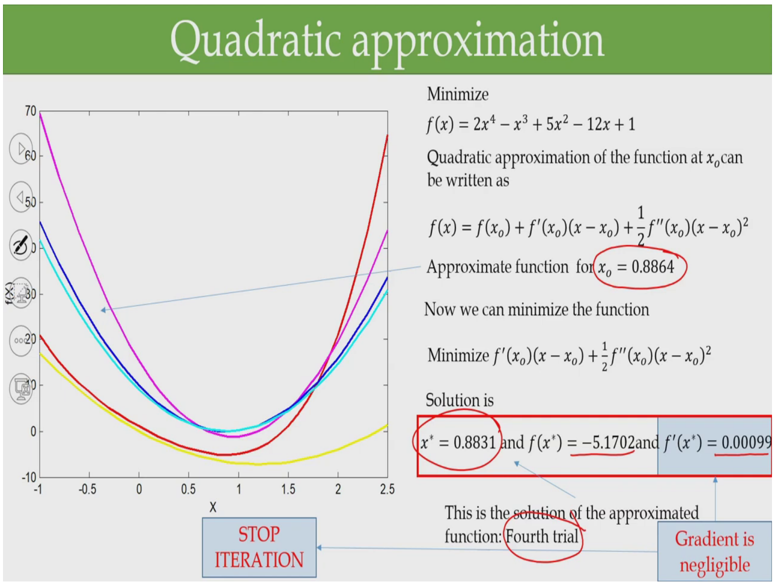
So, what we are doing? The in the third trial or third iteration now, we will approximate at 0.9456. So, if we do that so, we are getting a new solution and here, the blue color is the approximated function at 0.9456.

So, now, we are getting a new solution and that is 0.8864 and the function value at that particular point is minus 5.1701 and the derivative value has now reduced, and this value is

0.0785. So, it is not exactly 0, but it is approaching 0. So, at optimal point, this derivative should be equal to 0.

So, you can say that this is the third trial, and I am getting a new solution which is near the optimal solution because derivative value has reduced, and this is just this is near to 0.

(Refer Slide Time: 12:48)



So, let us continue our iteration again. So, this time we will approximate this function at 0.8864 and if I solve it, then I am getting this solution and which is 0.8831 and now, you can see the derivative value is almost 0 which is not 0, but it is almost 0 and the function value is minus 5.1702 ok. So, you can say this is the fourth trial.

Now, we can continue this one. So, gradient value is near to 0. So, if we want that ok, I can do another iteration and we can reduce or we can get a better solution, but if you are thinking that this gradient value is negligible depending upon what precision you want to use so, we can stop the iteration here and I can say that this is the optimal solution of this particular problem, the solution is 0.8831 and the function value at optimal point is minus 5.1702.

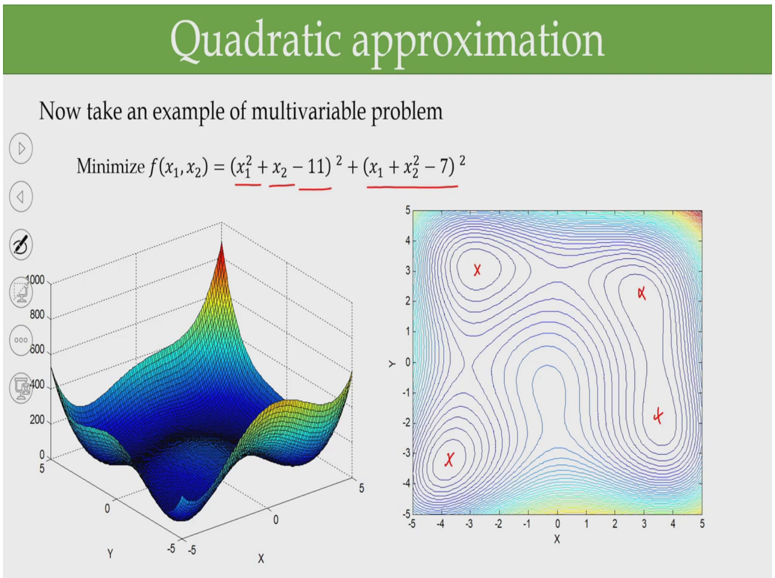
So, what we are doing here? In case of non-quadratic function so, we will approximate using Taylor series so, we will consider up to the second derivative and then, whatever approximated function you are getting that function will be a quadratic approximation at that particular point.

So, the idea is that so, you take a initial point, you approximate any function, any non-quadratic function at that particular point, solve it, you get a solution, again you approximate it at that new solution and then, again you solve it and finally, after some iteration so, you will get the x well optimal solution of the function.

So, when you will stop your iteration? After every iteration, you just take the gradient value. So, once the gradient value is near to 0 or if it is suppose 10 to the power minus 3 or 10 to the power minus 6 depending upon what precision you need, you can stop your iteration. So, this is an iterative base method.

So, what we are doing? We are just approximating a non-quadratic function as a quadratic function using Taylor series, then we are applying the method like Newton's method or conjugate direction method.

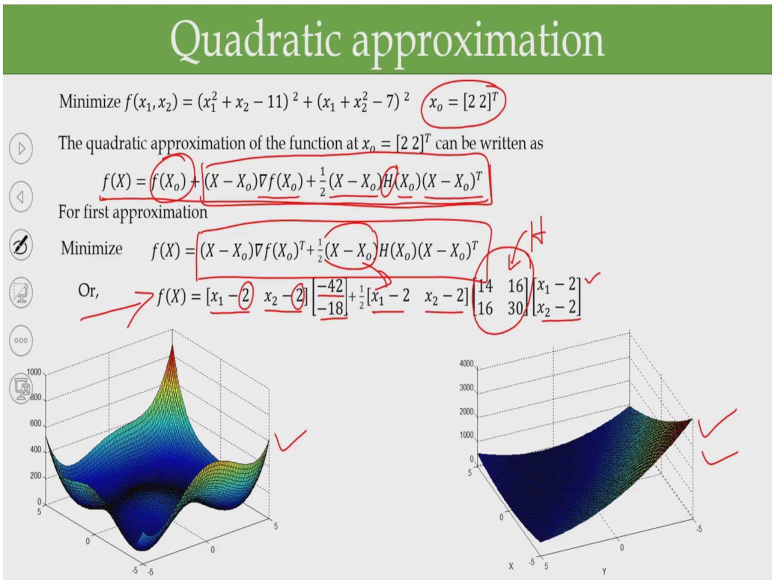
(Refer Slide Time: 15:27)



Now, let us take another example, its a multivariable problem. So, here, we have two variables that is x 1 and x 2. The function is x 1 square plus x 2 minus 11 whole square plus x 1 plus x 2 square minus 7 whole square so, this is a non-quadratic function, and it is a two-variable function. So, if I plot it so, I will get this is the surface plot, and you can see there are total four optimal solution of this particular function.

So, this is the control and you can clearly see, there is one optimal solution somewhere here, one is here, one is somewhere here and another one is here. Now, we will try to solve this problem using quadratic approximation method.

(Refer Slide Time: 16:31)



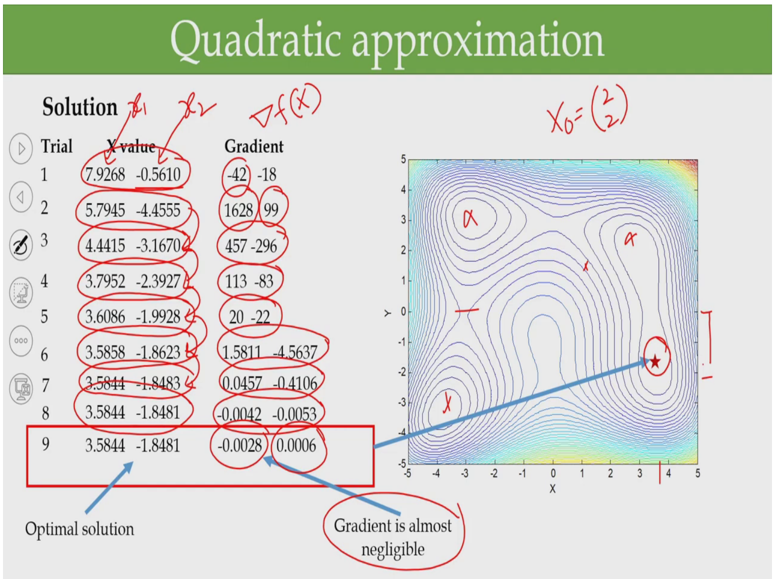
Now, what we have to do? We have to take a starting point. So, in this case, I have taken 2, 2 that means, the iteration will start from 2, 2. So, what is the first step? The first step is we have to approximate this function as a quadratic function at x naught equal to 2, 2 ok. So, I have written here the Taylor series up to the second derivative ok. So, that is f X equal to f X naught plus X minus X naught del f X naught transpose and this is X minus X naught, this is the Hessian matrix at X naught into X minus X naught transpose. So, using this, I can approximate the function as a quadratic function.

So, the first approximation is so, here, this is constant so, therefore, if I minimize this thing so, I can ignore that one so, I will only consider this part ok. So, I will minimize this particular part so, that is I am writing here. So, I can write it something like that this is x 1 minus 2 and x 2 minus 2 so, we have started from 2 n 2 that is x 1 minus 2, x 2 minus 2, then the gradient value is minus 42 and minus 18 ok, gradient value at x naught.

Then, this is, this term is this so, that is x 1 minus 2, x 2 minus 2 and the Hessian matrix, this is the Hessian matrix so, Hessian matrix is 14, 16, then 16, 30 and then, x 1 minus 2 and x 2 minus 2. So, this is the quadratic approximation of the original function at 2, 2. Now, this is the original function, and this is the quadratic approximated function. So, now, I can apply the Newton's method to find the solution of the approximated function. So, I can do that.

So, now, I have taken initial point 2, 2 so, I can also start from 0, 0 or any other point ah, but in this case, I have started my iteration from 2, 2. So, this is the original function, original your surface plot of the original function and right-hand side, this is the surface plot of the approximated function. Now, I can apply the method as I said. So, let us see.

(Refer Slide Time: 19:35)



So, I have done the iteration here. So, the 1st trial, this is this value is x 1 and this is x 2 ok. So, this is the 1st trial that means, the initial point, from initial point, the initial point was 2, 2 and once we are solving the approximated function so, I am getting a solution and that is 7.9268 and minus 0.5610.

Now, I have calculate the gradient, this is your del f X ok. So, the gradient value is minus 42 and minus 18. So, gradient should be 0 is not it so, at optimal point, the gradient will be 0 so, that means, this is not the x solution of the original function, but this is solution of the approximated function.

Now, what I will do? I will take this point as an initial point, and I will approximate the function again and I will solve it. So, in the second iteration when I solve it so, I am getting a new solution that is 5.7945 and minus 4.4555 and gradient value is now, 1628 and 99 that means, gradient is not 0 so, we have to continue our iteration.

So, now, I am considering this as the initial point, then the next solution is this one. So, I have approximated the original function now, the solution is 4.4415 and minus 3.1670 and gradient value is 457 and minus 296, then from that so, I am getting this solution, gradient value is 113 and minus 83 and from here, I am getting another solution now, gradient value is 20 and minus 22.

Then, I am getting another solution, from here, the gradient value is reducing you just see the gradient value is 1.5811 and minus 4.5637. Then, I am getting a new solution from that, the gradient value is this ok so, this is 0.0457 and minus 0.4106 and I am getting this solution and gradient value is 0.

So, I can continue the iteration, but if you are thinking that the gradient is sufficiently 0 depending upon your requirement so, in that case, I can stop my iteration so, I can say so, this is the optimal solution, optimal solution is 3.5844 and minus 1.8481 so, this is the optimal solution of this problem. So, if I consider the gradient is sufficiently small and I need not continue my iteration and I can declare that as a optimal solution. So, gradient is almost negligible. So, in this case so, I am get I am telling that this is the solution of this particular problem.

So, here I am getting this solution, this solution is this is 3.5844 so, somewhere here, this is 3.5844 and this is from here so, this is negative. So, you can see that from here, this value is minus 1.8481. So, this is the solution I am getting. So, in this case, I have started that means, the X naught I have taken 2, 2, but if I take different initial points so, I may get other solution.

So, this is one solution, this is another solution. So, here, I have started somewhere here from 2, 2 so, I am getting this solution, but if you sense your initial point, you may get different iteration. So, you may get different solution. So, if you are starting from different point so, you will get the other optimal solution of this problem.

So, in this class so, we have discussed mainly the quadratic approximation. So, what we are doing here? We are actually applying the Newton's method or conjugate direction method, but we are approximating a non-quadratic function as a quadrative function and iteratively, we are solving the problem. So, I have shown you some example problem so, using the Taylor series up to second derivatives.

So, I can apply this method for solving any non-quadratic function and I have to apply this method iteratively and can be applied to any non-quadratic function.

Thank you very much.