Plates and Shells
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Module-03
Lecture-09
Levy's Solution for Different Loading and Boundary Conditions

Hello everybody, today I am on the lecture 2 of module 3 and if you recall my last class, that I started the Levy's formulation for rectangular plate, which is by nature and provides an exact solution of the plate differential equation. And in that connection, we have seen that Levy's formulation can be done with the help of single series. And then, the convergence is repeat, but it is applicable for certain condition that when the two opposite edges of the rectangular plates are simply supported.

And other 2 opposite edges may have same condition or may have different conditions also. But, thing is that when this condition is not met directly, then Levy's condition cannot be applied. So, let us see how we can explore the Levy's formulation to solve other plate problems. That means plate subjected to other type of loading. In the last class, I have discussed the problem which was already solved by Navier's method, that four edges were simply supported.

So, in that case, we applied the Levy's condition, and we have found that the result obtained was closely in agreement with the Navier's method.

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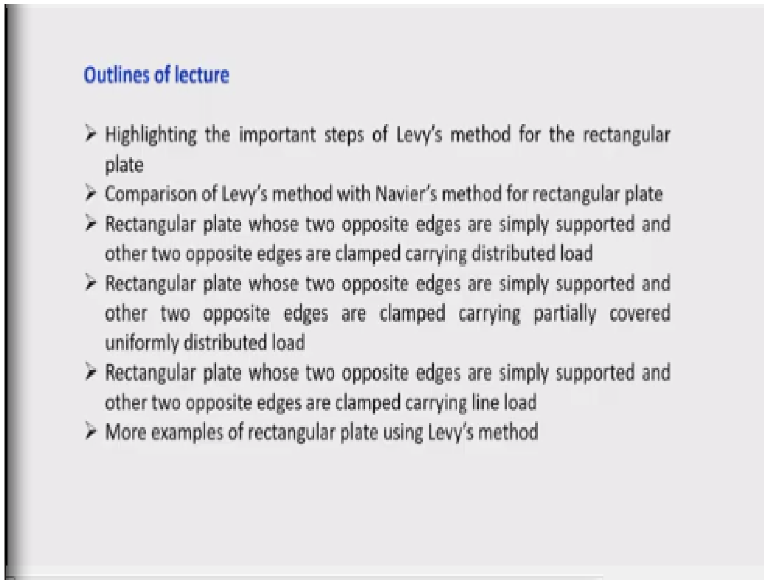

So, today's lecture that outlines will be highlighting the important steps of Levy's method for rectangular plate, comparison of Levy's method with Navier's method for rectangular plate. Then we will illustrate a problem of rectangular plate whose two opposite edges are simply supported, and other 2 opposite edges are clamped, which the plate carries uniformly distributed load. Then we will do this problem that 2 opposite edges are simply supported, and other 2 opposite edges are clamped carrying partially covered uniformly distributed load.

In the first case, the edges are clamped on 2 opposite edges, and here also the edges are clamped. But difference is that in the first case, the full plate was loaded; in the second case, the plate was partially loaded. Then we will study the plate with two opposite edges simply supported, and other 2 edges are clamped but carrying strip loading. So, that type of loading is sometimes encountered in case of brick wall that is constructed over this slab directly, not transferring the brick load on the beam.

So, in that case, the analysis of the plate for the line loading is important, and we will discuss this strip loading on the rectangular plate. Then some more examples of rectangular plate, I will try to show you and how it can be tackled in a different situation. So, Levy's method I have told that it gives the exact solution, Navier's method also gives the exact solution.

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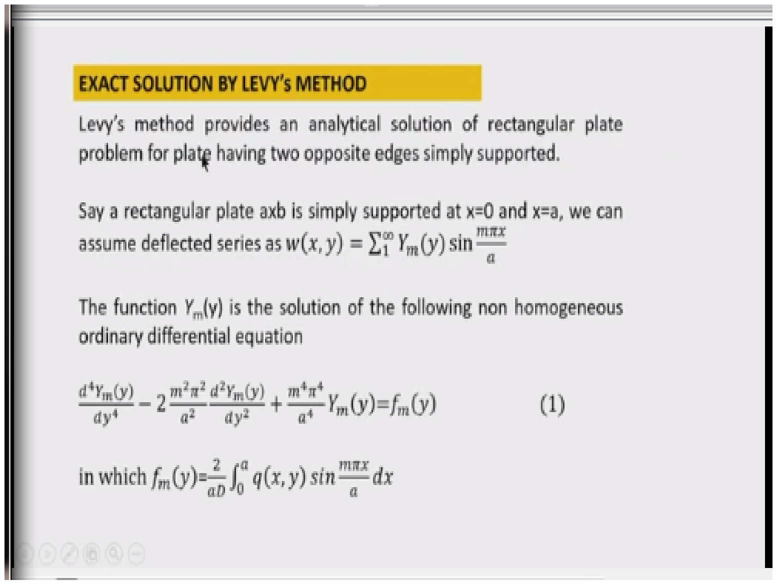

But the difference is that Navier's method the requirement is that all the 4 edges of the rectangular plate should be simply supported. But in case of Levy's method, some relaxation is made. That means only these 2 opposite edges required to be simply supported, and other 2 edges may have any boundary conditions, may have same boundary conditions or may have different boundary conditions or may have elastically supported edge or these supported by torsional spring edge support.

So, many conditions may exist, but it can be tackled by Levy's method when two opposite edges are simply supported. The most fundamental thing in Levy's method is that the deflection is assumed as a sine series. That you are seeing that we have to assume the deflection surface as single sine series, the summation extends from 1 to infinity. And the summation is done that the deflected surface is taken as a function of y multiplied by a sine function, $sin\frac{m\pi x}{a}$.

Here it is $sin\frac{m\pi x}{a}$ because the x edges, the x = 0 and x = a, that means edges parallel to y-axis are simply supported, so therefore sine function is taken. You can see that taking sine function it renders the boundary condition to be satisfied x = 0 and x = a edges, there is curvature along x direction is 0 and then your deflection is 0 at the edges x = 0, x = a. So, by taking these deflected surface this function, we have proceeded to satisfy the differential equation of the plate which is 4th order partial differential equation.

And you know that differential equation in compacted form is $\nabla^4 w = q(x, y)/D$, where D is the flexural rigidity of the plate, it is given by $\frac{Eh^3}{12(1-\nu^2)}$. Now, after substituting this deflected function in the differential equation $\nabla^4 w = q/D$. We actually arrive at this ordinary differential equation, $\frac{d^4 Y_m(y)}{dy^4} - 2\frac{m^2\pi^2}{a^2}\frac{d^2 Y_m(y)}{dy^2} + \frac{m^4\pi^4}{a^4}Y_m(y)=f_m(y)$, where $f_m(y)$ is a function of y.

How this function is produced? That can be understood by when I substitute this function to the original differential equation, and then after multiplying both sides by $sin\frac{m'\pi x}{a}$. And then using the arithmetic condition of the sine function, we can arrive; this differential equation with non-homogeneous term which becomes a function of y. Because the integration is actually carried out with respect to x, here you can see it.

Then because of integration with respect to x, the result will not contain any x variable. So, it will be purely a function of y. So, these differential equation you are seeing that it is the differential equation with y. So, that differential equation has been solved with repeated roots that are found for the characteristic equation for the homogeneous solution.

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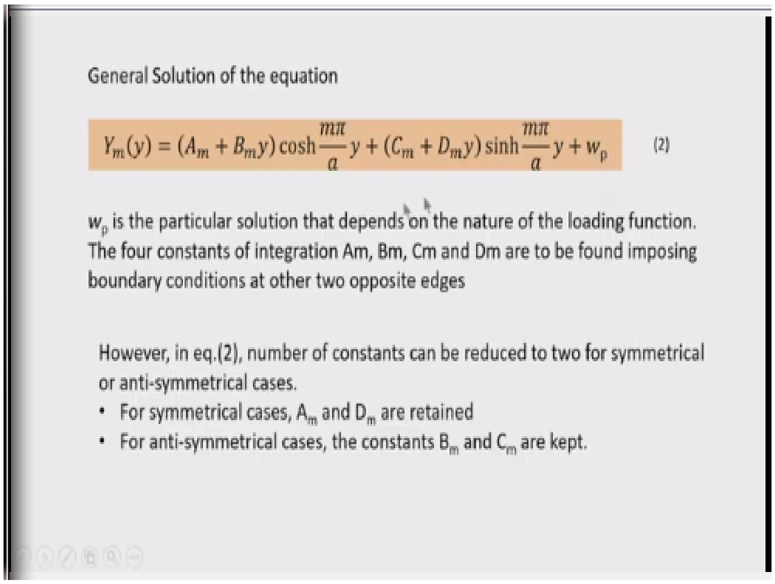

And then we have found that the solution is given for the homogeneous part as this function. That is $Y_m(y) = \left(A_m + B_m y\right) \cosh cosh \frac{m\pi}{a} y + \left(C_m + D_m y\right) \sinh sinh \frac{m\pi}{a} y + w_p$ or particular solution. That integral is to be found considering the forcing function in the left-hand side. If there is a no-load acting on the plate, then this function will be 0.

Now you can see the characteristic of the general solution of this equation number 1 here. You can see here that A_m, B_m, C_m, D_m are arbitrary constants that can be found by imposing the boundary condition on other 2 opposite edges. That means y = 0 and y = b edges if the side of the plate is $a \times b$. And then, this origin is taken at one of the corners on the left-hand side. So, w_p is the particular solution that depends on the nature of loading.

But one interesting thing you can see here, that $\cosh cosh \frac{m\pi}{a} y$, it is a symmetric function, and $\sinh sinh \frac{m\pi}{a} y$ is antisymmetric function. So, when the loading conditions and support conditions are symmetrical with respect to x-axis passing through the centre of the plate, then there is no need to retain any antisymmetric term on the deflection expression.

So, therefore, in that case, we can only retain the symmetric term, symmetric terms are say $\cosh cosh \frac{m\pi}{a} y$. And product of 2 antisymmetric term again is a symmetric, so we can take these 2 terms $A_m \cosh cosh \frac{m\pi y}{a} + D_m y \sinh sinh \frac{m\pi y}{a} + w_p$, if the support condition and loadings are symmetrical. And if there is a condition that anti symmetry exists in respect of loading, that means in one side the loading is $+ q$ in another side loading is $- q$ it varies linearly with 0 value at the centre, then it is anti symmetric loading.

And in case of antisymmetric loading, we can take only the antisymmetric term. The antisymmetric term here you can note that $y \cosh cosh \frac{m\pi y}{a}$ is antisymmetric. Why it is antisymmetric? Because y is a anti-symmetric function and $\cosh cosh \frac{m\pi y}{a}$ is a even function or symmetric function. So, product of antisymmetric and symmetric function is again anti symmetric function.

So, therefore in that case, we will retain this term and $\sinh sinh \frac{m\pi y}{a}$ is a antisymmetric term. So, $C_m \sinh sinh \frac{m\pi}{a} y$, this term and this term along with this coefficient $B_m y$ and C_m have to be kept in case of antisymmetric loading. So, for symmetric cases, A_m and D_m are retain, for anti-symmetrical cases the constant B_m and C_m are kept.

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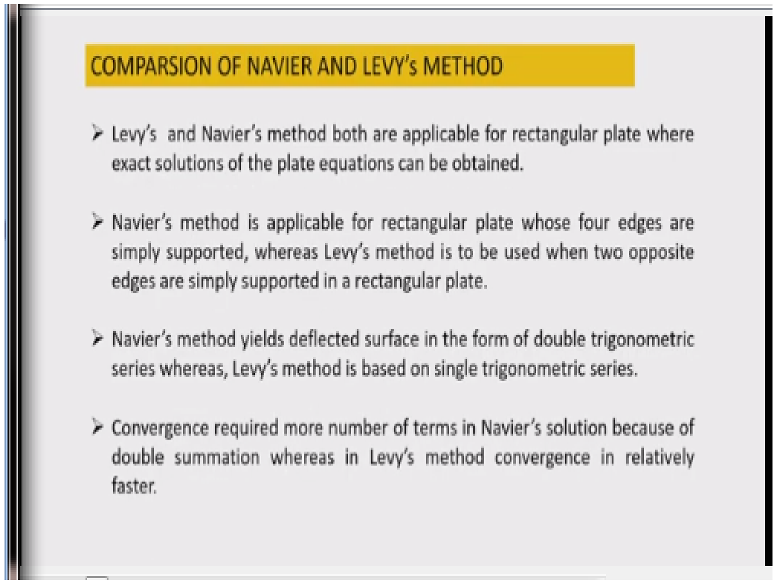

So, if we compare the 2 methods that we have learned so far, one is Navier's method, and another is Levy's method. Navier method is specifically applicable for a rectangular plate which has 4 edges simply supported. So, that is the condition that must be satisfied first; then we can use the Navier's as method. So, both the Levy and in Navier's method are applicable for rectangular plate that they have some general character.

And both of these methods actually use this trigonometric series, and both the methods yield the exact solution of the plate equations. So, that is some general character we have noted in case of Levy and Navier's method. But in Navier's method is applicable for rectangular plate whose 4 edges are simply supported, as I have repeatedly told this. Whereas Levy's method is to be used when 2 opposite edges are simply supported, so this is the difference.

But it is not that Levy's method cannot be applied for Navier's condition. Levy's method can be applied when 4 edges are simply supported. So, Levy's method is more general compared to Navier's method. Navier's method yields deflected surface in the form of double trigonometric series, but Levy's method is based on single trigonometric series. Because of double summation Navier's method is slowly converging method.

Whereas Levy's method it uses single trigonometric series, therefore it is first converging but the computation in Levy's method is slightly involved compared to maybe s method. So, these are the major difference between the Navier's method and Levy's method for rectangular plate.

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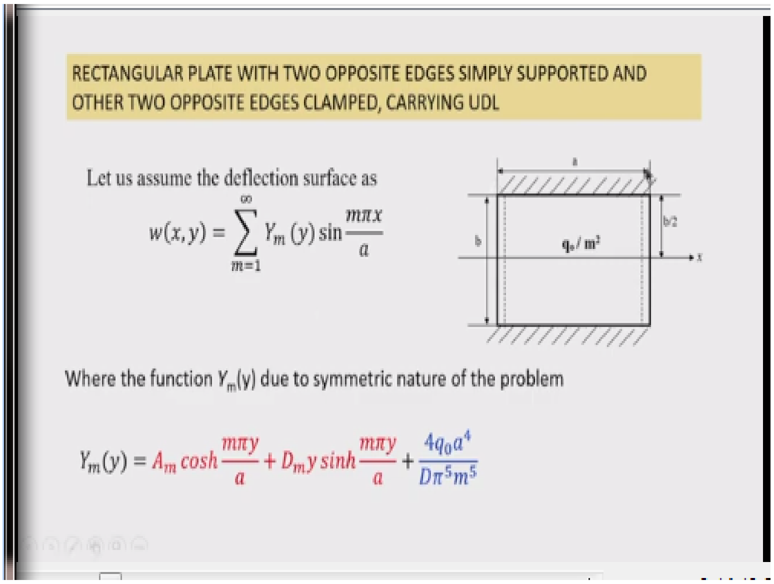

Now, let us see a problem when the rectangular plate a cross b is the dimension, and your thickness is h so that flexural rigidity D is $\frac{Eh^3}{12(1-\nu^2)}$. The plate is loaded with this q_0 is the loading intensity per unit area, and since it is symmetrical, so we pass the axis, x-axis passing through the centre of the edges, which is parallel to y-axis. So, the plate is symmetrical about the x-axis that you can see here.

And these 2 opposite edges are simply supported x = 0, x = a, and other 2 opposite edges y = 0 and y = b, are also I have the same conditions in the boundary and that conditions are clamped. Because clamped conditions are very common in case of plate, because in case of steel plate we weld the plate the component maybe the flange of a column or this wave of guarder. Then we have the reinforced concrete slab where the edges are the slab is supported by beam which is integrally built, that means monolithically cast.

So, therefore fixed condition develops at the end. So, fixed conditions are common, and therefore we cannot ignore this, and with the help of Levy's method, we can easily encounter this condition. So, let us assume the deflection surface as $Y_m(y) \sin sin \frac{m\pi x}{a}$. Now earlier, we have seen that solution is obtained in this form, but since in that case, the condition that I am now discussing that 2 opposite edges are clamped, and 2 opposite edges are simply supported. So, in that case symmetric condition is satisfied and therefore, we will take this term $A_m \cosh cosh \frac{m\pi y}{a} + D_m y \sinh sinh \frac{m\pi y}{a} + w_p$ and w_p will be there because the plate is loaded, so naturally, this particular solution will also exist.

So, now these solutions can be written in this form for $Y_m(y)$, that is $A_m \cosh cosh \frac{m\pi y}{a} + D_m y \sinh sinh \frac{m\pi y}{a}$. You can see the term that I have written with the red colour is a symmetric term specifying the homogeneous solution. But this term that is written with a blue colour is a particular integral, because the loading is here constant. So, particular solution is also assumed as a constant and after substituting in the differential equation, the solution is obtained as this $\frac{4q_0a^4}{D\pi^5m^5}$.

So, this particular solution is due to the load acting on the plate. Now, our requirement is that the boundary conditions have to be satisfied. Now on the boundary that is y = 0 and $y = \pm\frac{b}{2}$. You see now, the x-axis is passing through the edges parallel to y-axis, so, therefore, $x =+ \frac{b}{2}$ or $x =- \frac{b}{2}$, $x =- \frac{b}{2}$ is this edge if the direction of positive direction of y-axis upward here, then this edge specified as y = -b by 2 and this is the edge specified with $y =+ \frac{b}{2}$.

So, at both the edges, say $y =- \frac{b}{2}$., the boundary condition is same. So, now we will apply the boundary condition at the 2 opposite edges.

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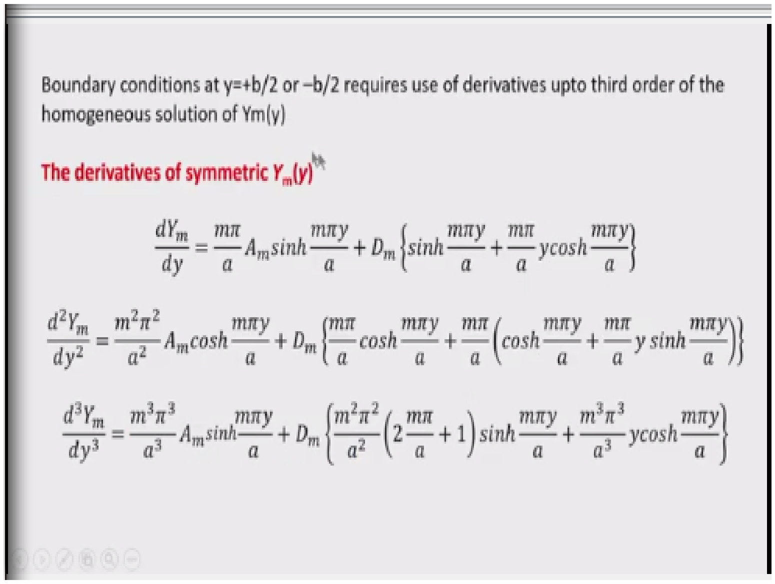

Now, boundary condition at edge $y = + \frac{b}{2}$ or $- \frac{b}{2}$ requires use of derivative; let us now physically understand the boundary condition of these 2 edges. Now, due to fixity, the slope will be 0 along the y-axis. So, this $\frac{\partial w}{\partial y} = 0$, and $\frac{\partial w}{\partial y} = 0$ that means $\frac{dY_m}{dy} = 0$. And because of clamped condition, the w is also 0; that is, y is also 0. So, at $y = + \frac{b}{2}$ and $- \frac{b}{2}$ boundary condition requires the function itself, and it is derivative up to third order maximum.

Because when we relate shear, that third-order derivative is required. So, in these 3 equations, I have given you the derivative of the function because derivative can be easily computed from the function this function $Y_m(y)$. First derivative you can see, because this is the constant term, so derivative of this term with respect to y is 0. But when I differentiate this term with respect to y, it will be A_m, $\frac{m\pi}{a}$ will come out as constant.

And $cosh$ will be converted to sinh $sinh \frac{m\pi}{a} y$. Now, here you are getting product of 2 functions of y, one is y, and another is sinh $sinh \frac{m\pi}{a} y$. Because the product of 2 terms when it is differentiated, it will give you say y. First, if I differentiate with respect to y for the second term, then $\frac{m\pi}{a}y.cosh\frac{m\pi y}{a}$ then I differentiate the first term, then differentiation of y with respect to y is 1.

So, then $sinh\frac{m\pi y}{a}$. So, therefore this condition you can see here that I have written, after differentiating the $Y_m(y)$. Then differentiating this quantity again, this $sinh\frac{m\pi y}{a}$, this $\frac{m\pi}{a}$ will come out, so it will be square term $\frac{m^2\pi^2}{a^2}$ then A_m. And $sinh$ will be converted to $cosh$ due to differentiation, so $cosh\frac{m\pi y}{a}$.

Then the terms inside the; second bracket that I will differentiate, now $sinh$ if I differentiate then $\frac{m\pi}{a}cosh\frac{m\pi y}{a}$. Then this term if I differentiate, this is the constant term, so I have taken it. And then this term $y.cosh\frac{m\pi y}{a}$ is differentiated term by term. First-term is differentiated with respect to y, so it is 1, then $cosh\frac{m\pi y}{a}$.

Then secondly, it is differentiated with respect to cos hyperbolic with respect to y this function, that $cosh\frac{m\pi y}{a}$is differentiated with respect to y. And therefore, $\frac{m\pi}{a}$ is coming out here, and then y remains as it is, and $cosh$ is transform to $sinh\frac{m\pi y}{a}$. So, this is the second derivative, it is required to impose the bending moment condition at the edges.

For simply supported edges, this is required, and also, for the spring supported edges, we required bending moment to be 0, so this derivative is important. Then if I differentiate further up to the 3rd order, then here you can see that I differentiated this, so $\frac{m^3\pi^3}{a^3}$, this term is coming. And then $\frac{m\pi}{a}$ already has come out inside the $cosh$ function, and then $cosh$ is transformed to $sinh$, $y.sinh\frac{m\pi y}{a}$.

$+D_m$, now inside this, you are getting so many terms, so you differentiate term by term. So, after differentiating and arranging some common terms, you will get this with $sinh\frac{m\pi y}{a}$ coefficient are this, and with $y\, cosh\frac{m\pi y}{a}$y coefficients are these. So, one important thing you can note here that when we take the odd derivative, you can see the odd terms are appearing.

So, $sinh\frac{m\pi y}{a}$is odd term; here also you can see $y\, cosh\frac{m\pi y}{a}$is also odd term. So, then when we are differentiating with respect to second derivative, that is, the even derivatives, so you can see the even terms are appearing. So, $cosh\frac{m\pi y}{a}$, then here you can see this is even term, and here this is also even, and this is odd, and this is also odd. So, product of 2 odd functions is again even function.

So, with even derivatives, the even terms are associated. Then when again we will go further higher derivative, say third derivative, then you can see that interestingly all the even terms appear. So, this $sinh\frac{m\pi y}{a}$ is odd term appears, when we do the 3rd derivative, the $sinh\frac{m\pi y}{a}$ is odd term. Then here you can see again $sinh$ is odd term. And here y is your odd term, but $cosh$ is even term, so product of odd and even is again odd term.

So, this interestingly, it is noted that with odd number of derivatives, odd terms are appearing, with even number of derivatives, even terms are appearing. Now apply boundary condition at $y = \ +\frac{b}{2}\ or\ -\frac{b}{2}$. So, I have chosen to apply the boundary condition at $+\frac{b}{2}$. So, Y_m at $\frac{b}{2} = 0$, now what is Y_m? If you see $Y_m = 0$ was this function that is a solution of this differential equation, complete solution.

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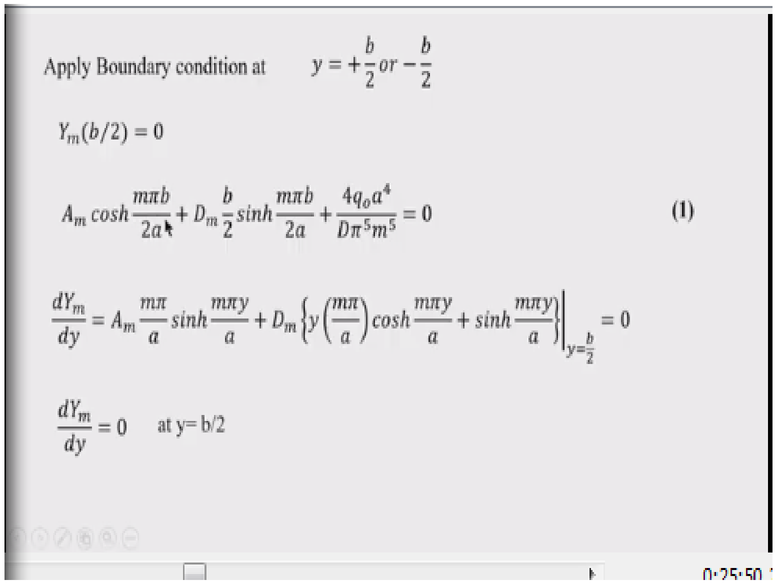

So, if I apply the deflection boundary condition at $y = \frac{b}{2}$, you will get that instead of y, I have substituted $\frac{b}{2}$, so $\frac{m\pi b}{2a}$. Similarly, on the other case with the D_m in terms of y was $y\, sinh\frac{m\pi y}{a}$. Now, instead of y, I am putting $\frac{b}{2}$, so it will be $D_m\frac{b}{2}$sinh $sinh\ \frac{m\pi b}{2a}$ this particular integral due to loading. So, that is the boundary condition at $y = b/2$, at one of the edges.

Then the clamped condition, that is, the slope, has to be 0 so, that I am now doing on the first derivative of the function. So, first derivative if this and that have to be 0 at $y = b/2$.

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$$A_m \frac{m\pi}{a} \sinh\frac{m\pi b}{2a} + D_m \left\{ \frac{m\pi b}{2a} \cosh\frac{m\pi b}{2a} + \sinh\frac{m\pi b}{2a} \right\} = 0 \qquad (2)$$

Equation (1) and (2) can be solved to find A_m and D_m

By Cramer's rule

$$A_m = \frac{\Delta_1}{\Delta} \quad , \quad D_m = \frac{\Delta_2}{\Delta}$$

Where

$$\Delta = \begin{bmatrix} \cosh\frac{m\pi b}{2a} & \frac{b}{2}\sinh\frac{m\pi b}{2a} \\ \frac{m\pi}{a}\sinh\frac{m\pi b}{2a} & \frac{m\pi b}{2a}\cosh\frac{m\pi b}{2a} + \sinh\frac{m\pi b}{2a} \end{bmatrix}$$

Now, if you substitute $y = b/2$, you will get A m this m pi a sine hyperbolic m pi b by 2a + D m m pi b by 2a cos hyperbolic m pi b by 2a + sine hyperbolic m pi b by 2a = 0. So, we have got this equation, this is number 1 equation and this is number 2 equations after applying the boundary conditions. After applying the boundary condition the 2 equations that we have got contains 2 unknown quantities one is A_m and another is D_m.

So, 2 equations are solved simultaneously, the easiest and systematic method is Cramer's rule for such few numbers of variables. The linear equations with few variables you can use up to 3 variables, you can use the Cramer's rule easily. So, using the Cramer's rule, we can write the solution for the 2 equations 1 and 2 that are obtained using the boundary condition. So, $A_m = \frac{\Delta_1}{\Delta}$ and $D_m = \frac{\Delta_2}{\Delta}$.

Now, the denominator Δ is the determinant form by the coefficient of this A_m and D_m of the 2 equations that are obtained applying the boundary condition. So, first coefficient for forming the determinant Δ, the coefficient of A_m in the first equation, is cosh $cosh\ \frac{m\pi b}{2a}$. So, therefore cosh $cosh\ \frac{m\pi b}{2a}$ that is the first term in the first row, first-term first-row first column that is that term.

Then this term, if you see $\frac{b}{2}$sinh $sinh\ \frac{m\pi b}{2a}$, so this will appear here. Similarly, systematically this will appear in this term, and this whatever inside this second bracket it will go in the position 2 by 2, that is, this second-row second column. So, these delta has to be evaluated, these delta evaluated, and this has to be non zero otherwise, solution will not be bounded.

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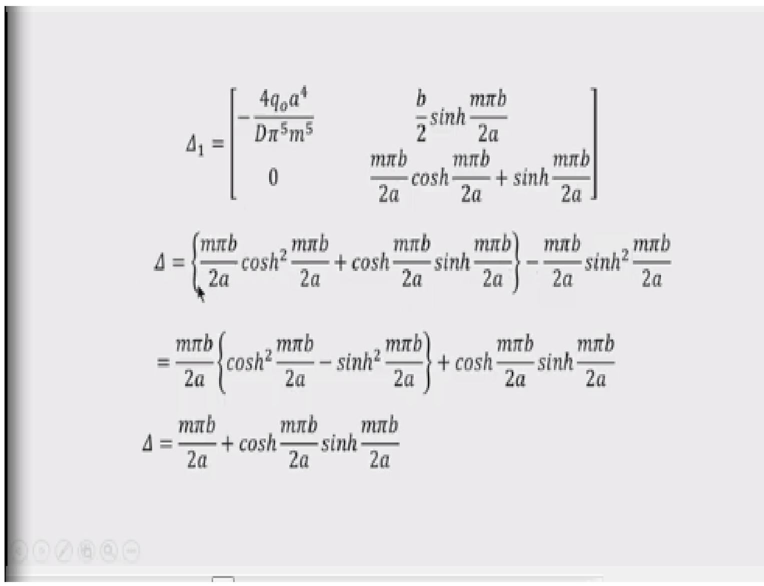

So, after evaluating this Δ, we can find this quantity as Δ. Then one interesting thing you can see that these 2 functions can be combined. Because you know that the relation $\frac{m\pi b}{2a} - \frac{m\pi b}{2a} = 1$, so, therefore we have combined this two function, this term and this term and this is the term, and other term is as this, so this is the delta. Now Δ_1 can be obtained very easily after expanding the determinant. If I expand the determinant then it is $\frac{4q_o a^4}{D\pi^5 m^5}$ and this is. So, this can be multiplied and this will be 0.

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$$\Delta_1 = -\frac{4q_0a^4}{D\pi^5m^5}\left\{\sinh\frac{m\pi b}{2a} + \frac{m\pi b}{2a}\cosh\frac{m\pi b}{2a}\right\}$$

$$A_m = -\frac{4q_0a^4}{D\pi^5m^5}\left\{\frac{\sinh\frac{m\pi b}{2a} + \frac{m\pi b}{2a}\cosh\frac{m\pi b}{2a}}{\frac{m\pi b}{2a} + \cosh\frac{m\pi b}{2a}\sinh\frac{m\pi b}{2a}}\right\}$$

$$\Delta_2 = \begin{bmatrix} \cosh\frac{m\pi b}{2a} & -\frac{4q_0a^4}{D\pi^5m^5} \\ \frac{m\pi}{a}\sinh\frac{m\pi b}{2a} & 0 \end{bmatrix}$$

$$\Delta_2 = \frac{4q_0a^4}{D\pi^5m^5}\frac{m\pi}{a}\sinh\frac{m\pi b}{2a}$$

So, Δ_1 is easily found out as this quantity, so A_m will be $\frac{\Delta_1}{\Delta}$. So, this term is there and you can see that this is the Δ, this determinant formed by the coefficient of the 2 equations coefficient of the variable A_m and D_m in 2 equation form after applying the boundary conditions. So, after finding A_m, you can find the D_m another constant by finding the Δ_2.

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Therefore, $$D_m = \frac{4q_0a^4}{D\pi^5m^5}\frac{\sinh\frac{m\pi b}{2a}}{\left[\frac{a}{m\pi}\left\{\sinh\frac{m\pi b}{2a}\cosh\frac{m\pi b}{2a} + \frac{m\pi b}{2a}\right\}\right]}$$

Hence we can completely find the deflection as,

$$w(x,y) = \sum_{m=1}^{\infty}\left\{A_{mn}\cosh\frac{m\pi y}{2a} + D_m y\sinh\frac{m\pi y}{2a} + \frac{4q_0a^2}{D\pi^5m^5}\right\}\sin\frac{m\pi x}{a}$$

Fixed end bending moment can be obtained by

$$M_x = -D\left\{\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right\}$$

So, when we find the Δ_2 by Cramer's rule, you can see that the coefficient or the elements have to be replaced by the non homogeneous term like that.

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$$M_x = -D\sum_{i=1}^{\infty}\left[-\frac{m^2\pi^2}{a^2}\left\{A_m \cosh\frac{m\pi y}{2a} + D_m y \sinh\frac{m\pi y}{2a} + \frac{4q_0a^2}{D\pi^5m^5}\right\} + \nu\frac{\partial^2 Y_m}{\partial y^2}\right] \sin\frac{m\pi x}{a}$$

$$M_y = -D\sum_{m=1}^{\infty}\left\{\frac{\partial^2 Y_m}{\partial y^2} + (-\nu)\frac{m^2\pi^2}{a^2}Y_m\right\}\sin\frac{m\pi x}{a}$$

At fixed edge, y=b/2 or –b/2. Substitute y=b/2 to get the bending moment M_x or M_y per unit length as the case may be .

So, Δ_2 is here, we are finding this D_m, coefficient D_m using the Δ_2. So, in place where the D_m appears that is the second place the non homogeneous term in the first equation is $-\frac{4q_o a^4}{D\pi^5 m^5}$. I mean the terms in the right hand side is put here, and in the second equation it is found from the slope condition, that means the forcing function which was constant does not appear in the slope equation. So, therefore this term is 0 here and it is appearing in the second column second row.

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$$\Delta_1 = -\frac{4q_0a^4}{D\pi^5m^5}\left\{\sinh\frac{m\pi b}{2a} + \frac{m\pi b}{2a}\cosh\frac{m\pi b}{2a}\right\}$$

$$A_m = -\frac{4q_0a^4}{D\pi^5m^5}\left\{\frac{\sinh\frac{m\pi b}{2a} + \frac{m\pi b}{2a}\cosh\frac{m\pi b}{2a}}{\frac{m\pi b}{2a} + \cosh\frac{m\pi b}{2a}\sinh\frac{m\pi b}{2a}}\right\}$$

$$\Delta_2 = \begin{bmatrix} \cosh\frac{m\pi b}{2a} & -\frac{4q_0a^4}{D\pi^5m^5} \\ \frac{m\pi}{a}\sinh\frac{m\pi b}{2a} & 0 \end{bmatrix}$$

$$\Delta_2 = \frac{4q_0a^4}{D\pi^5m^5}\frac{m\pi}{a}\sinh\frac{m\pi b}{2a}$$

So, this is Δ_2, and after expanding this you can see this goes to 0, and this is coming as this, this will be minus, but again minus, minus will be plus. So, this quantity is written as this. So, after finding Δ_1 and Δ_2, D_m is found. So, once you find A_m and D_m, you can easily find out the deflected Series as $w(x, y) = \sum_{m=1}^{\infty} \left\{ A_{mn} \cosh cosh \frac{m\pi y}{2a} + D_m y \sinh sinh \frac{m\pi y}{2a} + \frac{4q_o a^2}{D\pi^5 m^5} \right\} \sin sin \frac{m\pi x}{a}$, so, this is the complete deflection series.

So, when you find the deflected series, then you can go for finding the bending moment. The bending moment that are most important is the fixed end moment which will be more than the (()) (32:29) moment. So, here the negative moment that occurs in the fixed end can be found out that in the x-direction as well as in the y-direction by using this quantity the $M_x = -D\left\{\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right\}$, mu is the Poisson ratio.

So, M_x is found out; similarly, M_y can be found out, but in the expression of M_x you can substitute all the quantities and then you can find the value of M_x and M_y. So, at the fixed edge $y = \frac{b}{2} or - \frac{b}{2}$, we get the bending moment M_x and M_y per unit length as the case may be.

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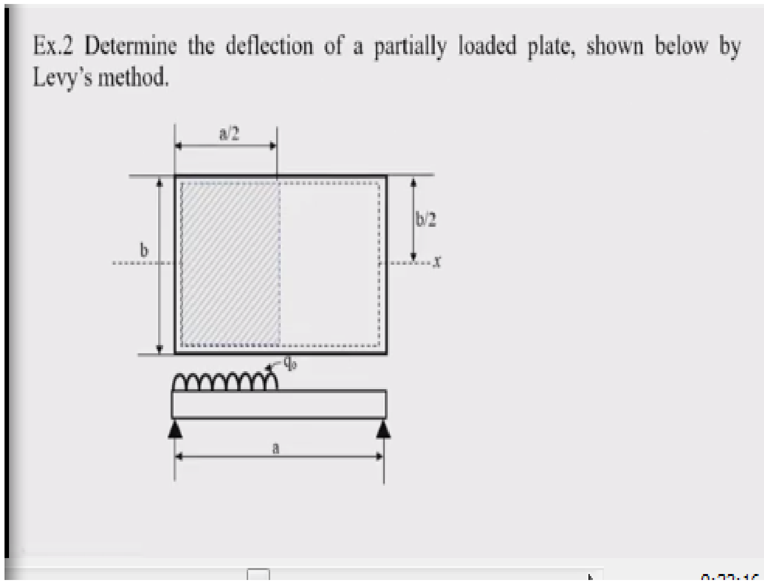

Now, let us discuss another problem of plate which has all the edges simply supported and it is loaded by UDL not up to the full extent of the plate, but up to the middle area. So, it covers only $\frac{a}{2}$ length of the plate, so plate length is a, and the width is b. Now again, you can note here that the plate is symmetrical about your x-axis. So, therefore I write this equation with the symmetric terms again.

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$$w(x,y) = \sum_{m=1}^{\infty} Y_m(y) \sin\frac{m\pi x}{a}$$

$$Y_m(y) = A_m \cosh\frac{m\pi y}{a} + B_m y \sinh\frac{m\pi y}{a} + w_p$$

w_p is particular solution of

$$\frac{d^4Y_m}{dy^4} - 2\frac{m^2\pi^2}{a^2}\frac{d^2Y_m}{dy^2} + \frac{m^4\pi^4}{a^4}Y_m = f_m(y)$$

So, symmetric terms are taken, and w_p is the particular integral.

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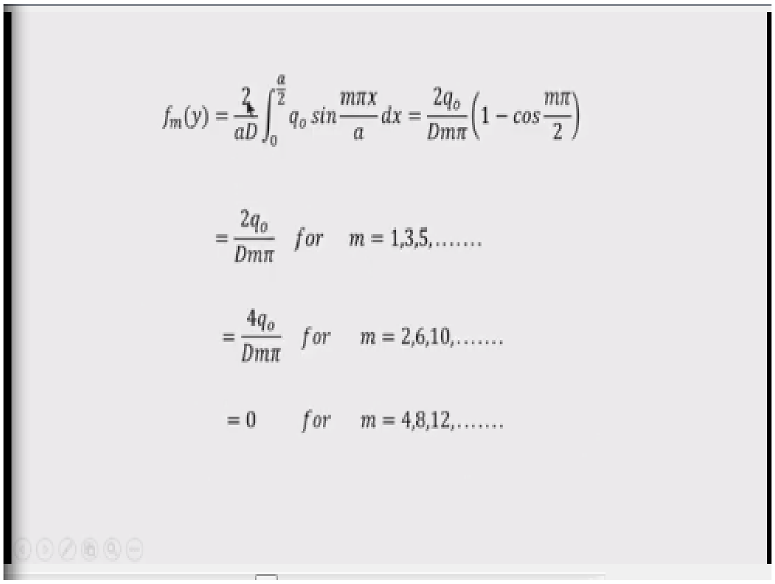

Now here, the w_p is obtained as the $\frac{2}{aD}\int_0^{\frac{a}{2}} q_o \sin sin \frac{m\pi x}{a} dx$. But the limit of integral you can see because of the presence of the load up to the middle portion of the plate, the upper limit of the integral is changed to $\frac{a}{2}$. So, putting this limit and evaluating the integral we get this, this $f_m(y) = \frac{2q_o}{Dm\pi}\left(1 - \cos cos \frac{m\pi}{2}\right)$. And you can see this, these integral gives you very interesting results.

Means when m = the odd integers 1, 3, 5 then these value will be $\frac{2q_o}{Dm\pi}$. And when the values are 2, 6, 10, then you will get $\frac{4q_o}{Dm\pi}$, and when the m is 4, 8, 12, you will get the value of the integral as 0.

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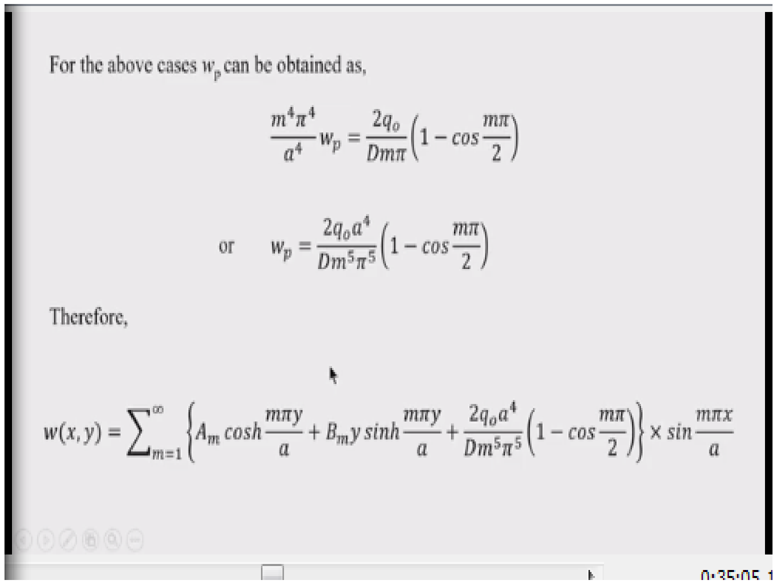

So, for the above cases, w_p is obtained as since again this $f_m(y)$ that is this function is again a constant. So, we can assume that particular solution is also constant. So, substituting the particular integral as constant in this differential equation, you can see this term has no significance, this term has also no significance, only here $\frac{m^4\pi^4}{a^4}$ some constant which is the particular integral w_p equal to this term that we have evaluated.

So, after carrying out this integral, we can find a particular integral in this term, $2q_o a^4$ will come here. And here it will be $Dm^5\pi^5$ will be there, and this term inside the first bracket will exist $\left(1 - \cos cos \frac{m\pi}{2}\right)$. Depending on the different values of m this will be evaluated.

So, after finding the complete solution $w(x, y) = \sum_{m=1}^{\infty}\left\{A_m \cosh cosh \frac{m\pi y}{a} + B_m y \sinh sinh \frac{m\pi y}{a} + \frac{2q_o a^4}{Dm^5\pi^5}\left(1 - \cos cos \frac{m\pi}{2}\right)\right\} \times \sin sin \frac{m\pi x}{a}$. So, this is the expression for the deflected series. Once you get the deflected series, you can go for finding this your bending moment and shear force, etcetera whatever you like.

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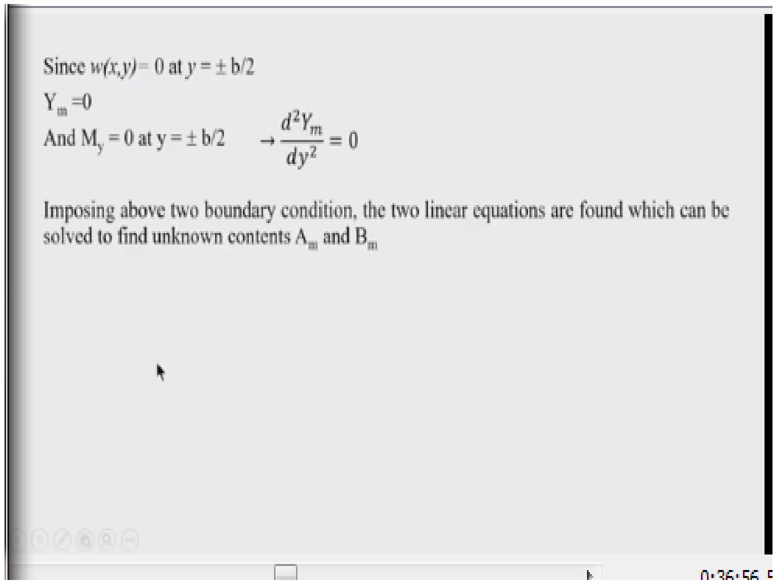

Now, let us come to another problem because the boundaries are taken as simply supported condition. So, 2 conditions have to be imposed one is your deflection condition, and another is curvature condition in this problem and since we have derived the second derivative of the function already in few occasions.

So, that result can be applied that the second derivative expression can be applied here to impose the boundary conditions on the other two edges, which are simply supported for this problem. And you will be able to find this constant A_m and D_m, and then all other quantities that is bending moment, shear force can be evaluated.

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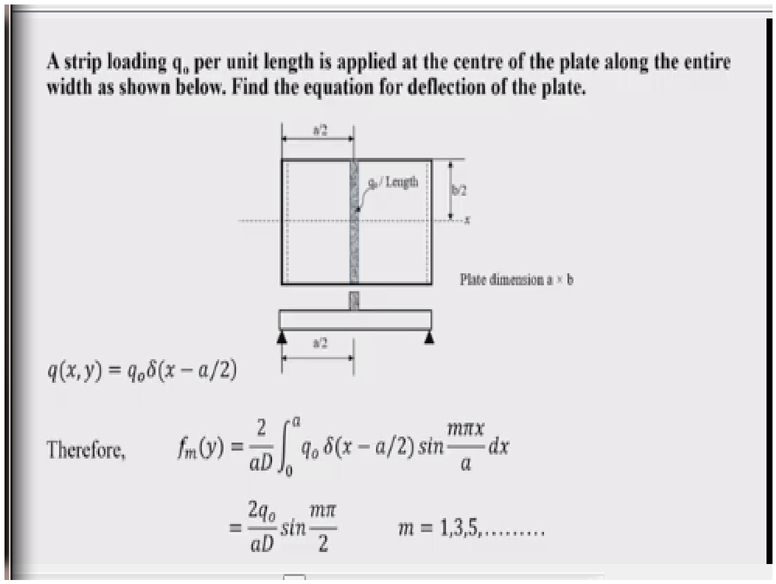

Now, let us consider another problem of plate which has a loading a q_0 per unit length, but in the sense that it is not distributed over the area; it is only a line load that you can call it strip loading. And these loading acts along the y-direction and you can see that this loading is situated at this a by 2, and it is continuous along the y axis. So, that means I can represent this loading with the help of Dirac delta function.

Because this is a line loading and this line loading is only meaningful when this x is only equal to $\frac{a}{2}$, in other points, there is no line loading. So, I can express this loading as $q_0\delta(x - \frac{a}{2})$. So, after substituting this loading function into this $f_m(y)$ that is required to find out the non-homogeneous term of differential equation. We get that this function and by virtue of this property of the direct delta function, we can now easily write that $\frac{2q_o}{aD}$, where q_0 is a constant and sin $sin\ \frac{m\pi}{2}$.

So, this result is again meaningful when m = 1, 3, 5, etcetera. For odd integers, this value will be not be in existence because this is again symmetrical plate problem.

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$$Y_m(y) = A_m \cosh\frac{m\pi y}{a} + D_m y \sinh\frac{m\pi y}{a} + w_p$$

w_p is the particular integral or particular solution of the equation,

$$\frac{d^4Y_m}{dy^4} - 2\frac{m^2\pi^2}{a^2}\frac{d^2Y_m}{dy^2} + \frac{m^4\pi^4}{a^4}Y_m = f_m(y)$$

Let $w_p = C$

Then

$$\frac{m^4\pi^4}{a^4}C = \frac{2q_o}{aD}\sin\frac{m\pi}{2}$$

$$C = \frac{2q_o a^3}{m^4\pi^4 D}\sin\frac{m\pi}{2}$$

So, I can now write this function as $Y_m(y) = A_m \cosh cosh \frac{m\pi y}{a} + D_m y \sinh sinh \frac{m\pi y}{a} + w_p$, w_p is the particular integral or particular solution of the equation. Now, again here you can see the $f_m(y)$ is a constant term which varies with the m. So, therefore, I have assumed this particular integral is also constant, so let w_p be C. So, after substituting this particular integral in this differential equation, we readily get what is the value of C. So, value of particular integral that is C is evaluated here.

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Hence

$$w(x,y) = \sum_{i=1,3...}^{\infty}\left\{A_m \cosh\frac{m\pi y}{a} + D_m y \sinh\frac{m\pi y}{a} + \frac{2q_o a^3}{Dm^4\pi^4}\sin\frac{m\pi}{2}\right\} \times \sin\frac{m\pi x}{a}$$

After applying Boundary condition as,

$Y_m = 0$ at $y = + b/2$

$\frac{dY_m}{dy} = 0$ at $y = + b/2$

Then solve two linear simultaneous equations to find A_m and B_m to find $w(x,y)$. Once $w(x,y)$ is found, other quantities can be obtained.

And after substituting this, we can get the full solution for Y_m and then series. And applying boundary condition, suppose I have taken here the fixed end, two ends are fixed. If I take the 2 ends are fixed, then I apply the boundary condition of the fixed end. Boundary condition of the fixed end is say $Y_m = 0$ and then $Y_m = 0$ at $y = + \frac{b}{2}$ or $- \frac{b}{2}$ and $\frac{dY_m}{dy} = 0$ at $y = + \frac{b}{2}$ or $- \frac{b}{2}$ any edge you can take because the same condition exist in both the edges.

So, when you substitute these 2 conditions, as usual, we obtained in the earlier cases, then we get 2 simultaneous equation linear equations with unknown variable A_m and D_m. So, it can be solved easily by any method. So, I have illustrated this systematic method known as a Cramer's rule that you can use by expanding the determinant that I have shown. And then, you can obtain the constant A_m and D_m completely.

So, when you obtain the A_m and D_m, your problem is known that means deflection series is completely known. And then, after obtaining the deflection, you can explore other quantities. That means if you want this further, say bending moment, shear force, you can derive it from the deflected quantities.

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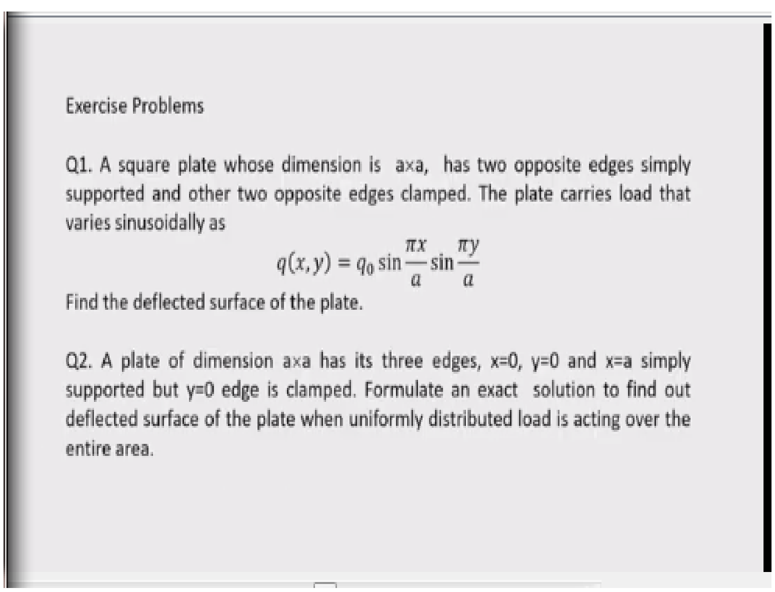

So, now I want to show you how some other problem can be taken with the help of this Levy's method. So, first problem is let us take a square plate which dimension is $a \times a$ and has 2 opposite edges simply supported and other 2 opposite edges clamped. The plate carries load that varies sinusoidally as $q(x, y) = q_0 \sin sin \frac{\pi x}{a} \sin sin \frac{\pi y}{a}$. So; our question is that deflected surface has to be found out.

Second question we will see, a plate of dimension $a \times a$; this has it is 3 edges x = 0, y = 0, and x = a are simply supported. But y = 0 edge is clamped, and other edge y = b edge is also clamped. So, let us formulate an exact solution using the Levy's method to find out the deflected surface of the plate when there is uniformly distributed load acting on the entire area. So, let us solve this problem and show you how this problem can be tackle using the method that we have learned in Levy's condition.

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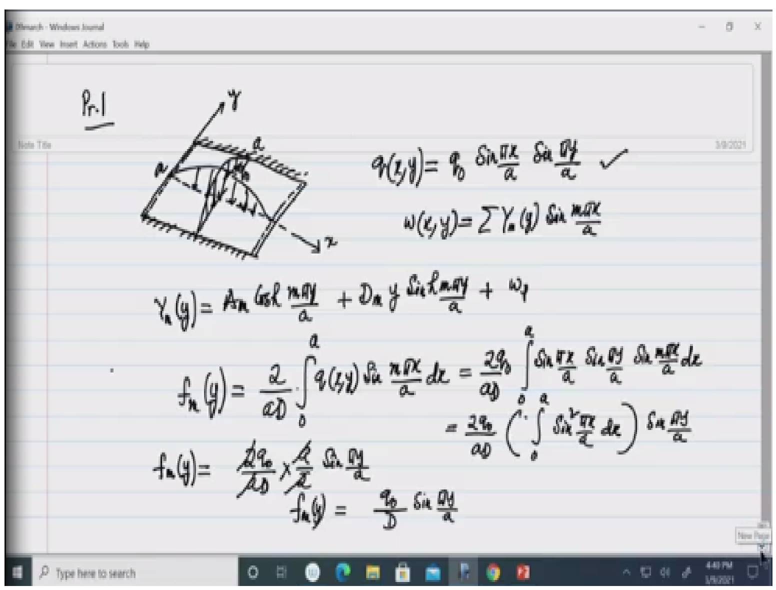

So, first problem is this, problem 1, we have a rectangular plate, but 2 sides are equal; for simplicity, we have taken 2 sides are equal. And 2 opposite edges are simply supported, but other 2 opposite edges are clamped. So, again this problem is a symmetric problem, so I can now take the x-axis here and y-axis, say this. The plate is loaded sinusoidally, that means if I see the variation along x-axis the variation is sinusoidal, the amplitude is q 0.

Then if I see the variation along y-axis again, it is sinusoidal. So, in the 2 direction load is sinusoidal. That means if I write the load function that can be written as $q(x, y)$ is equal to say intensity is q_0 sin $sin\ \frac{\pi x}{a}$ sin $sin\ \frac{\pi y}{a}$ because it is a square plate, so the dimension of the plate is $a\times a$. Now because of symmetrical problem, take the term in the deflection series Y_n, deflected series is this, summation of this $\frac{m\pi x}{a}$.

And $Y_m(y)$ you know because of symmetry that only symmetrical terms have to be taken. So, the differential equation that we need to solve that I illustrated and the solution is $Y_m(y) = A_m$ cosh $cosh\ \frac{m\pi y}{a}\ + D_m y$ sinh $sinh\ \frac{m\pi y}{a}\ +\ PI$. Now, particular integral has to be found in that case because it is not a uniformly distributed load, so the result is known to us.

The particular integral has to be found from this function that is given as $\frac{2}{aD}$, D is the flexural rigidity of the plate and the integration 0 to a, $q(x,\ y)$. sin $sin\ \frac{m\pi x}{a} dx$. Now, here the $q(x,\ y)$ is this function, $q(x,\ y)$ is given here as the distributed load in the form of the sinusoidal function. So, $2/aD$ amplitude of the loading is q_0, I have taken it as a constant and taken outside the integral sign.

So, inside the integral, we have this function; now, this integral can be easily carried out because this is a function of x, so only the integration with respect to x is meaningful, so sin $sin\ \frac{\pi y}{a}$ will appear as a constant. So, now if I integrate this sin $sin\ \frac{m\pi x}{a}$, sin $sin\ \frac{m\pi x}{a}$ that is 0, only it is meaningful non zero when m = 1. So, taking m = 1, that integration can be carried out twice aD and you will get this integration as 0 to a , and then sin $sin\ \frac{\pi y}{a}$ will appear as it is as a function.

So, ultimately that you are getting $f_m(y)$ is equal to this twice $\frac{q_0}{aD}$ into this will be $\frac{a}{2}$ and sin $sin\ \frac{\pi y}{a}$. So, this result will be this a will get cancelled, $\frac{q_0}{D}$sin $sin\ \frac{\pi y}{a}$. So, this is our function $f_m(y)$. So, now we can go forward to calculate this particular integral.

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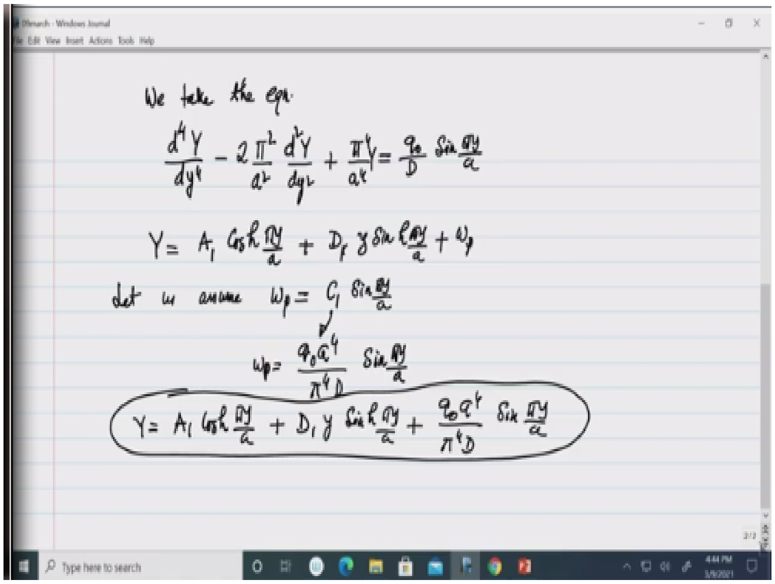

So, particular integral is now calculated; we take the differential equation, differential equation was this for m = 1. Now instead of $f_m(y)$, now we have got this particular integral. So, that value is substitute here in the earlier case we got $\frac{q_0}{D}$sin $sin\ \frac{\pi y}{a}$. So, solution for y is because it is symmetrical again. So, A_m cosh $cosh\ \frac{m\pi}{a}$, it is not necessary to take any integer value of this not to expand it because it is meaningful only when m = 1.

So, therefore A_1, I am writing A_1 directly, then $D_1 y\ sinh$ because this is to product of 2 odd functions plus particular integral, this is y. Now because this is a forcing function, again is a sinusoidal function. So, let us assume particular integral as also sum constant C_1 sin $sin\ \frac{\pi y}{a}$. Now, you can see that after substituting this value here, you will get that this particular integral will appear as a very simple term that I will give you this term to you.

So, w_p is now appearing is $\frac{q_o a^4}{D\pi^4}$sin sin $\frac{\pi y}{a}$. Because C_1 is now this quantity, how this C_1 is found that assume particular integral as this C_1 into this and substitute this in the differential equation and then equate the coefficient of like terms, so you will get C_1 as this value. So, total solution is; now, we can write A_1 cosh $cosh$ $\frac{\pi y}{a}$ $+ D_1 y$ sinh $sinh$ $\frac{\pi y}{a}$ $+ \frac{q_o a^4}{D\pi^4}$sin sin $\frac{\pi y}{a}$, so this is the complete solution of y.

Now, let us impose the boundary condition to determine the constant A_1 and D_1. That can be imposed by giving the 2 conditions that $y = \frac{a}{2}$ that deflection is 0, and slope is 0.

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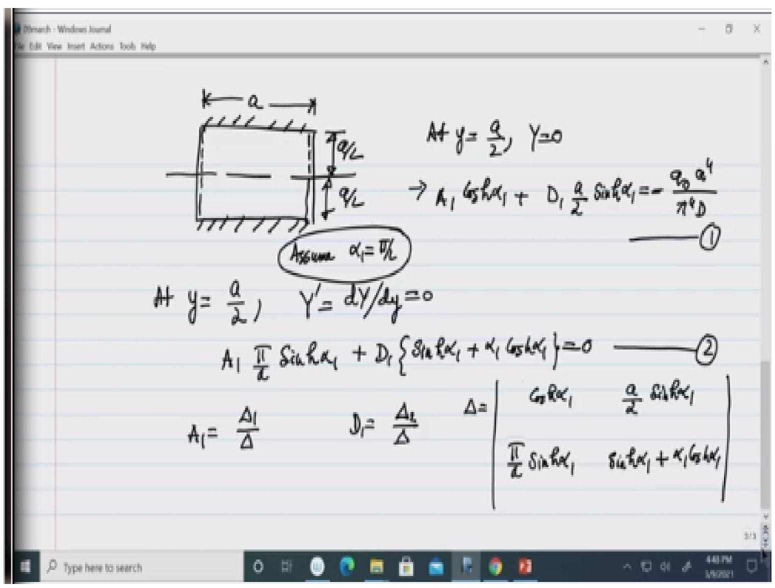

So, since the plate was clamped along 2 other edges, this is clamped condition, this is also clamped. And this $\frac{a}{2}$ and this is also $\frac{a}{2}$, and this is simply supported and this length is also I am taking as a, because it is a square plate. So, imposing the boundary condition at y = $\frac{a}{2}$,$y = 0$, that gives you that $A_1 cosh$. If you substitute this $y = \frac{a}{2}$, then cosh $cosh$ $\frac{\pi}{2}$ will come.

So, let us assume or assume take it $\alpha_1 = \frac{\pi}{2}$. So, just to simplify the calculation, I have taken this, so this is coming. So, after substituting this $y = \frac{a}{2}$, first term in this deflection equation is this. Then second term will come as $D_1 . \frac{a}{2}$ because y is there, so $\frac{a}{2}$ and sinh $sinh\ \frac{\pi}{2}$, so α_1 equal to this non-homogeneous term if you see here that sin $sin\ \frac{\pi y}{a}$ was there.

Instead of y, you put $\frac{a}{2}$, so sin $sin\ \frac{\pi}{2} = 1$, so therefore this term will be $-\frac{q_o a^4}{D\pi^4}$, so that is one equation. Second equation, we will get at $y = \frac{a}{2}$, Y' that is the first derivative $\frac{dY}{dy} = 0$. So, using the first derivative equation, we get here say $A_1 \frac{\pi}{a}$sinh $sinh\ \alpha_1$ derivative that will do is a product of 2 functions earlier the variable associated with D_1.

So, you get 2 terms, sinh $sinh\ \alpha_1 + \alpha_1$ cosh $cosh\ \alpha_1 = 0$, because the derivative of constant that requires there will be 0. So, this is your first equation, and this is your second equation after application of boundary condition. So, A_1 and B_1 can be solved A_1 is equal to say $\frac{\Delta_1}{\Delta}$ and D_1 will be $\frac{\Delta_2}{\Delta}$; Δ is a determinant that is formed by the coefficient of A_1 and D_1.

So, let us see coefficient of A_1 is cosh $cosh\ \alpha_1$ and coefficient of D_1 is $\frac{a}{2}$sinh $sinh\ \alpha_1$. Then coefficient of here A_1 in the second equation $\frac{\pi}{a}$sinh $sinh\ \alpha_1$ and coefficient of D_1 in the second equation will be sinh $sinh\ \alpha_1$, α_1 , so this is the Δ.

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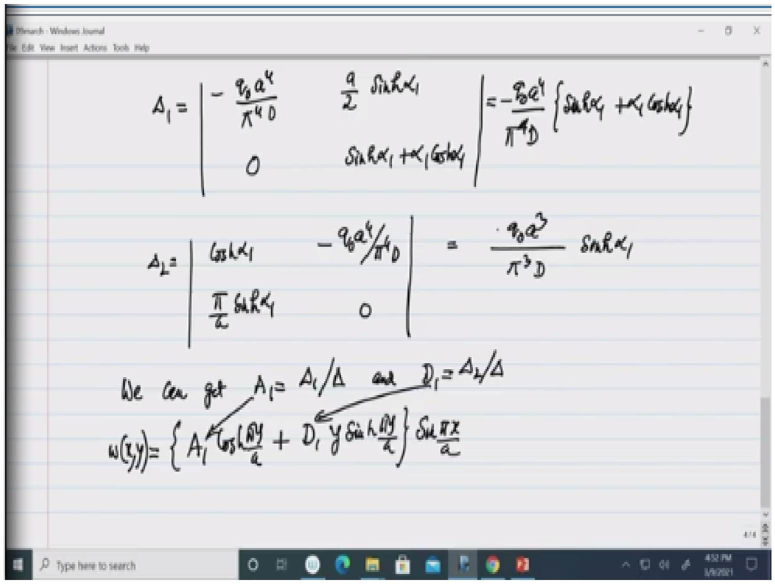

Similarly, you can find Δ_1 as this let us write down the Δ_1, Δ_1 will be $-\frac{q_o a^4}{D\pi^4}$. Here it will be 0, then here, in this case, it will be $\frac{\pi}{a}\sinh \mathit{sinh}\, \alpha_1$, and in that case, it will be $\sinh \mathit{sinh}\, \alpha_1 + \alpha_1 \cosh \mathit{cosh}\, \alpha_1$. Mind that $\alpha_1 = \frac{\pi}{2}$. So, Δ_1 is this; you can easily evaluate this, the determinant will be $\frac{q_o a^4}{D\pi^4}\left(\sinh \mathit{sinh}\, \alpha_1 + \alpha_1\right)$.

So, this is Δ_1, and Δ_2 can be similarly found out, Δ_2 will be $\left|\cosh \mathit{cosh}\, \alpha_1 \quad -\frac{q_o a^4}{D\pi^4} \quad \sinh \frac{\pi}{a} \mathit{sinh}\, \alpha_1 \quad 0\right|$. So, after expanding you will get this term as $\frac{q_o a^3}{D\pi^3}\sinh \mathit{sinh}\, \alpha_1$. So, we have got Δ_1 and Δ_2, now we can get A_1 as $\frac{\Delta_1}{\Delta}$ and D_1 as $\frac{\Delta_2}{\Delta}$.

Once these A_1 and D_1 is known, then deflection series is simply calculated as, because only m = 1 is important here and other term it does not exist. So, we will write this $\left(A_1 \cosh \mathit{cosh}\, \frac{\pi y}{a} + D_1 y \sinh \mathit{sinh}\, \frac{\pi y}{a}\right) \sin \mathit{sin}\, \frac{\pi x}{a}$. So, this is the deflected series, and substitute A_1 from here and substitute D_1 from here. So, complete deflection is known, maximum deflection will be at the centre of the plate when $x = \frac{a}{2}$ and $y = \frac{a}{2}$. So, this is one problem that I wanted to discuss, and that is solved with the help of Levy's method.

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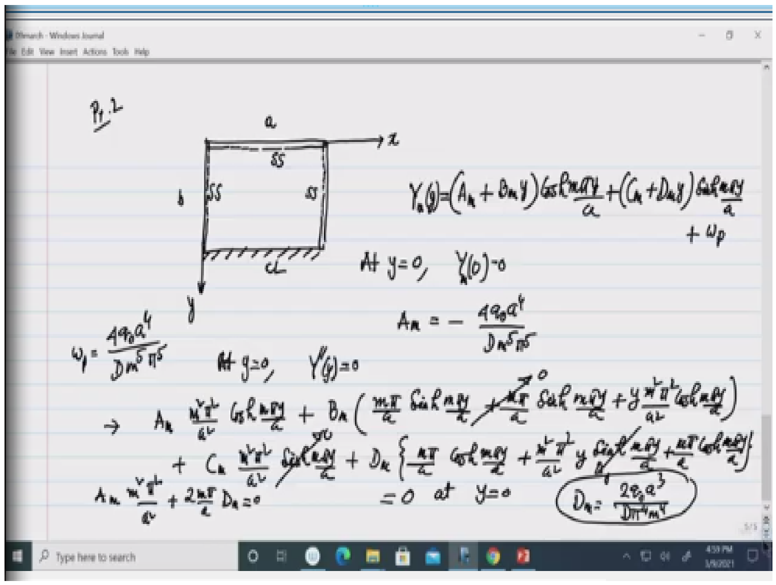

Second problem; let us consider a general problem which has completely unsymmetrical condition. So, unsymmetrical condition the problem 2. We have this rectangular plate, let the length is a and the width is b, and it has 2 opposite edges simply supported that must be there; otherwise, you cannot apply Levy's method. In addition, the boundary condition and other 2 edges at y = 0 edge again simply supported and here, we have taken fixed edge.

So, now you can see this is simply supported, this is simply supported, this is simply supported, and this is clamped. Because of the purely unsymmetrical nature, we cannot take any advantage of symmetry. That means omitting 2 constants and taking and retaining 2 constant that business cannot be done here. So, what we do here? We take the axis as it is, this is x-axis and y-axis, and we have to kept the full term of the deflected series.

That is $Y_m(y)$ was composed of $\left(A_m + B_m y\right)$ cosh $cosh\ \frac{m\pi}{a}y + \left(C_m + D_m y\right)$ sinh $sinh\ \frac{m\pi}{a}y$. So, these 4 constants we have to take, there is no other way because the boundary condition is purely unsymmetrical. Now we have to apply the boundary condition one by one. So, at y = 0, deflection is 0; of course, this particular integral will come, so particular integral value will be same for uniformly distributed load that we have found out earlier.

And this particular integral I am writing here for uniformly distributed load, it is $\frac{4q_0a^4}{D\pi^5m^5}$. So, this particular integral has to be written here. So, at y = 0, actually, $Y_m(0)\ is\ 0$, so substituting the value of this y = 0 because $\cosh cosh\ 0 = 1$ and $\sinh sinh\ 0 = 0$. So, therefore we get here this A_m and here you will get this term will vanish equal to $-\frac{4q_0a^4}{D\pi^5m^5}$.

So, one constant is known by application of this. And second constant, other constant we have to find out applying the boundary condition. So, applying the boundary condition on bending moment that at y = 0, second derivative of y is also 0. So, that condition gives you the equation that will be slightly larger equation, but it can be computed very easily, there is no doubt. So, $\frac{m^2\pi^2}{a^2}A_m cosh\frac{m\pi y}{a} + B_m\left(\frac{m\pi}{a}\sinh sinh\ \frac{m\pi y}{a} + \frac{m\pi}{a}\sinh sinh\ \frac{m\pi y}{a} + y\frac{m^2\pi^2}{a^2}\cosh cosh\ \frac{m\pi y}{a}\right)$.

So, this is expansion of B_m then on this C_m you will get $C_m\frac{m^2\pi^2}{a^2}\sinh sinh\ \frac{m\pi y}{a}$. And on D_m again, you will get the terms $\left(\frac{m\pi}{a}\cosh cosh\ \frac{m\pi y}{a} + \frac{m^2\pi^2}{a^2}y \sinh sinh\ \frac{m\pi y}{a} + \frac{m\pi}{a}\cosh cosh\ \frac{m\pi y}{a}\right)$. So, this is the second derivative, and it is 0 at y = 0. So, substituting these values you can easily see that this function exists.

But here you see sine hyperbolic 0 is 0 and this is again y into something is 0, y is 0, so this is going to be 0. Then here, this is going to be 0, and here you will find that with D m this is 1 and this is going to be 0, and this is 1. So, ultimately you will get $\frac{m^2\pi^2}{a^2}A_m + \frac{2m\pi}{a}D_m = 0$. Now since A_m is calculated earlier, we can now get the D_m.

After getting D_m and D_m of course I am writing this value of D_m, D_m will be $\frac{2q_0a^3}{D\pi^4m^4}$. So, value of D_m is this $\frac{2q_0a^3}{D\pi^4m^4}$. So, 2 constant A_m and D_m are known, other 2 constant will be found out applying the boundary condition at these edges, that is clamped edges.

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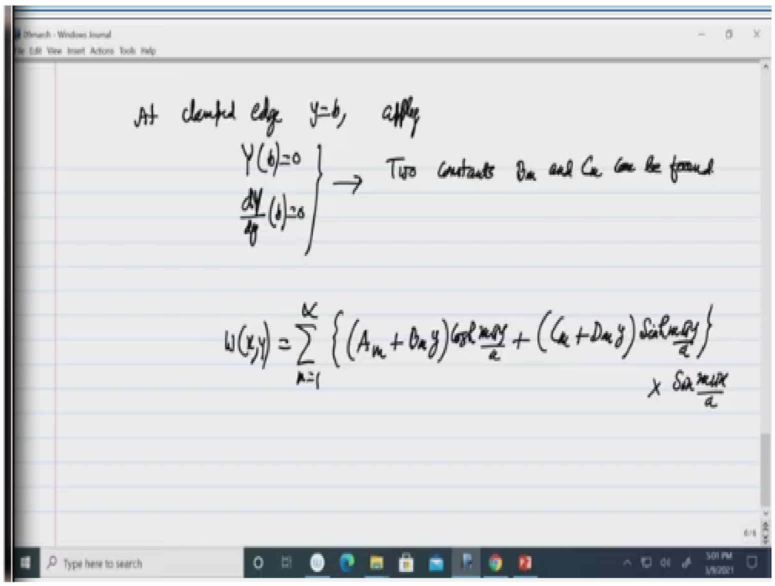

At clamped edges, there is y = b; apply this equation and $\frac{dY}{dy}$,of course, at b = 0. By using these 2 equations, you will be able to get these 2 constants A_m and B_m. So, using these 2 constants, other 2 constants say B_m and C_m can be found. So, these 2 conditions will give you 2 equations again with B_m and C_m, because A_m and D_m are already evaluated. So, now you get the 4 constants of integration that is resulted due to unsymmetrical condition of boundary.

So, therefore the solution is now known completely with this series. So, complete series is now known because A_m, B_m, C_m, D_m are calculated. So, in this way we can handle any unsymmetrical condition, only the computational effort may; definitely will increase because of 4 constants involved. So, what I want to tell you that in conclusion the Levy's method is applied for rectangular plate when 2 opposite edges are simply supported, and other 2 edges may have any boundary condition.

It yields the exact solution of the plate problem that is first thing what we learn from the Levy's method. But if we compare the other exact method, that is Navier method, that is restricted only to the simply supported boundary conditions along all edges. So, Levy's method is more general, and because of the use of only the single sine series, the calculative effort is less that is computational effort is less only the derivation part is slightly longer.

Because in the intermediate step, you have to solve one differential equation, ordinary differential equation, which will yield you the unknown function y but with 4 constants of integration that have to be found by using the conditions at the boundary on the other edges. So, 2 methods that we have learned now, Navier's method and Levy's method, can be successfully applied to the plate problem for specific boundary condition, and it yields the analytic solution.

In most of the cases, the loading is uniformly distributed load, and we have seen that for uniformly distributed load, the exact solution is very much; the solution obtained by Levy's method is in well agreement with the Navier's method. So, with this, I conclude, then we will see what are the other applications of Levy's method? So, we will investigate whether Levy's method can be used indirectly to solve the conditions not met in the Levy's boundary condition.

That is suppose the plate with all edges are fixed, can we use? The question is can we use the Levy's method for such plate? We will discuss this in the next class, thank you very much.