Plates and Shells
Prof. Sudip Talukdar
Department of Civil Engineering
Indian Institute of Technology-Guwahati

Module-02
Lecture-05
Simply Supported Plate Subjected to Distributed Loading

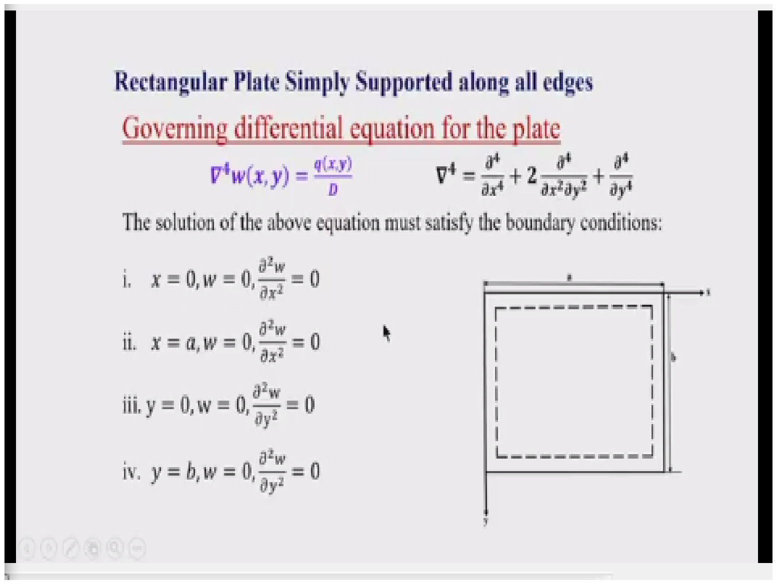
Hello everybody, today I am starting model 2, and this is my first lecture. In the last week, I discussed the plate theory and different assumptions required for analyzing of plate. We basically concentrate on the thin plate with a small deflection. And then, we have derived the differential equation of equilibrium of the plate. The fourth-order differential equation of the equilibrium of the plate has been derived.

And it has to be solved for given loading conditions and boundary conditions in the plate. The plate may have any geometrical shape, and boundary conditions may also be different. But the exact solution of the governing differential equation of the plate that we have derived in the last class cannot be achieved in all such cases; in very few cases, the analytical solutions are available, but these analytical solutions provide a benchmark result.

And it is also used as an auxiliary result in other numerical problems that you require to solve the complicated plate problem. So, today I will discuss a method for a rectangular plate that has been developed by Navier. And with the help of this method, we will try to find out the exact solution of the differential equation.

So, Navier's method that I have told you this has been developed by Navier, and it is applicable for plates simply supported along all edges, and it is especially applicable for rectangular plates. So, today I will discuss Navier's method for uniformly distributed load and also partially covered load.

(Refer Slide Time: 02:44)



So, let us see the governing differential equation for the plate; the governing differential equation for the plate is given as $\nabla^4 w(x, y) = \frac{q(x,y)}{D}$ where D is the flexural rigidity of the plate. If the thickness of the plate is uniform, we can assume that D is a constant, and it is given as $\frac{Eh^3}{12(1-v^2)}$, where h is the thickness of the plate, E is the modulus of elasticity of the plate material, and mu is the poisson ratio.

You can see here this is a 4th order partial differential equations, and it is also a non-homogeneous differential equation. So, if I consider the linear analysis, then I have to obtain the solution in two stages. First, I have to obtain the homogeneous solution taking the forcing function at 0, and then I have to obtain this particular integral or particular solution, and then superimposition of that will give you the general solution.

The process becomes lengthy if we follow this or the conventional rule of a solution of the differential equation. But Navier has overcome this difficulty by using a double trigonometric series. The double trigonometric series that it proposed is only applicable that it is validity is only for a rectangular plate having four edges as simply supported. Now here ∇^4 is the operator which is given by $\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$

We consider a simply supported plate, length is a, and breath is b. Now, here you can see that the boundary condition for simply supported is written here as equation 1, 2, 3, 4. So, let us see what their implications are. So, at x = 0, say this edge, we see that for simply supported condition, deflection is 0, and the bending moment has to be 0. Now bending moment equation, you know it is given by $-D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)$

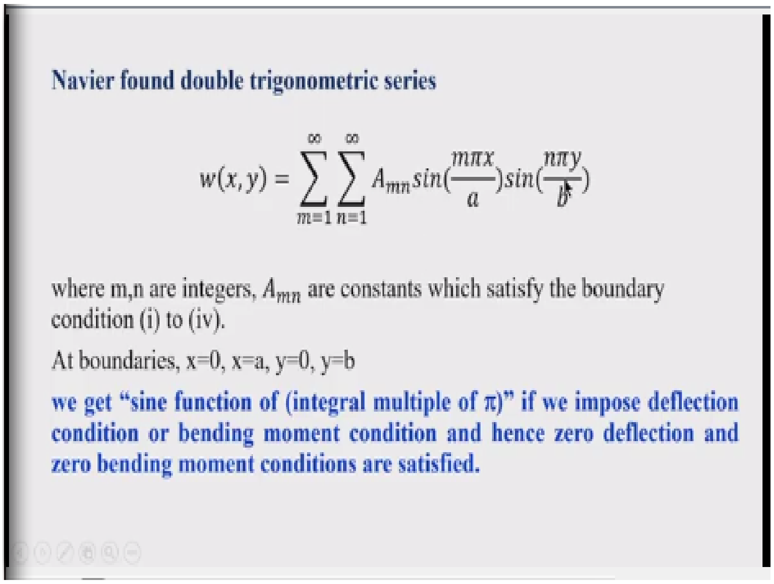
Now, since bending moment is 0 here and bending moment contains curvature in 2 directions in general. Now, here you can see this edge is supported along this edge x = 0 as that is along the y axis. So, there cannot be any curvature along the y axis at this edge. So, therefore the second term $\frac{\partial^2 w}{\partial y^2}$, which is multiplied by mu, does not exist. Hence, the condition for boundary at this edge x = 0 becomes w = 0 , $\frac{\partial^2 w}{\partial x^2}$.

Similarly, at x = a just opposite edge, you can see that deflection is 0 obviously, it is supported, so deflection is 0. And the bending moment $\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)$ is 0. But since it is supported all along the edges, $\frac{\partial^2 w}{\partial y^2}$ is 0, so the condition becomes simply, $\frac{\partial^2 w}{\partial x^2} = 0$. If we go other two edges, say y = 0 edge and y = b edge.

Now here w is 0 and similarly we get $\frac{\partial^2 w}{\partial y^2}$ is 0, because here the edge is supported all along the edge which is parallel to the x-axis. So, there cannot be any curvature along the x-axis. So, therefore the term that only exist in boundary bending moment equation is del square w by del y square. So, it is 0 because bending moment is 0, this edge permits rotation because it is simply supported end.

On the opposite edges y = b again deflection is 0 and $\frac{\partial^2 w}{\partial y^2}$ is 0. Now we have got this condition mainly the condition is deflection and it is second derivative. So, deflection function and at second derivative along x and along y-direction need to be 0 along the edges, since these edges are simply supported.

(Refer Slide Time: 07:46)



So, Navier has proposed say for such type of condition at double trigonometric series is the most suitable one for the solution of the differential equation. So, this equation you can see $w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} sin(\frac{m\pi x}{a}) sin(\frac{n\pi y}{b})$, where m and n are integers and this is a infinite series. So, summation index m varies from 1 to infinity, similarly n varies from 1 to infinity. This series that you are seeing is popularly known as Navier series and it satisfies the boundary condition.

You can see the boundary condition requires w = 0, at x = 0, at x = a and also w is 0, at y = 0, y = b. So, therefore if you put x = 0 here deflection becomes 0, if you put this x = A then sine m pi that is of course for any integral value of m, sin $sin\, m\pi$ will be 0. So, therefore deflection condition is satisfied and if you go to satisfy the bending moment condition, you differentiate this expression 2 times with respect to x as well as with respect to y.

So, if you see that if the satisfaction of boundary condition specially for bending moment along **y** x direction requires that curvature along the x-direction, that is $\frac{\partial^2 w}{\partial x^2} = 0$. So, if you take the second derivative of this equation, then obviously that $\frac{m^2\pi^2}{a^2}$ term will come, and again this sine function will appear with a proper sign. So, again this boundary this condition del square w by del x square is satisfied.

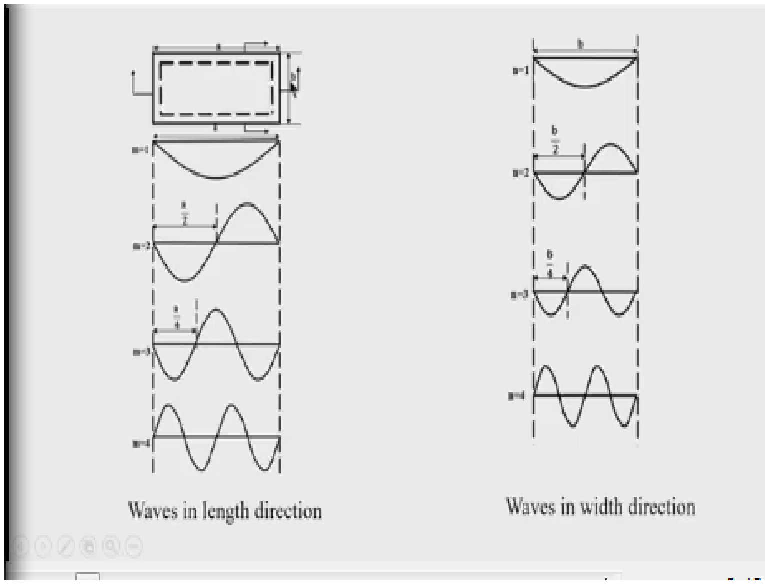
Similarly, if I differentiate this expression 2 times with respect to y, I will come back again the sine function, sin $sin\,\frac{n\pi y}{b}$ of course this coefficient $\frac{n^2\pi^2}{b^2}$ will come out. Now put y = 0 or y = b this bending moment condition will be satisfied, which means curvature will be 0, so the bending moment condition will be satisfied. So, this Navier series proposed as a trigonometrical series that is a double trigonometrical series completely satisfy the simply supported boundary condition.

It is a very well-known series, and it gives the value of deflection and other stress resultant the converse value of this deflection and stress resultant very rapidly. Of course, if you go this stress resultant with the higher derivatives, the convergence may be slow. But within, say maximum say 5 to 7 terms, you will get the convergence within the acceptable limit. Now the most important thing is the coefficient Amn, coefficient Amn is still unknown, that coefficient if I find out then I will get the complete deflection.

That is the deflection completely, which means A mn is still unknown. How can it be found? Now since it is a solution of the differential equation, we take it as a solution of the differential equation, and it satisfies the boundary conditions completely. So, if it is the solution, then it should satisfy the differential equation also. So, substituting this series, now in the differential equation, we have to obtain the A mn.

Now how will the deflection take place in this simply supported plate? Let us see the functions, the $\sin sin \frac{m\pi x}{a} \sin sin \frac{n\pi y}{b}$; these are waves, and we will call it half waves because this ranges from 0 to a and 0 to b, now m and n are integers.

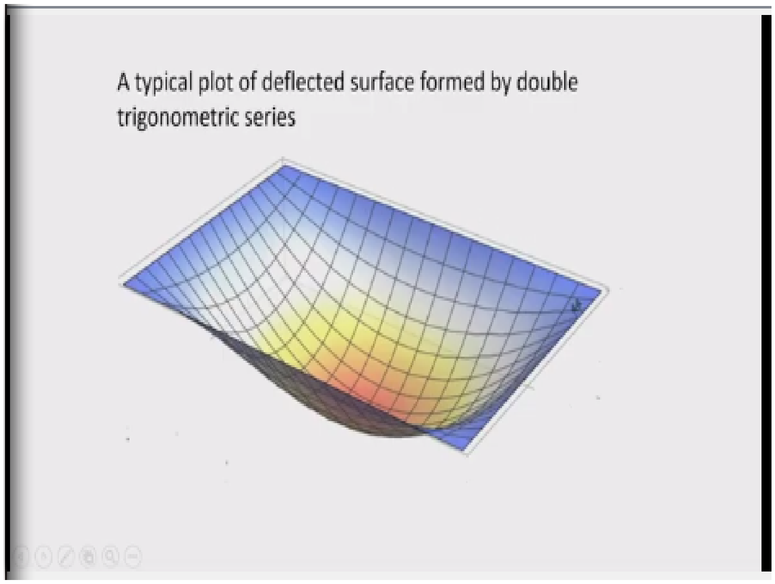
(Refer Slide Time: 12:36)



Now you can see that for m = 1 along the length direction, the first half-wave is this, second half-wave when we take m = 2, we get this second half-wave. Similarly, if you consider m = 3, you will get the third half-wave. Like that, even the number of half-waves will increase as you increase the integer. So, similarly, in the width direction, you will get the half-wave which is the width direction. The function is $\sin sin \frac{n\pi y}{b}$.

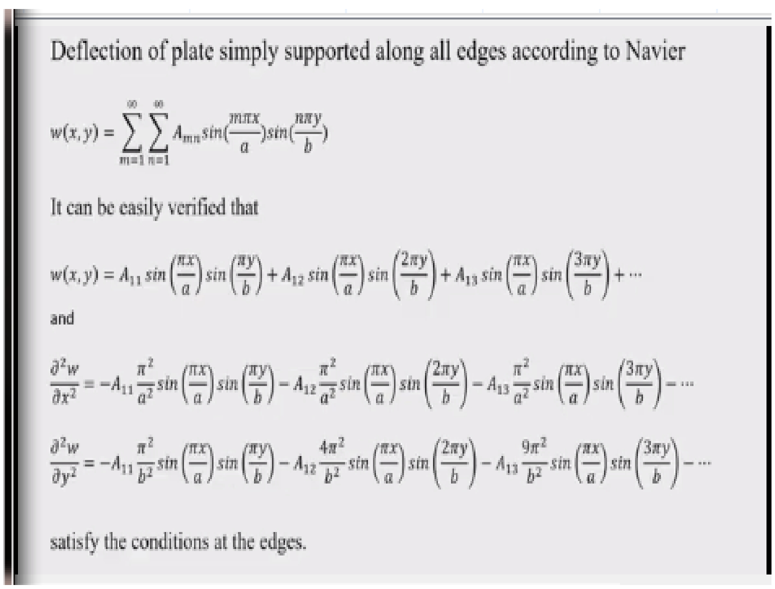
So, if you put n = 1, you will get one halfway; if you put n = 2, you will get two half-waves and so on. So, what is the Navier solution? The Navier solution is a product of 2 half-waves and their summation. So, the superimposition of this is the product of 2 half-waves, and if you have summed it, then you will get the deflected surface.

(Refer Slide Time: 13:42)



A typical plot of the deflected surface of a simply supported plate looks like that when you superimpose all the waves and if you know the A mn correctly. Then you will get the correct magnitude of the deflection function or deflection surface of the rectangular plate simply supported along all edges subjected to any type of load here.

(Refer Slide Time: 14:08)

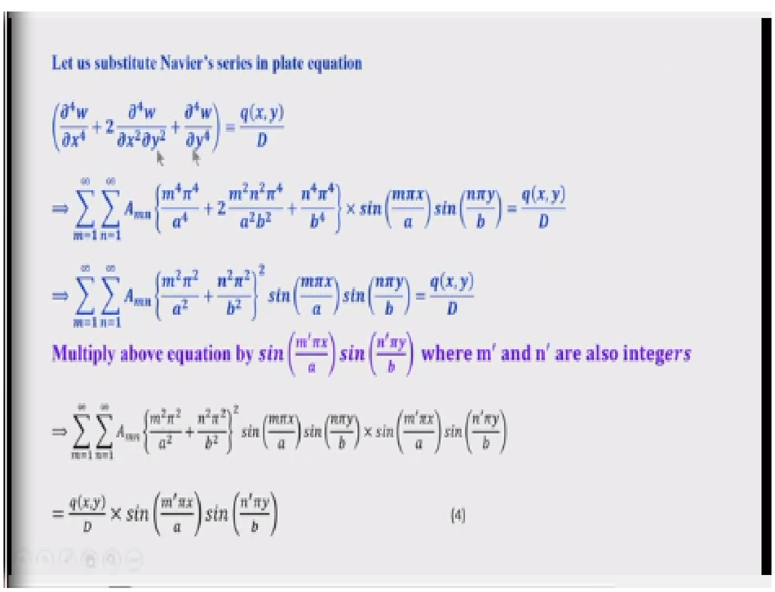


Now, let us come to the solution. So, w = double sum A mn sine m pi x by a sine n pi y by b. So, two terms you are seeing it. Now, if we expand this you can write it say a first term will be A_{11} sin $sin\left(\frac{\pi x}{a}\right) sin\left(\frac{\pi y}{b}\right)$, this is the first term; you can call it a generic term. Then the second term is m = 1, n = 2, $A_{12} \sin sin\left(\frac{\pi x}{a}\right) sin\left(\frac{2\pi y}{b}\right)$. Third term m = 1 and n = 3 you will get $A_{13} \sin sin\left(\frac{\pi x}{a}\right) sin\left(\frac{3\pi y}{b}\right)$and so on, so you will get an infinite series.

So, if you take the second derivative of this, which we require to calculate bending moment also if the deflection is completely known, then after taking the second derivative with respect to x, you will get $-A_{11} \sin \frac{\pi^2}{a^2} sin\left(\frac{\pi x}{a}\right) sin\left(\frac{\pi y}{b}\right)$. Similarly, here you will $-A_{12} \sin \frac{\pi^2}{a^2} sin\left(\frac{\pi x}{a}\right) sin\left(\frac{2\pi y}{b}\right)$, like that because we are taking this derivative with respect to x.

Similarly derivative with respect to y we get $-A_{11} \sin \frac{\pi^2}{b^2} sin\left(\frac{\pi x}{a}\right) sin\left(\frac{\pi y}{b}\right)$ and $-A_{12}$ because $\frac{2\pi}{b}$ will come out, so $-A_{12} \sin \frac{4\pi^2}{b^2} sin\left(\frac{\pi x}{a}\right) sin\left(\frac{2\pi y}{b}\right)$and so on. So, this derivative function can be evaluated. So, you can see that these derivatives satisfy the condition at the edges because bending moments have to be zero in the respective edges. Now we have to differentiate it up to 4th order because the differential equation is a 4th order equation.

(Refer Slide Time: 16:25)



So, after differentiating this w, 4 times, we will get $\frac{\partial^4 w}{\partial x^4}$. And differentiating w with respect to y 4 times we will get $\frac{\partial^4 w}{\partial y^4}$. Then the middle term, there is $2\frac{\partial^4 w}{\partial x^2 \partial y^2}$; this term is obtained. It can be easily verified that when we differentiate w with respect to x 4 times, we will get $\frac{m^4 \pi^4}{a^4}$ as a constant term that will be coming out as a process of differentiation.

But the function that is $\sin sin\left(\frac{m\pi x}{a}\right) sin\left(\frac{n\pi y}{b}\right)$will remain unchanged because it is an even number of derivatives. Then here, first, we differentiate 2 times with respect to x, and then the result is differentiated two times with respect to y. So, again it is an even number of derivatives, and the function remains as it is because there will be no change of sine function, only the coefficients that will be changed.

So, coefficient here we are getting $\frac{m^4 \pi^4}{a^4}$ plus due to differentiation of this quantity w with respect to x 2 times and the result is again differentiated with respect to y two times; we will get the coefficient $2\frac{m^2 n^2 \pi^4}{a^2 b^2}$. So, the coefficients are plugged in here, and the function remains as $\sin sin\left(\frac{m\pi x}{a}\right) sin\left(\frac{n\pi y}{b}\right)$

And the right-hand side remains as it is $\frac{q(x,y)}{D}$, and it is under summation because Amn is there, so the double sum is written here. Now we want to find out our target is to find the A_{mn}? So, A_{mn} can be found out by expressing the load q(x,y) as a Fourier series and then taking a general term and comparing the coefficients of like term we can find the A_{mn}. So, I will illustrate the process from the first principle.

So, this function or this constant that is written inside the second bracket $\frac{m^4 \pi^4}{a^4} + 2\frac{m^2 n^2 \pi^4}{a^2 b^2} + \frac{n^4 \pi^4}{b^4}$ can be conveniently expressed as $\left\{\frac{m^2 \pi^2}{a^2} + \frac{n^2 \pi^2}{b^2}\right\}^2$. So, that is of bias from this quantity.

Then this function is written here, sin $sin\left(\frac{m\pi x}{a}\right) sin\left(\frac{n\pi y}{b}\right)$and the right-hand side is the loading function $\frac{q(x,y)}{D}$. This loading is here a distributed loading, which means q is also a continuous function of x and y. Now multiply both sides of the above equation, this equation by sin $sin\left(\frac{m'\pi x}{a}\right) sin\left(\frac{n'\pi y}{b}\right)$, where m dash n dash are integers, m and n also integer and m dash, n dash are also integers.

So, we have taken different symbols with the intention to use a well-known principle of mathematics or sine series. So, therefore we have used this simple integer A mn integer with the difference sine simple m prime and n prime. So, if I multiply both sides by this function, I am writing this left-hand side which was this

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\left\{\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right\}^2 \sin sin\left(\frac{m\pi x}{a}\right) sin\left(\frac{n\pi y}{b}\right)\times \sin sin\left(\frac{m'\pi x}{a}\right) sin\left(\frac{n'\pi y}{b}\right)$$

Of course, this quantity is under summation. So, the summation is with respect to m as well as with respect to n; it is a double sum. And the right-hand side is $\frac{q(x,y)}{D}$, and due to the multiplication of both sides with this quantity that is sin $sin\left(\frac{m'\pi x}{a}\right) sin\left(\frac{n'\pi y}{b}\right)$, we also write sin $sin\left(\frac{m'\pi x}{a}\right) sin\left(\frac{n'\pi y}{b}\right)$ in the right-hand side.

(Refer Slide Time: 21:56)

Now, integrate both sides in the domain of the plate

$$\Rightarrow \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_0^a\int_0^b A_{mn}\left\{\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right\}^2 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{m'\pi x}{a}\right)\times\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{n'\pi y}{b}\right)dxdy$$

$$= \frac{1}{D}\int_0^a\int_0^b q(x,y)\times\sin\left(\frac{m'\pi x}{a}\right)\sin\left(\frac{n'\pi y}{b}\right)dxdy \qquad (4)$$

Let us simplify the equation(4) making use of orthogonality condition of sine function

$$\int_0^a \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{m'\pi x}{a}\right)dx = \frac{a}{2}\ if\ m=m'$$
$$= 0\ if\ m\neq m'$$

$$\int_0^b \sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{n'\pi y}{b}\right)dy = \frac{b}{2}\ if\ n=n'$$
$$= 0\ if\ n\neq n'$$

Then, equation (4) can be reduced to

$$A_{mn}\left\{\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right\}^2\frac{ab}{4} = \frac{1}{D}\int_0^a\int_0^b q(x,y)\times\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dxdy$$

Now integrate, the next step is to integrate both sides in the domain of the plate. So, I integrated the equation that we obtained in the last slide; this equation with respect to x and y, and the limit is 0 to a and 0 to b. We have taken the origin of the coordinate system as one of the corners of the plate; I have taken in the left-hand side top corner of the plate. So, in this x along the x-direction, the range of x is 0 to a, and along the y-direction, the range is 0 to b.

So, we put the limit 0 as the lower limit of the integration with respect to x, a as the upper limit of the integration with respect to x. Similarly, here 0 is the lower limit for integration with respect to x y, and b is the upper limit for integration with respect to y. So, we have written the limit, and now we can integrate it. So, we plugged the x term together and then try to integrate with respect to x.

Similarly, we plug this y term that is $sin\left(\frac{n\pi y}{b}\right)$and $sin\left(\frac{n'\pi y}{b}\right)$ and then try to integrate with respect to y. On the right-hand side, there is no problem, there is no sum, so we write sin $sin\left(\frac{m'\pi x}{a}\right) sin\left(\frac{n'\pi y}{b}\right)dx\,dy$, where m dash n dash are integer whether you put here m or m dash or n or n dash, it does not matter, it makes any difference. So, here this equation can be simplified.

This equation if I number it by equation number 4 here, it can be simplified by using the orthogonality condition. So, the orthogonality condition of the sine function is well known, and it is $\sin sin\left(\frac{m\pi x}{a}\right) sin\left(\frac{m'\pi x}{a}\right)dx$ 0 to a. So, if I integrate this, then for m = m', the value the integral is a by 2 to and for other values of m and m dash which is not equal to each other, there is m not equal to m', we get 0, so this is one condition.

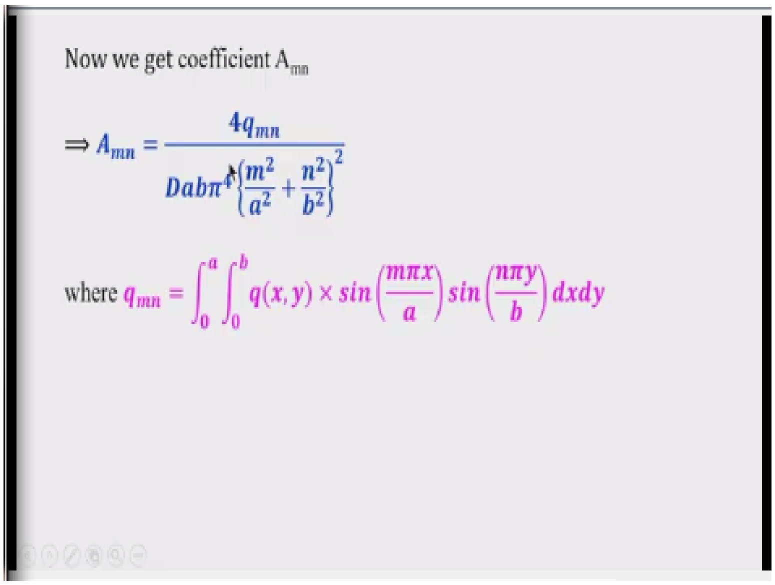
Similarly, along the y-direction, when I integrate the function with respect to y with a limit 0 to b, we get b by 2 for n = n dash and 0 if n is not equal to n dash. So, using this property, now it can be seen that all the terms containing the $\sin sin\left(\frac{m\pi x}{a}\right) sin\left(\frac{m'\pi x}{a}\right)$, where m and m' are different will be vanishing. Similarly, all the terms in the summation with $\sin sin\left(\frac{n\pi y}{b}\right) sin\left(\frac{n'\pi y}{b}\right)$ with the condition that n is equal to not n' again will vanish.

So, ultimately will be leftover only with the term to be integrated that $\int_0^a \left(\frac{m\pi x}{a}\right) dx \times \int_0^b sin^2\left(\frac{n\pi y}{b}\right)dy$. So, two integrations are needed. Now, here for, m = m' and n = n'. And obviously, the result will be $\frac{a}{2} \times \frac{b}{2}$. So, here now we can write because the summation term in now, not required because the only terms that is remaining m = m' and n = n' other terms will be 0 because of orthogonality condition.

So, that property orthogonality property is the q point here. So, without this orthogonality property, this Navier method cannot be formulated. So, the orthogonality property of the sine functions is utilized here, and then we get A mn coefficient as A mn into m square pi square by a square + n square pi square by b square whole square ab by 4, ab by 4 has come here, this term has come here due to product of a by 2 and b by 2.

As a result of the integration of $A_{mn}\left\{\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right\}^2 \frac{ab}{4}$, so as a result of this, you get here ab by 4. The right-hand side remains as it is $\frac{1}{D}$ integration of course 0 to a, 0 to b that integration is not completed, it depends on the nature of the function q x y, then only you can integrate. So, q x y is here, and the term sin $sin\left(\frac{m\pi x}{a}\right) sin\left(\frac{n\pi y}{b}\right)$ is here, so m and n are integers.

(Refer Slide Time: 28:03)



Now, from this equation, now it is possible to write A mn the coefficient of the deflected series or deflected function, deflected surface whatever you call is $A_{mn} = \frac{4q_{mn}}{Dab\pi^4\left\{\frac{m^2}{a^2}+\frac{n^2}{b^2}\right\}^2}$, how this term has come? Because this integral is taken as q mn and this is say π^2 is taken out, so it becomes π^4 and ab is there and 4 is there.

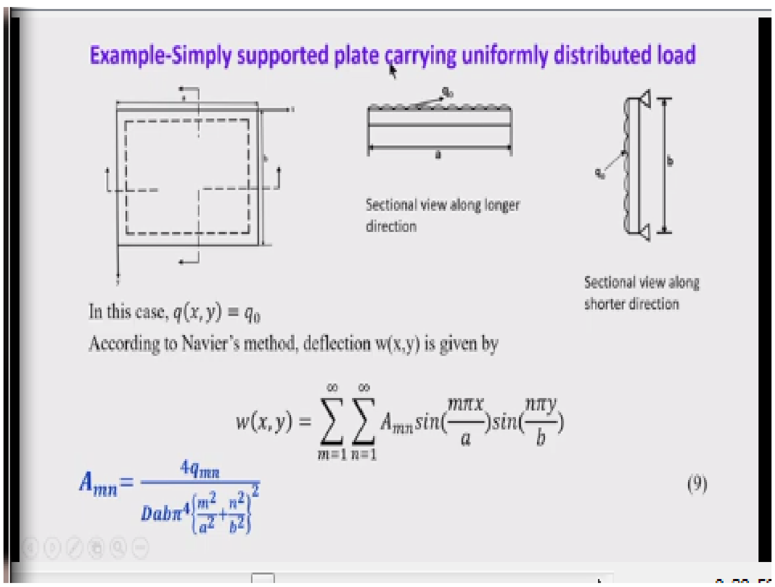
So, if this term is qmn then we can write $A_{mn} = \frac{4q_{mn}}{Dab\pi^4\left\{\frac{m^2}{a^2}+\frac{n^2}{b^2}\right\}^2}$. Now q mn here remains to be determined, you can see q mn can only be determined when you know the load function. So, given the loading condition in the plate, it may be a uniformly distributed load, it may be a uniformly varying load, or it may be any other variation of load that may not follow the linear pattern, and it may be a partially distributed load.

So, all these kinds of loads can be incorporated here. So, q(x,y) is the loading function that we have to know, and then after integration of this quantity, we can now find A mn. Now once A_{mn}

is found out, your deflection is known. So, deflector surface you can completely know if you can find the A_{mn}. So, what I see here the important steps in the Navier's method is first the Navier's method has to be applied only when all the edges of the plate are simply supported, so that is the condition.

Then for the given loading, you calculate the coefficient q mn; once the q mn is known, then you calculate A_{mn}. After knowing A_{mn} You can find the deflected surface. So, from the deflected surface or deflection function, you can go further to find out the slope, bending moment, shearing force, edge shear, corner reaction everything, you can find after finding the deflected surface.

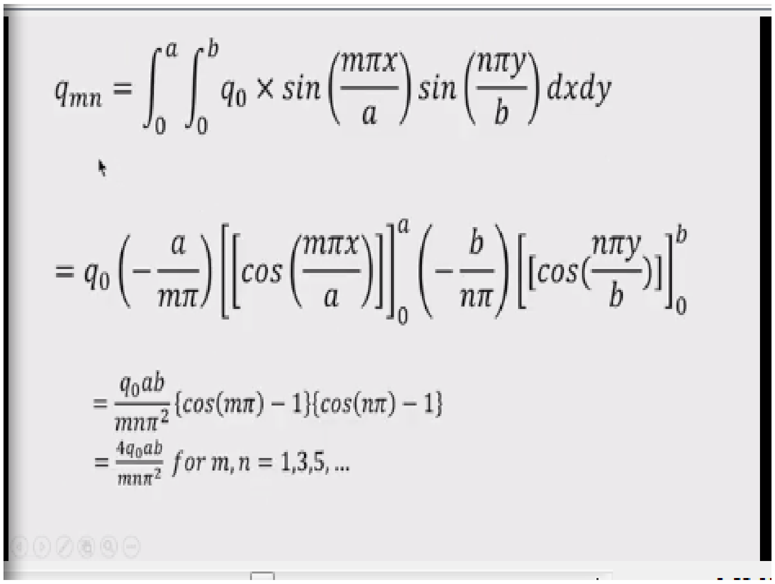
(Refer Slide Time: 30:55)



Now, let us take an example of a simply-supported plate carrying a uniformly distributed load. So, this is a simply supported plate that carries a uniformly distributed load. Now, you can see here the section is shown along the longer direction as well as along, the shorter direction that is the length direction that is a and the shorter direction you can see this width b. The aspect ratio of the plate is a by b.

Now here, the q x y, the loading function that we consider in the solution is now a constant quantity, which is a uniformly distributed load per unit area, so it is denoted as q0 so, as we have developed earlier the solution where the A mn is nothing but $A_{mn} = \frac{4q_{mn}}{Dab\pi^4\left\{\frac{m^2}{a^2}+\frac{n^2}{b^2}\right\}^2}$.

(Refer Slide Time: 32:01)



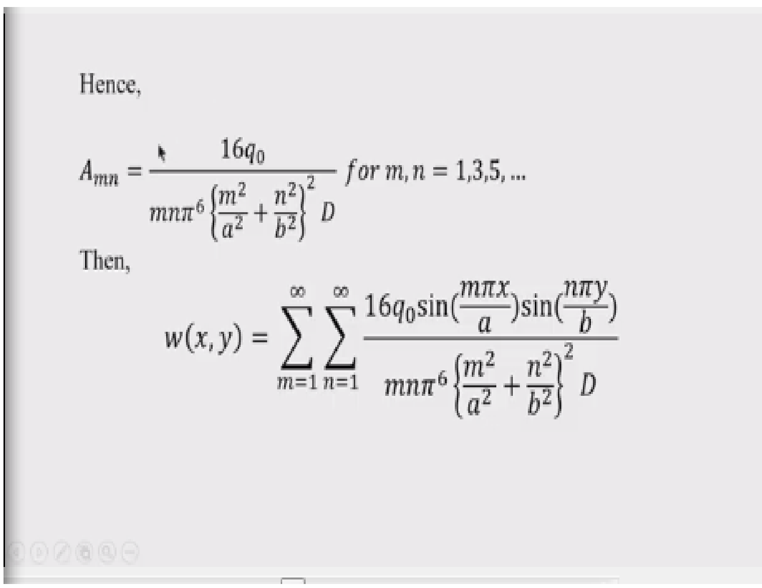
So, most important thing is to find q mn. So, $q_{mn} = \int_0^a \int_0^b q_0 \times \sin sin\left(\frac{m\pi x}{a}\right) sin\left(\frac{n\pi y}{b}\right)dxdy$. You can see here that q 0 is a constant quantity, so it can be taken outside the integral sign. And after integration with respect to x you will get this term as a constant $-\frac{a}{m\pi}$, and the integration of sin $sin\left(\frac{m\pi x}{a}\right)$ will be cos $cos\left(\frac{m\pi x}{a}\right)$.

And this constant appears after integration that the constant $\frac{a}{m\pi}$with a minus sign comes out here. And the limit of integration is shown here at 0 to a. Similarly, for the term $sin\left(\frac{n\pi y}{b}\right)dy$ an integrated within a limit 0 to b you will get $-\frac{b}{n\pi}$ as coefficient and $cos\left(\frac{n\pi y}{b}\right)$ and to be evaluated with a limit 0 to b. So, when we evaluate this quantity now, we get $\frac{q_0 ab}{mn\pi^2}$, this m, n are coming here π^2 is coming, and ab is there and minus minus again plus sign will be there.

And due to substitution of this upper limit and lower limit, you will now get $\{\cos cos\,(m\pi)\ -\ 1\}\{\cos cos\,(n\pi)\ -\ 1\}$. One interesting thing is that for any odd value, even value of m say for m = 2, this function will be 0. And for n = say 2, this function will be 0. So, if m = 1, n = 1, then you will get this as a -1 - 1 that is – 2, and here also you will get -2. So, ultimately the result is $4q_0ab$, and then you are getting $mn\pi^2$ where m and n are integers. These m and n are only odd integers.

So, that integral value is non zero, and otherwise, for the even value of integers, there is m and n are even integers, then the integral will be 0. So, our q mn is now $\frac{4q_0ab}{mn\pi^2}$.

(Refer Slide Time: 34:53)



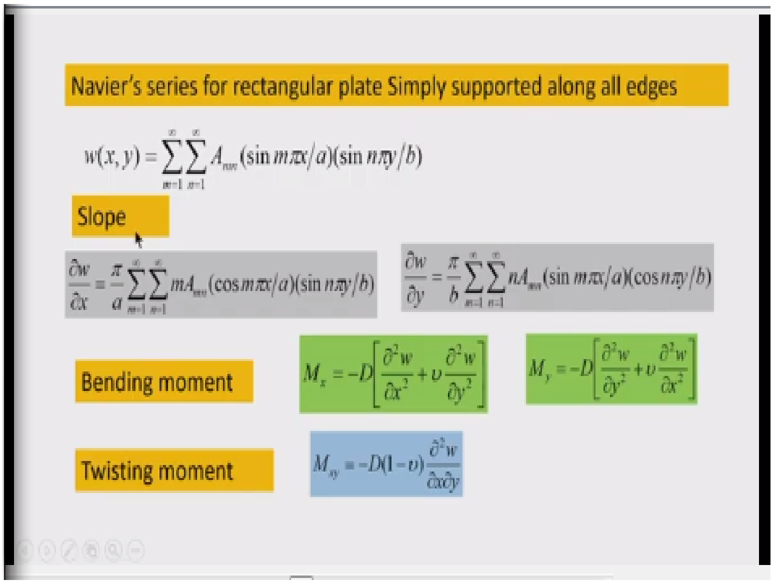
So, once the q mn is known, you can get A_{mn} So, A_{mn} after simplification

$$A_{mn} = \frac{16q_0}{mn\pi^6\left\{\frac{m^2}{a^2}+\frac{n^2}{b^2}\right\}^2 D}\ for\ m, n\ =\ 1, 3, 5, \dots .$$

One interesting thing you can see here that the deflection function is evaluated for uniformly distributed load covering the entire plate only for odd harmonics.

There is only odd half waves will contribute to the deflection, q 0 is the known quantity a and b are the size of the plate, and D contains the material property as well as thickness of the plate. Now once you know w, then you can go for calculating the other quantities.

(Refer Slide Time: 35:53)



So, other quantities of interest our slope, slope in the x-direction is given by $\frac{\partial w}{\partial x}$ and it is equal to $\frac{\partial w}{\partial x} = \frac{\pi}{a} summation$. And due to differentiation of this sine function this $\frac{m\pi}{a}$ term will come out. So, $\frac{\pi}{a}$ is a constant, we have taken outside this summation term and the integer m we have kept inside this summation term. So, it is written like that

$$\frac{\partial w}{\partial x} = \frac{\pi}{a}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} mA_{mn}(\cos cos\ \frac{m\pi x}{a})(\sin sin\ \frac{n\pi y}{b})$$

Similarly, the slope in the y-direction is calculated

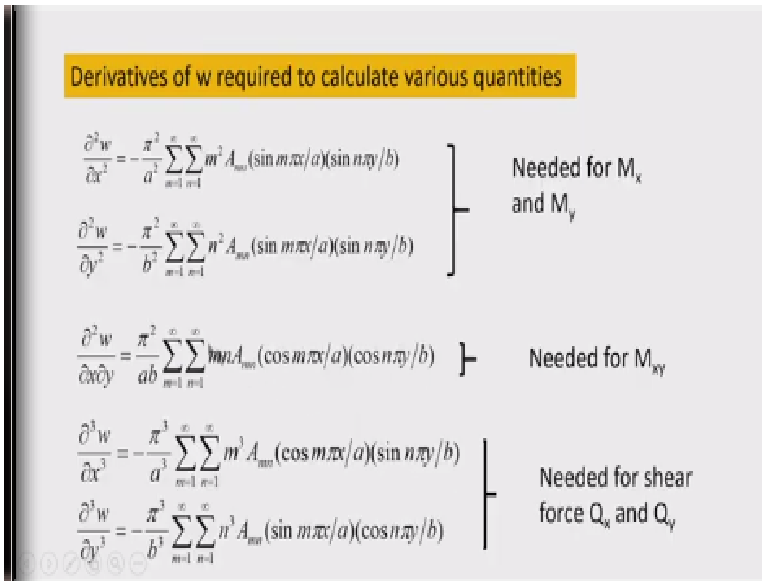
$$\frac{\partial w}{\partial y} = \frac{\pi}{b}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} nA_{mn}(\sin sin\ \frac{m\pi x}{a})(\cos cos\ \frac{n\pi y}{b})$$

Bending moment quantity has to be calculated utilizing the second derivatives. So, we write the bending moment expression as $M_x =- D\left[\frac{\partial^2 w}{\partial x^2} + \upsilon\frac{\partial^2 w}{\partial y^2}\right]$

Similarly, $M_y = - D\left[\frac{\partial^2 w}{\partial y^2} + \upsilon\frac{\partial^2 w}{\partial x^2}\right]$

Twisting moment is calculated by this cos derivative $\frac{\partial^2 w}{\partial x \partial y}$ and one of the deflection function is known; systematically all the derivatives can be calculated.

(Refer Slide Time: 37:42)



So, for convenience, I was just showing you the other higher derivatives $\frac{\partial^2 w}{\partial x^2}$ that is $-\frac{\pi^2}{a^2}$ term will come, and inside this summation, it will be $m^2 A_{mn}(\sin sin \frac{m\pi x}{a})(\sin sin \frac{n\pi y}{b})$. And it is needed for calculation of M x or M y in both bending moments expression M x, M y these derivatives are needed. So similarly, $\frac{\partial^2 w}{\partial y^2}$ is given by

$$-\frac{\pi^2}{b^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} n^2 A_{mn}(\sin sin \frac{m\pi x}{a})(\sin sin \frac{n\pi y}{b})$$

This mixed derivative, $\frac{\partial^2 w}{\partial x \partial y}$ which is needed for calculation of this twisting moment. Twisting moment is developed in the plate, so this twisting moment is sometimes important and it is calculated as with the help of this derivative. Of course, it has to be multiplied by $-D\times(1-\nu)$.

So, final result of this derivative is $\frac{\pi^2}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} mnA_{mn}(\cos cos\frac{m\pi x}{a})(\cos cos\frac{n\pi y}{b})$

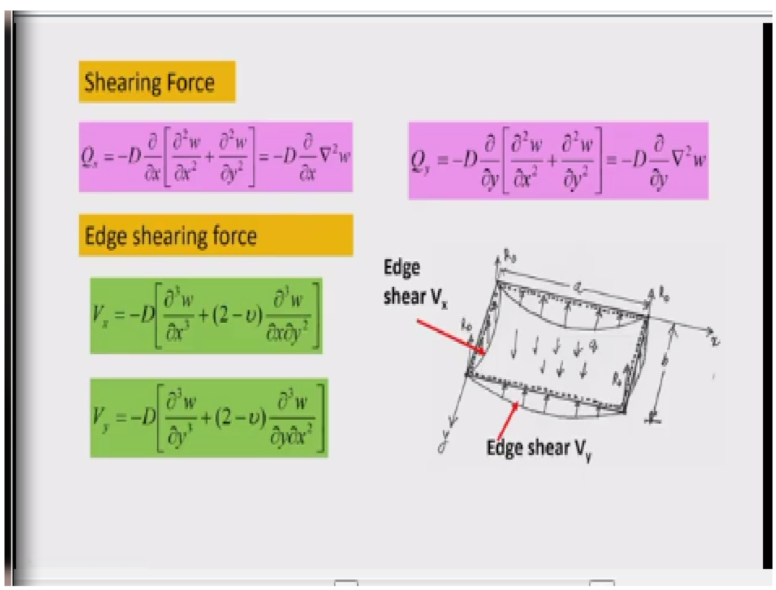
Shear force, you need to find out. So, shear force equate third derivative, so third derivative is calculated. Again you can see this function when it is differentiated with respect to x; you will arrive at the third derivative quantity that is

$\frac{\partial^3 w}{\partial x^3} =- \frac{\pi^3}{a^3}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^3 A_{mn}(\cos cos\frac{m\pi x}{a})(\sin sin\frac{n\pi y}{b})$.

Because the derivative is taken with respect to x, so this term remains untouched.

Similarly, $\frac{\partial^3 w}{\partial y^3} =- \frac{\pi^3}{b^3}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} n^3 A_{mn}(\sin sin\frac{m\pi x}{a})(\cos cos\frac{n\pi y}{b})$, and these quantities are needed for shear force calculation.

(Refer Slide Time: 40:04)

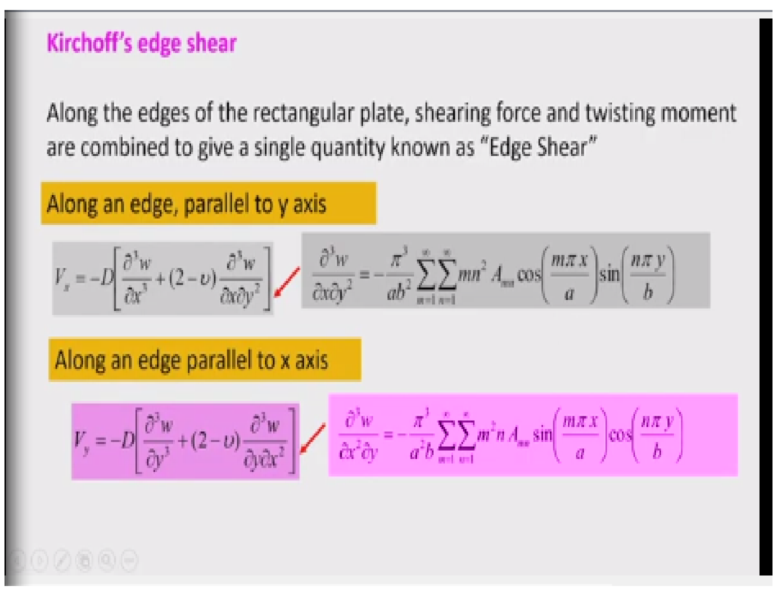


Now, here see these shearing force expressions, shearing force expression is given as this $\frac{\partial}{\partial x}\nabla^2 w$, ∇^2 is the Laplacian operator. Similarly, Q_y is also given as $-D\frac{\partial}{\partial y}\nabla^2 w$, where, ∇^2 here is a Laplacian operator. So, it is obvious from these two expressions; edge shear force is also used in plate problems because edge shear has been proposed by Kirchoff by combining this effect of the twisting moment and shear force along the edges.

So, he combined these two conditions to give a single quantity edge shear along the x-axis along the edges, which is running parallel to the x-axis or along the edges, which is running parallel to the y axis. So, edge shear force if you see it will be distributed like that. So, the edge, say y = 0 edge, here you are getting the edge shear force is it is for a simply-supported plate, of course, it is a sine function, and it is a function of x.

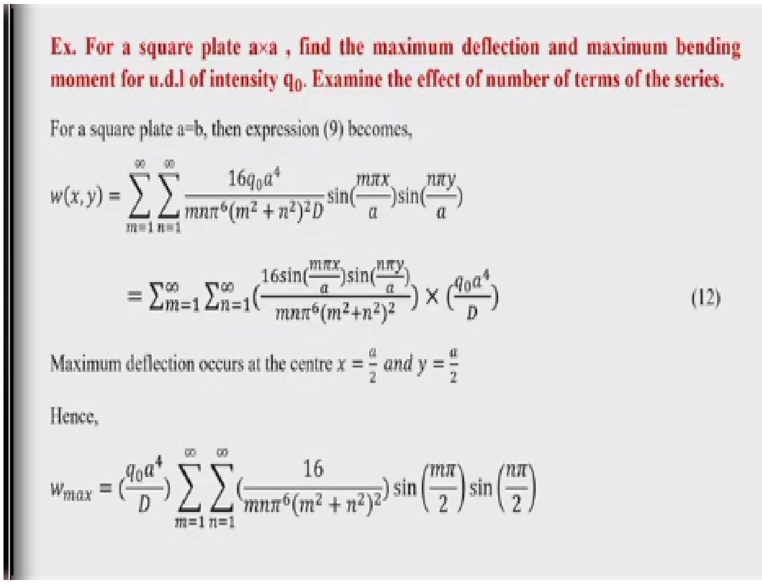
Similarly, if you see the edge shear force along, say x = 0 edge, the edge shear force is a function of y, and it is a sine function. And opposite edges you can also see. So, edge shear force when you integrate it, integrate this edge shear force you will get to total edge shear at these edges. Similarly, all other edges and this edge shear plus other reactions at the corner should be balanced by the vertical load or transverse load acting on the plate.

(Refer Slide Time: 42:08)



So, Kirchoff's edge shear, we require this third derivative and this. So, Kirchhoff edge shear is calculated, we require third derivative as well as the next derivative. So, all these quantities I have shown you in this slide in the previous slide so that you can utilize them here to calculate the edge shearing force, which is very important. And along an edge parallel to the x-axis and along the edge parallel to the y axis, the edge shears are calculated.

(Refer Slide Time: 42:45)



Now, let us see an example. We take a square plate dimension $a \times a$ we have to find the maximum deflection and maximum bending moment for u.d.l. So, we take the Navier series, we have already obtained this solution, and here we take a = b. The purpose of this problem is to find out the deflection quantity as well as to find the effect of a number of terms in the series.

Because we are using a series, so we have to see how it converges. So, w(x,y) is given as the summation of this term $16q_0a^4$ because a = b, so I have manipulated this such that this factor dimension factor a only remains here. So,

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{16q_0a^4}{mn\pi^6(m^2+n^2)^2D}sin(\frac{m\pi x}{a})sin(\frac{n\pi y}{a})$$

Of course, you should only sum with the odd number of terms. So, maximum deflection occurs at the center, so x = a by 2 and y = a by 2. So, when we put this x = a by 2 and y = a by 2, we get the maximum deflection as this. So, $\frac{q_0a^4}{D}$ is a constant term, and we have taken outside the summation term. And that summation term, we now have to evaluate and see how the maximum deflection converges with the number of terms.

So, here m and n are odd integers because we have seen that even integers have no contribution towards the solution for the deflection in the case of simply-supported plate subjected to a uniformly distributed load.

(Refer Slide Time: 45:03)

For m=1, n=1

$$w_{max} = \left\{\frac{16}{\pi^6 \times 1 \times 1(1^2+1^2)^2}\right\}\left(\frac{q_0 a^4}{D}\right) = 0.00416\left(\frac{q_0 a^4}{D}\right)$$

For m=3, n=1

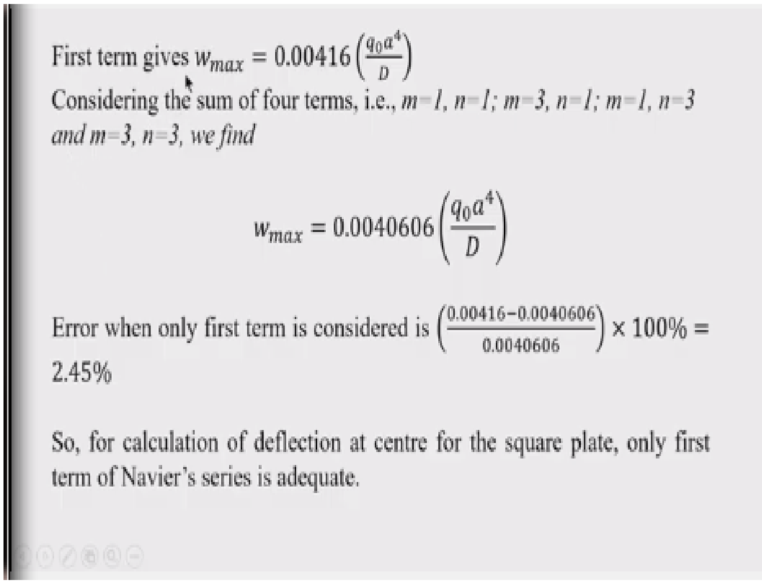
$$w_{max} = \left\{\frac{16(-1)(1)}{\pi^6 \times 3 \times 1(3^2+1^2)^2}\right\}\left(\frac{q_0 a^4}{D}\right)$$

$$= -5.5475 \times 10^{-5}\left(\frac{q_0 a^4}{D}\right)$$

So, for m = 1, n = 1, the center deflection is calculated, and it is found after substituting the value of integers, say; here, both the integers are 1. So, we get the result as 0.00416, and this is the load dimension and flexural rigidity. Similarly, m = 3, n = 1 when we increase the number of half-waves in the x-direction, but the number of half-waves in the y-direction is 1, then again we get the center deflection as this -5.5475 into 10 to the power -5 and $\frac{q_0 a^4}{D}$.

Because the integers power appears in the denominator, so, therefore, gradually, the deflected value or deflection will decrease when you go in the higher number of integers.

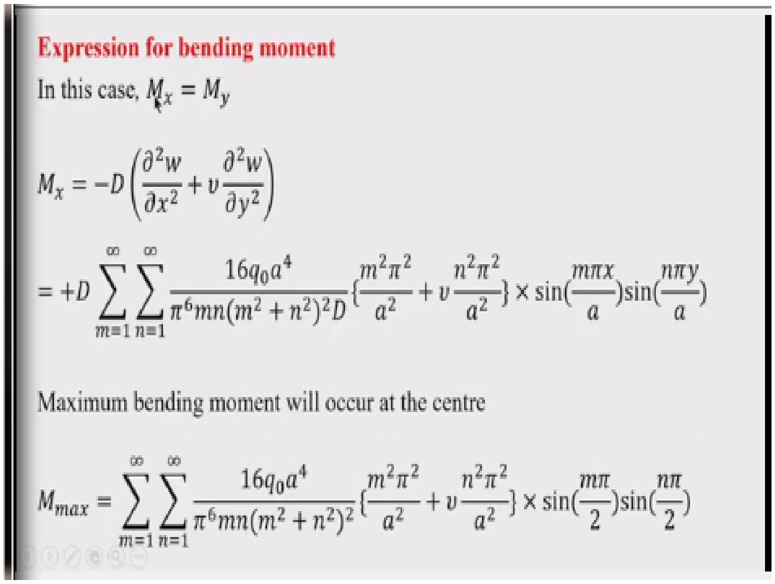
(Refer Slide Time: 46:12)



So, we have seen that considering only the first term; we get the deflection as 0.00416 the coefficient and other terms are there $\frac{q_0a^4}{D}$. When we consider the term m and n bending from 1 to 3 that is m = 1, n = 1; m = 3, m = 1; n = 3 and m = 3; n = 3 we get w max as $0.0040606 \times \frac{q_0a^4}{D}$. So, the difference between this 00416 - 0040606 if we evaluate and take the ratio of the improved deflection that we get after considering the higher number of terms in the summation.

Then we will get the percentage difference is 2.45% only. So, the first time is adequate to give a reasonable value of the deflection surface of the series. So, for calculation of deflection at the center of the square plate, the only first term of Navier's series is adequate.

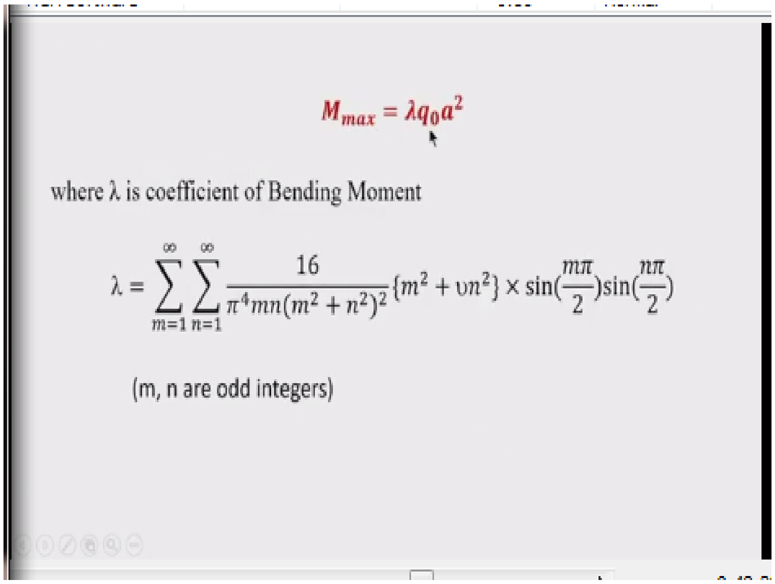
(Refer Slide Time: 47:30)



But if we go for the higher quantities, higher derivatives to find this test resultant like the bending moment shear force, etcetera. Then the first term may not be adequate. So, here the bending moment expression is this $M_x =- D\left(\frac{\partial^2 w}{\partial x^2} + \upsilon\frac{\partial^2 w}{\partial y^2}\right)$. And after taking $\nu = 0.3$ we have taken here mu = 0.3; later on, first, we arrange this equation, D, D will get canceled.

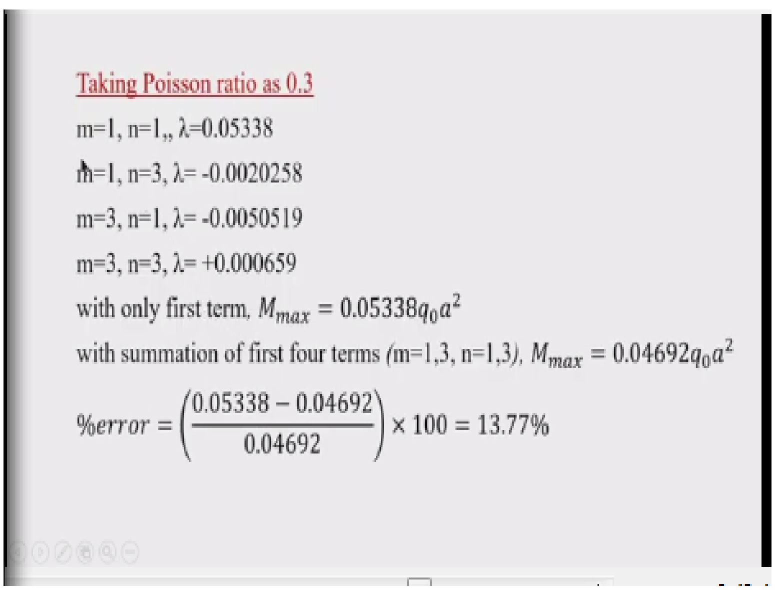
So, the maximum bending moment again will occur at the center. Maximum bending moment quantity is $M_{max} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{16q_0a^4}{\pi^6 mn(m^2+n^2)^2}\{\frac{m^2\pi^2}{a^2} + \upsilon\frac{n^2\pi^2}{a^2}\}\times sin(\frac{m\pi}{2})sin(\frac{n\pi}{2})$

(Refer Slide Time: 48:28)



So, the maximum bending moment is written like that, and considering this number of terms, we now see how it converges?

(Refer Slide Time: 48:38)



So, taking on the first term, we get the coefficient of bending moment as 0.05338. And with other terms, we calculate the coefficients, and then when we sum up the terms, then we get the difference. So, when we take the first four-term which is m varying from 1 to 3, n is also varying from 1 to 3, the maximum bending moment comes out to be 0.04692 coefficient. And here coefficient is 0.05338. So, the percentage difference you can see here is 13.77%.

That means the percentage error in bending moment is higher compared to deflection in Navier's method. So, more terms are needed to improve the accuracy.

(Refer Slide Time: 49:30)

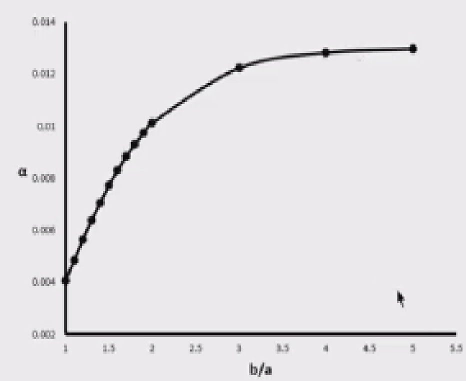
with summation of first four terms (m=1,3, n=1,3),

$$M_{max} = 0.04692 q_0 a^2$$

$$\%error = \left(\frac{0.05338 - 0.04692}{0.04692}\right) \times 100 = 13.77\%$$

The convergence of the series is not rapid, however, error can be reduced considering more terms of the series.

(Refer Slide Time: 49:33)

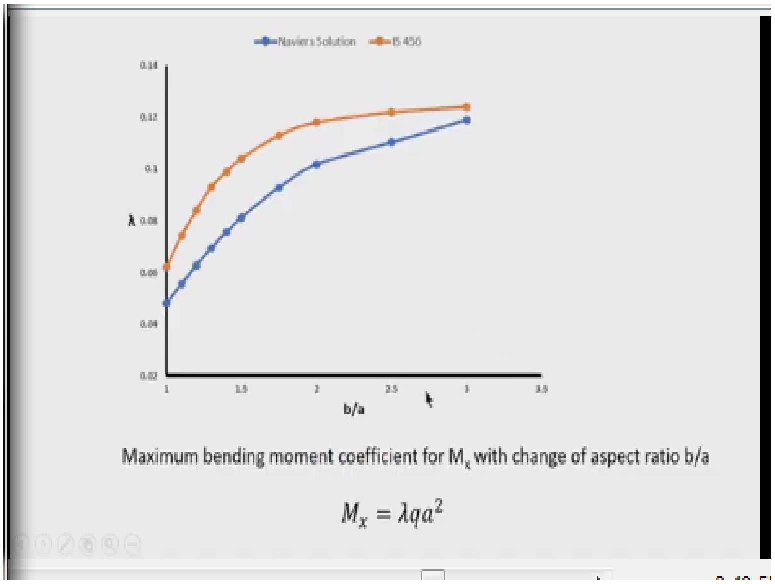


Maximum deflection coefficient Vs Aspect ratio

$$(w)_{max} = \alpha \frac{qa^2}{D}$$

So, if you see the deflection versus aspect ratio curve, you will see that deflection increases with the aspect ratio first and then after it does not increase much and it becomes almost the same, so with the higher aspect ratio.

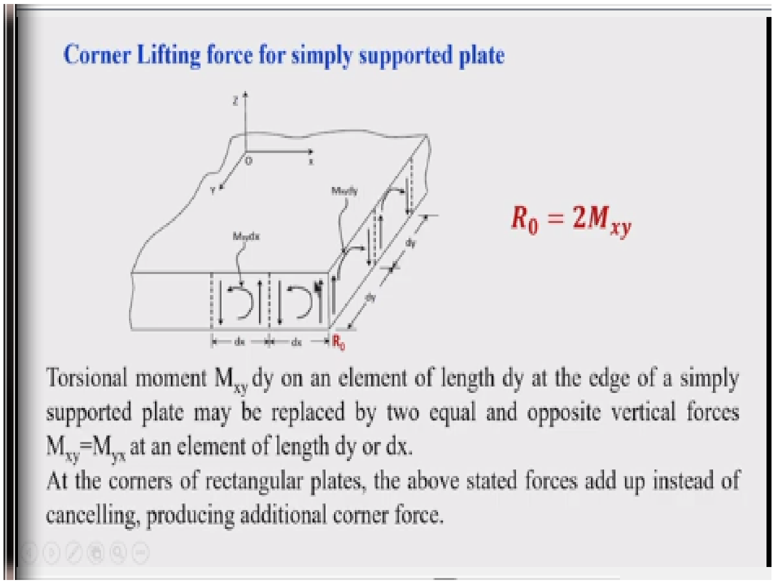
(Refer Slide Time: 49:51)



So, if we compute the bending moment, say I take the bending moment along the x-direction with b by a ratio. And we see that our Navier's solution gives a result this with a blue curve that is seen here, that is the variation of bending moment coefficient with the aspect ratio. But if I take the IS code value, because our Indian standard 456 has given the bending moment coefficient for simply supported slab for different aspect ratios.

So, here we can see that IS code values are slightly are conservative compared to this exact method that has been used in Navier's solution.

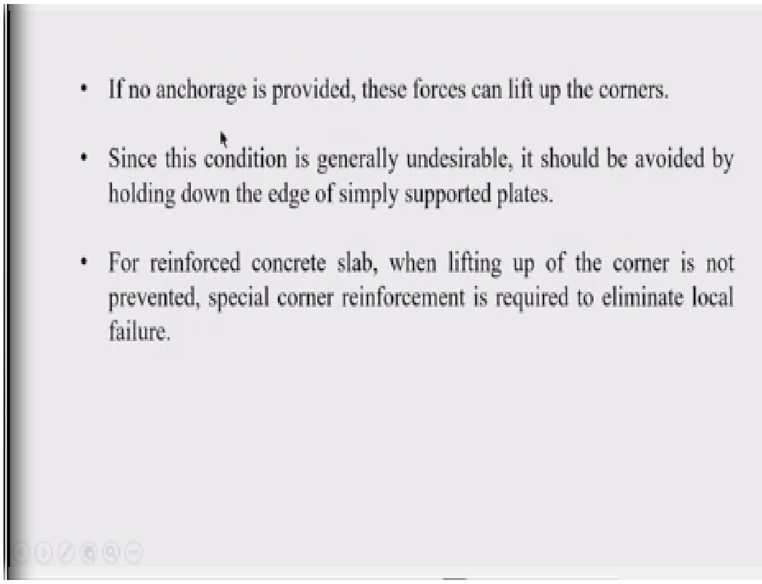
(Refer Slide Time: 50:39)



Now let us see what corner lifting force is. The corner lifting force is a force that is generated at the corner. How it generates? You can see that torsional moment $M_{xy}dy$ if we take an elemental length dy. Then torsional moment $M_{xy}dy$ can be replaced by 2 equal and opposite forces. So, if we do this at the corner, these forces will adopt and giving reactive force upward. So, in the length of dy, say here the total torsional moment $M_{xy}dy$ can be replaced by 2 equal and opposite forces M_{xy}.

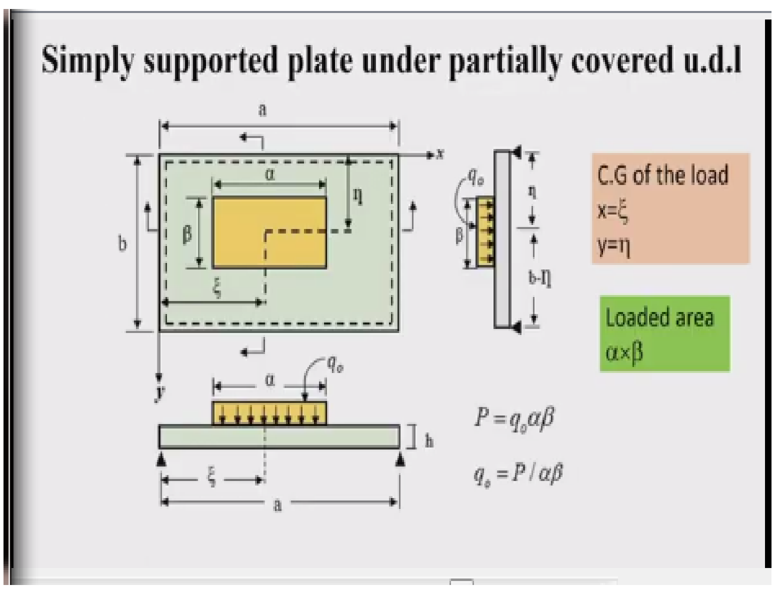
Similarly here also along the adjacent edges, the torsional moment M_{yx} or which is equal to M_{xy}, $M_{yx}dx$ will be replaced by 2 equal and opposite forces M_{xy}, and at the corner, you can see these 2 forces are added up, and the reaction is R_0. So, $R_0 = 2M_{xy}$ is the corner reaction. So, this happens in the case of the simply-supported plate.

(Refer Slide Time: 52:01)



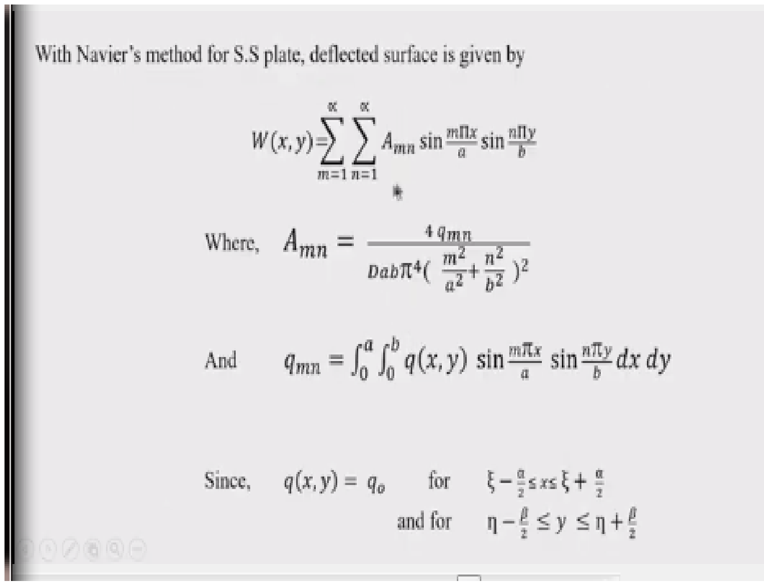
So, if no anchorage is provided, these forces can lift up the corners. Since this condition is generally undesirable, it should be avoided by holding down the edge of a simply supported plate. For reinforced concrete slab, provisions are given in IS code for providing the corner reinforcement to prevent the lifting up of the corner.

(Refer Slide Time: 52:27)



Now let us see a simply supported plate when instead of a fully covered load, a partially covered load is applied. So, again the load is uniform having an area of α×β, so that is the loaded area. And let us assume that the center of the load is located at the coordinates ξ and η. So, the x-coordinate is ξ, and the y-coordinate is η for the center of the C.G. of the load. So, you can see this section here; it is not fully loaded. So, in that case, how will we proceed?

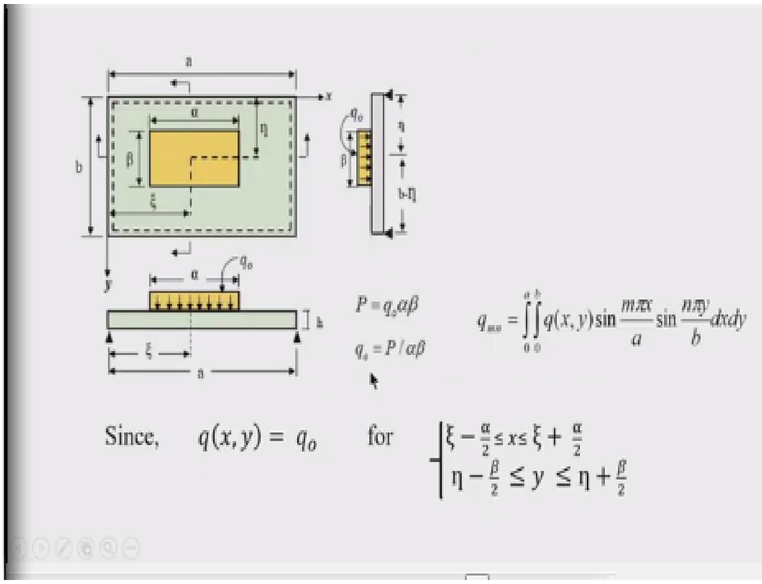
(Refer Slide Time: 53:09)



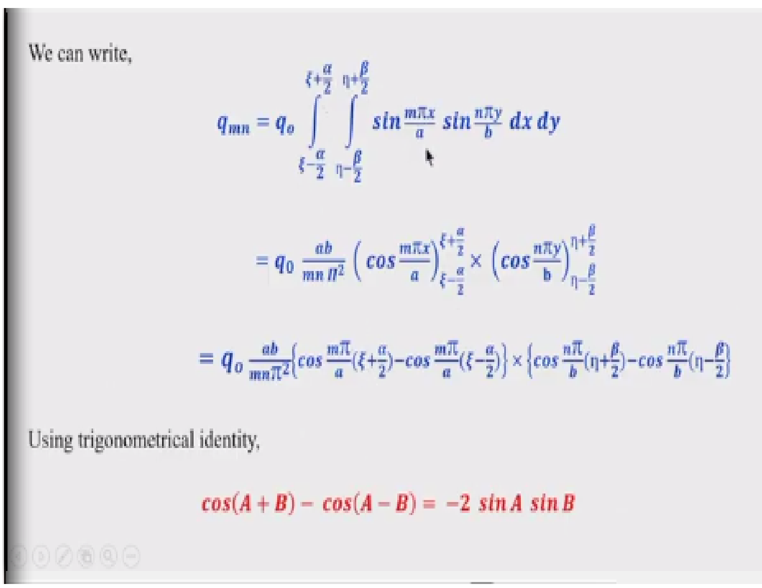
The procedure is the same, now here the Navier's method has been taken here again to find out the exact solution and $W(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin sin \frac{m\Pi x}{a} \sin sin \frac{n\Pi y}{b}$, this is the deflected surface equation, A_{mn} is this $\frac{4\, q_{mn}}{Dab\ ^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}$, q mn here will be different because q is not fully distributed fully covered load. So, this q is now containing or occupying a certain area of the or certain portion of the plate.

Now let us see what will be the limit of the integration? In that case, the lower limit of this integration will be instead of 0; it will start from here. So, obviously, it will be $\xi - \frac{\alpha}{2}$. And the upper limit is here, so it will be $\xi + \frac{\alpha}{2}$. Similarly, in the y-direction, the lower limit is your $\eta - \frac{\beta}{2}$ this point, and the upper limit is $\eta + \frac{\beta}{2}$. So, these limits are written here. So, the only thing is that you have to evaluate in the same way by changing the limit of integration.

(Refer Slide Time: 54:47)



(Refer Slide Time: 54:47)

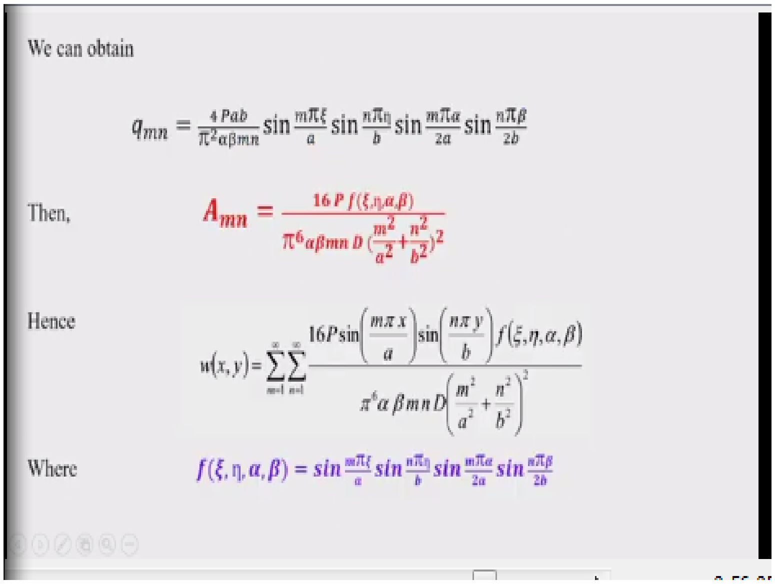


So, limit of integration has been changed here suitably, and integration is carried out. After carrying out integration, you can see that when the limits are substituted, you are getting a function like that $\left\{\cos cos \frac{m}{a} (\xi + \frac{\alpha}{2}) - \cos cos \frac{m}{a} (\xi - \frac{\alpha}{2})\right\}$. Similarly, for integration with respect to y, the result will be a $\left\{\cos cos \frac{n}{b} (\eta + \frac{\beta}{2}) - \cos cos \frac{n}{b}\left(\eta - \frac{\beta}{2}\right) \right\}$.

And coefficients are there which are known quantity, that is q 0 is known, a and b are known, m, n are integers and this pi is also known. So, using the well-known trigonometric identity cos (A +

B) - cos (A – B) = -2 sine A sine B. So, these two functions can be taken as cos (A + B) and cos (A – B), so this formula can be applied.

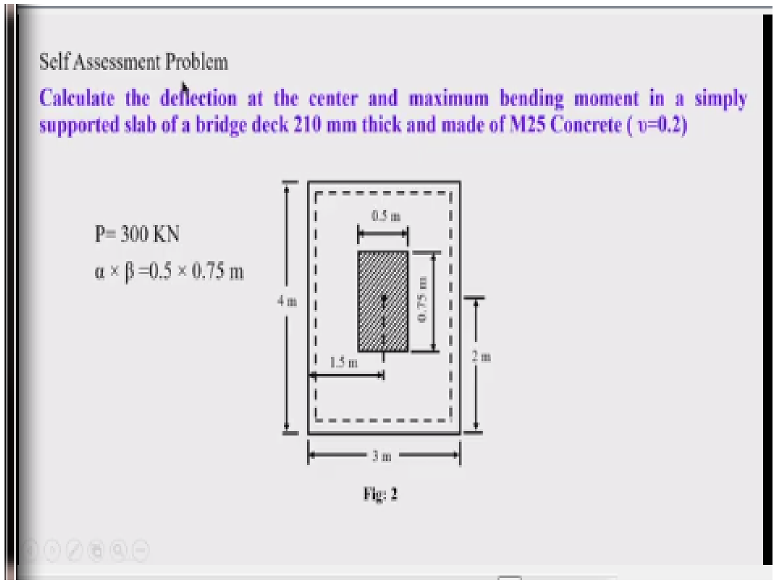
(Refer Slide Time: 56:04)



And after the application of this, you will $q_{mn} = \frac{4\,Pab}{^{2}\alpha\beta mn}\sin sin\ \frac{m\xi}{a}\ \sin sin\ \frac{n\eta}{b}\ \sin sin\ \frac{m\alpha}{2a}\ \sin sin\ \frac{n\beta}{2b}$. So, you get q_{mn} and hence A_{mn} is found out. So, A_{mn} you can see 16 P and $\sum q_{mn}$, so 16 P q_{mn} is this function, so 4 into 4 was there, so 16 is there and ab will get cancelled. So, ultimately we are getting that $A_{mn} = \frac{16\,P\,f(\xi,\eta,\alpha,\beta)}{\pi^6\alpha\beta mnD\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}$

And this f, the function of ξ, η, α, β; ξ, η are the coordinates of the center of the load, and α, β are the dimensions of the load. So, this function I have written separately so that this function is now this. So, f, $f(\xi, \eta, \alpha, \beta)$ is nowsin $sin\ \frac{m\pi\xi}{a}$ sin $sin\ \frac{n\pi\eta}{b}$ sin $sin\ \frac{m\pi x}{2a}$sin $sin\ \frac{n\pi\beta}{2b}$. So, knowing this function, we can completely know the A_{mn}, and then this q_{mn} and other quantities can be calculated.

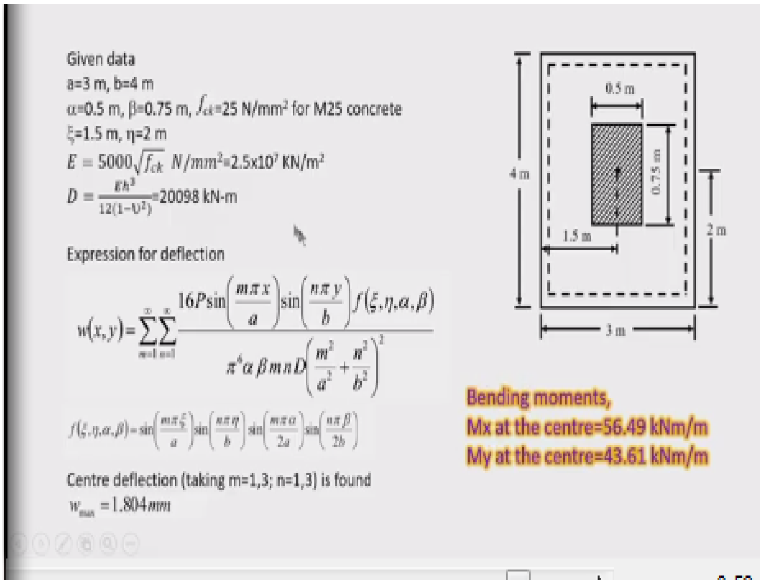
(Refer Slide Time: 57:41)



So, I am giving a self-assessment problem just application of this theory partially distributed load and Navier's method. Partially distributed loads are sometimes very useful, especially for bridge deck design. So, the bridge deck consists of full load, and full load may not be fully distributed over the deck slab panel. So, the contact area is distributed only on the tire contact area.

And different contact areas as specified in the course of provisions, course of practice in different countries. So, here I am taking an arbitrary contact area, say 0.5 meters by 0.75 meters just to illustrate the process. And the total load I have taken as P = 300 kilonewton. And for example, this is a panel of the bridge deck 4 meter by 3 meter simply supported along all edges. And concrete slab, we take the grade of concrete as M25 and whose characteristic strength is $25\frac{N}{mm^2}$, Poisson ratio of concrete is assumed to be 0.2. So, the loaded area is 0.5 + 0.75, and the total load is 300 kilonewton.

(Refer Slide Time: 59:09)



So, the solution is obtained; I am giving some important hints. You can complete and compare the results with the results given here. So, a = 3 meter, b = 4 meter and alpha = 0.5, beta = 0.75, load is placed symmetrically with respect to center. So, our ξ that is the x coordinate of the center of the load, is 1.5 meter, and y coordinate of the center of the load is 2 meter, modulus of elasticity because in this problem, only this characteristic strength that is grade of concrete is given.

Now, based on the grade of concrete, say M25 concrete is selected here. For M25, concrete characteristic strength is $25\ N/mm^2$. So, the modulus of elasticity is given by this formula Indian Standard Code. So, this is an empirical formula we take the help of this formula to calculate the modulus of elasticity. So, modulus of elasticity is here $5000\sqrt{f_{ck}}$. So, it is calculated as 2.5 into $10^7 kN/m^2$.

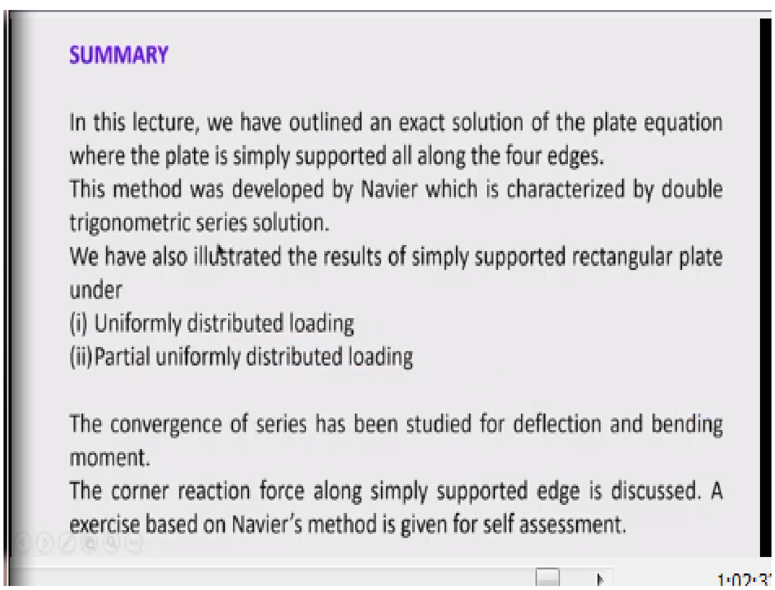
Because this quantity is newton per millimeter square, so converted into kilonewton per meter square. Flexural rigidity D is coming out as $20098\ kN - m$. So, the expression for deflection is this 16 P sine m pi x sine n pi y by b. And this function $f(\xi, \eta, \alpha, \beta)$ contains the coordinates of the load centroid of the load and the dimension of the loaded area.

And denominator contains this $\pi^6 \alpha\beta$ that are the dimensions of the loaded area, m, n are the integers, and D is the flexural rigidity we have already computed here. And these functions are there $(\frac{m^2}{a^2} + \frac{n^2}{b^2})^2$. So, after substituting these values and carrying out this summation, we take here these two terms, m = 1 to 3 and n = 1 to 3.

And we have calculated this w max center deflection; because it is a symmetrical problem, again the deflection will be maximum at the center. And the bending moment in both the directions, say M x and M y-direction, is calculated in the x-direction it is 56.49, in the y-direction, it is $43.61\ kN.m/m$. Remember one thing that all the stress resultant, that is, bending moment, shear force, edge shear, twisting moment, all are expressed in their respective units per unit length of the plate.

Suppose, for example, the bending moment you need is $kN.m$, so it has to be expressed as $kN.m/m$.

(Refer Slide Time: 1:02:30)



So, let us summarize what we have done in today's class. In today's lecture, we have outlined an exact solution of the plate equation, where the plate is simply supported all along the four edges. So, actually, Navier's method is an exact solution, and it is a very well-known analytical method used everywhere, say when you take a composite plate or any other thing. This Navier series has to be taken for such a boundary condition.

And simply supported boundary condition is very common, and it provides the benchmark result for various completed problems. So, therefore this problem that we have discussed in today's class has a significant impact on various other results that will obtain in the future or will be used in a research project. So, in this lecture, we have outlined this exact solution, Navier's method is used, and it is characterized by double trigonometrical series.

And the method is illustrated by a simply supported rectangular plate with uniformly distributed loading and also partially uniformly distributed loading. The convergence of the series has been tested for deflection and bending moment. And it is concluded that for deflection, the convergence is repeated in the case of Navier's method. Only first term gives reasonably accurate results.

The corner reaction force along simply supported as is also discussed, and lastly, an exercise, a self-assessment problem based on Navier's method, is given for solution. So, you can solve the problem and compare the results with the results that have been given in the slides. Thank you very much.