Plates and Shells

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Module-02
Lecture-07

Simply Supported Plate Resting on Elastic Foundation and Other Examples

Hello everyone, today I am giving lecture number 3 of module 2. Based on the formulations developed in my earlier lecture on simply supported rectangular plate, I will go further to take up some problem which are frequently encountered in our practical application.

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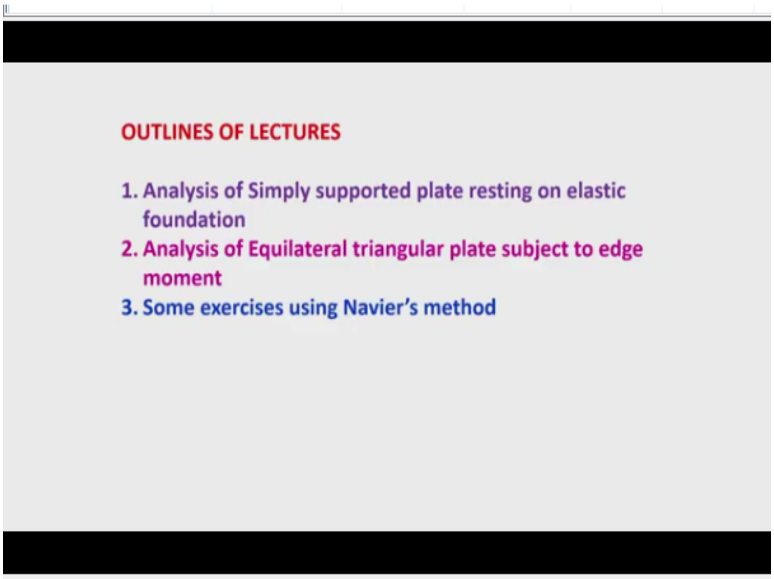

So, today what I want to discuss is analysis of simply supported plate resting on elastic foundation that will be my first topic, then I will go to analysis of equilateral triangular plate subject to edge moment. And we will see how some other exercises based on Navier's method that we are continuing till today can be solved. So, simply supported plate that I have discussed is best possible way to find a solution, is taking a double trigonometric series.

And it gives you the exact results, because it satisfies the geometrical as well as fourth boundary condition on the edges and therefore, it is possible to get the exact solution for the deflection surface. Once the deflection surfaces are obtained, one can find the bending moment in 2 direction x and y, twisting moment M_{xy}, edge shear force, V_x and V_y and shear force also at any point Q_x and Q_y as well as your this corner reaction. That is very important in case of simply supported plate where the corners are not resting.

Because in that case corners have a tendency to lift up, then these elastic foundations are actually the material which offers the resistance to the deflection, just it acts like a spring. So, when a plate is resting on a spring, the spring is uniformly distributed over the area; it is not at the discrete point. So, naturally the load acting on the plate will be reduced by the spring forces, the upward spring forces.

So, that has to be taken into account to modify the governing differential equation of the plate. The most important parameter for such type of problem there is problem number 1 simply supported plate on elastic foundation is the value of the subgrade modulus of reaction, that value is sometimes very difficult to find out accurately. The example of such type of problem is found in case of foundation design.

So, when we consider a foundation slab resting on the soil, there soils of our say upward reactions that is soil pressure is distributed over the plate area and in that case the subgrade modulus of soil is very important to be given as an input to the problem, otherwise our analysis will be wrong and subgrade modulus in case of soil generally is found by plate load test and the value that we take from geotechnical engineer will be used for our analysis in structural component.

Then we have type of plate which is triangular in shape and I have idealized this equilateral triangle, because such type of slab is also our plate is also common. It is not necessary that always the plate should be rectangular, but most common application is rectangular and circular, but sometimes other than rectangular and circular. Other shapes are also adopted in practical application.

So, problem I have selected are, they are triangular plate, equilateral triangular plates subjected to edge moment of equal amount, equal magnitude acting along the edge uniformly distributed. So, that problem we will see and later on 2 exercises I will discuss and I will give important steps to find out the solution for that exercises.

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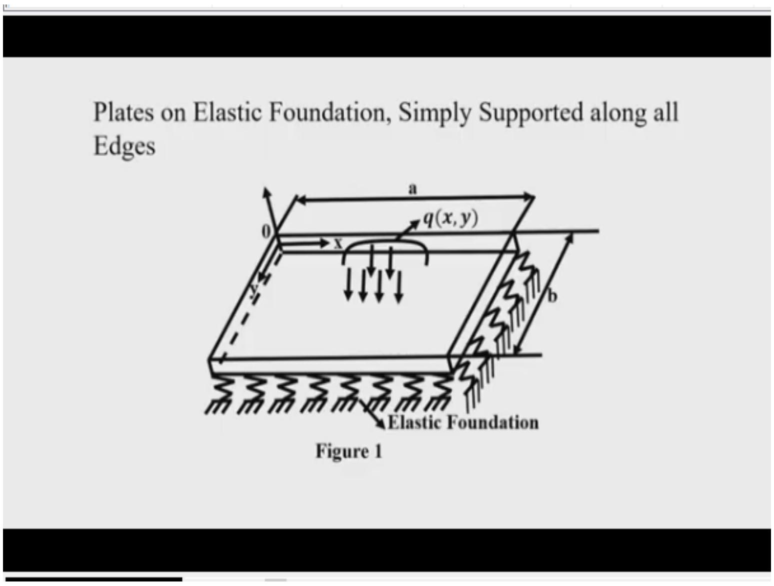

Now, let us see this plate which you can see a rectangular plate. The length is a and the breadth is b. The load acting on the plate is q(x,y), it may be a uniformly distributed load or it may be concentrated load or any other type of load uniformly varying load, so many cases may arise and it is resting on the soil subgrade or elastic subgrade that is we call the elastic foundation which is model as a spring. That is called Winkler foundation model.

So, spring is representing the soil here or elastic subgrade which offers upward reaction to the deflection of the plate. So, at any point on the plate the spring force will be the modulus of subgrade reaction if the value is k and the deflection at that point $k \times w$ will be the upward spring force and that force will reduce the loading that is downward load that is applied on the plate. So, net loading will be $q - k \times w$.

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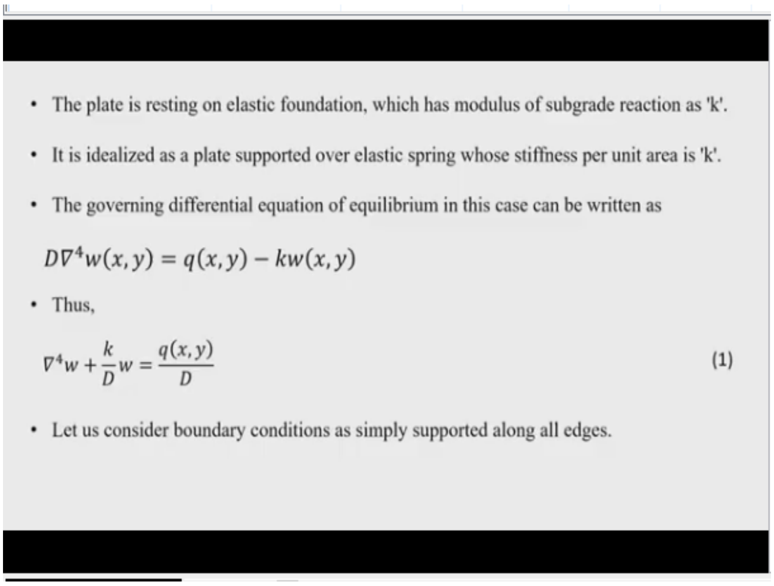

So, you have seen that due to reduction of the net loading $q - k \times w$, we can write the differential equation in modified form; D is the flexural rigidity of the plate which is equal to $Eh^3/12(1-v^2)$ where v is the poisons ratio of the plate and $D\nabla^4 w = q - kw$. So, right hand side of the equation is now modified. If you see the ordinary plate equation where no elastic foundation is there, then k can be put to zero. Then the equation of the plate becomes $D\nabla^4 w = q$, where $\nabla^4 w$ after expansion we can write it as a $\partial^4 w/\partial x^4 + 2\partial^4 w/\partial x^2 \partial y^2 + \partial^4 w/\partial y^4$. So, these by harmonic operator I have explained you earlier and after expanding we can write this differential equation with all the terms. So, thus $\nabla^4 w + \frac{k}{D} w = \frac{q(x,y)}{D}$. Now, here one thing you should note that the thickness of the plate is uniform throughout the area.

And also the soil stiffness that is the subgrade modulus is also constant, but in some problem it may vary also, then the k will be a function of the x and y. So, similarly, the thickness is varying with the space variable x and y then you will get this D as a variable in the differential equation. Now, let us consider the boundary condition as simply supported. Now simply supported boundary condition why I am discussing because we have already formulated this problem and will apply this to rectangular plate resting on elastic foundation.

But it is also true that except the simply supported boundary condition other boundary conditions are also available in the practice. For example, if I consider the foundation which is acted on by column load and by this upward soil pressure, then I can treat this foundation is fixed at the edge of the column and the edges of the foundations may be taken as free. So, in that case boundary condition will be different. Now, let us see the boundary condition in that case is our simply supported boundary condition.

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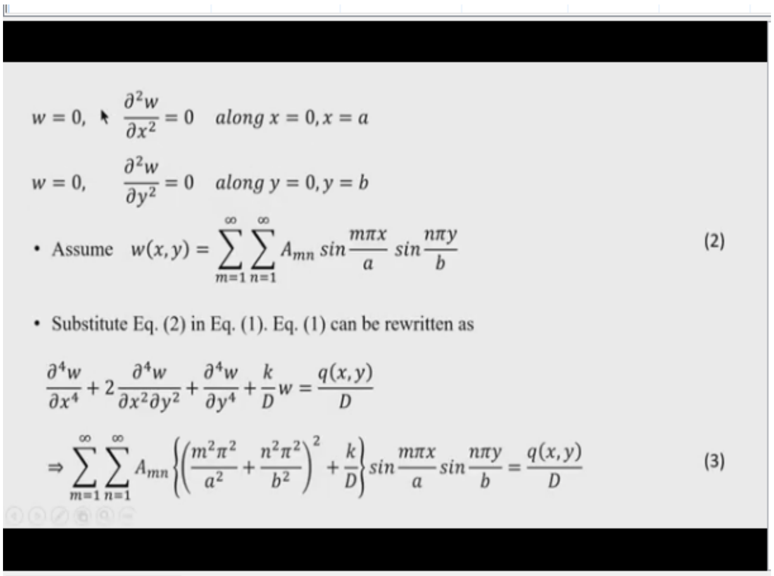

So we will see that the curvature along the x direction is 0, along x = 0 and x = a, deflection is 0 of course, because it is supported. And here again y = 0, y = b, deflection is 0 and curvature in y direction is 0, because it is supported along the y direction. So, the curvature along x reaction, you can see that plate where y = 0 edge is y = 0 edge means it is parallel to x axis.

So, if it is parallel to x axis, then there cannot be any curvature along the x axis. So, the curvature term with respect to x is omitted here. Similarly, here also you can see this condition. Now, according to Navier we assume a double trigonometric series, because these series satisfy the differential equation completely and boundary conditions fully. So, depending on the value of A_{mn} the correct value of *w* or deflection surface can be obtained.

So, $A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$, where m and n are integers and you have noted that in my first lecture while I was discussing the simply supported conditions of a rectangular plate, deflection is composed of so many waves, half waves in x direction and y direction. So, deflection surface is composed of half waves in x as well as y directions. Now, substitute 2 in equation 1.

Equation 1 is our this differential equation. This equation is 1. So, if I substitute these in the differential equation, then equation one can be rewritten as this and after substitution you will get this term that is after 4 derivatives you can see $m^4\pi^4/a^4$. This term will come and for 4 derivatives with respect to y again $n^4\pi^4/b^4$ terms will come.

And the middle term that is the cross derivative twice with respect to x and twice with respect to y, then you will find that the $m^2n^2\pi^4/a^2b^2$ term will come and since it is multiplied by 2 we can arrange this the result of differentiation in this compact form $(m^2\pi^2/a^2 + n^2\pi^2/b^2)^2$ plus the additional term that is due to your elastic foundation reaction that is upward reaction of soil that is entering here.

So, you can see k/D term is present here and that term that when after conducting these even differentiation second order or fourth order you will get again this sine term is coming and the sine will be positive because the another negative terms will appear for differentiation with respect to your y then the negative multiplied by another negative quantity will be positive.

So, this quantity inside the second bracket is positive quantity and you are getting the deflection surface. Now, you can note here that summation is still present. So, we have to get rid of the summation term and want to find the value of the constant A_{mn}.

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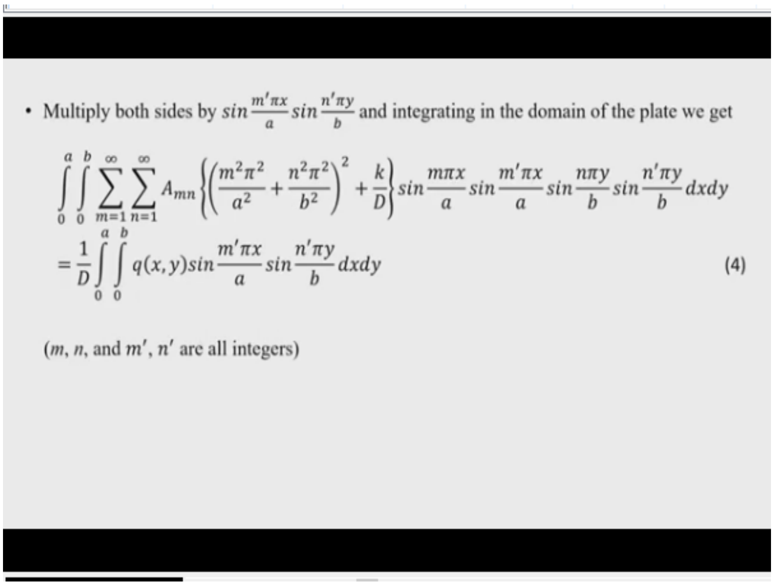

So, multiply both sides by $\sin\frac{m'\pi x}{a}\sin\frac{n'\pi y}{b}$ and integrate in the domain of the plate what do you understand by domain of the plate? Because it is a rectangular plate, it extends from 0 to a in the length direction and 0 to b in the width direction. So, area of the plate is $a \times b$, $a\,b$ is the area of the plate. So, in the domain of the plate if I integrate the equation after multiplying the full equation with another $\sin\frac{m'\pi x}{a}\sin\frac{n'\pi y}{b}$.

This term is originally multiplied here with this original term. So, that we can make use of the orthogonality condition, that is the intention for multiplying this and integrating over the domain of the plate. Because of this operation, we can use the orthogonality condition by which we can get rid of all the terms which are not involving $m = m'$ or $n = n'$. So, all the terms which contains says $\sin\frac{m\pi x}{a}\sin\frac{m\pi x}{a}$ will be only written here other terms, if $m \neq m'$ or $n \neq n'$ then it will be removed, it will be vanished, because of orthogonality conditions that will be 0. So, right hand side remain as it is and q(x,y) is our loading function, these loading function may differ for a uniformly distributed load, it will be one form, for a uniformly varying load it will be in another form and for concentrated load it will be in other form that I have shown you that it can be represented by Dirac delta function.

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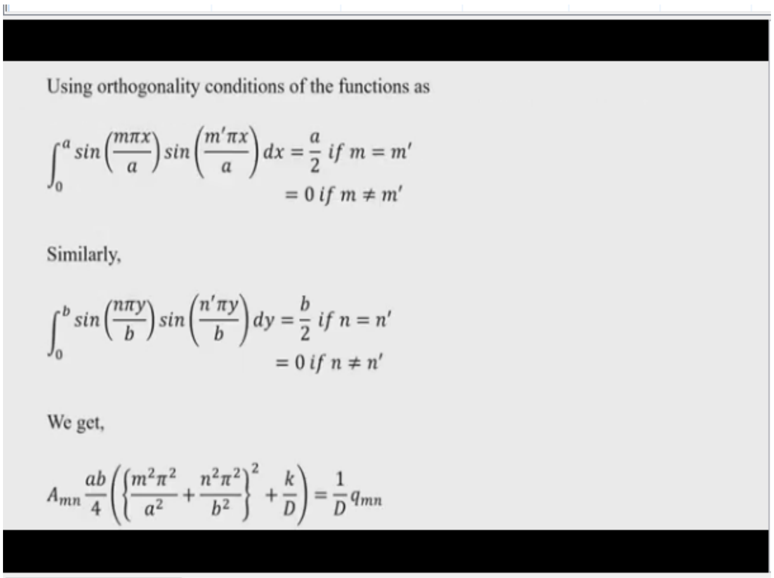

So, now, after doing this operation and using the orthogonality condition, ultimately we get rid of the summation term and we get A_{mn} that is the most important coefficient of our deflection series in general term A_{mn} and A_{mn} may vary up to infinity, it is integer A_{mn} are integer different combination of A_{mn} are possible, say m = 1, n = 1, m = 1, n = 2, m = 1, n = 3, like that different combinations may be possible.

So, there will be infinite terms in the series and we have seen that for deflection only first term is sufficient for this reasonable result, but for bending moment and shear force you have to consider more number of terms. However, we get the accuracy with that series. The series is rapidly converging series. So, after integration we get this *ab*/4, why *ab*/4 is there? Because this integration gives you *a*/2 when m = m' . And this integration with respect to y give you *b*/2. So, naturally it will be *ab*/4 and other times remain as it is and the right hand side I have written as 1/D q_{mn}, where q_{mn} is the all loading function q_{mn} is this function.

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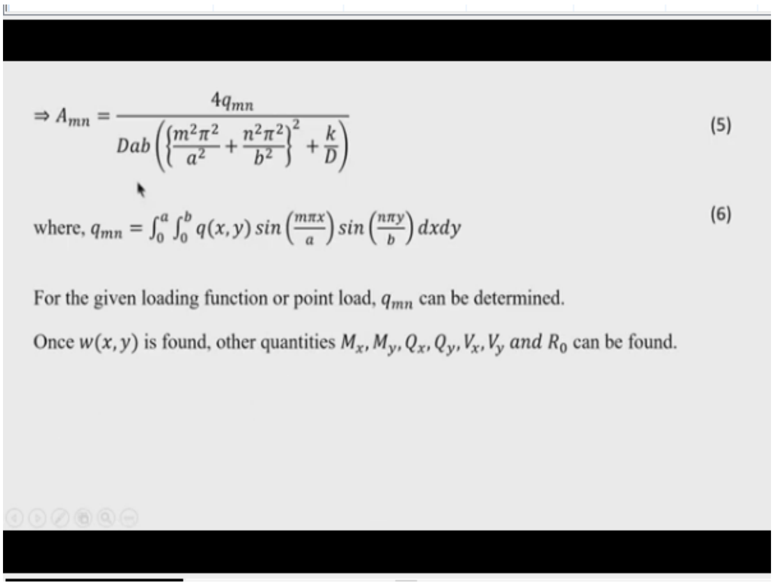

So, now one can get the value of A_{mn}, so, A_{mn} is $4q_{mn}/Dab$, where D is the flexural rigidity *a* and *b* are the dimension of the plate and you are getting here these terms which contain the harmonic numbers *m* and *n* and also the dimension of the plate *a* and *b* and it is squared quantity + *k*/D, where *k* is also positive quantity, D is also positive quantity. So, this term is always positive inside the bracket.

Now, you can note that if *k* is 0, which actually indicates that there is no elastic foundation, so, just like it is a supported plate that we have studied earlier. So, equation for A_{mn} coincides with the earlier equation. So, this formulation is correct because we have verified it after putting *k* is equal to 0, it goes back to our earlier equation that signifies that plate is not supported.

So, q_{mn} is that integration of $q(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dx\,dy$, where the integration is carried out with respect to x in the limit 0 to *a* and with respect to y in the limit 0 to *b*. For the given loading function or point load q_{mn} can be determined. Now, once we determine *w* that *w* is here; once we determine *w* the complete series after finding the A_{mn} then our other plate quantity can be found.

So, what are the other plate quantities, bending moment in the x direction, bending moment in the y direction, shear force along the x axis, shear force along the y axis and edge shear along the x axis, edge shear along the y axis and corner reaction are not, because there are 4 corners. So, at 4 corners we will get lifting up force. So, with that formulation, that specialty is here then modulus of subgrade reaction is entering into the expression.

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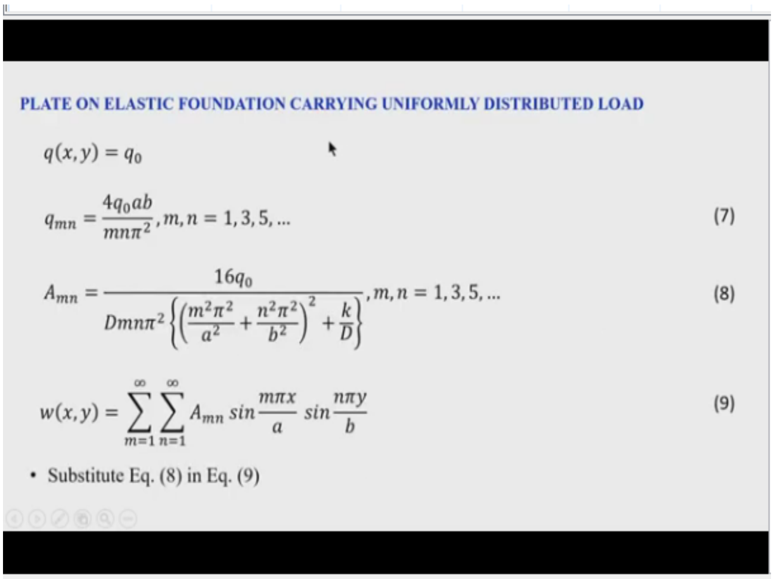

Now, let us discuss a case. Where the plate is carrying the UDL, plate is carrying the uniformly distributed load. So, in that case the q(x,y) is q_0 is a constant quantity and you will get that $q_{mn} = 4q_0ab/mn\pi^2$, where mn is varying from 1, 3, 5 etcetera, only odd integers have to be considered for this q_{mn}. So, q_{mn} is nothing but actually it can be looked at as a Fourier series expansion of the load.

And the process is that when the loading function is expressed as a Fourier series, double Fourier series or here in that case double Fourier series, then we can compare the like terms and find out the coefficient of the unknown coefficient of the deflected surface, then full deflection of the plate can be known. So, A_{mn} after substituting q_{mn} in the expression for A_{mn} that is the expression for A_{mn} we get the A_{mn} as this.

So, once the A_{mn} is found then $w(x,y)$ is $A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$. So, equation 8 into equation 9 we can get full deflected series.

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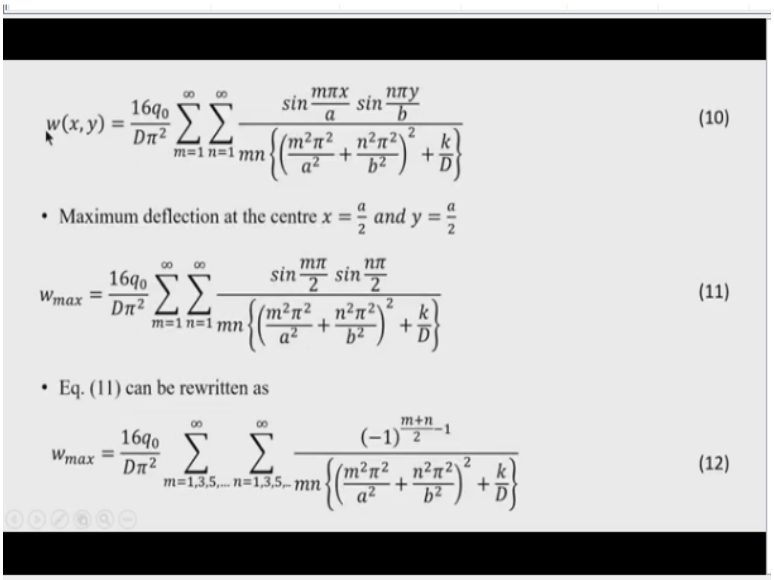

Maximum deflection will be at the center of obviously because it is symmetrically loaded and the support is also symmetrical. So, $x = a/2$, $y = b/2$ it is the center and we will get the w_{max} is equal to this. So, w_{max} you can see here $16q_0/D\pi^2$ that is a constant term and under summation that A_{mn} terms are there and k/D, that is the extra term that is appearing here because of elastic foundation reaction.

So, you can see this function can be written as in a convergent form as $(-1)^{\frac{m+n}{2}-1}$. So, m varies from 1 to 3, 5, n is also 1, to 3, 5. So, one can find out the convergent of the series with the first few times the exact result will be obtained, there will be no difficulty.

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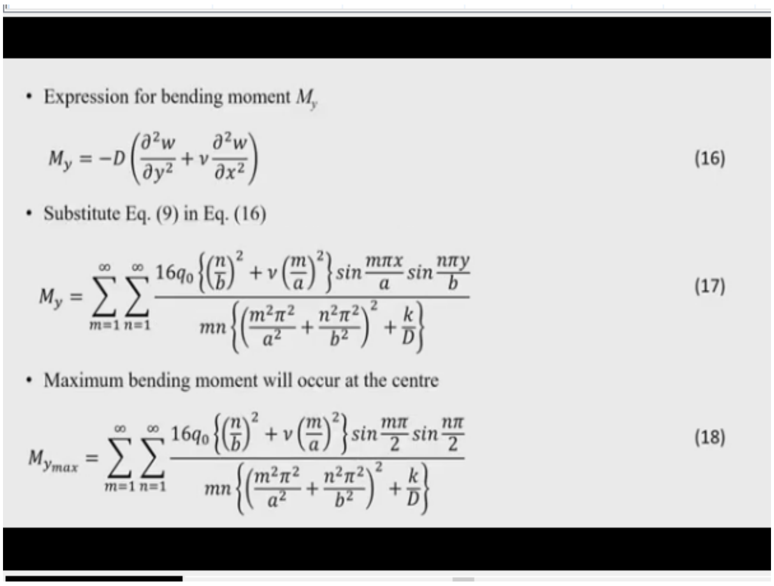

And let us see here the bending moment in y direction. So, bending moment in y direction is –D, curvature in a y direction $\partial^2 w/\partial y^2 + \nu\, \partial^2 w/\partial x^2$. So, it means that Poison ratio is important for finding the bending moment. So, knowing the deflected curve or deflected surface w, here we call it surface because it is a two dimensional problem.

So, knowing the deflected surface *w*, we can find out the second derivative with respect to x and with respect to y and then after substituting this in the expression that here expression 16, we can find out the expression for M_y. So, expression for M_y is given like that, where n and m are integers and here also again this value is important for m is equal to these odd values of integers are also taken here for calculating the bending moment.

Maximum bending moment will be occurring at the center and therefore, we get the maximum bending moment in 2 directions by putting $x = a/2$ and $y = b/2$.

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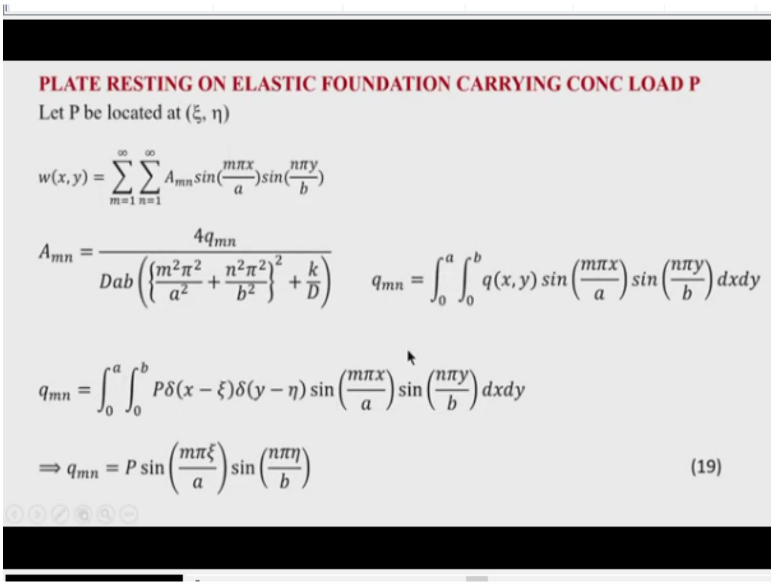

So, let us see another case where the rectangular plate is resting on elastic foundation, but instead of uniformly distributed load it carries a concentrated load. So, concentrated load is a discrete quantity and therefore loading function becomes discontinuous. So, because of discontinuity the loading function is not differentiable. So, that is the problem. So, this type of loading case has been handled in my class by limiting process.

And also I have illustrated a special mathematical function Dirac delta which can be conveniently used to find out to analyze the plate for discrete loading that is consolidate load using the properties of Dirac delta function, because of special property of Dirac delta function which after integration when this is associated with a function and it is integrated in the limit between any limit in the real line - ∞ to + ∞ you will get the value of the function at the point where Dirac delta function is applied or concentrated load is applied.

So, by this rule, we can find that q_{mn} is $q(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dx\,dy$, but mind one thing that q(x,y) is not continuous function, because of discontinuity due to concentrated load, we have to express the q(x,y) in this form. So, q(x,y) is expressed as P into the location of the concentrated loading given by the coordinate ξ and η. So, the q(x,y) is represented by P into Dirac delta function, index is the $x-\xi$ and again Dirac delta function, the index is $y-\eta$.

So, ξ and η represents the location of the concentrated load and other variables $\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$ are also written here. So, by the property of the Dirac delta function, we can immediately evaluate the integral without going too lengthy process of limit. So, in that case, we have to put $x = \xi$ and $y = \eta$. Then our integration is found out. So, integration is $q_{mn} = P\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}$.

So, this is the value of q_{mn}. The main thing is here that for any problem of simply supported plate using the Navier's method we use the deflection series as same as this in every problem, only thing is that due to loading condition, which are different from problem to problem, we get different values of q_{mn}. That is equivalent to expressing the external loading in terms of Fourier series for different cases. So, we get here q_{mn}, this $P\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}$.

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$$\Rightarrow A_{mn} = \frac{4\,P\sin\left(\frac{m\pi\xi}{a}\right)\sin\left(\frac{n\pi\eta}{b}\right)}{Dab\left(\left\{\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right\}^2+\frac{k}{D}\right)}$$

$$w(x,y) = \frac{4P}{Dab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{m\pi\xi}{a}\right)\sin\left(\frac{n\pi\eta}{b}\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)}{\left(\left\{\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right\}^2+\frac{k}{D}\right)} \qquad (20)$$

So, once the q_{mn} is obtained, then you can get the A_{mn} as A_{mn} contains q_{mn}, it also contains the value of flexural rigidity dimension of the plate and for this case the modulus of subgrade reaction is also appearing in the expression and these are the wave numbers *m* and *n* that is appearing due to the assumption of double trigonometric series for the deflection function.

So, A_{mn} is known. After knowing A_{mn}, we can find the deflection of the plate, a complete expression of the deflection of the plate is found out. So, 2 sine functions are appearing, but these are determined because of location of the concentrated load. Now, in a problem where there is more than one concentrated load, it is possible because when the bridge deck is subjected to vehicle loading, vehicle has several actions (28:00).

So, each action will impose, each action has 2 wheels right and left and therefore, it will impose the 2 concentrated load while it is traveling for production generally. So, you will get these multiple action load that is multiple concentrated loads. So, in that case, the deflection at any point can be found out by the principle of super imposition, because we are only dealing with the linear analysis.

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Expression for bending moment

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)$$

$$\Rightarrow M_x = \frac{4P}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{m\pi\xi}{a}\right)\sin\left(\frac{n\pi\eta}{b}\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\left(\frac{m^2\pi^2}{a^2} + \nu\frac{n^2\pi^2}{b^2}\right)}{\left(\left\{\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right\}^2 + \frac{k}{D}\right)} \quad (21)$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right)$$

$$M_y = \frac{4P}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{m\pi\xi}{a}\right)\sin\left(\frac{n\pi\eta}{b}\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\left(\frac{n^2\pi^2}{b^2} + \nu\frac{m^2\pi^2}{a^2}\right)}{\left(\left\{\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right\}^2 + \frac{k}{D}\right)} \quad (22)$$

So, once the deflection series is known, then it is possible for us to calculate the bending moment expression. So, bending moment expression M_x, the bending moment in the x direction equal to -D $\partial^2 w/\partial x^2$, that is the curvature along the x direction + $\nu\,\partial^2 w/\partial y^2$. So, that condition we can apply and the derivative of this function can be now evaluated because this is a constant, this is also constant and these are constant.

So, we have to deal with these functions $\sin\frac{m\pi x}{a}$ and $\sin\frac{n\pi y}{b}$ for differentiation purpose. So, once you differentiate and put the value then you will get the quantity like that. So, this is the complete expression for bending moment. Similarly, bending moment in x direction, similarly bending moment for y direction can also be obtained as $-D\left(\partial^2 w/\partial y^2 + \nu\,\partial^2 w/\partial x^2\right)$.

So, partial derivative of deflection function with respect to y 2 times and partial derivative of the deflection function with respect to x 2 times have to be taken and added with in one case multiplied by poison ratio. So, that you should remember. So, if it is a steel plate, the poison ratio is generally varying from 0.25 to 0.3 or around this and if it is a concrete slab, then we can take position ratio 0.15 to 0.2.

So, M_y can be determined, M_x is determined and you can see the difference here, there is the curvature interchange here only and others are same.

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Maximum values occur at the centre, $x = \frac{a}{2}$ and $y = \frac{b}{2}$

$$w_{max} = \frac{4P}{Dab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{m\pi\xi}{a}\right)\sin\left(\frac{n\pi\eta}{b}\right)\sin\left(\frac{m\pi}{2}\right)\sin\left(\frac{n\pi}{2}\right)}{\left(\left\{\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right\}^2+\frac{k}{D}\right)} \quad (23)$$

$$(M_x)_{max} = \frac{4P}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{m\pi\xi}{a}\right)\sin\left(\frac{n\pi\eta}{b}\right)\sin\left(\frac{m\pi}{2}\right)\sin\left(\frac{n\pi}{2}\right)\left(\frac{m^2\pi^2}{a^2}+\nu\frac{n^2\pi^2}{b^2}\right)}{\left(\left\{\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right\}^2+\frac{k}{D}\right)} \quad (24)$$

$$(M_y)_{max} = \frac{4P}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{m\pi\xi}{a}\right)\sin\left(\frac{n\pi\eta}{b}\right)\sin\left(\frac{m\pi}{2}\right)\sin\left(\frac{n\pi}{2}\right)\left(\frac{n^2\pi^2}{b^2}+\nu\frac{m^2\pi^2}{a^2}\right)}{\left(\left\{\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right\}^2+\frac{k}{D}\right)} \quad (25)$$

And you can see the maximum bending moment occurs at the center that is obvious. So, $x = a/2$ and $y = b/2$ is the point where the maximum bending moment occurs and substituting back this $x = a/2$ $y = b/2$, here also $x = a/2$ $y = b/2$, we ultimately get the expression for maximum deflection, expression for maximum bending moment in x direction, expression for maximum bending moment in y direction.

So, these are the expressions that are very important especially for design of deck slab in the bridge because we are dealing with several concentrated load, UDL is there you can take it as a superimposed dead load or self weight, but the concentrated load that is the vehicle imposed load can be treated by this formulation if the deck slab boundary condition is taken as simply supported.

In bridge design handbook that is generally charts are given for bending moment coefficient in x direction and y direction. And in this chart you will get that bending moment expression is similar to the expression that we have discussed here that is the curvature in 2 directions are taken into account. But, these expressions in the handbook were obtained using the simply supported boundary condition.

So, generally the charts are available in bridge design book and known as Pigeaud's curve (32:31) but it is also written by the authors of bridge design that for continuity of the deck slab one can assume a reduction factor of bending moment as 0.8. That means, simply is about a case the bending moment is more. So, a reaction that is the bending moment can be reduced up to 20% due to continuity of the edges.

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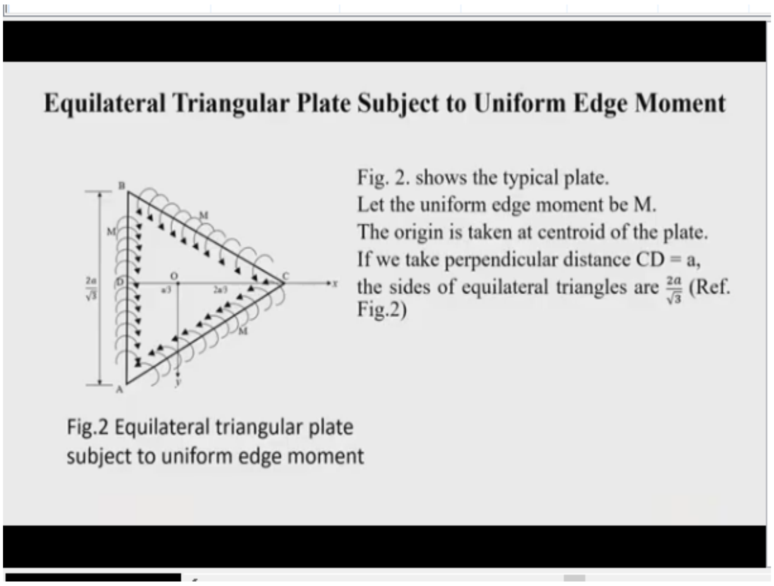

Now, I shall discuss another problem that is sometimes it is encountered and you can see how we have developed this solution, solution is developed by taking a deflection function or assume a deflection function reasonably that can satisfy the boundary conditions. Now, here the problem is a triangular plate, but for simplicity I have taken an equilateral triangle. So, each angle of the triangle is 60°.

So, and we consider the uniform moment distributed along the edges of the triangle are applied, symmetrical moments are applied, the origin is taken at this centroid of the plate. So, that means from the base it is $a/3$ and $2a/3$ if their altitude that is this length say CD is a, if CD is a then, the centroid is situated at $a/3$ from the here from the side AB and from the apex C it is at a distance of $2a/3$.

So, after getting the CG now, let us formulate the problem. So, we can see that at the boundaries since it is simply supported again we are taking this plate as simply supported because we are dealing with only simply supported condition. So, in the simply supported case, along the boundary deflection is 0 that is obvious. And another condition is that bending moment is 0, so bending moment 0 means here say this edge is inclined. It is neither parallel to X axis not parallel to Y axis. So, we have to take the bending moment that is normal to the plate is 0, but here it will be not 0 because the moment M is applied. So, one boundary condition is that along the all edges that deflection is 0 and second boundary condition is that bending moment normal to the edge is equal to M not 0 because the moment is applied only. Now, let us see how we can formulate the problem.

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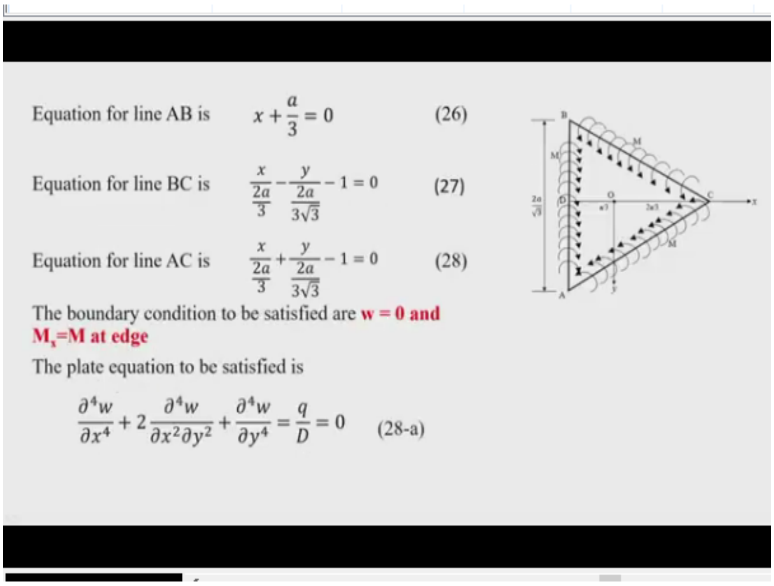

Now, you can see this side say AB, what will be the equation of AB, line AB, equation of line AB is nothing but x = $a/3$. It is very easily written the equation the line $x + a/3$, equation of the line BC is written in the intercept form and we can write that $\frac{x}{\frac{2a}{3}}-\frac{y}{\frac{2a}{3\sqrt{3}}}$. This factor $\sqrt{3}$, $1/\sqrt{3}$ is coming because this angle is 30. This angle is 30° and this is tan(30) when I find this intercept BD for that line with the y axis then BD will be this length into tan(30).

Therefore, this condition is appearing. Now, one important thing you are seeing here that, since the y axis taken downward, so, intercept is this. Intercept is here this length so, obviously it is $2a/3\times\tan(30)$. Since it is taken downward positive therefore, minus sign is coming here and in the intercept form say the standard intercept form of the straight line is $x/a + y/b = 1$.

So, here is also the intercept along the x axis is $2a/3$, that is obvious OC is $2a/3$ intercept along y axis is this side that is $2a/3\times\tan(30)$. So, therefore, this is coming but it is negative because it is in the negative direction, equal to 1. So, 1 I have transferred to the left hand side equal to 0.

Equation of the line AC is similarly written. Now, here the intercept is found as positive. So, positive intercept is found. Now, you can see that the plate equation can be formed or can be composed by taking the equation of the line because equation of the sides of the triangle because along this side that is along the edges of the triangle deflection is 0 and when you take the equation of the line or equation of this side of the triangle that is the side of the plate as a term of deflected surface then we can get that along the edges deflection is satisfied.

So, that condition is satisfied by taking w is 0, because the along the boundary w is 0 and M_x = M at edges. Now, M_x = M at edge only it is possible for edge AB, but here this edge is inclined. So, we have to find the normal the moment along the normal so, M_n. Now, let us see the plate equation. Plate equation is $\partial^4 w/\partial x^4 + 2\partial^4 w/\partial x^2 \partial y^2 + \partial^4 w/\partial y^4 = q/D$. Now, since no load is applied here, so, right hand side is 0.

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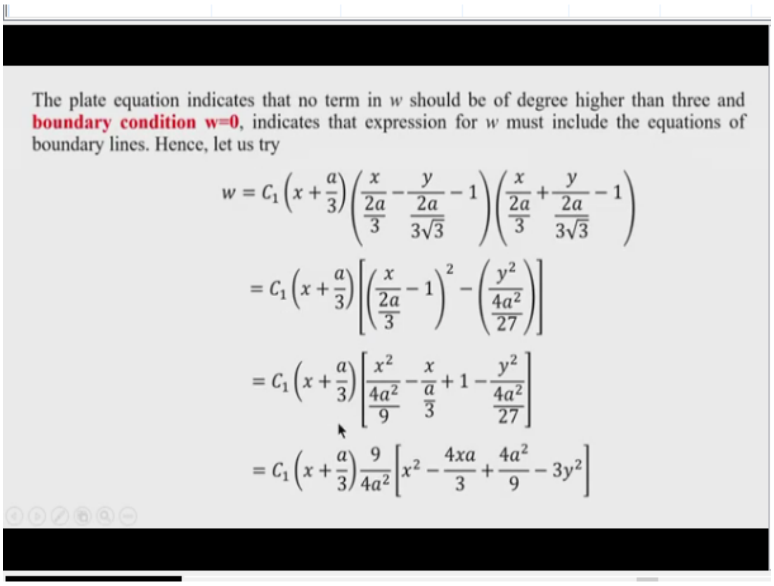

So, let us see in the next slide, the plate equation indicates that no term in w should be degree higher than the 3 because it is a fourth order equation and boundary condition w is 0 indicates the expression for w must contain the equations of the boundary lines. So, hence let us try the equation of the line as C_1. This is the equation of the line AB. First we have written the equation of the line AB, then equation of the line BC and the equation of the line AC.

So, if you see that equation of the 3 sides of the triangle are written here, and it is multiplied with the unknown constant C, because of this type of expression, you can see that w equal to 0 is immediately satisfied. There is no doubt about that because at the boundary this expression has to be 0, this expression has to be 0, this expression has to be 0. Now, this term can be arranged because $x + a/3$, I am keeping here.

And this can be treated as your say this term can be taken here say this -1 $-y/2a/3\sqrt{3}$ and this term can be taken as it is plus. So, that means $a + b$ into $a - b$ this formula if you apply you will get $a^2 - b^2$. So, I have written like that and after simplification, one can write this expression inside the third bracket as this. Now further simplification of this expression will give you w is $C_1(x + a/3)(9/4a^2)[x^2 - (4xa/3) + (4a^2/9) - 3y^2]$.

So, this is the expression for deflected surface provided C_1 is known, because this w is found to satisfy the boundary condition, but still the value of C_1 is not determined because the second boundary condition on the bending moment is not the imposed till now. So, let us see what is the next step.

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$$= C_1 \frac{9}{4a^2}\left[x^3 - \frac{4ax^2}{3} + \frac{4a^2x}{9} - 3xy^2 + \frac{ax^2}{3} - \frac{4xa^2}{9} + \frac{4}{27}a^3 - ay^2\right]$$

$$= \frac{9C_1}{4a^2}\left[x^3 - 3xy^2 - ax^2 - ay^2 + \frac{4}{27}a^3\right]$$

$$= \frac{9C_1}{4a^2}\left[x^3 - 3xy^2 - a(x^2 + y^2) + \frac{4}{27}a^3\right]$$

$$\therefore \quad \frac{\partial^4 w}{\partial x^4} = 0 \qquad \frac{\partial^4 w}{\partial x^2 \partial y^2} = 0 \quad \text{and} \quad \frac{\partial^4 w}{\partial y^4} = 0$$

Hence the plate equation is satisfied.

So, after simplification we have got this and we can see that if you take this expression of the deflected series and take the derivative for derivative of w with respect to x, then this derivative and again this you will find these are all 0s. So, differential equation is also satisfied, because we have found $\nabla^4 w = 0$. In that case, because no distributed loading is there, no applied loading is there except the edge moment.

So, q is 0. So, therefore, differential equation is the homogeneous equation and by taking the expression of deflected series with this form, it is easily verified that the differential equation is satisfied. So, plate equation is satisfied, this is our inference.

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$w = 0$ along the edges is satisfied, since, at any point on edge either $\quad x + \frac{a}{3} = 0$

$$\text{or} \quad \frac{x}{\frac{2a}{3}} - \frac{y}{\frac{2a}{3\sqrt{3}}} - 1 = 0 \quad \text{or} \quad \frac{x}{\frac{2a}{3}} + \frac{y}{\frac{2a}{3\sqrt{3}}} - 1 = 0$$

Another boundary condition to be satisfied is

$$M_n = M$$

$$i.e \qquad -D\left[\frac{\partial^2 w}{\partial n^2} + v\frac{\partial^2 w}{\partial t^2}\right] = M$$

Since, $\quad \frac{\partial^2 w}{\partial t^2} = 0, we get \quad -D\frac{\partial^2 w}{\partial n^2} = M$

Next, w is 0 along the edges that is we have verified. So, these are the equation of the line. So, naturally it will be 0 at the boundary for the respective edges. So, another boundary condition to be satisfied is moment M_n along the normal to this side is M. Now, M_n is given as $\partial^2 w/\partial n^2 + \nu\, \partial^2 w/\partial t^2 = -M/D$. Now, here you can see their derivative is taken with respect to Cartesian coordinates x and y.

But it is taken a normal and tangential coordinates n and t and that transformation from Cartesian derivative to normal derivative or tangential derivative we have covered earlier and given the expression that we will utilize here. Now, since it is supported along the edges. So, naturally this derivative with 0, curvature is 0 and hence we get the condition of bending moment as $-D\,\partial^2 w/\partial n^2 = M$.

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$$\text{or} \quad -\frac{M}{D} = \left(\frac{\partial}{\partial x}\cos\alpha + \frac{\partial}{\partial y}\sin\alpha\right)\left(\frac{\partial w}{\partial x}\cos\alpha + \frac{\partial w}{\partial y}\sin\alpha\right)$$

$$= \frac{9C_1}{4a^2}[(6x - 2a)\cos^2\alpha + 2(-3 \times 2y)\sin\alpha\cos\alpha + (-3x \times 2 - 2a)\sin^2\alpha]$$

$$= \frac{9C_1}{4a^2}[6x(\cos^2\alpha - \sin^2\alpha) - 2a(\cos^2\alpha + \sin^2\alpha) - 12y\sin\alpha\cos\alpha]$$

$$= \frac{9C_1}{4a^2}[6x\cos 2\alpha - 2a - 6y\sin 2\alpha] \qquad \textbf{(29)}$$

Now, the bending moment that is the curvature with respect to the direction n that is not with respect to direction x or y. So, here the operator that is the differential operator with respect to n where α is the angle that makes with the x axis that the normal makes with the x axis. So, that operator is written and this is nothing but the differentiation of w that is the deflected surface with respect to your the normal direction.

So, this is the derivative and this is the operator. So, we get the second derivative. So, term by term multiplication and after putting the value of all the quantities w that we have obtained earlier we get ultimately this expression. This expression can be further simplified and written here in this form. So, this is a general expression for -M/D. Now, we can see what happens along the 3 sides.

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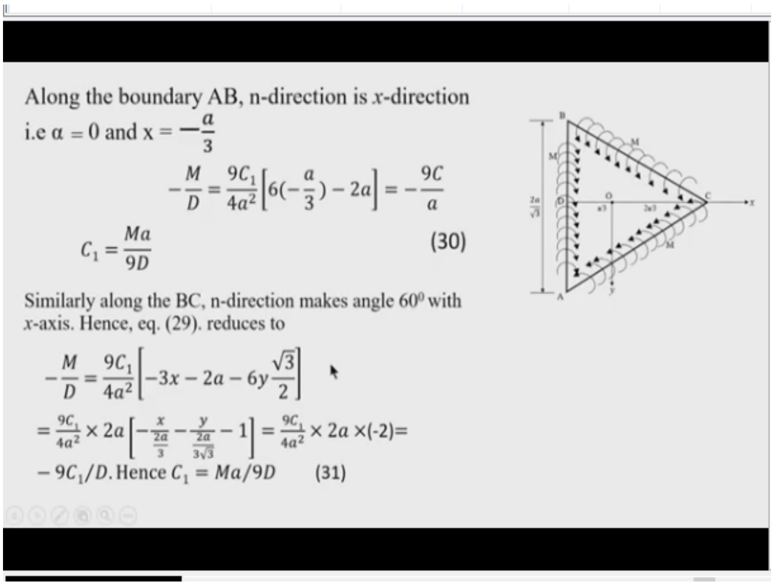

Now, along the side AB if you see this side AB that is n direction is nothing but edge direction. So, α is 0 because α is the angle which the normal makes with the x direction. So, naturally here α a is 0. So, we get this α is 0, and x coordinate is – a/3, x coordinate along this line is always – a/3. So, therefore, the previous equation, we can now put α = 0 and x = – a/3.

Then we can get this form of the equation here and then we can get the unknown constant C_1, so unknown constant C_1 can be determined in terms of applied moment M, the side of the plate and flexural rigidity. So, now the constant is known, but we have applied the boundary condition, along only one edge, AB. Let us now apply the boundary condition, along the other edges. So, other edges are your BC edge and AC edge.

And that edge makes an angle 60°. That is the equilateral triangle so this makes an angle of 30° with the x axis. So, now taking the derivative of that function that is with respect to n and putting this α and x properly for that line, say for BC, we can get this expression. And here this expression can be rearranged, taking these 2a as common and it can be rearranged like that, from where we can get this, because these quantities again -1.

This quantity is -1 from the equation of the line, so it will be -1, -1, -2, so it is coming as $(9C_1/4a^2) \times 2a \times -2$. So, it is equal to $-9C_1/D$. So, hence $C_1 = Ma/9D$. So, same value is obtained, again, applying the boundary condition, moment condition along the edge, BC.

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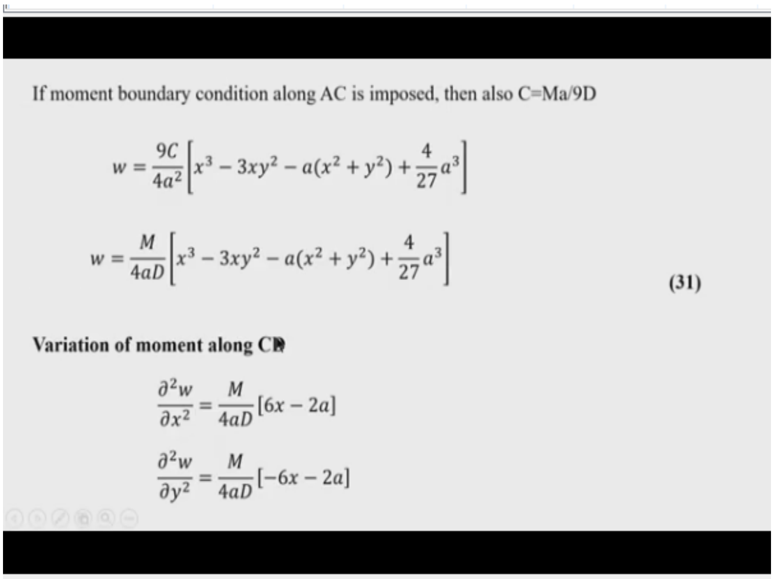

If moment boundary along AC is imposed, then also we get C = Ma/9D. So, after substituting now C_1 or the constant associated with the deflected series, we now get this $w = (9C/4a^2)\left[x^3 - 3xy^2 - a(x^3 + y^3) + (4/27)a^3\right]$. This is the deflection series and putting the value of C, as C is found as Ma/9D. So, after putting the value of C, we get the w is this in terms of all known parameters, M is also known, edge condition is known, edge bending moment is known.

Size of the plate is known, flexural rigidity which contains the thickness of the plate and material properties are also known. Now let us see what will be the variation of bending moment along CD. So, we will see the variation of bending moment along this line. Because when you move along the edges are known. So, let us see if we calculate the bending moment along CD that is along the x direction that is the derivative along the y direction that is the curvature.

So, taking the second derivative of this expression with respect to x and second derivative of the expression with respect to y, we get these 2 equation $\partial^2 w / \partial x^2 = (M/4aD)[6x - 2a]$ and another case $\partial^2 w / \partial y^2 = (M/4aD)[-6x - 2a]$.

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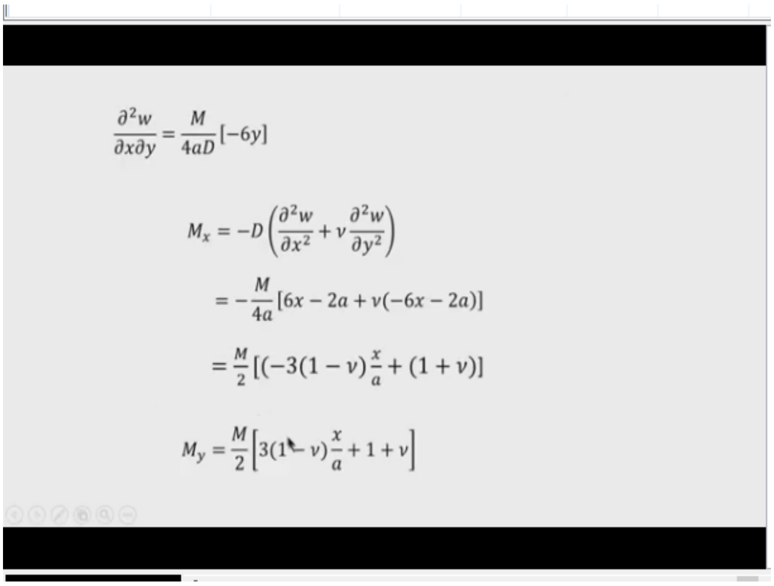

So, bending moment can be written conveniently like that, of course the twisting moment is also find out. So, derivative is calculated by taking the differential coefficient of *w* with respect to *x* and with respect to *y*. Now, after substituting these values, you can see, the bending moment along the x direction is this and bending moment along y direction is this. You can see that both are varying with x.

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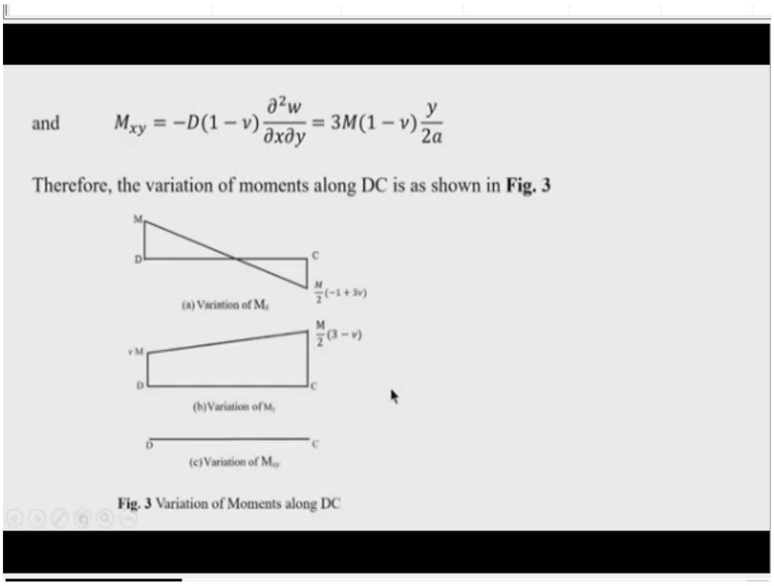

And this twisting moment along this direction is basically a function of y. So, if I plot this variation of M_x along the CD, we get this shape and variation of M_y along the CD, we get this shape. The value of the extreme edge or this quantity and this extreme has the 2 values are equal, so (M/2) × (-1 + 3 ν). And here you will get the (M/2) × (3 - ν) and here you will get νM.

But since the torsional moment or twisting moment does not contain any x. So, when we plotted the variation of M_{xy} along the x axis, it represents a straight line. So, that is the problem that I have discussed the special cases where the triangular plate is subjected to edge moment. The main philosophy of solution of this type of problem is to take the deflected surface as a function of x and y in the form of polynomial, such that it contains the equation of the line.

So, equation of the line when it is associated in the deflected surface obviously satisfies the deflected boundary condition at the edges and other condition, when it is imposed this gives the unknown coefficient associated with the deflected function, then we get the complete expression for deflection and then we go for what to calculate the other quantities.

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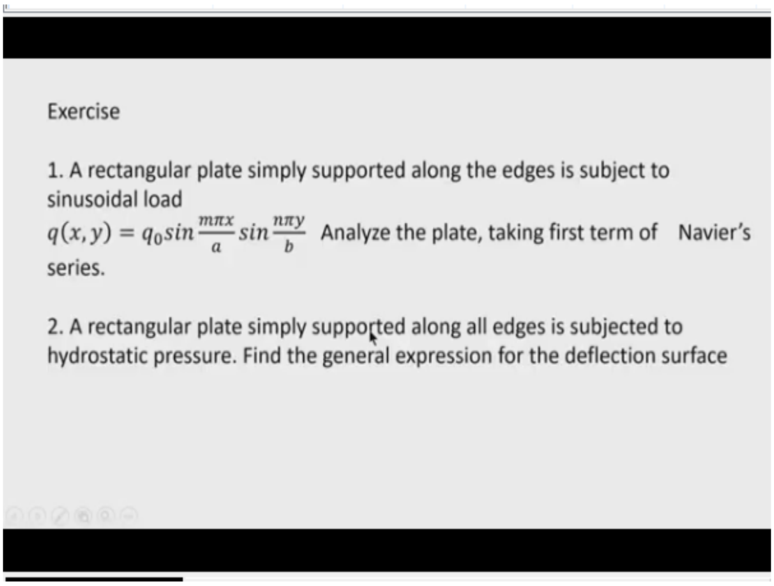

Now lastly let us discuss some true problems. One is rectangular plates simply supported along the edges is subjected to sinusoidally varying load. Let us say this is a very interesting case and easy also. Analyze the plate, taking first term of Navier's series. Now for simplicity let us illustrate it with the only one term of the Navier's series that is m = 1 and n = 1. So, first problem will be solved, like that.

And second problem, we have taken a rectangular plate, simply supported along the edges, and is subjected to hydrostatic pressure; find the general expression for the deflected surface. So, these 2 problems can be solved by Navier's method. That is why I have selected.

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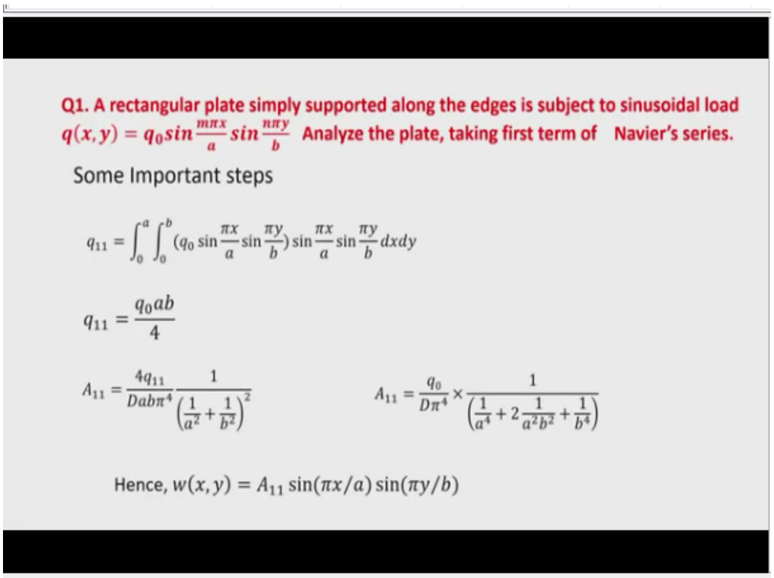

So, first problems some important steps I am giving. The procedures are saying whatever load is there, but only thing is that q_{mn}. That is the loading function have to be now changed because of the different type of loading. Since loading is this, so q_{11} is nothing but this is the loading and multiplied by $\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$. Here m = 1 and n = 1 is taken, so therefore no m and n is appearing in this expression.

Now, that integration can be done very conveniently because $\sin^2\frac{\pi x}{a}dx$ will be $a/2$ and $\sin^2\frac{\pi y}{b}dy$ will be $b/2$. So, net result will be $q_0ab/4$. Once you find the q_{11}, then you can find A_{11} as $4q_{11}/Dab\pi^4$. And here actually m^2/a^2 was there, but m is 1, so $1/a^2 + n^2/b^2$ where there, but since n is 1, so $1/b^2$ and whole square.

So, this is the expression for A_{11}, after knowing A_{11} we can find the complete deflection series. So, hence the deflected surface is given by this function. So, this solves the first question.

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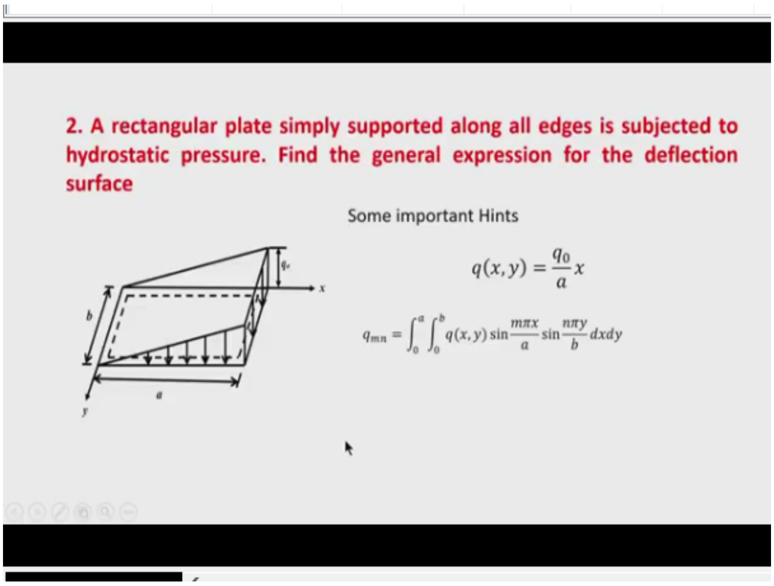

Let us come to the second question. Second question is a rectangular plate again. The length is a, and side is b and it is simply supported along the edges, but it is subjected to uniformly varying load, that is this type of load is hydrostatic pressure and it is very common, especially when the plate is used to resist the earth pressure or resist the say water pressure in a sewage gate used in flood control works.

Then the pressure will vary uniformly linearly from base to the top and the maximum and the pressure will be linearly varying with the depth or distance. So, therefore the maximum pressure that you will get at the one extreme edge and here it is 0. So, this pressure is q_0 and some important hints I am giving you the loading function is expressed as $(q_0/a) \times x$.

Because the variation is only along the x axis at any section along the y axis, you will find the same intensity of the load. So, y is not coming in this expression. So, therefore, q_{mn} is written as an integral 0 to a. And also, 0 to b $q(x, y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dx\,dy$.

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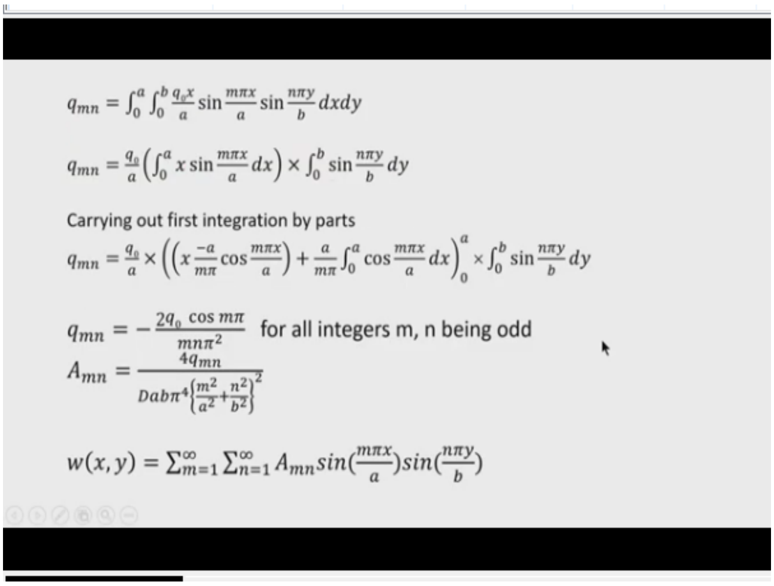

So, therefore, this integral is carried out. And here you will get this constant term is taken out, and this integral is done with the rule of integration by parts and this is the integration of sine integration, $\sin\frac{n\pi y}{b}dy$. So, 2 integrations when it is carried out that integration by parts the first integral is this and after putting limit, you will get q_{mn} is $-\frac{2q_0\cos m\pi}{mn\pi^2}$.

For all integers m and n being odd. So, once you got get q_{mn}, then you can find A_{mn} again for the value of odd integers. So, therefore, after getting your value of A_{mn}, you can get the deflected series $w(x,y)$. Now here because of this uniformly varying load, you will not find the deflection exactly at the center, maximum deflection will occur along the line $b/2$. This at the center of the plate.

But this, the x coordinate will be different. So, x coordinate has to be found by differentiating or making the slope to be 0, because at the point of maximum deflection we will get slope 0. So, finding this location x, where the slope vanishes along the x direction, you will find and put in the deflection surface, again, the value of x, y is equal to $b/2$and value of x has to be found, after making the slope to be 0 you will get the magnitude of the maximum deflection.

Similarly, you can find the magnitude of the maximum bending moment and in the x and y direction. So, what we have done, today's class is we have illustrated the problem of plate on elastic foundation, then I have discussed a special problem, where a triangular shaped plate is taken and the plate is equilateral and subjected to moment along the edges, this moment of course the magnitude is taken equal and it is symmetrical.

That it is trying to set the plate. So, under the action of this edge moment, we have found the deflected surface of the triangular plate and from that we have found the expression for bending moment and twisting moment. Then we have taken 2 problems that can be solved, applying Navier's method that we have learned so far in our module 2 and applying Navier's method that I have shown that procedure is saying only up to get the value of q_{mn} for each case of different types of loading.

The uniformly distributed load is more common, but any other load appears on the plate or acting on the plate. Then you have to find q_{mn} based on the loading function. So, that I discussed and 2 problems are particularly important. That is sinusoidally bearing load on the plate and the hydrostatic pressure acting on the plate, have been discussed and solution have been followed. So, thank you very much.