Optimization Methods for Civil Engineering
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Convex Function
Lecture - 11
Line Search Methods for Multi-variable Problems

Hello student, welcome back to the course on Optimization Methods for Civil Engineering. So, in today's class we will discuss about Convex Function.

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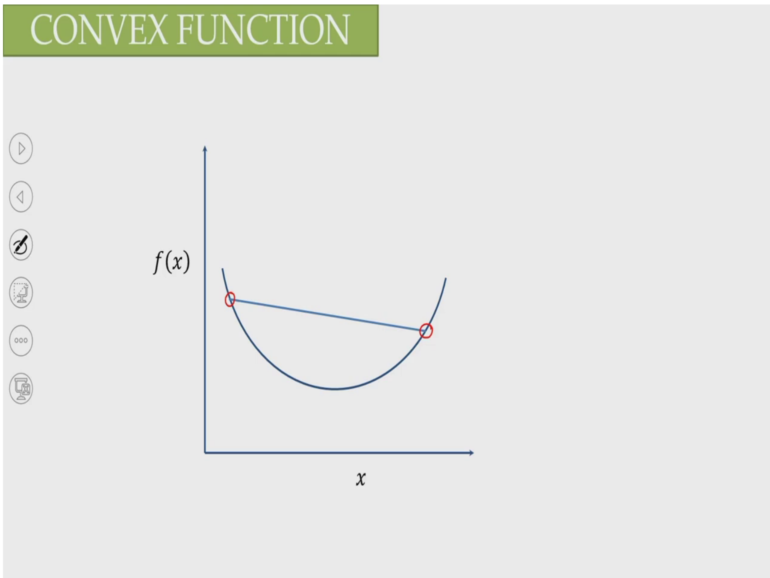

Now, what is convex function? Let us take a function, now if I take any two points on this function. Suppose, in this case I have taken these two points and if I join these two points and if I draw a line so, you can see that this line is above this particular curve ok. So, this line is not cutting this particular graph in any other location.

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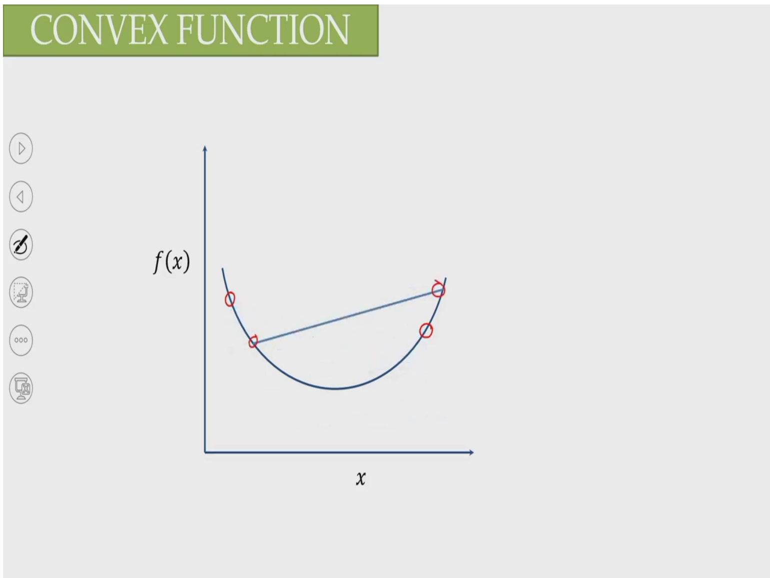

Similarly, if I take these two points then also that is true ok, and if I take these two points then also this is true.

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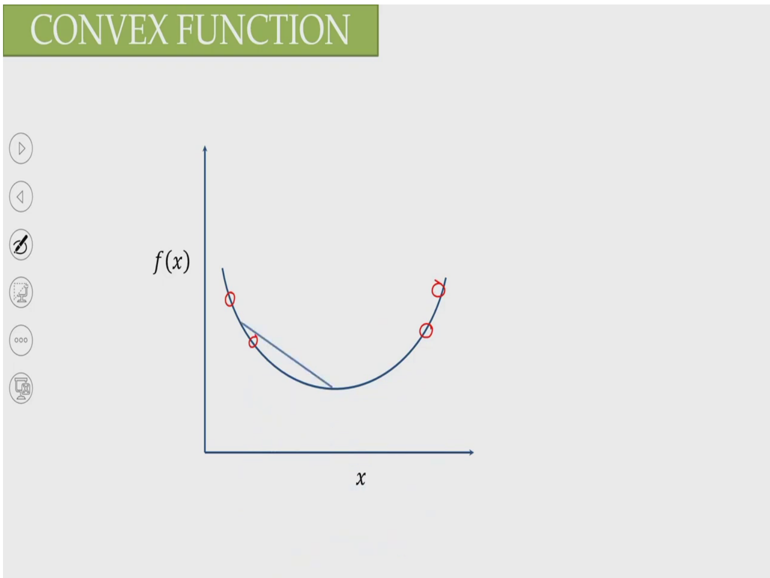

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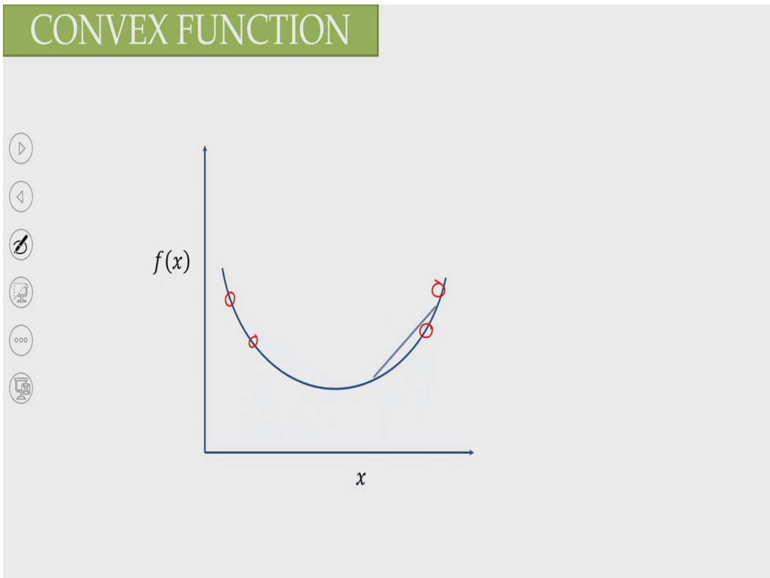

And, if I take this two point then also this is true; that means, if I take any two points on this particular function or on this particular curve. So, and if I join the line then the line will not cut this curve, in any other third location ok. So, in that case this function is called a convex function.

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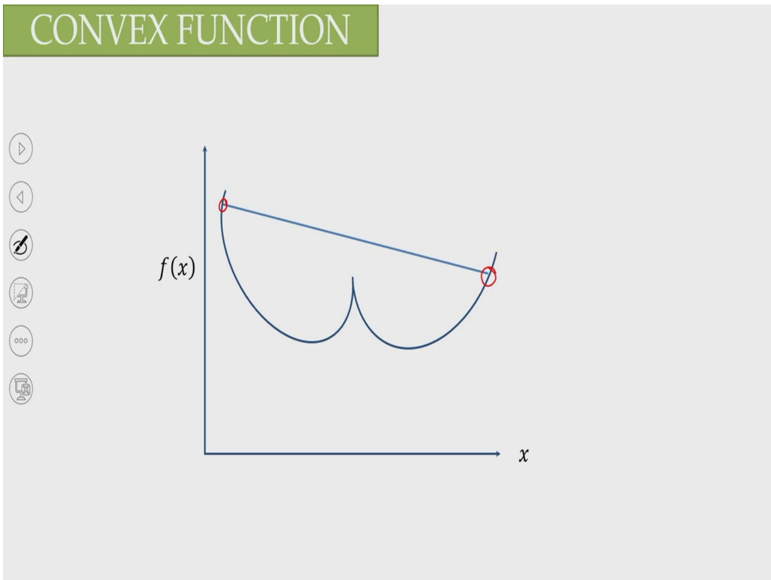

Now, you take this particular function. So, here if I take these two points ok, that is that is not, that will not cut this particular curve in any other location.

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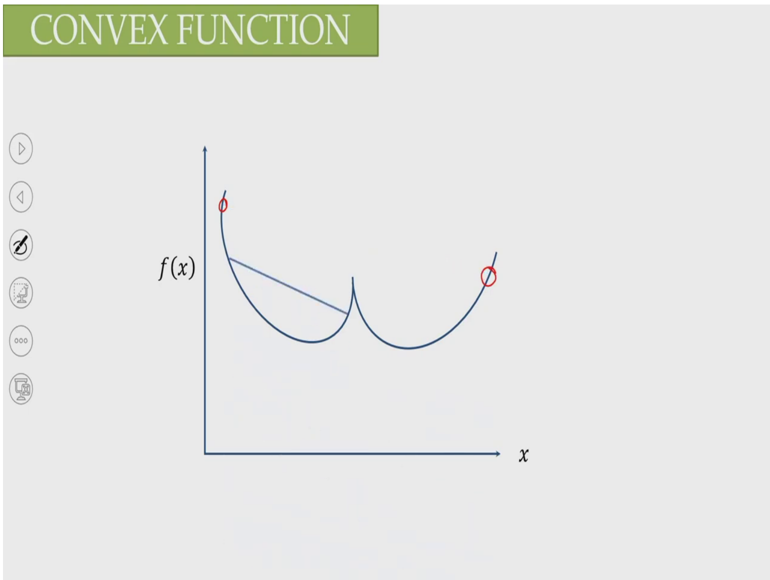

Similarly, if I take this then also this is true.

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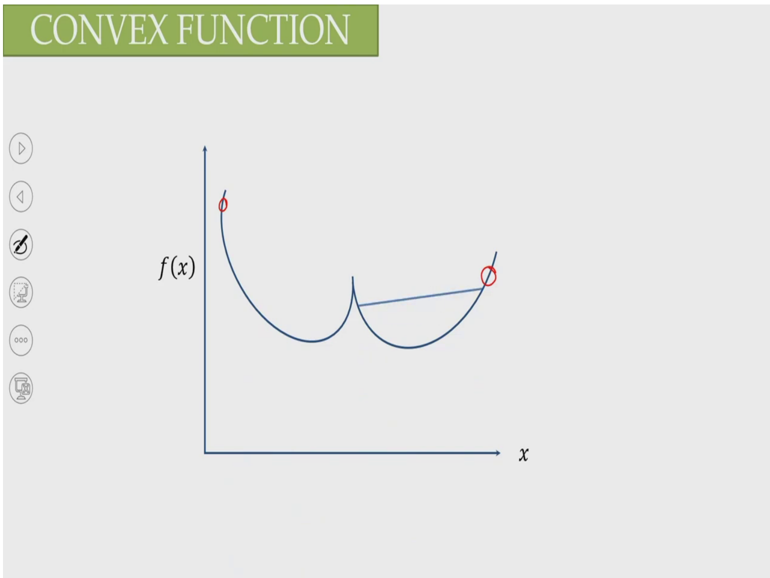

But, if I if I take if I take this one then also this is true but, if I take these two points ok.

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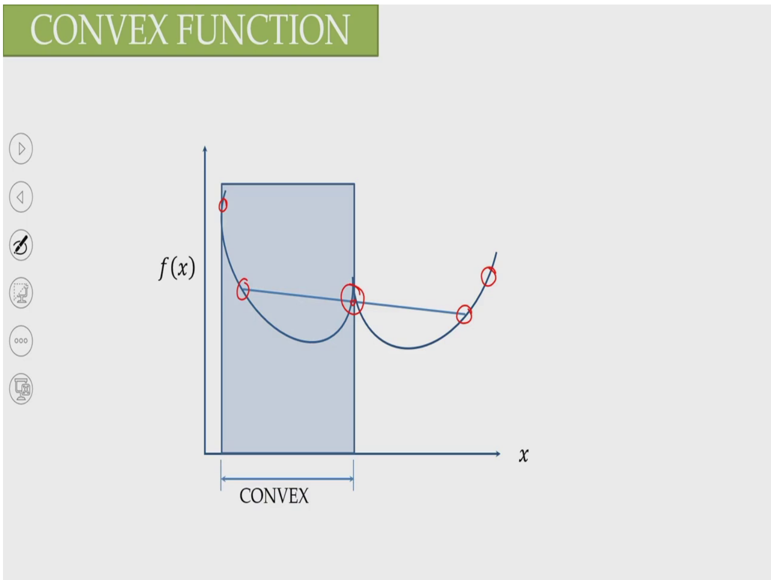

So, in that case this line is cutting this particular curve here ok. And, therefore, this function is not a convex function. But, if I consider this as lower bound and this is upper bound ok. Then, in that case suppose in that case this function is a convex between these two lower bound and upper bound ok.

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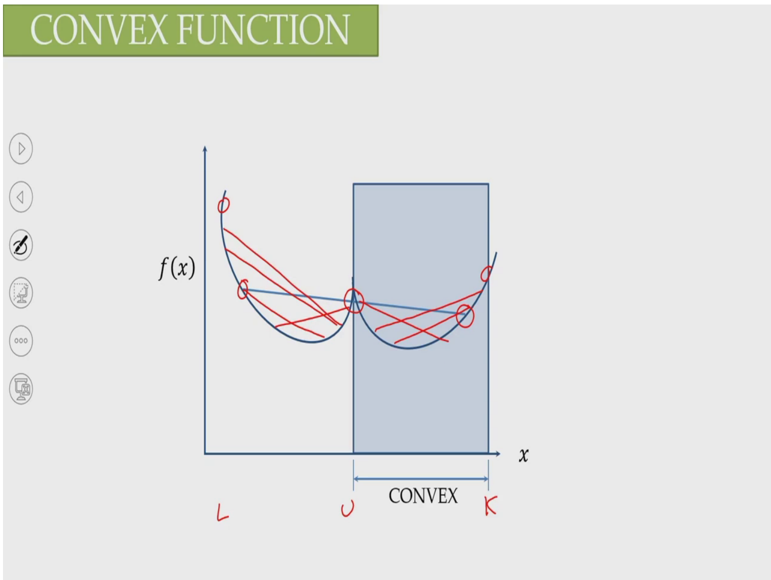

Now, if I also consider suppose this lower bound as U and upper bound as K, so, in that case also the function is convex function. Because, if I take any two points you just see that that line is not cutting this curve any two points you take on this particular region. Similarly, here also you take any two points ok, then that will that will not cut this particular curve ok.

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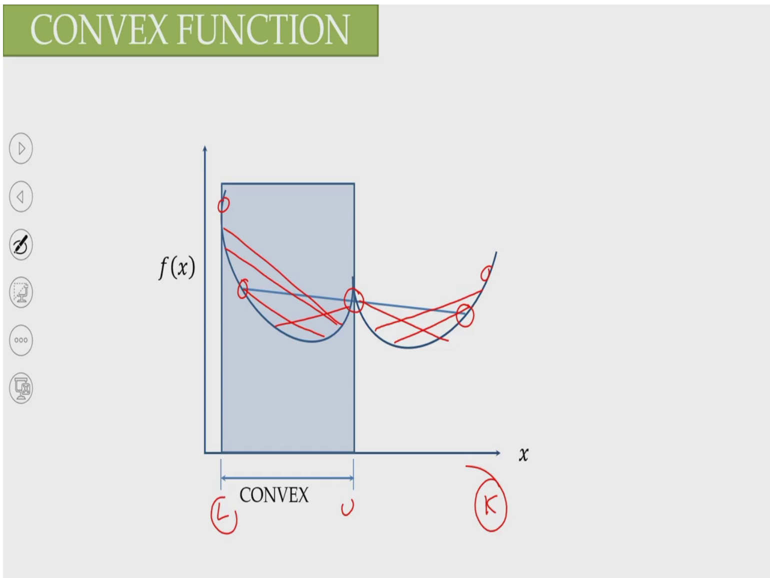

So, therefore, this is convex between U and K and, this is convex between L and U ok. So, but if I consider the enter source space enters this is lower bound as L and upper bound as K. So, in that case this particular function is not a convex function.

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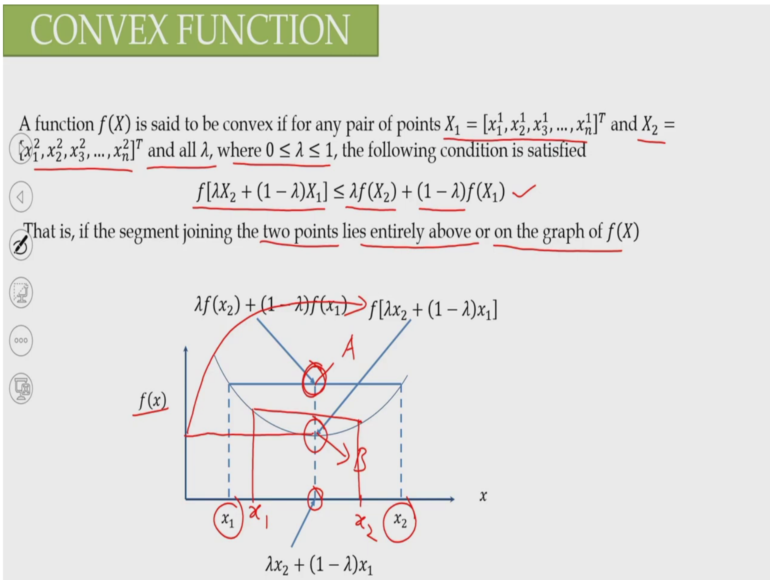

Now, what is the definition of convex function? A function f X is said to be convex if for any pair of points. So, if I take two points, as I said that X 1 and X 2 ok. So, if I take any two points, any pair of points. And, all lambda where lambda is between 0 and 1 ok, the following condition is satisfied. If it is satisfied, then we will say that the particular function is a convex function.

So, what is this condition? That the function value at lambda X 2 plus 1 minus lambda X 1 is less than lambda f of X 2 plus 1 minus lambda f of X 1. So, in that case if this condition is satisfied, if this condition is satisfied, then we will say for any value of lambda ok so, for any value of lambda. Then, we will say that this particular function is a convex function. That is if the segment joining the two points ok, lies entirely above or on the on the graph of f X.

So, what is this condition? That, the line segment what you are drawing from X 1 to X 2 lies entirely above or on the graph of f X. So, in that case we will we will say that this particular function f X is a convex function. So, let me explain with this figure now this is a one dimensional function one variable function. So, this is this is a one variable function f x. Now, if I take two points x 1 and x 2 ok. Then, if I join these two points, so, I will get this particular line, now what is this point?

So, this particular point is lambda x 2 plus 1 minus lambda x 1 ok. This particular point is lambda x 2 plus 1 minus lambda x 1. Now, what is this? This particular point is that, f of lambda x 2 plus 1 minus lambda x 1 ok. So, this is the value. This is the so, this value is equal to this one ok, that is f of lambda x 2 plus 1 minus lambda x 1.

Now, what is this? What is this value? This value is this value is lambda f of x 2 plus 1 minus lambda f of x 1 ok. So, now, if this point, if this particular point is above this point so, in that case this function is called a convex function. That means, for any value of lambda if this particular so, I am marking here this is A and this is B. So, if the point A is above B for any value of lambda.

So, if you take any value of lambda between 0 and 1, if this condition is true; that means, the point A is always above point B. So, in that case the function will be a convex function. So, this is the definition of the convex function. So, if you take any two points. Suppose, if I take this is x 1, but I can also take this is x 1 and somewhere here this is x 2 ok. So, this is x 2 and if I draw this thing then also that will be true.

That particular point will be above B basically. So, if I take means for any value of lambda that A will be above B. So, in that case the function will be called a convex function.

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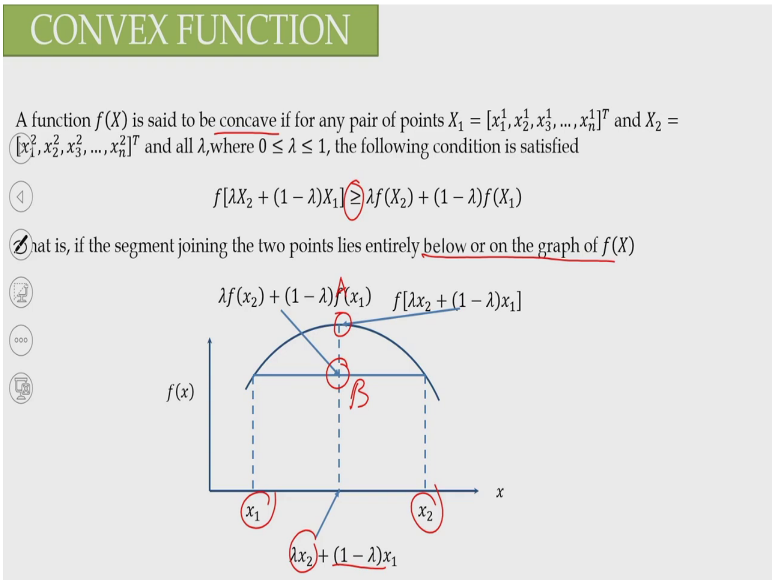

Now, the opposite of convex is concave ok. So, what is the definition of concave function? A function f X is said to be concave if for any pair of points X 1 and X 2 and for all lambda where lambda is between 0 and 1 the following condition is satisfied. So, in this case this is greater than equality ok. So; that means f of lambda X 2 plus 1 minus lambda X 1 ok that point is above lambda f of X 2 plus 1 minus lambda f of X 1 ok.

So, in this case in case of concave if the line segment joining the two points lies entirely below or on the graph ok. So, it is because this is greater than equality, if it is equal then also it is a concave function. So, it is entirely lying below or on the graph of f X. So, let me explain this thing. So, this is again this is the function and this is I say that this is a concave function.

Now, if I take two points that is x 1 and x 2 and join this particular line and this particular point is lambda x 2 plus 1 minus lambda x 1 and this particular point is. So, I can say this is this is your A and which is equal to lambda f of x 2 plus 1 minus lambda x 1, and this particular point is B that is lambda f of x 2 plus 1 minus lambda f of x 1.

So, in this case the point A that is which is on the on the on this particular function is above the point B. So, in that case the function will be called A concave function ok. So, if this condition is true we will call it concave. Now, if the condition is not satisfied then we will call the function is a non convex function ok.

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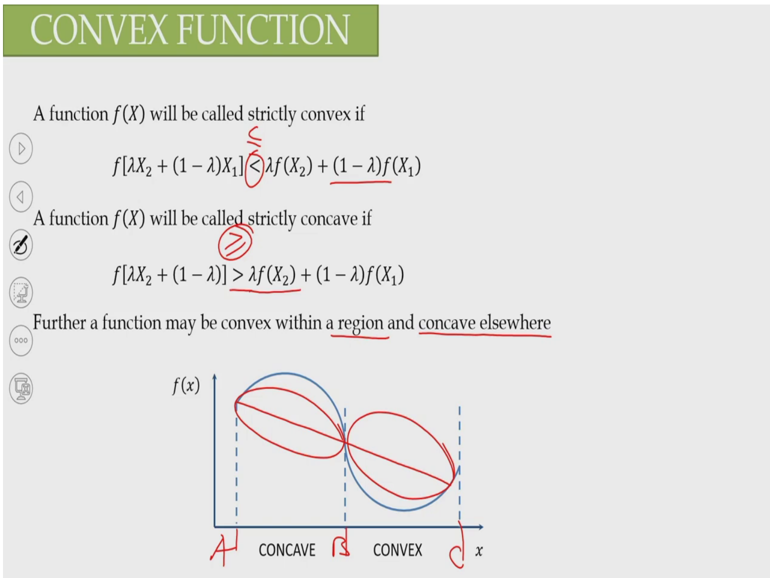

A function f X will be called strictly convex if this is less than. So, in case of convex so, it was less than equality type, but if it is less than then we will call it strictly convex. That means f of lambda X 2 plus 1 minus lambda X 1 is less than lambda f of X 2 plus 1 minus lambda f of X 1.

If it is only less than then it is strictly convex function or otherwise if you if it is less than equally type then it is a convex function. Similarly, a function will be called strictly concave. So, if it is greater than type ok. So, if it is greater than then it is called strictly concave ok. Or it is if it is greater equality it will be called a concave function, but if it is greater than then it will be called strictly concave function. Further a function may be convex within a region and concave elsewhere ok. So, this is also possible.

So, as I have shown in the second slide, that if I take a particular region the function is convex, but if I take the enter zone then that particular function is a non convex function. Similarly, a function may be convex in one region and concave may be another region. So, this is the function. So, here between these two points this is concave and between these two points suppose if I say A, B and C.

So, between A B the function is concave and between B and C the function is convex, but if I consider the region A C, then this is a non convex function. So, this is neither concave neither convex. So, this is a non convex function, because if I take any two points you just see for this portion, this is above the function and for this portion, this is below the function. So, therefore, if you take the entire region this function is a non convex function. This is neither concave nor convex.

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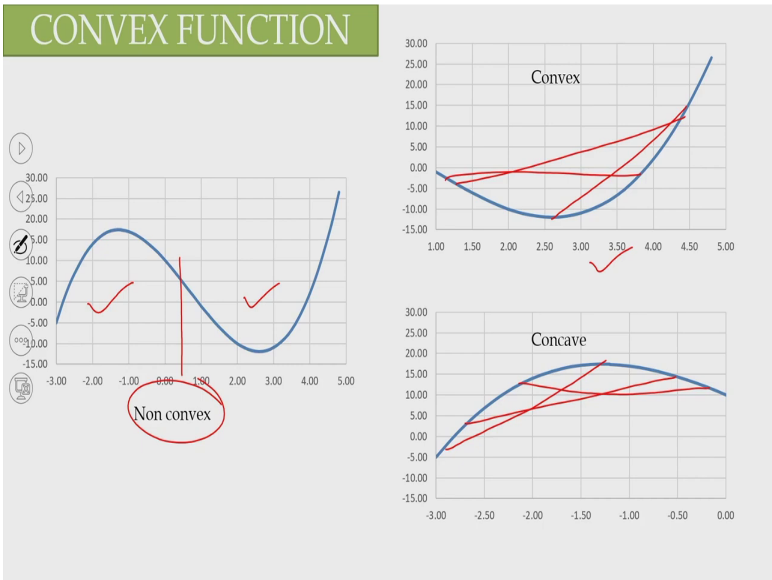

Now, these are some of the example of convex and non convex function, you can see that this is a convex function as you so; I can take any two points ok. So, this function is a convex function. Similarly, this function is a concave function, because if I take any two points the and then join the line, the line will not cut this function in anywhere else.

But, this function is a non convex function ok. Suppose, if I consider this region, then this is a convex function and this is a concave function, this is a concave function, but for between minus 3 and plus 3 between minus 3 and plus 5 this function is a non convex function

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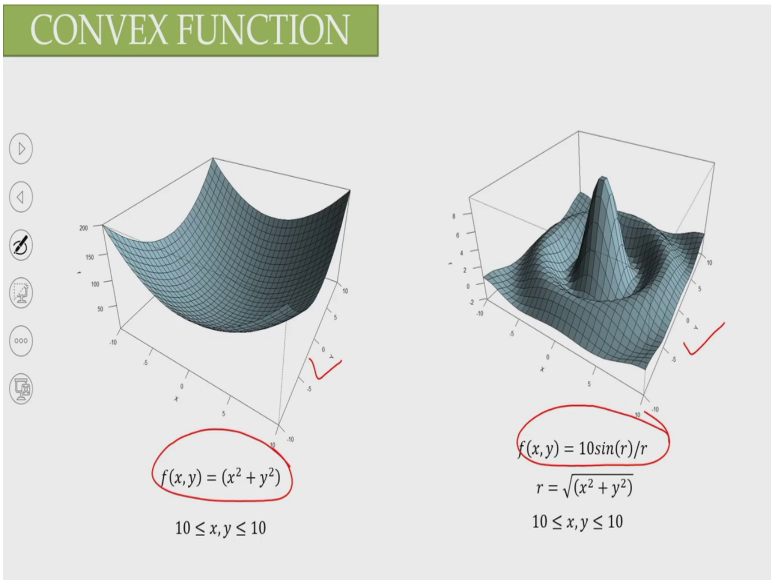

So, similarly this is a 2D function two variable function and you can see that this function is a convex function. But, if you look at this particular function so, this is a non convex function ok.

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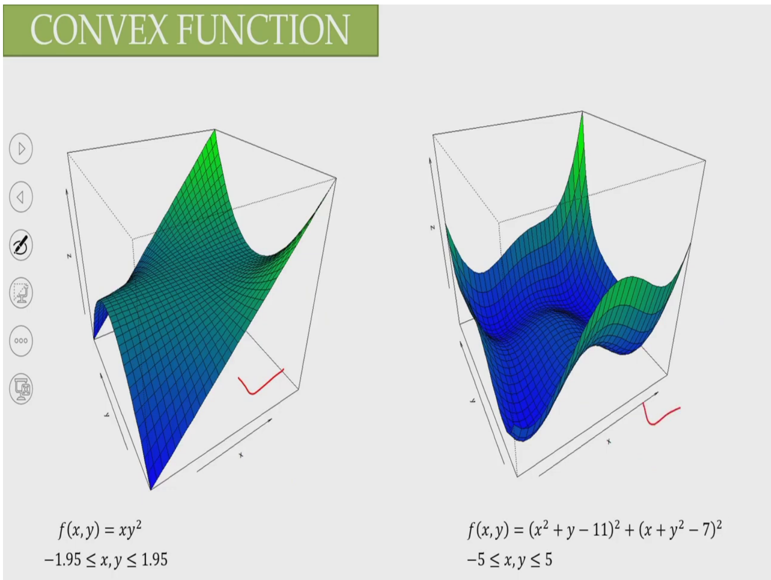

Similarly, there are some other examples. So, this is also a non convex function and this is also a non convex function. Why we are looking at whether the function is a convex function or a non convex function? So, you can see that if the function is a convex function, then there is only a one optimal solution, either it is a minima or it is a maxima. So, therefore, the solution process will be very simple.

So, I will get so, there will be only one solution and if you can say that is a local optimal solution as well as that is also a global optimal solution of this particular function. So, therefore, the solution procedure will be very simple. So, I can easily find out, the local as well as global optimal solution of the problem, but if your function is a non convex function, then the solution procedure will be complex.

Because, if you are taking a particular initial point you may get one optimal solution, if you take another points you may get a different optimal solution. So, there will be more than one optimal solution or you may have alternate optimal solution. So, therefore, if I know the nature of the function that, if I know that nature of the function is a convex function, then I think you need not worry. So, you can apply your classical method for solving the problem.

Similarly, if I know that my function is a non convex function then you have to plan accordingly. So, your classical method may not work. So, in that case we will discuss those algorithms also, there are some materialistic algorithms which you can you can apply for solving complex problem. Suppose, when you have more than one optimal solution or you have multiple optimal solution and one of them is the global optimal solution.

So, therefore, you have to know the nature of the function whether your function is a convex function or non convex function ok.

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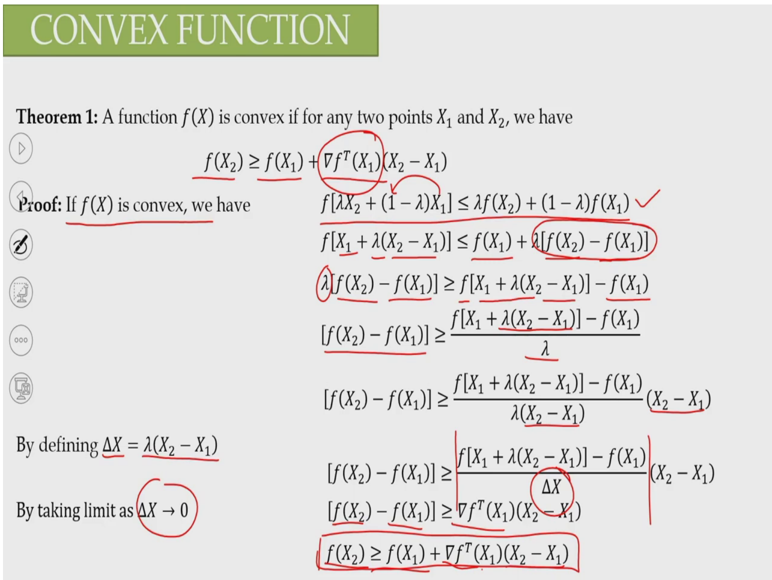

Now, let us see the theorem ok. So, theorem 1; a function f X is convex if for any two points X 1 and X 2 we have that f of X 2 is greater than equal to f of X 1 plus del f X 1 into X 2 minus X 1. So, if this condition is true ok, this condition is true. So, this is the first derivative at X 1, first derivative at X 1 into X 2 minus X 1.

If, this condition is true then the function will be a convex function. So, if a function f X is convex, if for any two points X 1 and X 2 we have this condition ok. So, if this is true then the function is a convex function. Now, let us prove it if f X is convex. So, we have ok. So, this is coming from the definition of convex function.

So, what is the definition? The definition is that function value at lambda X 2 plus 1 minus lambda X 1 is less than equal to lambda f of X 2 plus 1 minus lambda f of X 1. So, we got this relation from the definition of a convex function ok. Now, if we rearrange this function.

So, if we rearrange this and I can write it like that, that f of so, f of X 1 so, this X 1 is f of X 1 plus lambda X 2 minus X 1 less than equal to f of X 1. So, f of X 1 plus lambda f of X 2 minus f of X 1 so, I can write like this.

So, I if I rearrange the terms so, if I rearrange this term then let us let us take this term on the left hand side. So, I can write it lambda f of X 2 minus f of X 1 is greater than equal to f of X 1 plus lambda X 2 minus X 1 minus f of X 1 ok.

If, I take lambda on the right hand side so, I can write that f of X 2 minus f of X 1 is greater than equal to f of X 1 plus lambda f of X 2 f of X 1 plus lambda X 2 minus X 1 minus f of X 1 divided by lambda. Now, if I multiply X 2 minus X 1 here and X 2 minus X 1 here. And, by definition del X equal to if I take lambda X 2 minus X 1. So, if I take that one then I can write this is as a del X ok. So, this is that lambda X 2 minus X 1. So, if I put it del X. So, I can write like this.

And, if del X tends to 0 ok. So, del X tends to 0 and then this particular term these terms will be the first derivative is not it. So, this will be the first derivative this term del f X 1 ok. So, this will be the first derivative. So, what I am getting, that f of X 2 minus f of X 1 is greater than equal to del f T X 1 into X 2 minus X 1 ok. So, if I rearrange the term. So, I will get f of X 2 greater than equal to f of X 1 del f T X 1 X 2 minus X 1.

So, I am getting. So, from the definition of convexity ok, definition of a convex function. So, we can derive this equation that f of X 2 is greater than equal to f of X 1 plus del f X 1 into X 2 minus X 1 ok. So, we have proved it; that means, if this condition is true then the function will be called a convex function.

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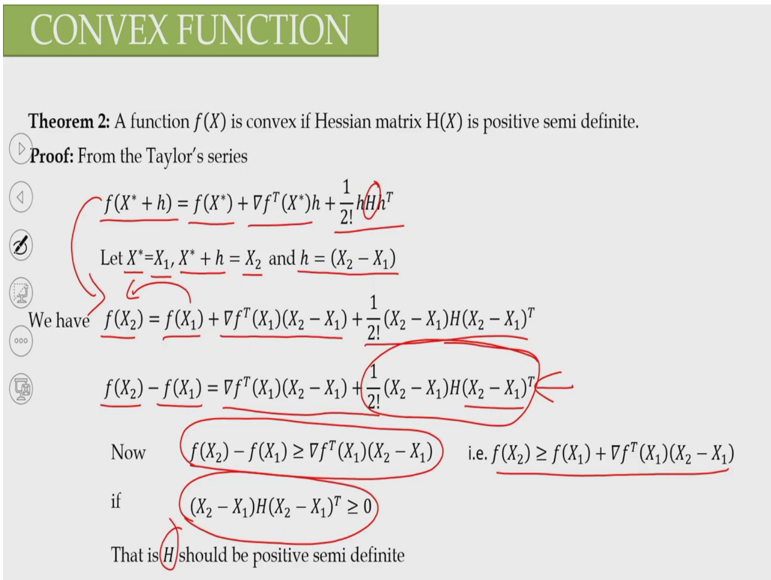

Now, let us look at the theorem 2; a function f X is convex if the hessian matrix is positive semi definite ok. So, if the Hessian matrix is positive semi definite then the function will be a convex function ok.

So, let us start from the Taylor series. So, I can write that f of X star plus h which is equal to f of X star del f T X star h into 1 by factorial to small h capital H h T, so, where the h is the Hessian Matrix. So, if I take X star equal to X 1, X star plus h equal to X 2 and h equal to X 2 minus X 1 ok. So, that is the your step length you can say that h equal to X 2 minus X 1.

So, if I do that then I can write this Taylor series something like that, f of X 2 equal to f of X 1 del f T X 1, X 2 minus X 1 1 by factorial 2 X 2 minus X 1 H X 2 minus X 1 transpose ok. So, I can I can write this one. Now, if I rearrange the term. So, this is f of X 2 minus f of X 1.

So, I am taking on the other hand side, then I can I can write like this del f T X 1, X 2 minus X 1, 1 by factorial 2 X 2 minus X 1 H X 2 minus X 1 transpose.

Now, what is the definition of convex function? So, as for theorem 1; if this is true ok, if this is true then the function is a convex function; that means, f of X 2 is greater than equal to f of X 1 plus del f T X 1 into X 2 minus X 1. If, this is true then the function will be a convex function. Now, when this will be true, if you look at this particular line so, this will be true when this is positive ok.

So, this term is positive; that means the 1 by factorial to X 2 minus X 1 H X 2 minus X 1 transpose. So, if this term is positive, then this condition will be true ok. So, this will be true when this particular term is positive. So, this is positive means, this is greater than equal to 0 ok. So, this is greater than 0. Now, when this will greater than 0 that H should be a positive semi definite, if H is positive semi definite, then this particular terms will be 0.

So, therefore, you can say that if function will be convex when the H the Hessian Matrix is positive semi definite. So, we can evaluate the Hessian Matrix. So, already we have discussed how to check whether the Hessian Matrix is the positive definite or negative definite or positive semi definite. So, I can do that and by looking at the Hessian Matrix.

So, you can tell whether your function is a convex function or a non convex function. If, it is positive semi definite in that case that function is a convex function.

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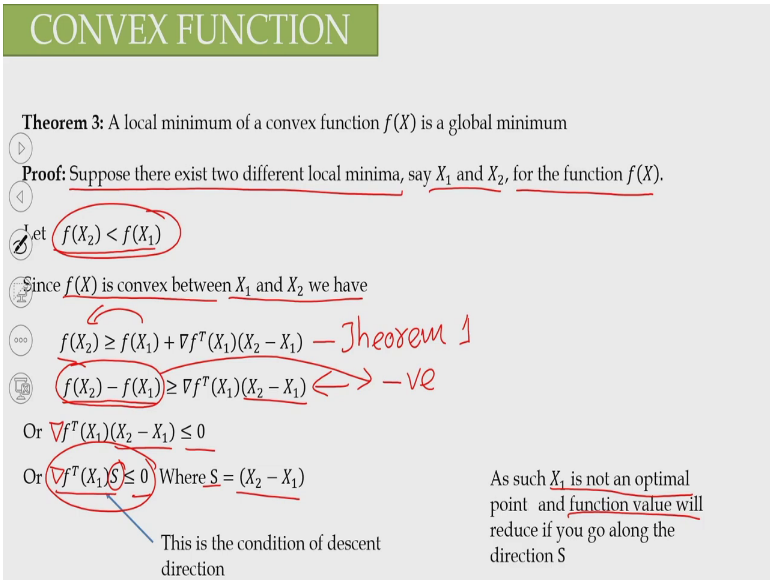

Now, let us go to theorem 3; the theorem 3 is saying a local minimum of a convex function is a global minimum. So, what is theorem 3? So, a local minima as I said earlier, that if your function is a convex function. So, in that case you have only one optimal solution.

So, you can say that that is the local optimal solution as well as global optimal solution, because you have only one optimal solution, either it is a minima or it is a maximum point. So, therefore, if your function is a convex function, then whatever local minima you have that will also be a global minima. So, with theorem 3, so, we will prove that a local minima of a convex function is also a global minima ok.

So; that means, there is only one optimal solution that is local as well as global optimal solution. So, let us prove that one. So, we will prove it by contradiction. What we will do basically, we will initially say that there are two different local minima. So, we will say that let us consider that there exists two different minima. So, this is basically we have stated so, we have stated. So, we have to prove that this is not correct ok.

So, we have to prove that this is not correct. So, what we have assume initially, that there exist two different local minima. So, we are saying that so, we are saying that there are two minimas of a convex function. And, now we have to prove it that this statement is not correct ok. So, that we are going to do that. So, it is basically by contradiction we are trying to prove, that there is only a one minima and that minima is also a global minimum point ok.

But, initially so, as I said we are stating that there are two local minima say X 1 and X 2. So, there are two points ok and these two are minimum points X 1 and X 2 for the function f X. So, we will we are assume that f of X 2 is less than f of X 1; that means, if I say that there are two minimum point that, X 2 is the global minima and X 1 is the local minimas; that means, f of X 2 is less than f of X 1. Since, f X is a convex between X 1 and X 2.

So, we already said that the function is a convex function. Since, f of X is convex between X 1 and X 2 we have so, this is already we proved it. That is the theorem 1. So, as per theorem 1 that f of X 2 is greater than f of X 1 plus del f T X 1 into X 2 minus X 1 ok. So, this is already we have proved that ok. So, as per theorem 1 so, this is theorem 1 ok.

So, as per theorem 1 already we have proved it that f of X 2 is greater than f of X 1 plus del f T X 1 into X 2 minus X 1. So, that is as per theorem 1. Now, if I rearrange the terms so, f of X 2 minus f of X 1. So, I am taking on the other side is greater than equal to del f T X 1 into X 2 minus X 1. So, I am rearranging this term. Now, here as assume so, what we have assumed? That f of X 2 is less than f of X 1.

So, f of X 2 is less than f of X 1 so; that means this particular ok. So, left hand side, left hand side of this equation is basically is negative is not it, f of X 2 is less than f of X 2 is less than f of X 1 so, therefore, f of X 2 minus f of X 1 is negative ok. So, this particular this particular term is negative, this particular term is negative ok. So, this is negative. So, therefore, that

that del f T X 1 into X 2 minus X 1 is less than equal to 0, because this part f of X 2 minus f of X 1 is negative ok.

So, sorry I did not write del here. So, now, what is this, if I take X 2 minus X 1 as a direction, it is direction X 2 minus so, difference between these two points is giving you a direction. So, if I write the direction S, which is equal to X 2 minus X 1. So, I can write that del f T X 1 into S ok. So, S is the direction and which is less than equal to 0. Now, what is this condition? So, this condition, this condition is basically condition of descent direction ok.

So, this is the conditions of descent direction; that means, what is descent direction? So, as we have already discussed descent direction means, if we go along that direction S, that is X 2 minus X 1, then the function value will reduce ok. So, that is called descent direction; that means, if you go along that direction, that function value will reduce. So; that means, whatever optimal solution you are getting ok.

So, X 1, X 1 is not a local minima because from X 1 you can go to X 2 and the function value will reduce and this condition is telling that one; that means, at X 1 if you go along s direction the function value will reduce. So, therefore, X 1 cannot be a local optimal solution ok. As such X 1 is not an optimal point and function value will reduce if you go along the direction X ok. So, therefore, X 1 cannot be an optimal point.

So, we have proved it that X 1 cannot be an optimal point. So, therefore, a local minima of a convex function f X is also a global minima ok. So, that we have proven.

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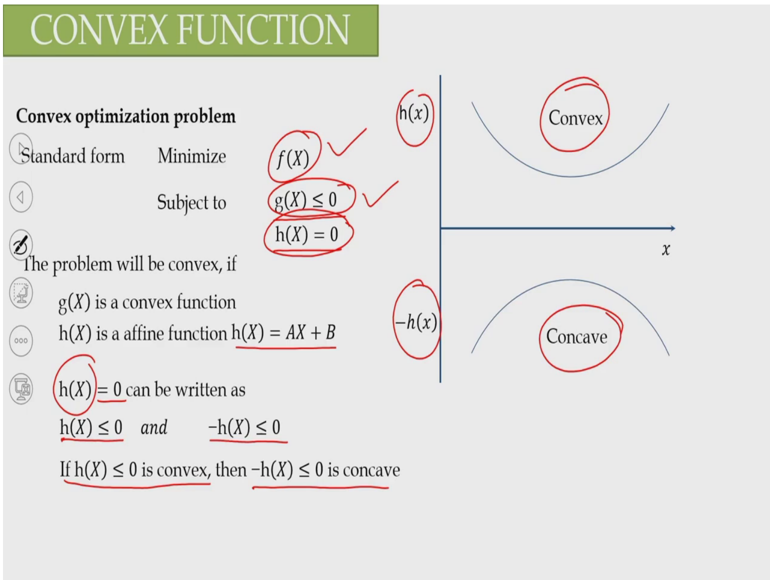

Let us see, what is convex optimization problem? A standard form of optimization problem suppose if I write minimize f X subject to g X is say less than equally type constraint and h X is greater than equally type constraint; that means, g X is less than equal to 0, h X is equal to 0 ok. So, this is the standard form of a problem ok, standard form of a of an optimization problem.

So, you have an objective function, you have an objective function, you have a constraint less than equally type constraint and you have also a equality type constraint. The problem will be convex if the function f X is convex and g X is also convex and h X is a fine function; that means, h X is linear that is h X equal to a X plus B.

Now, suppose h X is not linear. So, it is a non-linear function and this is a equally type constraint. So, what we can do? We can write this particular function as a h X less than equal to 0 and minus h X is less than equal to 0; that means, I can replace this equality type constraint by using 2 less than equally type constraint; that means, h X less than equal to 0 and minus h X less than equal to 0 ok.

Now, if h X is convex, then minus h X will be concave. So, therefore, if you are putting a convex function along with concave so, this will not be a convex. So, let me check that one suppose this is h X and this is negative of h X. So, if this is a convex function, then this will be a concave function. So, therefore, the problem will be a non convex problem. So, if you have a equality constraint in your problem and the constraint is non-linear.

So, in that case your entire problem will be a non convex problem. A problem will be a convex problem, if your objective function is convex, if your constraint that inequality type constraint is a convex function. And, if you have equality type constraint that must be a linear function ok. So, if your function is linear equality type function, then your problem will be a convex problem, but if you have any non-linear equality type constraint, then your problem will be a non convex problem.

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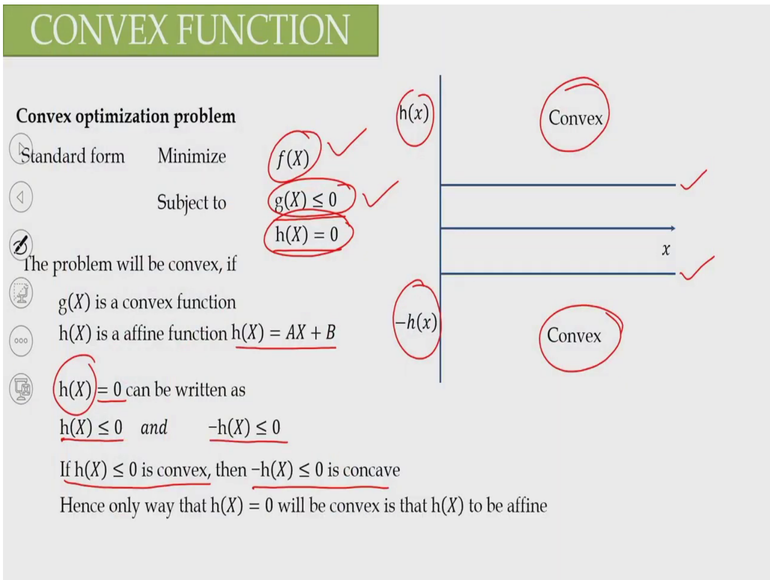

So, therefore, so, only possible is that if it is a fine function. So, in that case this will also be convex and this will also be convex. So, in that case your entire problem will be a convex problem. So, in today's class we have discussed about convex function. So, it is an property of a function and if your function is a convex function, then you will have only one optimal solution.

So, then you can apply your classical method for finding the optimal solution of the problem, but if your problem is a non convex problem, then you have to give some special treatment to your function to find the global optimal solution of the problem. So, therefore, before designing an algorithm that, which algorithm you would like to apply for solving your problem.

So, you first look at whether your function is a convex function or a non convex function. So, if your function is a convex function or if your problem is a convex problem so, in that case solution procedure will be simpler and you can apply the standard classical method for finding the solution of the problem, for finding the solution of the problem, but if you problem is a non convex problem. So, in that case you have to look for a specialized algorithm for finding the global optimal solution of the problem.

Thank you.