Plates and Shells

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Module-03
Lecture-08
General Formulation for Rectangular Plate with Two Opposite Edges Simply Supported

Hello everybody, welcome to you all in massive open online course and today, I am starting module 3 first lectures; I will be delivering today for module 3.

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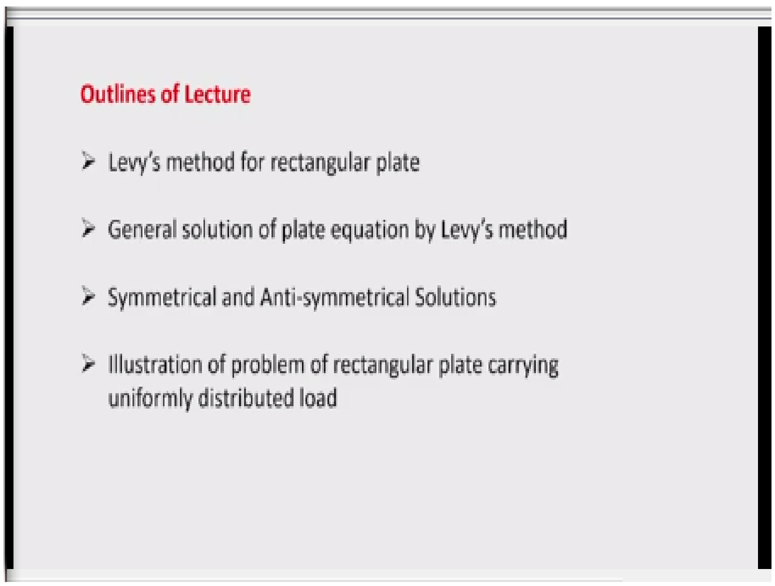

Let us see what are the topics that I intend to cover in this lecture. The topic that I intended to cover is Levy's method for rectangular plate, then general solution of the plate equation by Levy's method, symmetrical and anti symmetrical solutions, illustration of problem of rectangular plate getting uniformly distributed load using Levy's method. So, you have understood that today our main focus is to analyse a rectangular plate using another method, whichever we learned earlier was the Navier's method.

Now we will use the rectangular plate using Levy's method. So, let me see what is the difference between these two methods; if we recall that Navier's method, then we used the double trigonometric series for the solution of the plate deflection. Now, this is possible only when all the edges of the plates are simply supported. So, in a rectangular plate four edges are simply supported, then we can handle the problem by Navier's method, which gives a very simple solution with double series.

Of course, the double series increases the complexity in summation, but the series is rapidly convergent. Here we use another method Levy's method, for the rectangular plate, but the difference is that. In Levy's method, two opposite edges must be simply supported, so Levy has proposed a solution considering two opposite edges to be simply supported for which he proposed a close form exact solution.

Now, what about the other two edges? Other two edges may be simply supported also or may have any other conditions. That means other two edges may have one edge may be fixed, the other edge may be free, or one edge may be simply supported, other edge may be supported on elastic beam, the Levy's method only is possible when two opposite edges are simply supported. But a plate when all edges are, say, for example, is clamped, then directly using Levy's method or Navier solution is not possible.

We will discuss it later by using indirectly this Levy's method; how can we obtain the solution for a clamp plate? So, our intention is to discuss the Levy's method for rectangular plates, general solution we will formulate. And then symmetrical and anti symmetrical conditions, how it arises and what are the simplification possible in case of these two special conditions? And for general conditions, what we will do? So, that will be discussed.

Then I want to illustrate a problem that we had already solved by Navier's method, when the all edges of the plate was simply supported. So, this problem now I will solve it by Levy's method.

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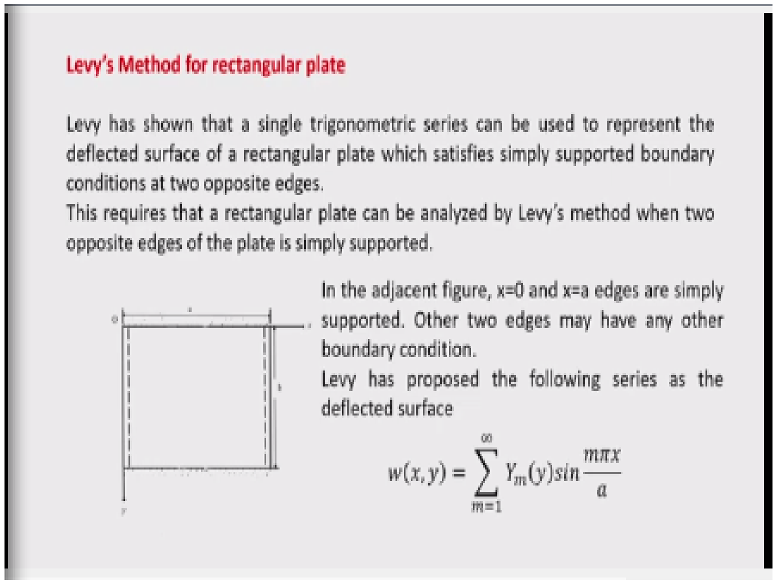

And the plate that I consider in the last section here that will be carrying a uniformly distributed load. So, Levy has shown that a single trigonometrical series can be used to represent the deflected surface of a rectangular plate which satisfied simply supported boundary conditions at two opposite edges. Now, consider this plate; here, these edges say x = 0 and x = a; these edges are simply supported. The dotted line shows that it is simply supported, this symbol we generally use for simply supported condition.

The other two edge y = 0 and y = b may have any other conditions, so I have not specified it. The condition will be imposed to evaluate the constants of integration that we will see later on. So, you have seen that x = 0 and x = a, the plate is simply supported and therefore, Levy proposed a solution $w(x, y)$ single summation a function of y that is called capital Y_m as a function of y sin $sin\, \frac{m\pi x}{a}$

So, the sum is over with only one term. So, that means, when I expand the series, it will be say $Y_1 \sin sin\, \frac{\pi x}{a} + Y_2 \sin sin\, \frac{2\pi x}{a} + Y_3 \sin sin\, \frac{3\pi x}{a}$ like that it will go. So, the function Y_m has to be evaluated because this is still unknown. Because, did not satisfy the boundary condition and other two edges. But these edges say x = 0, x = a the boundary condition is satisfied by taking the sine function to represent the boundary condition at the two opposite as edges.

Because immediately we can see that when x = 0, w is 0 and y when x = a, again the w is 0, so in both the edges deflection is 0. So, when you consider the bending moment condition, that is the bending moment is given by -D then multiplied by the curvature in x-direction +($\nu\times$ curvature in y-direction). So, since the plate is supported along the y-direction, there will be no deflection along the y-direction at the edge.

So, therefore curvature is 0 along this x edge, x = 0 or x = a edge. So, therefore the boundary condition for bending moment only reduces to the curvature $\frac{\partial^2 w}{\partial x^2} = 0$. And we can readily see, you can verify that by taking this function $w(x, y) = Y_m \sin sin \frac{m\pi x}{a}$ the boundary condition is easily satisfied, all the boundary condition, geometric as well as this force boundary condition.

Now I think you have understood the difference between the Navier's method and Levy's method. Navier's method requires that all edges have to be simply supported, so that means here I left this edge y = 0 and y = b edge as unknown boundary conditions. But in Navier's method these boundaries are also have to be simply supported, so this is the difference. So, the generality of the approach has been improved in case of Levy's method.

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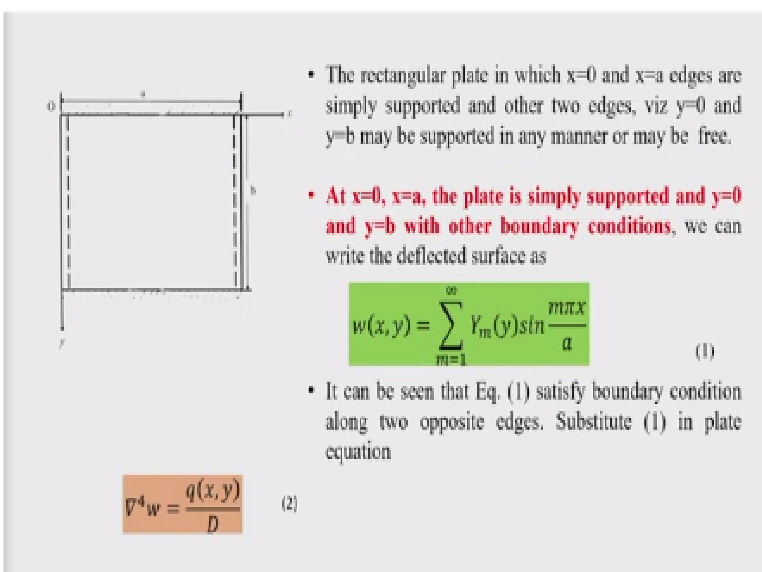

So, taking a single trigonometrical series, we have seen that boundary condition is satisfied at x = 0, x = a and other boundary condition y = 0, y = b is not specified yet. And depending on the different edge conditions, we will be able to evaluate the unknown constants of the integration that we will see later. Now it is the plate equation that equation number 2 is well known to you, $\nabla^4 w = \frac{q(x,y)}{D}$.

Where ∇^4 is the biharmonic operator that is this partial derivative operator that is $\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}$. So, you know this operator, and this is operating on w. And on the right-hand side, $q(x, y)$is that distributed load, that means q is a continuous function of x and y. But sometimes, the discontinuity arises because the concentrated load may also act, and there are also the steep loading or line loading.

So, in this case, we have to deal with in separately the Levy's method. And capital D is flexural rigidity of the plate; what is the flexural rigidity of the plate? Flexural rigidity of the plate is nothing but $\frac{Eh^3}{12(1-\nu^2)}$, where E is Young's modulus of elasticity, h is the thickness of the plate, and ν is the Poisson ratio. Now, one thing here can be noted that while solving the differential equation, we have taken the uniform thickness of the plate.

So, therefore D is not variable in x and y, so that is the characteristics of this differential equation. When D is variable then plate equation will be complicated and different techniques have to be used.

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$$\Rightarrow \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q(x,y)}{D} \qquad (2)$$

$$\frac{\partial^4 w}{\partial x^4} = \frac{m^4\pi^4}{a^4} Y_m(y) sin\frac{m\pi x}{a}$$

$$\frac{\partial^4 w}{\partial y^4} = \frac{d^4 Y_m(y)}{dy^4} sin\frac{m\pi x}{a}$$

$$\frac{\partial^4 w}{\partial x^2 \partial y^2} = -\frac{m^2\pi^2}{a^2}\frac{d^2 Y_m(y)}{dy^2} sin\frac{m\pi x}{a}$$

Now, on expansion, the $\nabla^4 w$ looks like that, that is $\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q(x,y)}{D}$. So, you have understood that there are three terms and right-hand side this is the load function and flexural rigidity and left-hand side all are actually 4th order derivative. So, if we see the function $w(x, y) = Y_m(y) sin\frac{m\pi x}{a}$ is the proposed solution of the differential equation because it satisfied the boundary condition on 2 opposite edges completely.

So, if it is the possible solution of the differential equation, then it must satisfy the differential equation also. So, substituting this series $w(x, y) = Y_m(y) sin\frac{m\pi x}{a}$ into the differential equation. We get an equation like that, say this is the first term that is after differentiation of w 4 times with respect to x, we get $\frac{m^4\pi^4}{a^4} Y_m(y)$. Because capital Y_m is a function of y, so it remains undifferentiated.

And then sin $sin\, m\pi x$ after even number of differentiation in (()) (11:42) says $sin\frac{m\pi x}{a}$. Now one thing is that the m is the half wave numbers, so m can vary 1 to infinity. And for you have seen for different half wave numbers, the waves are formed, and the superimposition of all waves actually contributes to the total deflection of the plate. Then $\frac{\partial^4 w}{\partial y^4}$, now here we are taking the differentiation with respect to y.

So, therefore this quantity has to be differentiated 4 times, but this is treated as a constant, so it remains as it is. So, $sin\frac{m\pi x}{a}$ remains as it is but this function is differentiated 4 times with respect to y. Then second derivative, this middle intermediate-term or middle term, is $\frac{\partial^4 w}{\partial x^2 \partial y^2}$. That means the second derivative of w with respect to x square is first taken, then second derivative of this result is again evaluated 2 times with respect to x or vice versa.

Because the operators are interchangeable, it is a linear operator, so you can proceed in either way. So, after doing this operation, you will get here actually you will get the minus sign will come appear in this equation, and you will get $-\frac{m^2\pi^2}{a^2}\frac{d^2Y_m(y)}{dy^2} sin\frac{m\pi x}{a}$, so the minus sign appears here.

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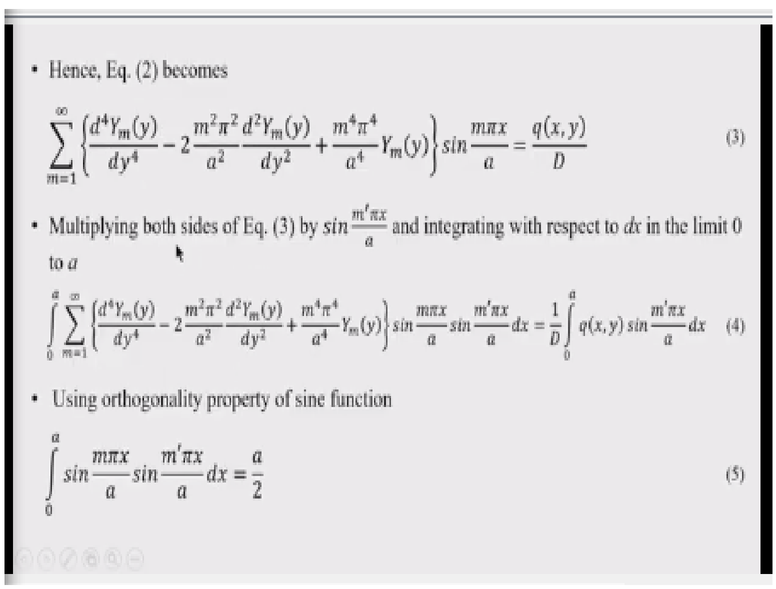

And then, substituting this derivative in this equation completely, we get this term and it is under the summation term. So, it is under the summation term, and $sin\frac{m\pi x}{a}$ is there in $\frac{q(x,y)}{D}$. Now, our intention is to solve for this differential equation, but we have to get rid of this summation term. That means there are 2 ways you apply here orthogonality condition to get rid of the summation term, or you can express $q(x, y)$ in terms of Fourier series and then comparing the like terms, you will get the coefficient.

So, both are the same, and we will give the same results. Now here we have proceeded from the first to work in principle. So, multiplying both sides of the equation 3, that equation I am numbering it as three by $sin\frac{m'\pi x}{a}$integrating with respect to dx in the limit 0 to a. So, why I have taken this $sin\frac{m'\pi x}{a}$because the m' is also an integer. So, when $sin\frac{m\pi x}{a}$ is multiplied by with another sine function of $sin\frac{m'\pi x}{a}$, when m and m' are integers.

And when it is integrated from 0 to a with respect to x, that value will be 0 if m is equal to not m'. Means, when we carry out summation, we will encounter here that sine say m = 1, and m = 2, so we will get sin $sin\ \frac{\pi x}{a}$sin $sin\ \frac{2\pi x}{a}$, so it will be 0. Then we will get sin $sin\ \frac{2\pi x}{a}$, so it will be m, and m' are equal, so it will not be 0. So, according to orthogonality condition, that $sin\frac{m\pi x}{a}sin\frac{m'\pi x}{a}dx = \frac{a}{2}$, if m = m'.

But for other values of m, this is never the non zero value; when m = m', it will be $\frac{a}{2}$; when $m \neq m'$ then this integral will be reduced to 0.

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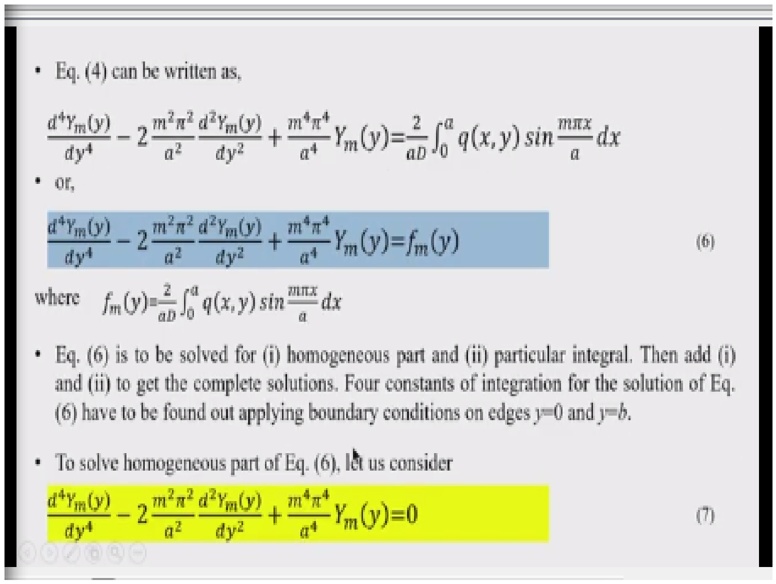

So equation 4, now can be written because after integration, only one term will remain, and it will be $\frac{a}{2}$. So, this is written as like that because the $\frac{a}{2}$ will now transfer to the right-hand side, so here it will be $\frac{2}{aD}$, and this term will remain as it is. So, naturally, we get here this is the ordinary differential equation because the only one function that is Y_m which is a function of y, is differentiated.

So, Y is a function of y, so 4th derivative appears here; then here it is second derivative and here the function itself. So, right-hand side, you are getting $\frac{2}{aD}\int_0^a q(x,y) sin\frac{m\pi x}{a}dx$, $q(x,y)$ is the distributed load on the plate, this may be any form, it may be uniformly distributed load, it may be partially distributed load, it may be as a hydrostatic load that is uniformly varying in a linear manner, or it may be a load which is nonlinearly varying.

Any type of variation of load may be there in practical cases, and this is the general term for evaluating the load function that can be easily incorporated when $q(x,y)$ have continuous distribution. So, here you can see that right-hand side now becomes a function of y because the integration is carried out with respect to x, clear. So, integration is carried out with respect to x, and it is a definite integral, so the x variable will be replaced by the constant term, which is a, the limit of the integration.

So, therefore there will be no x variable in the integration result, and it will be simply a function of y. So, these differential equation turns out to be $\frac{d^4Y_m(y)}{dy^4} - 2\frac{m^2\pi^2}{a^2}\frac{d^2Y_m(y)}{dy^2} + \frac{m^4\pi^4}{a^4}Y_m(y) = f_m(y)$ and this is the function of y, $f_m(y)$. So, this differential equation we need to solve completely.

So, that we know the function Y and then the full deflected surface is known. Because now at this moment, we are only knowing this function $sin\frac{m\pi x}{a}$, but still this function $Y_m(y)$ is remaining unknown till now, so let us see how we can solve it. So, this function of y as I explained that it is integrated with respect to x and the limit is substituted, and then you will get a function where x variable does not appear.

The equation 6, you can see this equation is an ordinary 4th order linear differential equation with constant coefficient, but this equation is non-homogeneous because of the presence of the forcing function, which is appearing as a function of y. So, because of this non-homogeneous equation as per the theory of linear differential equation, we have to find the solution in 2 parts; one is homogeneous solution, and second is particular integral or particular solution.

Homogeneous solution is obtained when there is no load function that is right-hand side is 0, and particular integral has to be obtained when the right-hand side represents the forcing function whatever force may be. So, equation 6 has to be solved, first homogeneous part and particular integral. Then will superimpose one over other because it is a linear differential equation, so complete solutions will be obtained.

Now you can see here it is a 4th order differential equation. So, naturally, in 4th order and differential equation when you get the complete solution of homogeneous equation, we get 4 constants of integration; that 4 constants have to be evaluated using the boundary condition at the 2 other opposite edges. We have not test the boundary condition or we have not spelled out any boundary condition on other 2 edges.

In Levy's condition when we started the formulation, we stated that the 2 edges x = 0 and x = a are simply supported. So, 2 opposite edges, other 2 opposite edges y = 0, y = b, the boundary condition is still now not specified. To solve the homogeneous part of the equation, we will consider the right-hand side to be 0, so right-hand side is considered to be 0.

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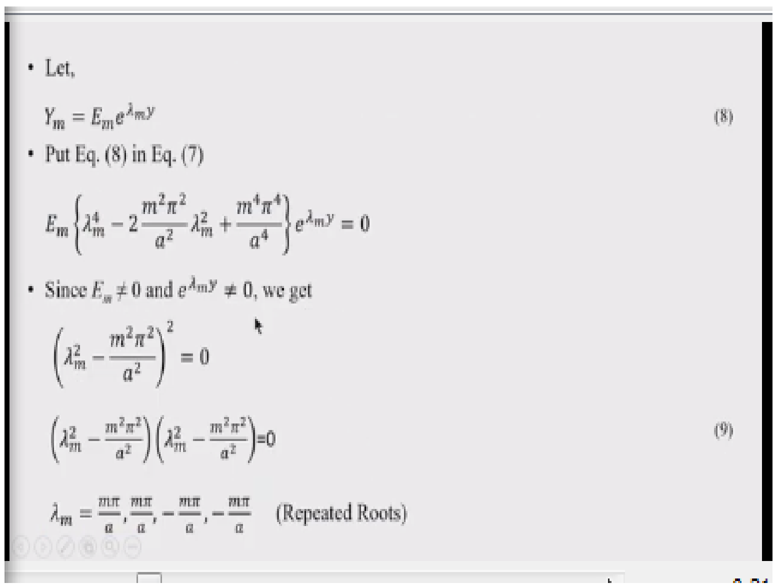

So, now using the theory of linear differential equation, we assume that solution is in the form of $Y_m = E_m e^{\lambda_m y}$, where lambda m is the characteristic parameter. So, substituting $Y_m = E_m e^{\lambda_m y}$ in this equation. Say here it is 4 derivative, so naturally, we will get λ_m^4 and other constant will appear as it is E_m, and when you differentiate e to the power exponential function, again exponential function will appear.

So, exponential function will be common to all the derivative terms and here also, because the $Y_m(y) = E_m e^{\frac{m\pi}{a}y}$. So, that means after substituting the derivatives of this Y_m assumed Y_m, here we get a characteristic equation of this form. That is E_m is there, and this will also appear here, now you can see that E_m should not be 0 for non-trivial solution.

Otherwise, you will not get a meaningful result, or we cannot go forward for the solution of the problem, and $e^{\lambda y_0}$ is also non zero. So, this quantity inside the bracket expression inside the bracket is 0. Now you can see the expression inside the bracket is nothing but the $\left(\lambda_m^2 - \frac{m^2\pi^2}{a^2}\right)^2$, so it can be written as this form.

That means when we expand this, you will get again $\lambda_m^4 - 2\frac{m^2\pi^2}{a^2}\lambda_m^2 + \frac{m^4\pi^4}{a^4}$. Now, this function can be factorised as this, because it is a square, so we have written like tat. So, you will get the repeated root as per the theory of equations, so we get $\lambda_m = \frac{m\pi}{a}$ again $\frac{m\pi}{a}$ repeated roots, $-\frac{m\pi}{a}$ and $-\frac{m\pi}{a}$.

So, after getting this the roots of the characteristic equation, characteristic equation is this, whatever inside the second bracket this is the characteristic equation. So, we have solved this 4th order polynomial and we have got this the 4 roots.

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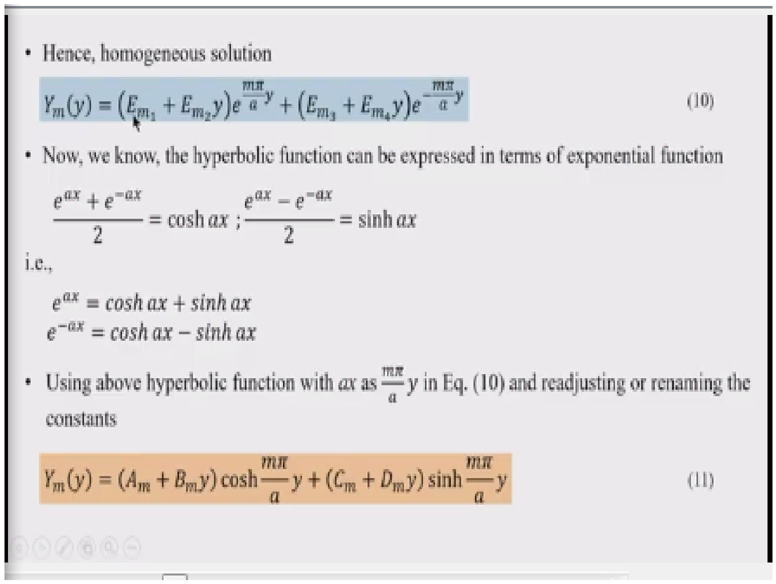

And then we can write the solution simply as this E_{m1}, E_{m1} is a constant E_{m2} because of repeated roots here we are assigning a function $\left(E_{m_1} + E_{m_2}y\right)e^{\frac{m\pi}{a}y} + \left(E_{m_3} + E_{m_4}y\right)e^{-\frac{m\pi}{a}y}$. You can see this is the repeated root $\frac{m\pi}{a}$, and $-\frac{m\pi}{a}$ is also another repeated root. So, because of these 2 repeated roots, we have written the solution like that. If the roots are not repeated, then individually, this say the exponential function will be associated with this each of this constant, and there will be no variable Y.

Now we know that exponential function and hyperbolic form of it is very useful in mathematical physics or any problem in engineering. So, therefore I want to express this exponential function in terms of hyperbolic function. Now the hyperbolic function can be expressed in terms of exponential function. We know that $\frac{e^{ax}+e^{-ax}}{2} = \cosh cosh\, ax$, that is known to us.

Similarly, $\frac{e^{ax}-e^{-ax}}{2} = \sinh sinh\, ax$, so this is also known to us. So, you can see now, if I add these two, I can get e^{ax}, so adding these 2 say $\frac{e^{ax}+e^{-ax}}{2} = \cosh cosh\, ax$ and here $\frac{e^{ax}-e^{-ax}}{2} = \sinh sinh\, ax$. Now, when these 2 are added, we get e^{ax}. And when these 2 are subtracted, when one is subtracted from another, we get e^{-ax}.

So, e^{ax}, you can see cosh $cosh\, ax\, +$ sinh $sinh\, ax$ and e^{-ax}, cosh $cosh\, ax\, -$ sinh $sinh\, ax$. So, our intention is to express the solution in terms of hyperbolic function. So, using the hyperbolic function with ax suppose here we are having $\frac{m\pi}{a}y$. So, let us substitute ax by $\frac{m\pi}{a}y$, then we can write this equation in terms of hyperbolic function replacing ax by $\frac{m\pi}{a}y$.

And then arranging the constant terms because you will get $cosh$, $sinh$ common with associated with other constant then we can plot these terms and constant can be renamed as other constants. So, here the constants are renamed as $A_m\, B_m\, C_m\, D_m$ etcetera. And you will find that the solution can be written in this form $\left(A_m + B_m y\right)$ cosh $cosh\, \frac{m\pi}{a}y\, +\, \left(C_m + D_m y\right)$ sinh $sinh\, \frac{m\pi}{a}y$, so this is what is a homogeneous solution.

But because the differential equation itself is a non-homogeneous equation, you must find the particular integral also. That means the solution for the loading function; if loading function is non zero, then you have to again obtain the solution of that differential equation, then you can superimpose these two solutions to get the complete solution. So, this is our solution 11 one for this rectangular plate, which has two opposite edges; what are the two opposite edges simply supported? x = 0 and x = a, these 2 edges are considered to be simply supported, other 2 edges at y = 0 and y = b the boundary conditions are not specified till now. Let us see what we can do with this solution.

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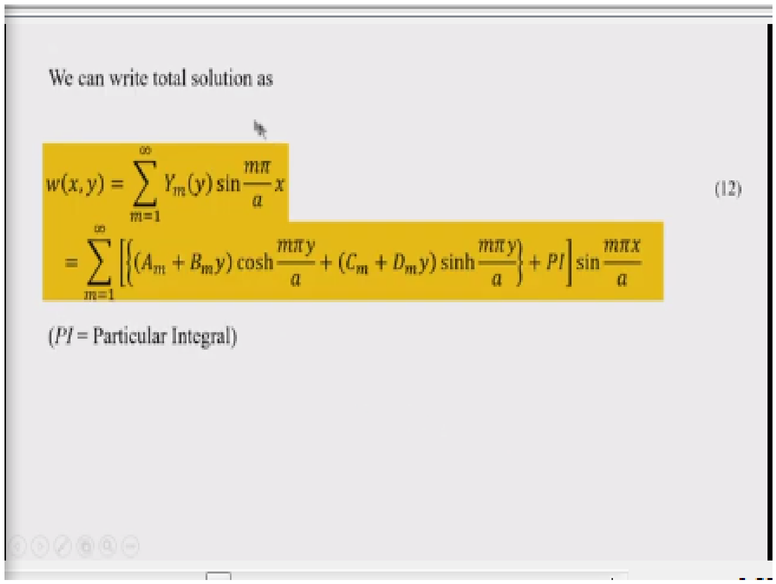

So, we can write the total solution is $Y_m(y)$ sin $sin\ \frac{m\pi}{a}x$, this. Now you can see the earlier this solution is homogeneous solution, here I have added another term say PI, PI is the particular integral or particular solution due to forcing function. If there is a no forcing load, for example, a plate is subjected to edge moment, then there will be no force though in that case PI will not appear. But since the force is existing, so, therefore, I have taken PI also along with this solution.

So, total $Y_m(y)$ is this function inside the 3rd bracket, and it is multiplied by the function sin $sin\ \frac{m\pi}{a}x$, which represents the waves in the x-direction, half-waves in the x-direction in the deflected series. Now sine function is used in the deflection series because it is obvious they are 2 opposite edges are simply supported. So, only sine function is the appropriate algebraic or trigonometry function that can satisfy both geometric as well as force boundary condition.

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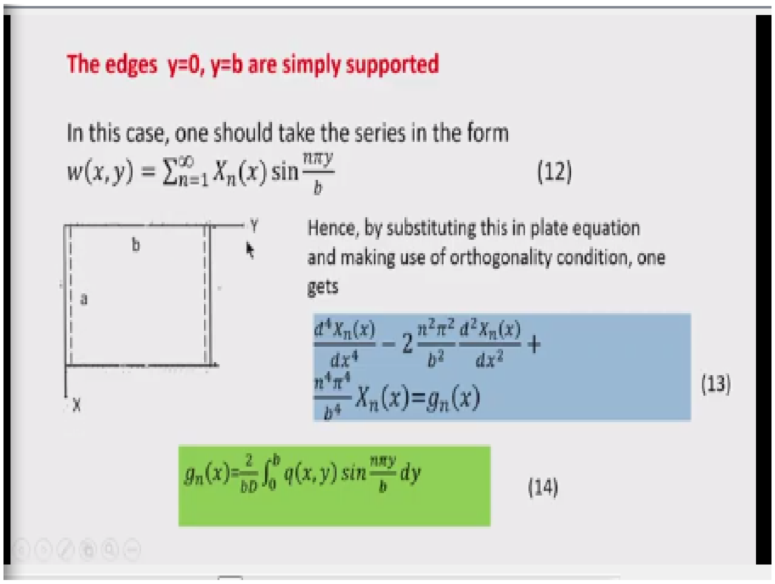

Now, what will happen? If suppose, here the y = 0 and y = b, these edge are simply supporting. So, it is the y-axis, and it is the x-axis, x-axis taken here vertical and y-axis horizontal. Only to see what will be the difference with the expression that we have earlier obtained. So, one should not confuse that only x = 0, x = a should be simply supported, it is not like that y = 0 y = b that the other 2 opposite edges, why we did not touch here till now that may be also simply supported.

So, if this edge is simply supported, that is y = 0, y = 0 edge and y = b edge. So, these 2 edges are simply supported then we can write this solution just by reversing the function. So, earlier it was $Y_m(y)$, now here it is written as $X_n(x) \sin sin \frac{n\pi y}{b}$, so this is the difference in series. Because it will be expanded with function of x, and function of y is now a sine function.

Because the deflection and moment are 0 along the y-direction, so that is why we have taken this $\sin sin \frac{n\pi y}{b}$, here represent the length of the plate in the y-direction. So, by substituting in plate equation again similarly and making use of orthogonality condition as we have done earlier, we now arrive at this equation. So, that is nothing but $\frac{d^4X_n(x)}{dx^4}$; now, here, all derivatives of the functions are with respect to x.

Previously we have obtained the function with respect to y, because this function was assumed as a function of y. Now similarly, we will get the forcing integral, that is, a forcing part is a function of x, because when we multiply these by $sin\frac{n\pi y}{b}$, this is the function, and it is integrated with respect to y. The variable y has to be replaced by constant, which are the limits of the integral; so, therefore, the result will be a function of x. Hence this is the differential equation right-hand side; this non-zero part is a function of x.

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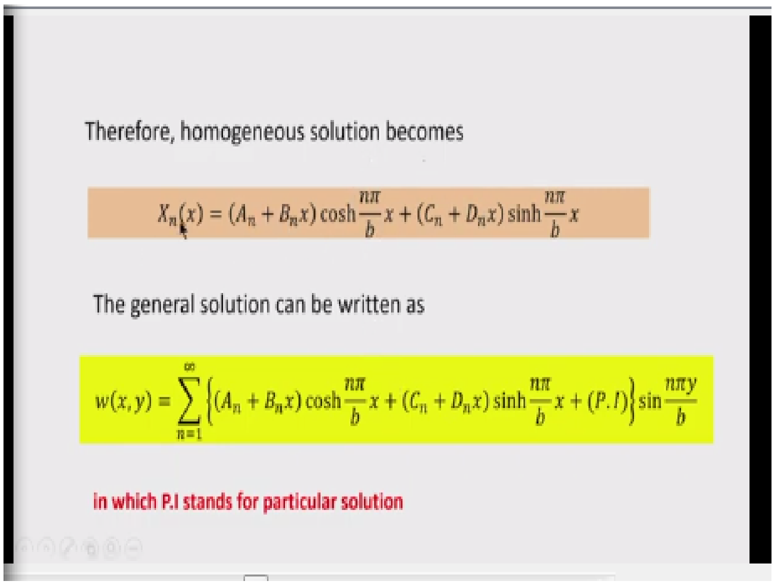

So, therefore homogeneous solution becomes this. So, these are the constants again, but with the variable n, the harmonic number is here n instead of m. The general solution now can be written as $\left\{\left(A_n + B_n x\right) \cosh cosh \frac{n\pi}{b} x + \left(C_n + D_n x\right) \sinh sinh \frac{n\pi}{b} x + (P.I)\right\} \sin sin \frac{n\pi y}{b}$. So, that we have seen previously, it was a function of y completely; the expression inside the 2nd bracket was function of y because the x = 0, x = a simply supported.

But in that case, y = 0 and y = b are simply supported. So, therefore the simplest about a s function is satisfied by this sine function, and other function must be a function of x. In which P I stands for particular solution.

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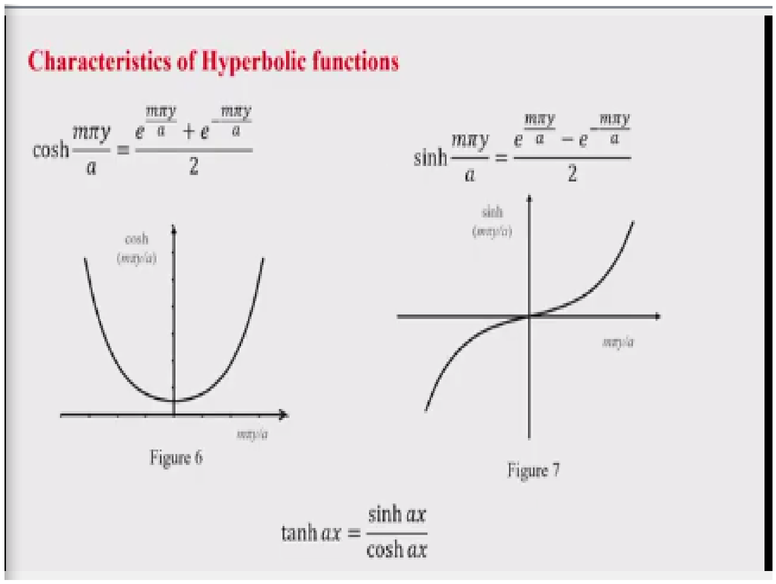

Now let me show you the characteristics of hyperbolic function, how it looks? You can see the cosh $cosh\ \frac{m\pi y}{a}$ or cosh $cosh\ \frac{n\pi x}{a}$, b whatever we encounter in the solution, total solution in the complete solution of the Levy's method. There we have to deal with this function that we have seen cosh $cosh\ \frac{n\pi y}{b}$, or you have seen that it is cosh $cosh\ \frac{m\pi x}{a}$, whatever maybe.

So, here you can see that if I plot this graph with $\frac{m\pi x}{a}$ with this index, then you can see the cos hyperbolic is a symmetric function; it is symmetrical with respect to your x and y-axis. But if we plot, say sinh $sinh\ \frac{m\pi y}{a}$, we will see that it is anti-symmetric function, so that is the difference. Here cos hyperbolic is a symmetric function and sine hyperbolic is anti-symmetric function, tanh $tanh\ ax$ is nothing, but just like your trigonometric quantity, it is $\frac{\text{sinh}sinh\ ax}{\text{cosh}cosh\ ax}$.

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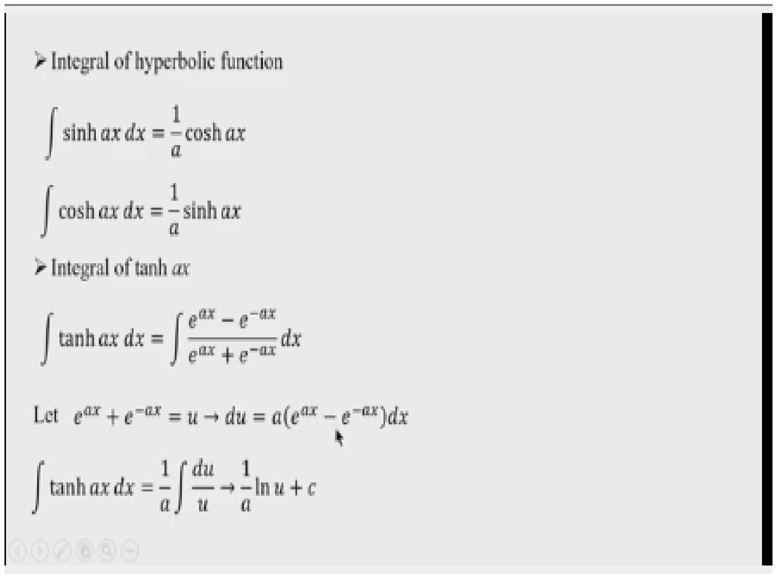

So, we will be requiring to use several times this integration of $sinh$ is $cosh$ or derivative of $sinh$ is $cosh$, so you must be familiar with that. So, integration of $sinh$ is cosh $cosh\, ax$, cosh $cosh\, ax$ the integration is sinh $sinh\, ax$, integral of tanh $tanh\, ax$ is found out like that. Because tan hyperbolic first I expressed in terms of the ratio of sinh $sinh\, ax$ to the cosh $cosh\, ax$, then after method of substitution, I obtained the integral.

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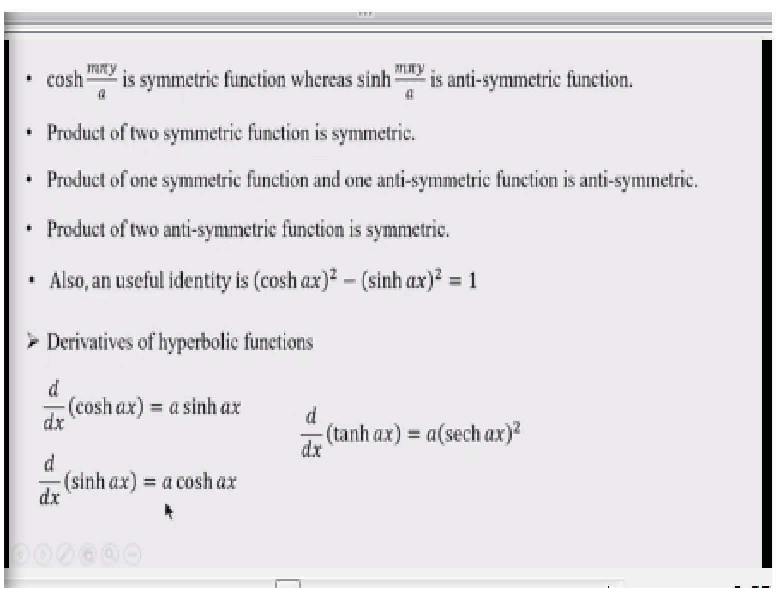

So, derivatives are also important. So, derivative of cos hyperbolic is sinh $sinh\, ax$, derivative of sinh $sinh\, ax$ is cosh $cosh\, ax$, tanh $tanh\, ax$ is like that. Now, the characteristic of the function is $cosh$ function is symmetric function, and sine hyperbolic function is anti symmetric function. So, that statement is very important to solve the problems using the Levy's method.

Because in symmetric function, if the loading is symmetrical or boundary condition is symmetrical, then there will be no question of any anti symmetrical term in the deflection expression. So, naturally, we can omit from physical reasoning. Then also, we know the product of 2 symmetric functions is symmetric, that is also true. Then product of 1 symmetric function and one anti-symmetric function is again anti-symmetric.

So, suppose the x square is a symmetric function, and sine x is anti-symmetric function, naturally, so x square sine x is again anti-symmetric function. Similarly, here say we have got when we found the solution, we have seen that the terms $y\times$ cosh $cosh\,\frac{m\pi y}{a}$ appears. So, in that case, $y\times$ cosh $cosh\,\frac{m\pi y}{a}$, what will be the nature of the function? It will be anti-symmetric, because y is anti-symmetric function and cosh $cosh\,\frac{m\pi y}{a}$ is symmetric.

So, product of one symmetric and one anti-symmetric function is again anti-symmetric. But when we see that y is multiplied by sinh $sinh\,\frac{m\pi y}{a}$, then y is anti-symmetric, sinh $sinh\,\frac{m\pi y}{a}$ is anti-symmetric. So, product of 2 anti-symmetric function is again symmetric function, so that we will use. Then useful identity for simplifying the condition or solution there $ax\ -\ ax\ =1$.

So, there is a difference that you can note, in trigonometrical function, say if θ is the angle $\theta + \theta = 1$. But here, you can see $ax\ -\ ax\ = 1$, so that identity can be used in simplifying the solution.

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So, integral of this tanh $tanh\ ax$ is given here.

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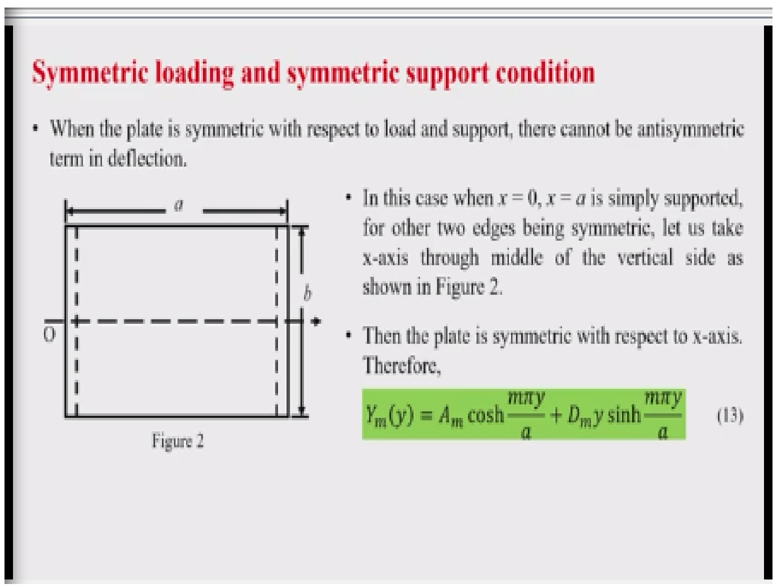

Then we can see what happens when there is a symmetric loading and anti-symmetric and symmetric support condition. Now, in many cases, we adopt the symmetric type of construction because symmetric structure or symmetric element is easy to construct and also easy to analyse also and load resistance response to load is also favourable. Because when there is a anti-symmetric condition, where there is an eccentric loading or anything, then in addition to your this vertical deflection, the twisting moment will be also of significant amount.

So, therefore, in practical conditions, we will try to keep the structure symmetric as far as possible. Now, we call this structure symmetric when the loading and support conditions are completely symmetric. Now, here see this plate, this plate has length a and width b, and two opposite edges are simply supported x = 0 and x = a are simply supported if this represents the x-axis along the length direction and along the width direction the y-axis is shown.

It is not shown in the diagram but considers that along the edge parallel to y here, parallel to width, the y-axis exist. So, the plate here is symmetrical; if I take the x-axis or origin at the centre of the left-hand edge, then we draw a horizontal line. Then we can see that plate is symmetrical about that line provided the other two opposite edges have also same boundary condition. So, for example, the boundary condition may be simply supported on two opposite edges, or it may have this fixed clamp condition on two opposite edges or it may have free condition at two opposite edges.

For example, a slab culvert, you have seen the slab culvert or slab bridge, slab is also there for small span; only the slab is resting on the abutment. So, in that case, you can see that if this line these are the abutment support, so it is supporting the slab on the two opposite edges just like it simply support. And the other two edges are free if there is no stiffening beam or girder. So, then these two edges are free, so this condition may also represent a slab bridge supported by abutment at the two ends.

So, in this case, at x = 0 and x = a, simply supported for other two edges being symmetric. Symmetric means condition we have not specified yet, let us take x-axis through the middle of the vertical side as shown in figure, then the plate is symmetric. And we have seen the solution of this w(x,y), homogeneous solution, here; you can see that when this cos hyperbolic m pi by a into y is multiplied by A_m, then cosh $cosh\,\frac{m\pi y}{a}$ is a symmetric term, and sinh $sinh\,\frac{m\pi y}{a}$ is anti-symmetric term.

But when the anti-symmetric term is multiplied by another anti-symmetric term, it will be a symmetric term. So, when the support condition and the loadings are symmetrical, then we can take the symmetrical boundary conditions. And therefore, the equation becomes simplified. That means keeping only the symmetric term of the solution $cosh$ and $y \times sinh$. Because sine hyperbolic is anti-symmetric and y is anti-symmetric, so product of 2 anti-symmetric term is symmetric term.

So, therefore for this type of condition, we have kept on these symmetric terms. But when anti-symmetric condition exists, then we will keep $B_m \sinh sinh \frac{m\pi y}{a} +$ other terms say $C_m y \cosh cosh \frac{m\pi y}{a}$. Because in that case y is anti-symmetric and cos hyperbolic is also symmetric, so product of anti-symmetric and symmetric function is again anti-symmetric.

So, accordingly, after knowing the nature of the function, now you have completely known because graphical representation also I have shown. So, you can take appropriately the symmetric terms or anti-symmetric terms.

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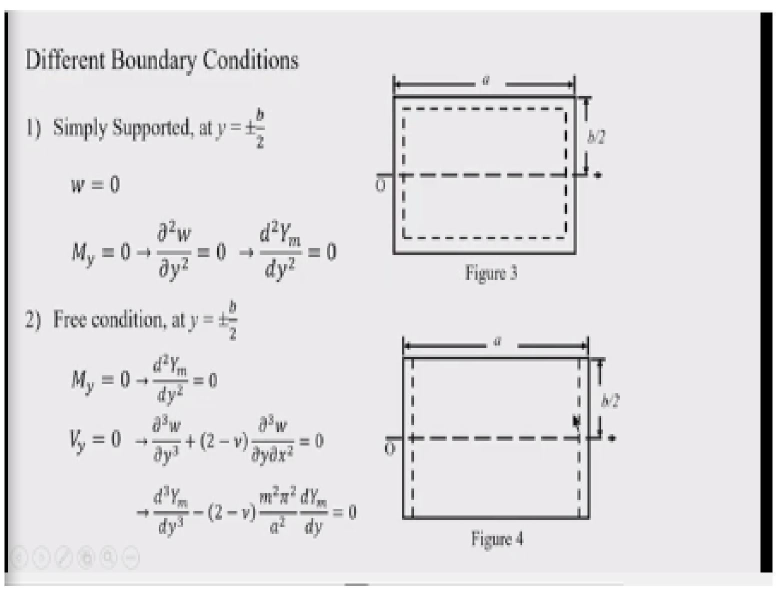

Now, different boundary conditions for this condition say 2 edges are simply supported. So, if I take the x-axis here, running through the middle of the vertical edges, then at $y = \pm b$.

Suppose, this is the positive direction of x-axis, so it is $y = +\frac{b}{2}$ and here it will be $y = -\frac{b}{2}$, that simply supported boundary condition again have to be satisfied. So, this conditions are $M_y = 0$, and this condition $M_y = 0$, that means $\frac{\partial^2 w}{\partial y^2} = 0$, that means $\frac{d^2 Y_m}{dy^2} = 0$.

Free condition for example, $y = \pm b$. So, for example, this is a slab bridge or a slab culvert, so these two edges are free. So, in that case the condition here the bending moment is 0, so this curvature term is 0, and edge shear is 0. So, edge shear quantity is this quantity and when it is differentiated with respect to this function, then m square pi square by a square come because of $\frac{\partial^2 w}{\partial x^2} = 0$, because w contains sinh $sinh\ \frac{m\pi x}{a}$ also.

So, this is the two boundary conditions at the free edges. So, naturally imposing two boundary conditions at the 2 edges, we will get the 2 equations and that can be solved for 2 unknowns constants associated with the homogeneous solution.

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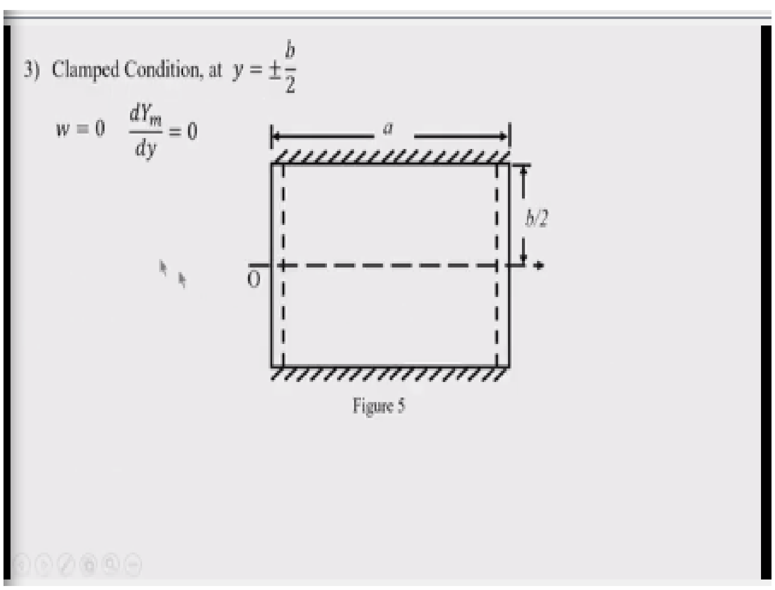

Then at the clamped condition, we have w = 0, and the slope is 0, which requires that derivative of Y_m with respect to y = 0, because this edges along the y the edges clamped. So, the slope is 0 along the y-direction.

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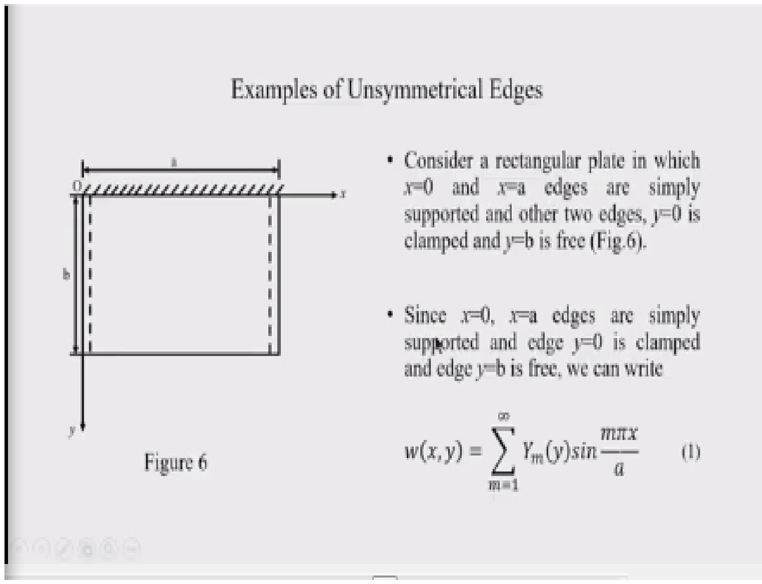

Now examples of unsymmetrical edges.

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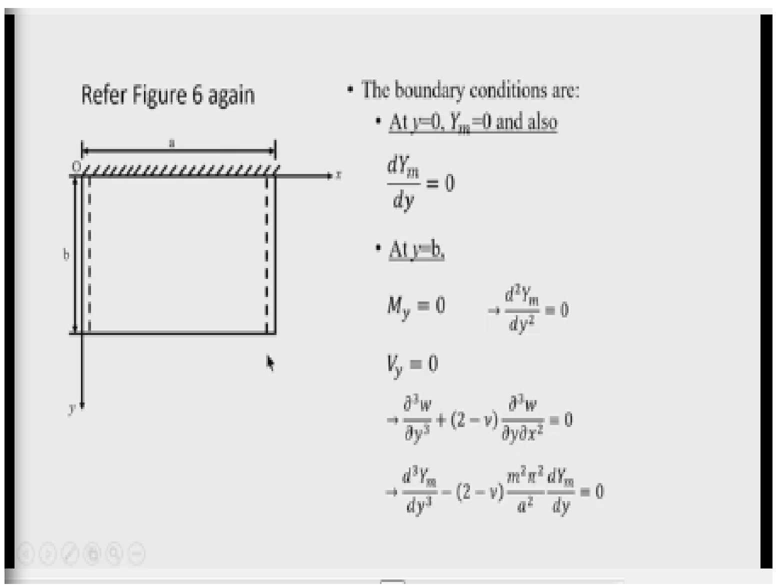

So, unsymmetrical edges, again we know that these boundary conditions will be different, say boundary condition say if it is unsymmetrical say this figure, we have shown this figure, this shows that y = 0 edge is fixed, and y is b edge is free. So, because of these boundary conditions, the plate is unsymmetrical plate. So, in that case, the requirement is that at y = 0, that means here along this edge, the slope is 0, and deflection is 0.

So, y is 0 and Y_m is 0, that means Y_m is 0 and slope that is the first derivative of Y m is also 0. At y = b that is this edge $M_y = 0$, that is $\frac{d^2 Y_m}{dy^2} = 0$ and edge shear 0, edge shear 0 means $\frac{\partial^3 w}{\partial y^3} + (2 - \nu)\frac{\partial^3 w}{\partial y \partial x^2} = 0$. So this obviously reduces to this equation, and imposing this, we can find the constants of integration.

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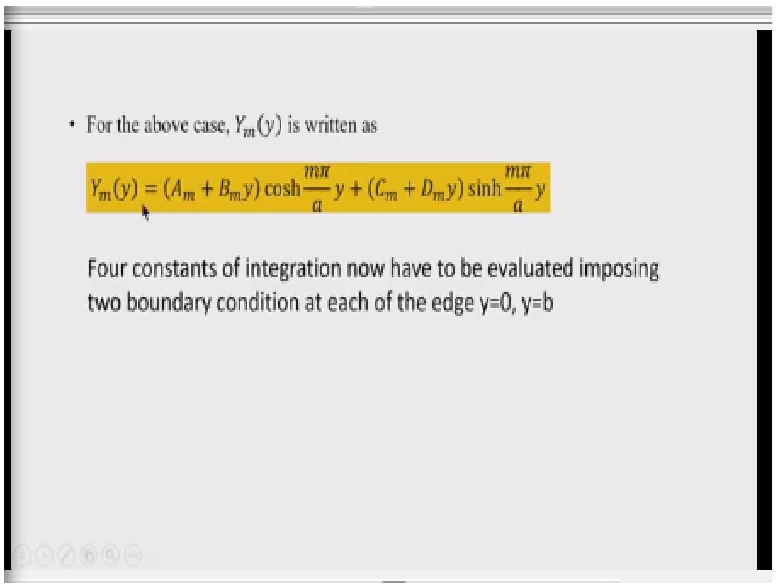

So, in that case, the full equations have to be retained, that means we cannot omit any terms. So, 4 constants are retained here, and therefore 2 condition at each edge need to be imposed to find out these 4 constants of integration from 4 algebraic equations. After applying the boundary condition, the problem will be these boundary value problems, so you can calculate the constants of integration.

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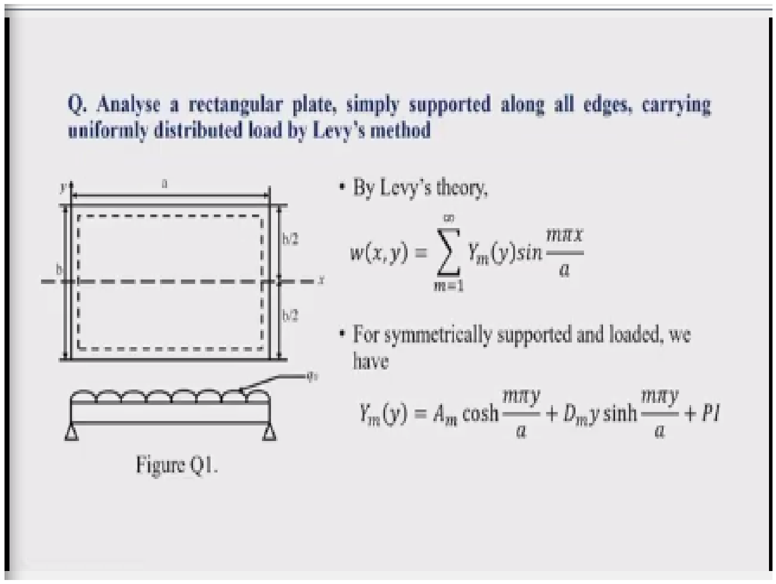

Now let us illustrate the Levy's method in a rectangular plate, which simply supported edges in the 2 opposite edges simply supported. But there is no restriction that Navier's method can only be used for all edges simply-supported. Levy's method is more general, that means when all 4 edges are simply supported, you can also use the Levy's method. But Levy's method relaxes more boundary condition that means it is also applicable when other 2 edges are also differently supported.

But one essential thing is that 2 opposite edges must be simply supported. Now, here in this plate problem that I have for this class, the x = 0 and x = a are simply supported this edge and y = 0 and y = b; also, I have taken simply supported. And I have taken the x-axis running through the middle of the vertical edges. So, plate is symmetrical about this x-axis, and therefore symmetrical condition exists.

And we can take now only this $A_m \cosh cosh \frac{m\pi y}{a} + D_m y \sinh sinh \frac{m\pi y}{a} + PI$, because the plate is acted upon by uniformly distributed load of intensity q_0 per unit area. So, the load generally expressed in the plate or per unit area when it is distributed, so q_0 is expressed as load per unit area. And because of the loading, the particular integral is non zero, and we have to consider this particular integral in this expression.

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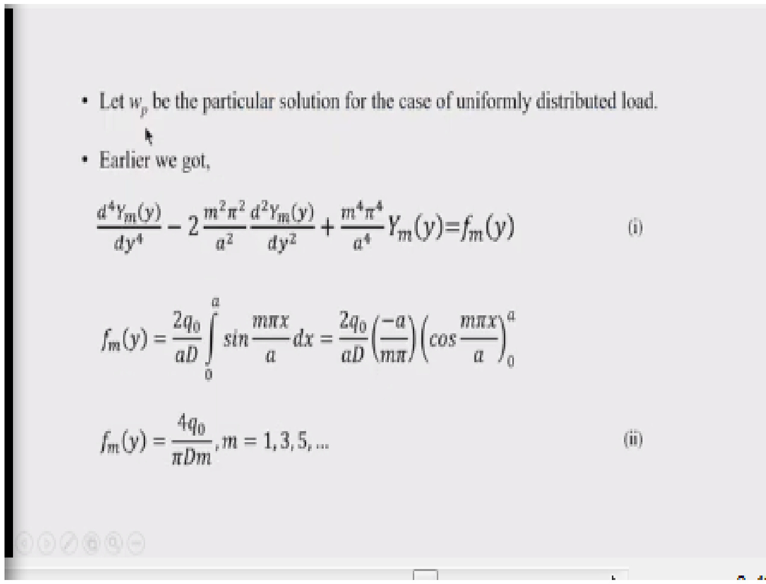

Now let w_p the particular solution for the case of uniformly distributed load. Now, here there are different ways of solving the non-homogeneous differential equation. One very attractive method for our physical problem that is the problem encountered in structural mechanics is the method of undetermined coefficient. So, in this method, observing the nature of the forcing function or this non-homogeneous part, we can assume the possible solution.

For example, if the forcing term is a constant, then obviously the particular solution is also constant. If the forcing function is a harmonic, say sine function, the particular solution should also contain functions of sine and cosine; that is harmonic, it may be cos; also, it may be sine also. So, there may be phase difference, of course, so a harmonic forcing function may require or may necessitate the particular solution as also harmonic.

So, here seeing the nature of this particular integral, this forcing function as q_0, we first evaluate $f_m(y)$. And $f_m(y)$ is evaluated by this integral that I have shown in the beginning and that it becomes this value. After integrating sine with respect to cos and then putting the limit, we get this $\frac{4q_0}{\pi Dm}$, where D is the flexural rigidity of the plate and m is the integer.

And you can see that integral is valid only for the odd number of m; for even number of m, this function will disappear. So, you can see that when this m = 2, or 4, 6, etcetera, then this function will be 0, the integration, after putting the limit, it will give you 0 result. So, only the result that is obvious it is -2 when m is odd number that is 1, 3, 5, etcetera. So, we get this $f_m(y)$ as this function, and this is obviously a constant.

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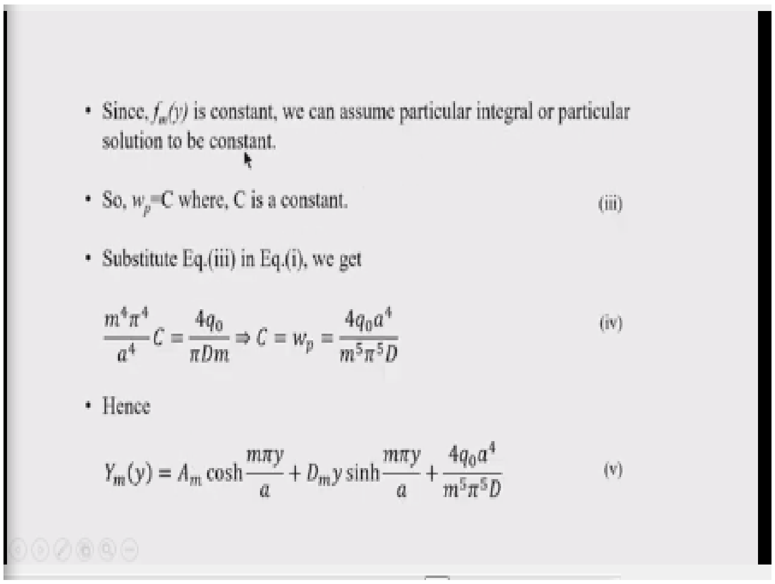

So, therefore, we assume that particular integral or particular solution is also a constant. So, let us assume that w_p=C, where C is a constant. Substituting these 3 in equation 1, equation 1 is this. That means if this is a solution, then it must satisfy the differential equation that is the philosophy. So, after substituting this, we get when the constant is substituted $Y_m = Constant$, then this term goes to 0, this also goes to 0.

So, here it will be $\frac{m^4\pi^4}{a^4}C = \frac{4q_0}{\pi Dm}$. So, after doing this, we get this equation from where C is obtained as, C is nothing but particular integral is $\frac{4q_0a^4}{m^5\pi^5D}$. Where π and m appears with a 5th power, you can see, so it will be rapidly converging.

So, m contains the power of 5, so when this first term is taken for deflection, you will get almost accurate result. When the second term that is 3 is taken, 3 to the power 5 in the denominator will decrease the value drastically, so the series will rapidly convergent after taking a certain terms. Hence the complete solution is for $Y_m(y) = A_m \cosh cosh \frac{m\pi y}{a} + D_m y \sinh sinh \frac{m\pi y}{a} + \frac{4q_0 a^4}{m^5 \pi^5 D}$, D is the flexural rigidity.

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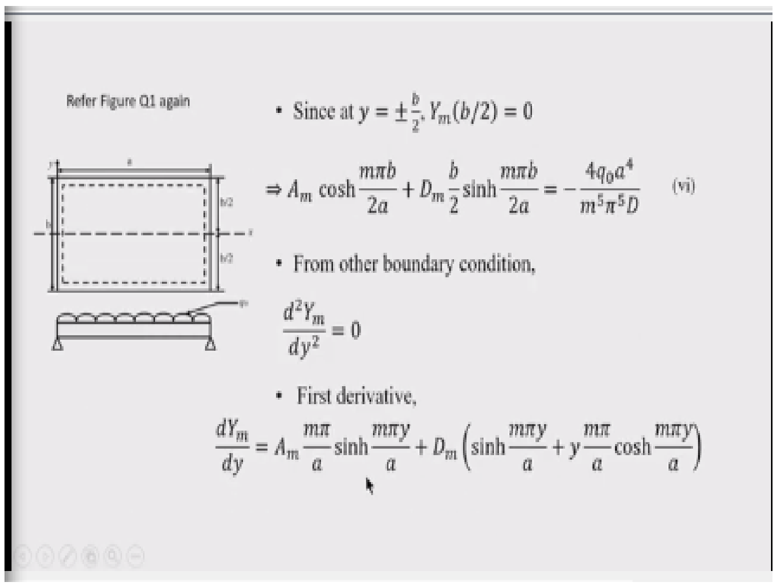

So, now let us see how we can impose the boundary condition. So, refer the figure of the question that they simply supported plate here again, and this is the x-axis, and this is the loading that applied on the plate. Since at $y = \pm\frac{b}{2}$, when $y = +\frac{b}{2}$, that means this edge or $-\frac{b}{2}$ is this edge, then the boundary condition is same. So, first applying the boundary condition on this edge, $y = \frac{b}{2}$ on deflection.

So, deflection series that means this expression will come and when you substitute $y = \frac{b}{2}$, this function will be 0 of course, because at $y = \frac{b}{2}$ deflection is 0, so Y_m is 0. So, $A_m \cosh cosh \frac{m\pi b}{2a} + D_m \frac{b}{2} \sinh sinh \frac{m\pi b}{2a} + \frac{4q_0 a^4}{m^5 \pi^5 D} = 0$. So, this term is written, and this constant term is transferred on the right hand side with q 0.

Then from other boundary condition, that is the second derivative of this y is 0. So, first derivative is taken, very carefully you have to take the derivative otherwise, there is a chance of committing mistake in the solution, so first derivative is taken.

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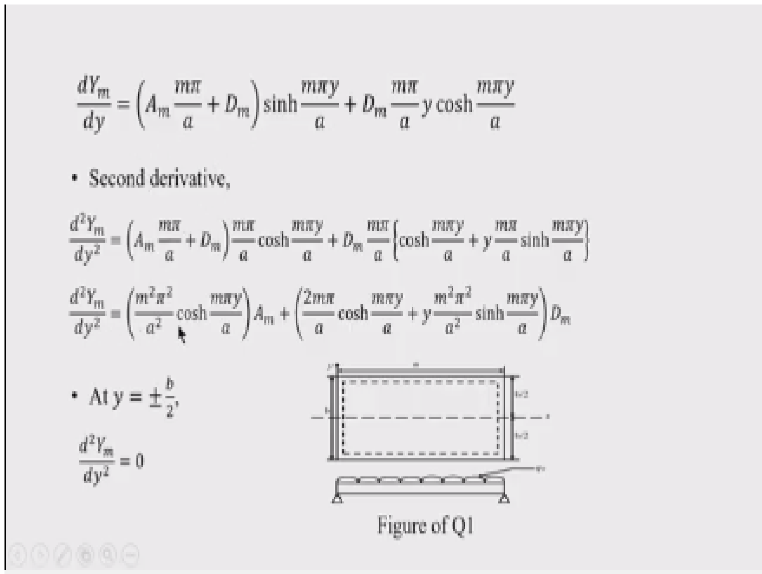

Then second derivative is again taken of this function. And you can see now imposing the boundary condition at $y = +\frac{b}{2}$, or $-\frac{b}{2}$, that curvature in the y-direction is 0, because the edge y = 0, y = b is also taken as simply supported. That means here not y = 0, y = b because the axis is taken running through the middle of the vertical sides. So, that means $y = +\frac{b}{2}$, or $-\frac{b}{2}$, the edge is simply supported.

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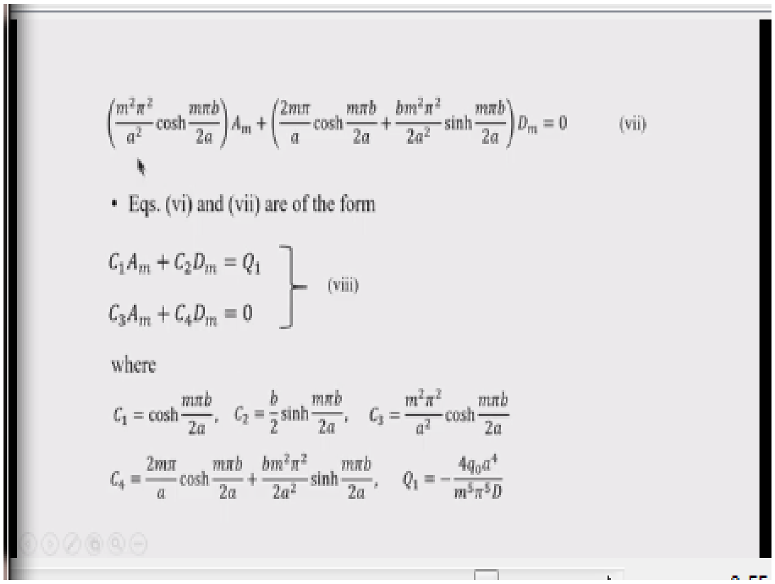

$$\left(\frac{m^2\pi^2}{a^2}\cosh\frac{m\pi b}{2a}\right)A_m + \left(\frac{2m\pi}{a}\cosh\frac{m\pi b}{2a} + \frac{bm^2\pi^2}{2a^2}\sinh\frac{m\pi b}{2a}\right)D_m = 0 \qquad \text{(vii)}$$

- Eqs. (vi) and (vii) are of the form

$$\left.\begin{aligned} C_1A_m + C_2D_m &= Q_1 \\ C_3A_m + C_4D_m &= 0 \end{aligned}\right\} \text{(viii)}$$

where

$$C_1 = \cosh\frac{m\pi b}{2a}, \quad C_2 = \frac{b}{2}\sinh\frac{m\pi b}{2a}, \quad C_3 = \frac{m^2\pi^2}{a^2}\cosh\frac{m\pi b}{2a}$$

$$C_4 = \frac{2m\pi}{a}\cosh\frac{m\pi b}{2a} + \frac{bm^2\pi^2}{2a^2}\sinh\frac{m\pi b}{2a}, \quad Q_1 = -\frac{4q_0a^4}{m^5\pi^5D}$$

So, substituting the boundary condition on second derivative, we get this. Now you can see that 2 unknown constants and 2 equations are obtained. One condition is obtained using the deflection condition that is the equation. Second equation is obtained by imposing the curvature condition that is bending moment equated to 0. So, these 2 equations are of the form, say C 1 A m + C 2 D m = Q 1 this is one equation. And another equation is C 3 A m + C 4 D m = 0.

So, because of the loading, we get this as non zero term Q_1. So, 2 equations are obtained and 2 unknown, and these are algebraic equation, and this can be very easily solved. Where they C_1 are the coefficient that can be collected from this expression, and you can see that this is the second equation for after imposing the boundary condition on bending moment. So, therefore, C 1, we get this C_3 that we get is written here C_3 is $\frac{m^2\pi^2}{a^2}\cosh cosh \frac{m\pi b}{2a}$.

And C_4 is this quantity that is written here. Q_1 is the right-hand side of the first equation, this is. So, these 2 algebraic equations are written in equation number 8 and coefficients are identified clearly. So, this can be solved by using any rule.

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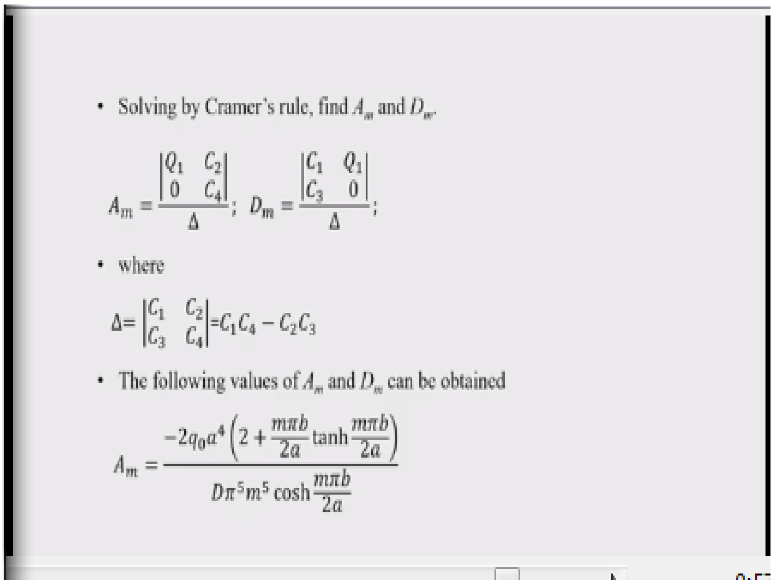

The easiest rule is Cramer's rule using the determinant, it is a systematic procedure, and you can find that using Cramer's rule the coefficient, the variable A m, the unknown constant A_m becomes this determinant of Q_1 C_2 I am telling you row wise. So, Q_1 C_2 0 C_4 divided by Δ, what is Δ? Δ is the determinant formed by the coefficients associated with the A_m D_m C_m, A_m and D_m, 2 variables are there, so A_m and D_m

So similarly, we get there is a first coefficient is A_m, so first column is replaced by the right-hand side value, so right-hand side value is Q_1 due to forcing term, in second equation it is 0. And the second column contains the coefficient C_2 and C_4. Similarly, for the second variable that is D_m, we get this second column as the forcing term Q_1 and 0 and the first column with the coefficient associated with A_m C_1 and the second row coefficient associated with A_m is C_3.

And delta is the determinant that is form by the coefficient, and it can be found out by expansion. So, after doing this algebra, we can get the value of A_m as this quantity and value of D_m as this quantity.

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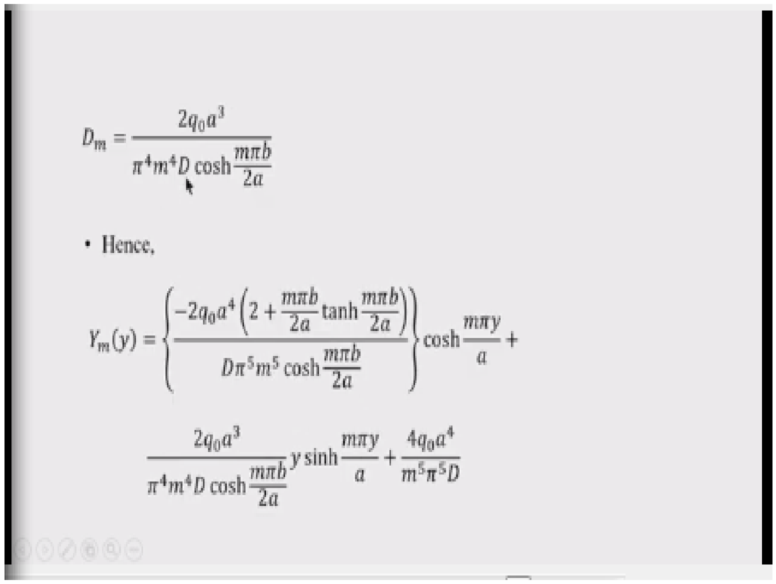

So, once the value of A_m and D_m is found, then you can completely write the equation for $Y_m(y)$. So, $Y_m(y)$ is written completely because this is known, and this is known, and these quantities are known. So, after writing this equation, we get now the complete expression for $Y_m(y)$ function; it is given like that.

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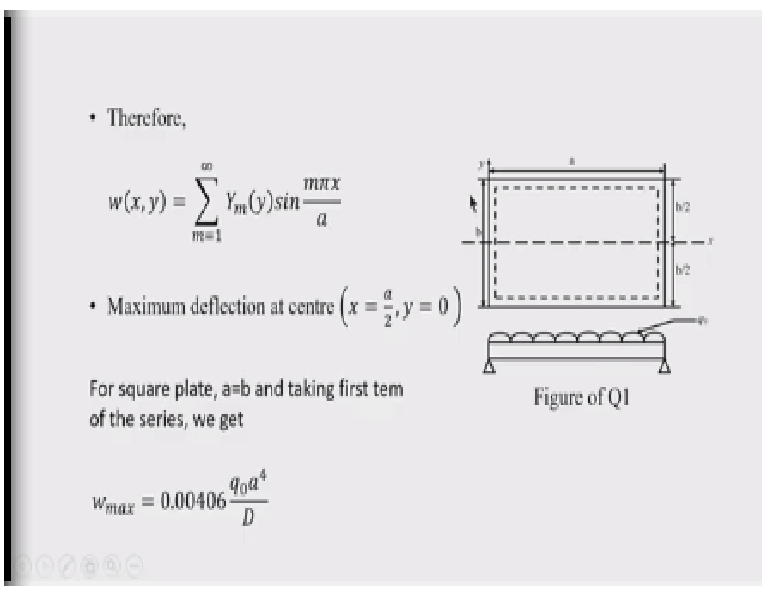

And the deflection is now $Y_m(y)sin\frac{m\pi x}{a}$. Now, a rectangular plate carrying infinitely distributed load. And we see this a symmetrical support condition simply supported along all edges, the standard result we know from Navier's solution already. And we can compare our result that we obtained by Levy's condition. So, maximum deflection is obtained at the centre of the plate, that is x = $\frac{a}{2}$ and y = 0.

Now, for example, the plate is a square plate that is a = b, then after putting this a = b in this expression of deflection. Because now the quantities are completely known and taking the summation with any number of terms. Say if we take first term, you will get the accurate result, enough accurate result for the deflection. Of course, for bending moment, you require more terms to get the convergence of the series for shear still more terms.

So for deflection quantity, we take on the first term and taking first term and a = b; this series is evaluated because Y_m is completely known, m is taken here as 1. And whatever value here, that for Y_m, m is substituted as 1. So, after calculating this, the deflection is found to be $0.00406\frac{q_0a^4}{D}$. So, this quantity coincides with the value that we obtained earlier by Navier's method.

So, Levy's method has application in more general cases compared to your Navier's method. Because Navier's method has severe limitations because all the edges must be simply supported for a rectangular plate; otherwise, double trigonometrical series cannot be used. But Levy's method relaxes one restriction, that is 2 opposite edges only need to be simply supported; other 2 opposite edges may have any condition, may have same condition or may have different condition, does not matter.

Classical or non-classical boundary condition, anything can, but requirement is that 2 opposite edges must be simply supported; without that, Levy's method cannot be used. So, let us see what we have done today.

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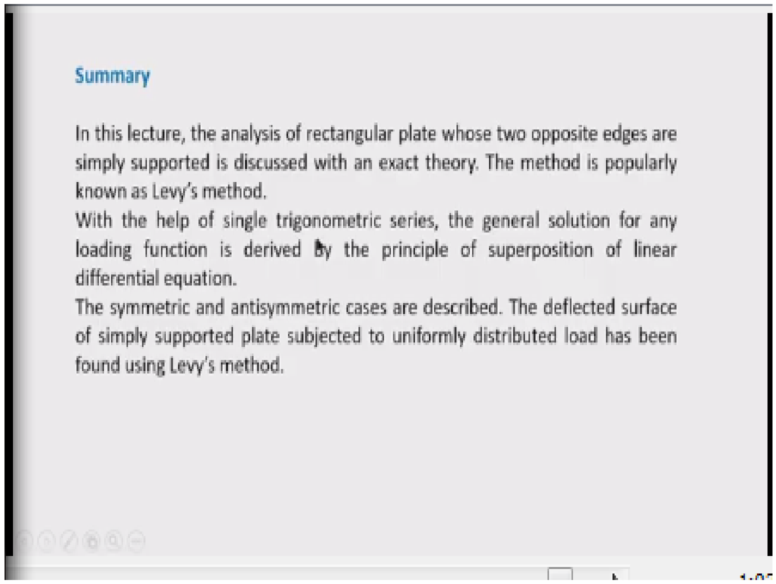

In this lecture, the analysis of rectangular plate whose 2 opposite edges are simply supported is discussed within an exact theory. Because Levy's method is an exact theory, it yields the closed-form solution, closed-form expression for the differential equation of the plate; the method is popularly known as Levy's method with the help of single trigonometric series because the 2 boundary conditions are known only.

So, one sine function is taken in the deflected surface assumption first. So, with the help of single trigonometric series, the general solution for any loading function is derived. We have derived this general solution for other functions, that is, y, for any loading condition. Particular integral, of course, have to be found out depending on the nature of the load, that I have told you that best method to apply in the plate problem is the method of undetermined coefficient in the theory of non-homogeneous linear differential equation.

So, the general solution is derived by the principle of superposition that means homogeneous solution is first obtained, and particular integral is added to this to get the complete solution. But the constant of integration are evaluated, imposing the boundary condition. In general, there are 4 constants of integration, and 4 constants of integration require 2 boundary conditions on each edge, so that have to be applied.

But this requires only if the conditions are completely unsymmetrical. But if this is anti symmetrical or symmetrical, we can use only the 2 constants appearing in the homogeneous solution of y because the other 2 constants can be drop. If the conditions are symmetric, anti-symmetric a term in the deflection function can be dropped. Similarly, if the condition is anti-symmetric, then this symmetric function can be drop.

Anyway, the results are simplified based on the condition of symmetry. So, that we have utilised, and we have compared the results of simply supported plate means rectangular plate, simply supported along all edges. And this is analysed for deflection is in Levy's method, and we knew the results obtained from Navier's method, and it is also available in the textbook. So, we have seen that 2 results coincide and are in well agreement.

So, Levy's method, you have understood, it is more general compared to this Navier's method. And in our next few classes, we shall explore this method to obtain the solution for other boundary condition, and for other load conditions. Thank you very much.