Plates and Shells

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Lecture-04
Exercises on the Plate Bending Theory

Hello everyone, so, today I am delivering my lecture on module 1 on the course plates and shell. Now, in the last few lectures I have given you some basic concepts of the elasticity, three dimensional elasticity and from there I reduced it to a two dimensional problem, which will be used for plate analysis. Then I told you about these assumptions in the theory of plate in bending specially for thin plate, which is popularly known as Kirchhoff-Love hypothesis for thin plate bending.

Where the assumptions I have mentioned very important assumption is that the deflection is small, it is limited to one fifth of the thickness of the plate, then I have discussed another important assumption that normal to the plane before bending remains normal after bending. That means length of the normal does not change after bending. So, this indicates that ε_z in this vertical direction where the z axis is aligned in the vertical direction is 0.

Then I have discussed this stress strain relationship and how the stresses are expressed in terms of vertical deflection, then the stress resultants, bending moment, shear force and edge shear for the plate I have derived. After that I have gone to the equilibrium equations of the plate, there I have taken a small element of a plate and I have shown the free body diagram of this element under the action of external loading, distributed loading and internal forces that is developed.

All internal force quantities that is the bending moment, shear force, twisting moment etcetera are expressed in case of plate as unit of per unit length. Thereafter, I told you the characteristics of plate equations and how it can be solved in special cases. One of the rare cases I have discussed is the circular plate with clamped edges and general technique for the solution of the plate problem that involves a homogeneous solution and a particular integral.

So, up to that we have discussed in boundary condition that we encounter in a plate of straight edges I have mentioned and in course of that discussion, I have also covered that these slope and curvature of the plate in any direction, which is at an angle of inclination α with the x axis, then we have derived the differential operator in the new direction, normal direction and tangential direction.

With that knowledge, I think we will be able to apply to some problems that today I have brought for you. Today I intend to solve some problem based on the lectures that I have delivered in the first modules. So, what are the problems? Let us see.

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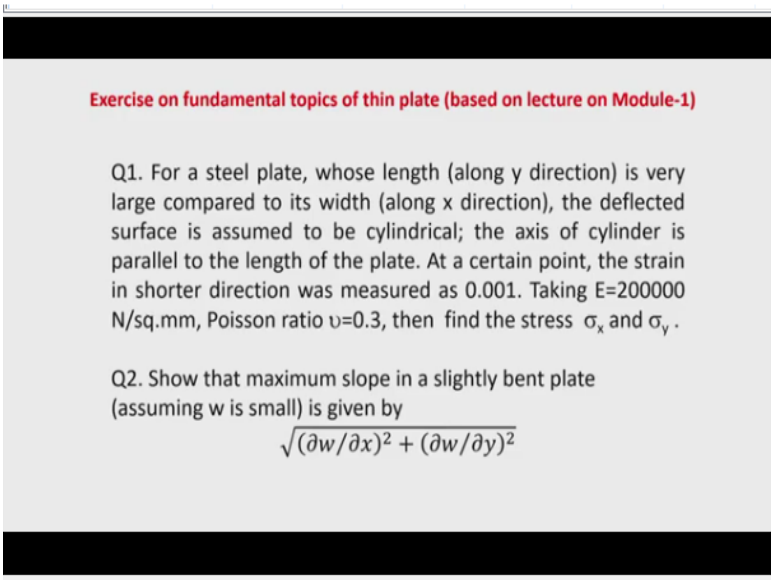

Five problems I have brought for you. The first problem is let us read this problem. For a steel plate, whose length along y direction is very large compared to its width along the x direction, the deflected surface in that case is assumed to be a cylindrical, the axis of the cylinder is parallel to the length of the plate. At a certain point, the strain in shorter direction was measured as 0.001.

Now, taking modulus of elasticity Young's modulus of elasticity as 2×10^5 N/mm^2, Poisson ratio as 0.3 we required to find the normal stresses, σ_x and σ_y. So, that is the first problem, second problem let us see. The slope in x direction and y directions are known as the first derivative of the deflected surface with respect to x and with respect to y, respectively.

Now, I want to prove or whether you will be able to show this with the knowledge that you have developed. The maximum slope in a slightly bent plate is given by $\sqrt{(\partial w/\partial x)^2 + (\partial w/\partial y)^2}$. So, that is 2 problems here.

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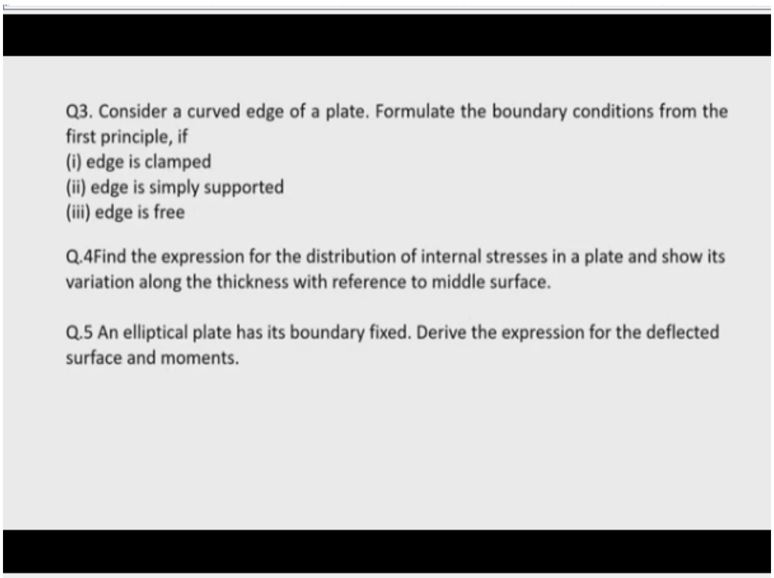

Next problem let us see, we have derived the boundary condition for straight edges. Now, let me derive the boundary condition for curvilinear edge. A plate may have a curvilinear edge say all the edges may not be perfectly straight line. For example, you have a quarter of a circle required for any purpose for resisting the flow load or live load, then in that case the two edges are straight, but one edges are curve.

So, we want to formulate the boundary condition along the curved edges for 3 different edge conditions that is one is clamped, another is simply supported and another is free. And I want to show the formulation from the first principle, if you can derive the equation from the first principle then we can substitute our known expression and we can get the full boundary equations after simplification.

Then question number 4, find the expression for the distribution of internal stresses in a plate and show its variation along the thickness with reference to the middle surface. Now, here the internal stresses in thin plate is σ_x, σ_y, τ_{xy}, which other stresses that is σ_z then your τ_{xz} and τ_{yz} we neglected, but in general they also exist. So, their influence is small because compared to others, but they are also existing.

So, we will try to find the expression for all stress components and when the thin plate example come then we can show that these components are negligible compared to the free component of stresses σ_x, σ_y and τ_{xy}. Then we will go for one problem that we want to solve exactly exact solution we want to find and it is possible because of the boundary condition taken as fixed.

So, for that condition, let us derive the expression for deflected surface and one of the deflected surfaces found you can find the expression for a moment. So, let us solve this problem one by one and if some steps are not completed, you can complete it.

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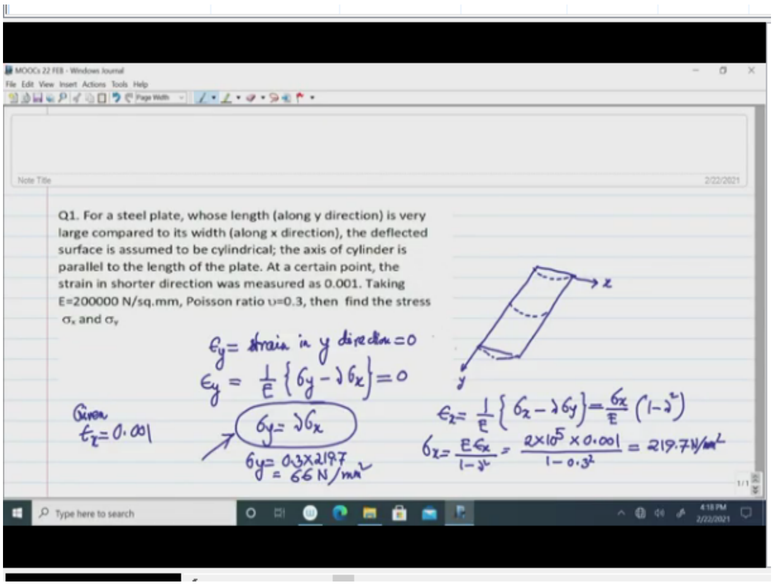

Now, let us see the first problem that I have told you, a steel plate whose length in y direction is very large compared to it is width. So, that means, if I have a plate like that, say this is the x axis and this is our y axis. So, along the y direction the length is very large such that the plate deforms in the form of a cylinder. So, deflected surface of the plate is a cylinder. Now, due to this assumption, we can say that ε_y that is the strain in y direction is 0.

But strain in y direction, we know that from expression is $\frac{1}{E}\{\sigma_y - v\sigma_x\}$. In this equation, σ_y is the stress along the y direction normal stress and σ_x is the stress along the x direction, v is the Poisson ratio. Because of this condition this is equal to 0. So, that means we get $\sigma_y = \nu \times \sigma_x$. So, this equation we will use later on. Now, we are aiming to find the σ_x and σ_y.

Now, let us write the expression for ε_x. $\varepsilon_x = 1/E\ \{\sigma_x - \nu\ \sigma_y\}$. Now, since σ_y is given as $\nu \times \sigma_x$ this equation can be simplified as $\sigma_x/E \times (1 - \nu^2)$. Based on that because our major quantity is ε_x given the $\varepsilon_x = 0.001$. Now, this is given. So, we are aiming to find σ_x. So, σ_x can be found out $\sigma_x = (E \times \varepsilon_x)/(1 - \nu^2)$.
So, substituting this value that is E is 2×10^5 ε_x is 0.001 and ν is 0.3. So, $1 - 0.3^2$, after calculation these becomes 219.7 Newton per millimeter square. So, once σ_x is found, then σ_y from this relation can be easily obtained $\sigma_y = 0.3 \times 219.7$ and this is equal to 66 Newton per millimeter square. So, this is the answer of this problem based on the stress strain relationship and for the condition that in one direction strain is negligible. So, by that way we could get the result because of this assumption. Now, let us go to next problem.

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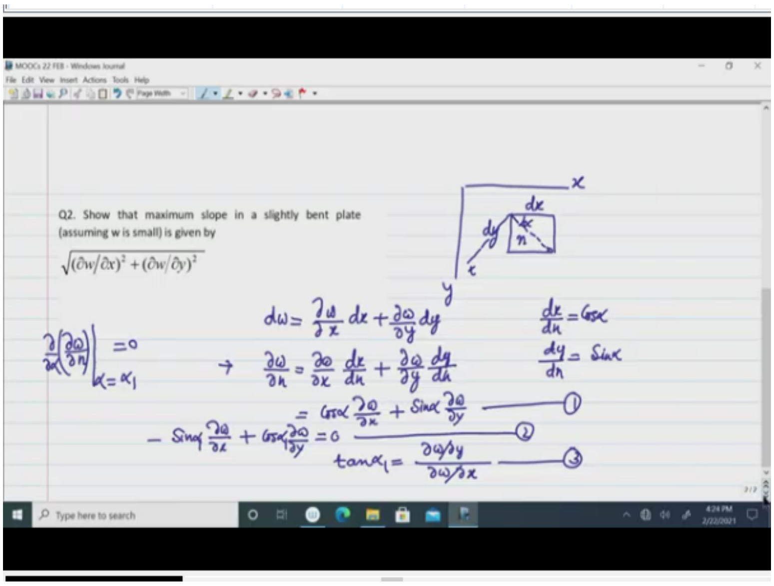

Next problem let us see, we have this regarding slope. So, the problem is that the maximum slope in a slightly bent plate assuming w is small is given by $\sqrt{(\partial w/\partial x)^2 + (\partial w/\partial y)^2}$. Now, let us show the coordinate axis x, this is y. So, we know the slope, in x and y direction. Now, for an element if I take an element of plate of length dx and width dy. So, in any other normal direction say this direction is denoted by n direction.

And other direction perpendicular that is *t* direction tangential direction and say normal is making an angle α with the x axis. So, we expressed this deflection in this *dw* is the contribution of this plus. So, we have shown this as in our last classes that means, slope in x direction which is increased by the length ∂x and slope in y direction increased by the length ∂y. From that we got this $\frac{\partial w}{\partial n} = \frac{\partial w}{\partial x}\frac{dx}{dn} + \frac{\partial w}{\partial y}\frac{dy}{dn}$.

Now, from this triangle we can see that *dx*/*dn* is cos(α) and *dy*/*dn* sine(α). So, accordingly I can write this $\cos\alpha \frac{\partial w}{\partial x} + \sin\alpha \frac{\partial w}{\partial y}$. So, this is the expression for slope in any direction. For the condition that the slope in that direction is maximum, we have to use the maximization principle of calculus that this is equate to 0 at some angle α equal to α_1 with taking the derivative with respect to α.

So, this equation is say 1, if I take the partial derivative of that equation with respect to α we will get this $-\sin\alpha \frac{\partial w}{\partial x} + \cos\alpha \frac{\partial w}{\partial y} = 0$ and α_1 will substitute because I have taken a particular angle $\alpha = \alpha_1$ the slope will be maximum. From this equation number 2 we get that tan(α_1), tan(α_1) will be your equal to this $\frac{\partial w/\partial y}{\partial w/\partial x}$.

So, this equation we have obtained that is the direction of maximum slope is given by this tan(α_1) that is the slope in a direction α_1 and we got this as $\frac{\partial w/\partial y}{\partial w/\partial x}$. Now, let us see the expression of maximum slope. Now, if I substitute the angle α in this expression equation number 1 then we will able to get for that particular direction the slope which will be maximum because we are imposing the condition.

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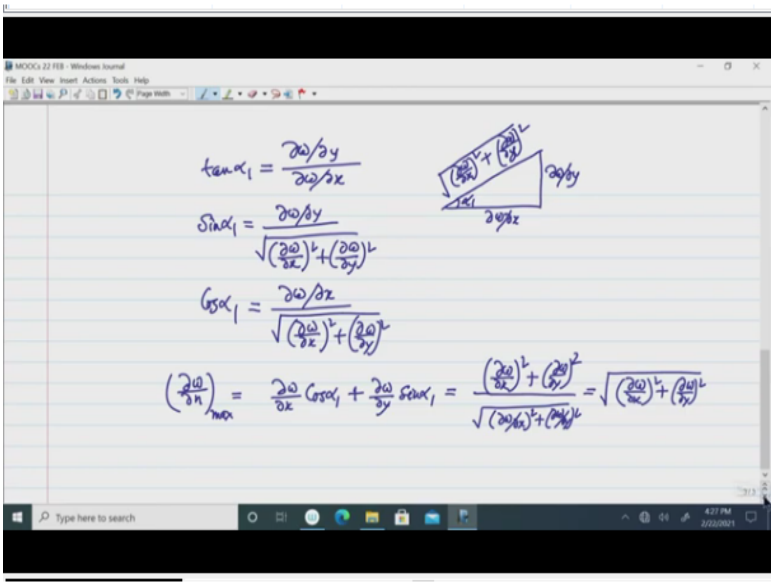

Now, before substituting this expression, let us first express the tan(α_1) in terms of sine and cosine. So, tan(α_1) we obtain there $\frac{\partial w/\partial y}{\partial w/\partial x}$. So, suppose this is right angular triangle and we have this angle is say α_1. So, this is say for example, $\partial w/\partial y$ this side and this side is $\partial w/\partial x$. So, naturally hypotenuse of this triangle is $\sqrt{(\partial w/\partial x)^2 + (\partial w/\partial y)^2}$.

So, we get this sine(α) as sine(α_1), sine(α_1) will be $\frac{\partial w/\partial y}{\sqrt{(\partial w/\partial x)^2 + (\partial w/\partial y)^2}}$. Similarly, cos($\alpha_1$) we will get $\frac{\partial w/\partial x}{\sqrt{(\partial w/\partial x)^2 + (\partial w/\partial y)^2}}$. So, this is the thing that we obtain for this sine(α_1) and cos(α_1).

Now, let us put in this expression. The maximum slope is given by this for the particular angle we have found that it is given as $\frac{\partial w}{\partial x}\cos\alpha_1 + \frac{\partial w}{\partial y}\sin\alpha_1$. Now, substituting these expressions that I have got here for cos(α_1) and sine(α_1) we can see that the denominator of this expression is this; numerator of this expression will be this.

So, one can get from this expression very easily that maximum slope is nothing but root over. So, this is proved. So, whatever we have given in the question it is now proved. So, we get the maximum slope in this slightly bent plate is under root of the square of the slope in x direction plus square of the slope in y direction. Let us go to the third problem.

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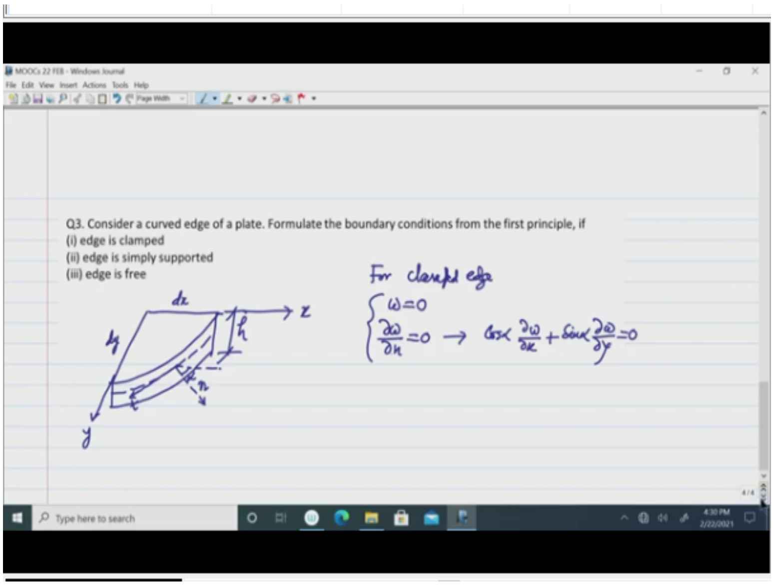

Now, third problem we are having about the boundary condition. This is regarding the boundary condition and we will see that curved boundary condition having the different expression, different expression because of this curvilinear coordinate that is necessary to be involved here. So, here what do we mean that actually we have a plate say a portion of the plate certain plate and this is curved, this edge is curved.

Thickness of the plate is h as usual, as usual this is the middle plane and say for example, this is x axis and this is your y axis. Now, this length is taken as elemental length *dx* and *dy*, this length is the arc length. Now, we define the direction in which we want the derivative or the quantities of moment and 2 directions, one is normal direction that is normal to the edge and another is tangential direction *t*.

And this normal makes an angle for example with the x axis as α. So, that things are given to us and now we go for deriving the quantities for this plate if the edge is clamped, if the edge is simply supported and if the edge is free. For clamped edge there is no difficulty, clamped edge we have w = 0 and $\partial w/\partial n = 0$. Now, in terms of this the Cartesian derivative you can write it $\cos\alpha \frac{\partial w}{\partial x} + \sin\alpha \frac{\partial w}{\partial y} = 0$. So, these 2 equations represent the boundaries along the curved edge if it is clamped. Now, let us go for the simply supported edge.

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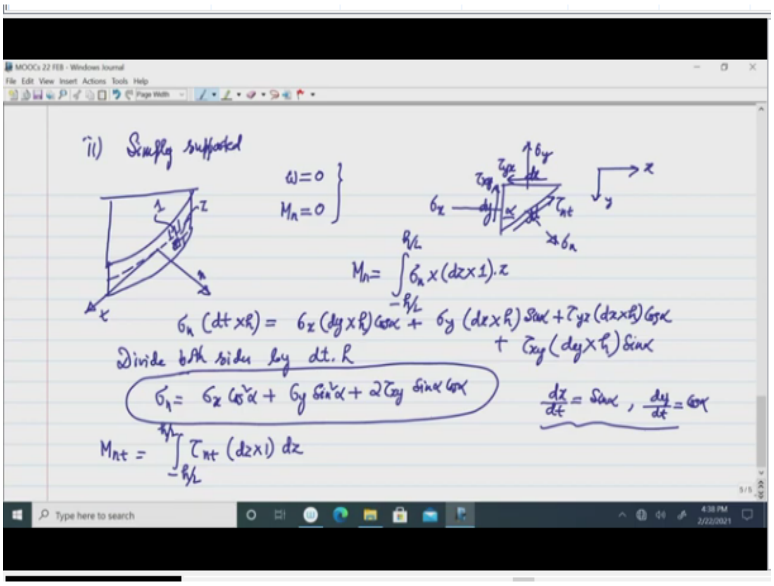

Second is simply supported. In the simply supported edge, for example, again I am drawing this curved edge. This is curved edge; this is the middle plane. So, this is normal direction and this is your tangential direction as usual. Now, it is given that we have to obtain the boundary condition from this first principle. Now, if you see that along the normal direction the normal stress is σ_n.

So, if I want to calculate because in simply supported condition the moment and deflection is 0. So, one condition at the boundary at the curvilinear boundary is w = 0 that is easily satisfied, but second condition is that $M_n = 0$, bending moment along the normal direction that has to be 0. Now, let us obtain the expression for the new moment M_n from the first principle. Now how M_n is generated?

M_n is generated M_n and M_{nt}, M_{nt} of course, it is not necessary here, but we will still evaluate this. Now consider a small element, small elements say this angle is α, this is dx, this is dy and say this slanting side is dt, this side is dt. Now, we have a small element here situated at any point on the plate at a distance z from the middle surface. Then we can write the moment of this element force on this element is σ_n × area of the element $dz \times 1$.

If I take the unit width, then this is the force. So, force into a distance z and if it is integrated from – h/2 to + h/2, we will get the moment along the normal direction. Now we have to obtain this σ_n from the first principle. Now, if you see this condition then along the h which is if this is the direction x, this is direction y, along the h where the x axis is normal.

So, this is your σ_x and along the y axis is normal, this is σ_y. Then you have shear stress in the 2 phases τ_{xy} and another shear stress along these phases τ_{yx} or τ_{xy} because these are same. So, let me write τ_{yx}. So, this is any element on the curve boundary because *dt* is for a small element the curve length and the straight length is same.

So, I have taken dt is approximately equal to your curve length. So, now, I want to obtain the expression for σ_x and σ_y when they are resolved in the normal direction. So, expression of normal stresses along the normal to the inclined plane, tangential plane is σ_n and σ_n can be found by equating the sum of the forces to 0 along the normal direction.

So, that is the principal. So, and of course, the shear stress will also act shear stress say here tau nt. So, these 2 stresses will generate the moment along the normal direction and twisting moment with reference to *n* and *t* direction. Now, let us find out what is σ_n? Now, if you see, if you resolve the forces along the normal direction, what do we get $\sigma_n \times dt$ is the length of this slanting side into h, h is the thickness of the plate.

So, this is the force in the normal direction and this is equal to the component of the force on the *x* and *y* direction that we are showing here in the free body diagram of the element. So, I can write $\sigma_x\ dy$ × thickness h and after resolving it normal direction it will have the term $\cos(\alpha)$. Similarly, σ_y and dx × h and if I resolve to the normal direction it will have $\sin(\alpha)$.

Then we have τ_{yx}, τ_{yx} is acting on the face *dx*, so dx × h is the area and when I resolve it to the normal direction it will be $\cos(\alpha)$, + τ_{yx}, it is acting on the face *dy*. So, dy × h into $\sin(\alpha)$. Dividing both sides by your this dt × h, we will be able to get this very interesting expression we also know this $\sigma_n = \sigma_x \cos^2(\alpha) + \sigma_y \sin^2(\alpha) + 2\ \tau_{xy} \sin(\alpha) \cos(\alpha)$.

Because we know from this triangle there dx/dt is $\sin(\alpha)$ and $dy/dt = \cos(\alpha)$. So, making use of these 2 relationships in this expression we arrived at the normal stress component. Similarly, if I want to find out the tangential stress component τ_{nt}, which generates the normal the twisting moment M_{nt}. M_{nt} generated by τ_{nt} and similarly here $dz \times 1$, if I take a unit width of the plate and integrated from – h/2 to + h/2. We need τ_{nt}. τ_{nt} is found after resolving the forces.

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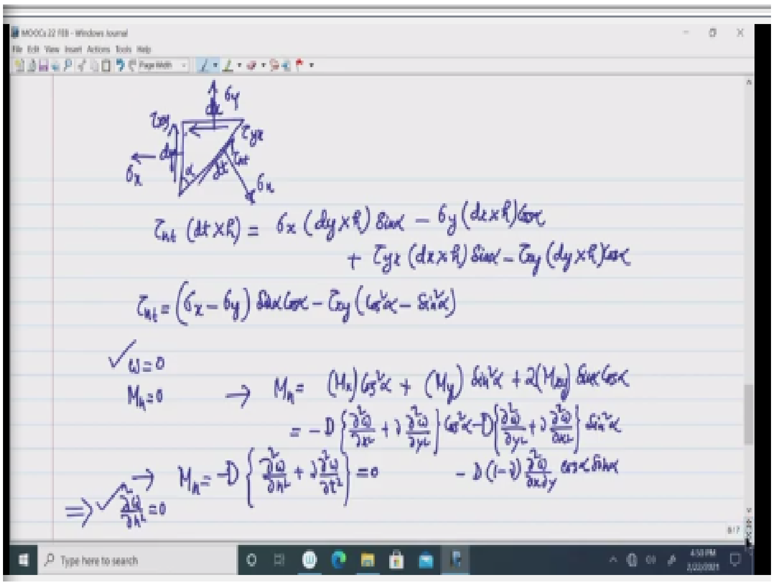

Say this is my triangular element that I have shown you, here it is α and this is dt, this is dx, this is dy. So, this stress is σ_x this normal stress is σ_y and there is shear stress τ_{yx} and there is also shear stress τ_{xy}. So, resolving tangentially along say tangent direction, this is σ_n and this is tau nt. So, τ_{nt} is found after making the equilibrium equations for all the forces that are shown in the tangential direction.

So, if I resolve the forces in the tangential direction, then we have $\tau_{nt} \times dt \times$ h, there is the forces in this tangential direction on this inclined face is equal to $\sigma_x\ dy \times$ h $\sin(\alpha)$ then we have this $-\ \sigma_y\ dx \times$ h $\cos(\alpha) + \tau_{xy}\ dx \times$ h $\sin(\alpha)$ and - $\tau_{xy}\ dy \times$ h $\cos(\alpha)$. Now divide both sides by $dt \times$ h that we have done earlier. So, ultimately we will get τ_{nt} after simplification, $(\sigma_x - \sigma_y)$ $\sin(\alpha)\cos(\alpha)$ - τ_{xy} $\{\cos^2(\alpha) - \sin^2(\alpha)\}$.

So, the expression for τ_{nt} is now known. So, based on that expression what we describe in the previous slide we can obtain this M_{nt} just by substituting τ_{nt} that we have obtained here in this equation. Now the boundary condition along the simply supported edges, this is 0 and M_n = 0. M_n now is obtained after substituting the expression for σ_x σ_n, that we have obtained in the expression of this moment there you substitute here, you will be able to get M_n.

So, expression for M_n is obtained as very simple term it is obtainable (M_x) × $\cos^2(\alpha)$ + (M_y) × $\sin^2(\alpha)$ + 2 (M_{xy}) $\sin(\alpha)$ $\cos(\alpha)$, why I wrote this term inside the bracket? Because I know the expression for M_x, M_y and M_{xy} in terms of derivative. So, if I want to write this in terms of derivative instead of keeping this as M_x I can write it say -D and this is $\cos^2(\alpha)$ + and again minus will be there -D then for M_y I can write and then this M_{xy}, M_{xy} expression also we know it.

That is -D then 1 - ν then $\partial^2 w/\partial x\partial y$, these two will get cancelled. Now, after rearranging this you will find that this gives you a very interesting expression M_n equal to we will get -D and you will get $\partial^2 w/\partial n^2$. So, that is the Cartesian derivative $\partial^2 w/\partial x^2 \cos^2(\alpha)$. And then $\partial^2 w/\partial y^2 \sin^2(\alpha)$.

And then $\{\partial^2 w/\partial x\partial y\}\cos(\alpha)\sin(\alpha)$ that is there in this term. You will get this is transformable in terms of the second differentiation in the normal direction. That operator I have shown you in the earlier classes you can see these notes. Then with another terms that is once you get the second differential operator in the normal direction, second differential operator in the Cartesian reaction is obtained only by changing the angle α.

So, instead of α you put 90 + α. So, you will get operator in the tangential direction. So, you will be able to find this expression is simply reduced to like that, so M_n is having very simple expression like that equal to 0. Now since the plate is supported along the boundary, naturally there cannot be deflection in the boundary. So, ultimately with the same reasoning that we have done earlier in the straight edges, we have for simply supported edges along the curvilinear boundary, is this one condition.

And first condition is this, second condition is this. So, this is the condition along the simply supported edges**.** Now when it comes to the free edge then difficulty arises.

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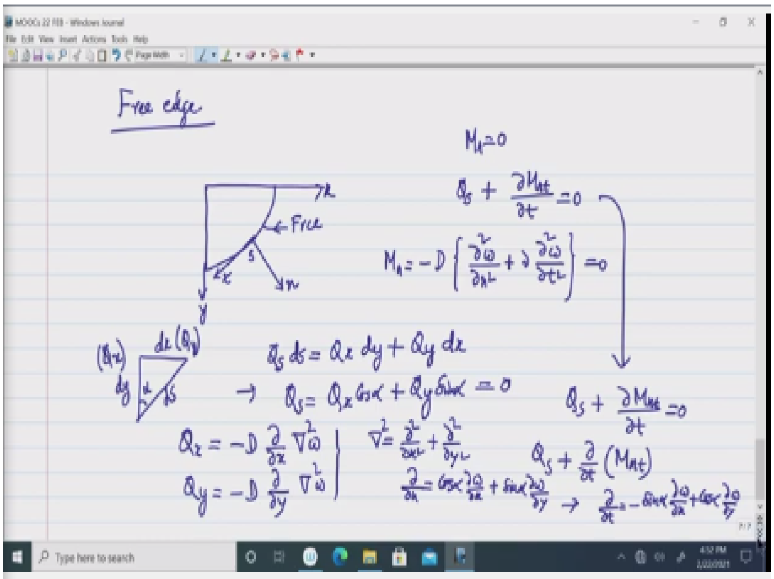

Now let us see curvilinear free edge**.** Curvilinear free edge has your different obligations. Because of the fact that the free edge bending moment is 0, so this edge is free and say this is my x axis and this is the y axis. So, along the free edge you have this if this is taken as an arc length *s*. So, Q_s there is the shear force along this *s* direction plus this twisting moment and it is derivative with respect to tangential direction.

So, we have at any point on the curve boundary, we have 2 directions along the tangent *t* and normal to the tangent *n*. So, this is equal to 0 is one condition and another condition is $M_n = 0$. So, first condition can be easily substituted that we have earlier obtained but second condition we want to derive Q_s and M_{nt} is known and the $\partial/\partial t$ operator has to be found out.

So, in this connection the M_n is that we have found earlier -D. So, this is M_n and this is equal to 0 because it is not supported. So, I cannot take this as one curvature is 0**,** that is not possible alright, so we have taken this is $M_n = 0$. Second equation is coming from this $Q_s + \partial M_{nt}/\partial t = 0$. So, that is the edge shear Kirchhoff edge shear along the curved edge. So, that means it is not necessary to consider the twisting moment and shear force along the curved edges to be 0 separately 2 effects can be combined to give the single equation.

Now if I see an element, this is *dx*, this is *dy*, now here I am telling it say *ds* whatever maybe and this is angle, is α. So, for the equilibrium if the forces here, these shear forces here are Q_y along this edge *dx*, edge along the *dy* shear force is Q_x. And these are taken upwards, so I can write $Q_s\, ds = Q_x\, dy + Q_y\, dx$, that I can write. Now dividing both sides by *ds* we now get $Q_s = Q_x\, dy/ds + Q_y\, dx/ds$, dy/ds we can write $\cos(\alpha)$ and dx/ds we can write as $\sin(\alpha)$.

So, this is the expression for Q_s and it is equated to be 0 actually. Now expression for Q_s, Q_x and Q_y unknown in terms of Cartesian derivative, where we have derived earlier $Q_x = -D\frac{\partial}{\partial x}\nabla^2 w$ where ∇^2 is the Laplacian operator that is $\partial^2/\partial x^2 + \partial^2/\partial y^2$. So, this is Q_x similarly $Q_y = -D\frac{\partial}{\partial y}\nabla^2 w$.

So, these two equation can be substituted here to find the Q_s. Now question coming here is $\frac{\partial}{\partial t}(M_{nt})$. Whereas M_{nt} expression is already obtained, τ_{nt} is obtained, so naturally you can obtain M_{nt} alright. Once you know the τ_{nt} you can obtain the M_{nt} by simply integrating across the thickness. So, that is possible and based on that we can write this the second equation that is this equation.

Only knowing these operator, so $Q_s + \partial M_{nt}/\partial t = 0$, Q_s we have substituted whatever we got here and $\frac{\partial}{\partial t}(M_{nt})$. Now, $\partial/\partial t$ is operator which is actually found from this we know that $\partial/\partial n = \cos\alpha\, \partial w/\partial x + \sin\alpha\, \partial w/\partial y$. Now to find $\partial/\partial t$, we only substitute, we only change the α. So, because the normal direction and tangential direction are mutually perpendicular.

So, if I substitute $\alpha = \pi/2 + \alpha$ or 90 degrees + α, then I get simply $\partial/\partial t$ as this operator $-\sin\alpha\, \partial w/\partial x + \cos\alpha\, \partial w/\partial y$. So, substituting M_{nt} that we have obtained from τ_{nt} after integrating across the thickness and then differentiating it. That means, you use this operator, we will be able to get the expression for the second term and Q_s term is already given here $Q_s = Q_x \cos(\alpha) + Q_y \sin(\alpha)$.

And Q_x and Q_y expressions are written here. So, you have to substitute and write down the full expression for the curved boundary.

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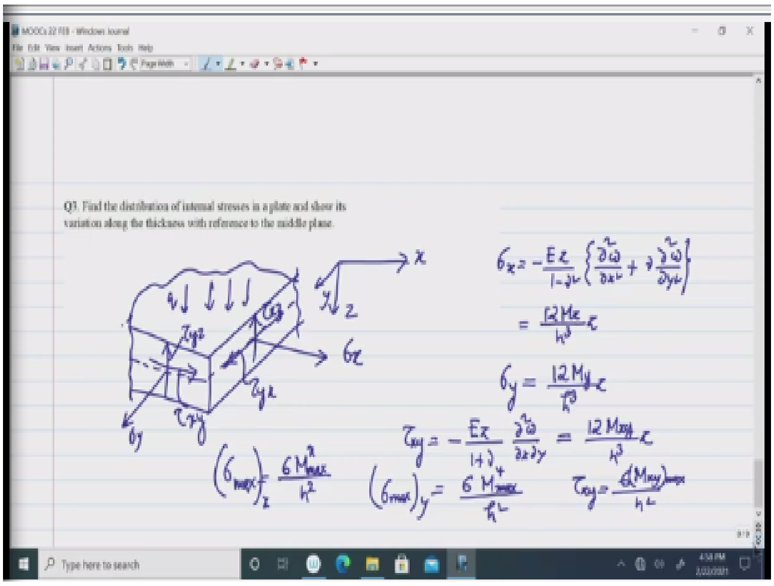

So, let us now go to the fourth problem, fourth problem we will go here, we find out the picture. So, 4th problem is to find the distribution of internal stresses in a plate and show its variation along thickness with reference to the middle surface. So, we have, say a rectangular plate. So, I have shown a part of the plate subjected to load q.

And suppose this is my Cartesian axis x, y and z. So, along the middle surface let me show the middle surface, I am showing the stresses this along the x direction normal stresses along the y direction. And then τ_{xy}, τ_{yx} here, τ_{yx} and here τ_{xy}, other stresses are shear stress though we neglected, let us find the expression also. And then when the τ_{xz}, τ_{yz} are considered this is not a thin plate problem but let us find this from the first principle.

Now σ_x is known to us, it is equal to $-E \times z/(1 - \nu^2) \times$ curvature this, that we have derived earlier. Now we also know the expression for M_x, so M_x is -D and this curvature. So, this term can be written as $\{12\ M_x/h^3\} \times z$. Now why it is $12/\ h^3$? Because in the denominator

$h^3/12$, and if I take a unit width of the plate, $1 \times h^3/12$ is nothing but moment of inertia of the plate of unit thickness.

So, you can see it is similar with the beam, so that means the stresses varies linearly, if I measure the stresses from the middle plane, then it varies linearly with z along the depth. So, h thickness is known, and the quantity M_x that will be found out and then you can find the variation of M_x. Similarly, we can find the variation of σ_y, variation of σ_y is this, τ_{xy} you can similarly find τ_{xy} is $\frac{-Ez}{1+\nu}\frac{\partial^2 w}{\partial x \partial y}$.

Now what do we know that M_{xy} is nothing but -D $(1 - \nu)$ and twisting curvature. So, based on that we can simplify it and bring it in terms of M_{xy}, twisting moment. So, twisting moment M_{xy} is responsible to produce the shear stress τ_{xy}. So, 3 expressions are known σ_x, σ_y and τ_{xy} and τ_{yx} is similar. And you can see here that all the three stresses that we assign for thin plate problem is linearly varying along the thickness.

So, maximum will be when the z = + - h/2. So, maximum value can be found out, σ_{max} = $6M_{max}/h^2$. Similarly, σ, this is say for example, this is x similarly σ_{max}, y is also found as, this is also bending moment in the x direction. Similarly, this is bending moment in y direction, I am writing as a subscript to distinguish between the bending moment along x and y direction, and this is.

Maximum τ_{xy} is again given as at z = h/2, so $6(M_{xy})_{max}/h^2$. So, these are the quantities, so readily we can know the variation, let us find the variation of other stresses.

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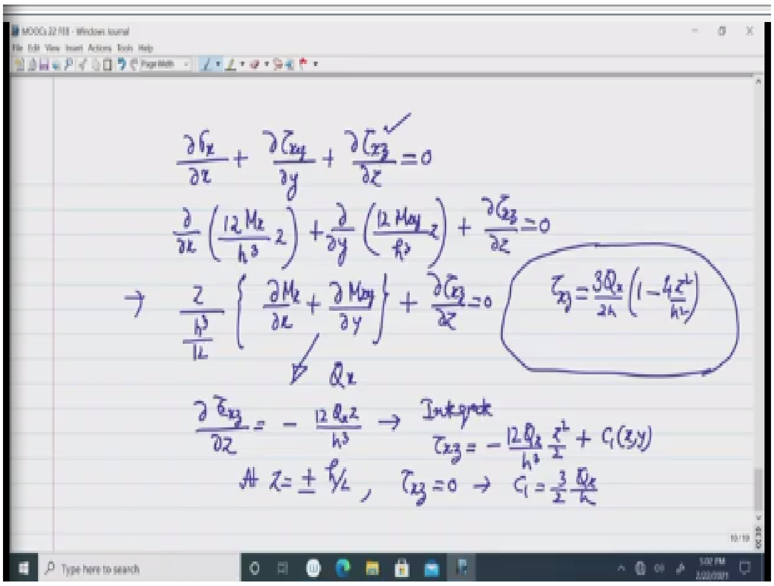

Variation of other stresses, we have to find out from equilibrium equation. So, let us take the equilibrium equation in *x* direction, in general 3D elasticity. So, we have this, in absence of body forces this is written as like that. Now whatever we get in σ_x τ_{xy} and etcetera we will substitute. Our aim is to find τ_{xz}, so we substitute this, this is we found it as $\{12\ M_x/h^3\} \times z$, then here, we found it as $\{12\ M_{xy}/h^3\} \times z$.

So, this quantity remains now as it is. So, what we actually find it here from this equation? I can write as z, z I can take common and $12/h^3$ I can take common, so I can write like that, that represents moment of inertia of plate of unit thickness. So, there you can see that 2 things can be arranged like that $\frac{12z}{h^3}\left\{\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}\right\}+\frac{\partial \tau_{xz}}{\partial z}=0$. Now recall the equilibrium equation, that is nothing but our Q_x, if you recall this, this is equal to Q_x.

So, I can write like that. Now, I can write $\partial \tau_{xz}/\partial z = -12Q_x z/h^3$, integrate this equation will be able to obtain the τ_{xz}. So, after integrating $\tau_{xz} = \frac{-12Q_x}{h^3}\frac{z^2}{2} + C_1(x,y)$. Because you we are integrating with respect to z, so it will be a function of x and y. But we know at surface the τ_{xz} is 0, so at z = + - h/ 2 we get τ_{xz} = 0.

So, imposing this condition, we get C_1 = 3/2 Q_x/h. Hence, the expression for τ_{xz} now I am writing here, the final expression here I am writing $\tau_{xz} = \frac{3Q_x}{2h}\left(1 - \frac{4z^2}{h^2}\right)$. So, this is very known expression suppose at z = 0, we get the maximum shear stress, there is the vertical shear stress is 3/2 Q_x/h, that is shear force divided by 1.5 into nominal shear stress Q_x/h, that quantity is known. In a rectangular section, the maximum shear stress, the average shear stress, maximum shear stress is 1.5 times of the maximum shear stress. So, that the expression we have obtained for τ_{xz}.

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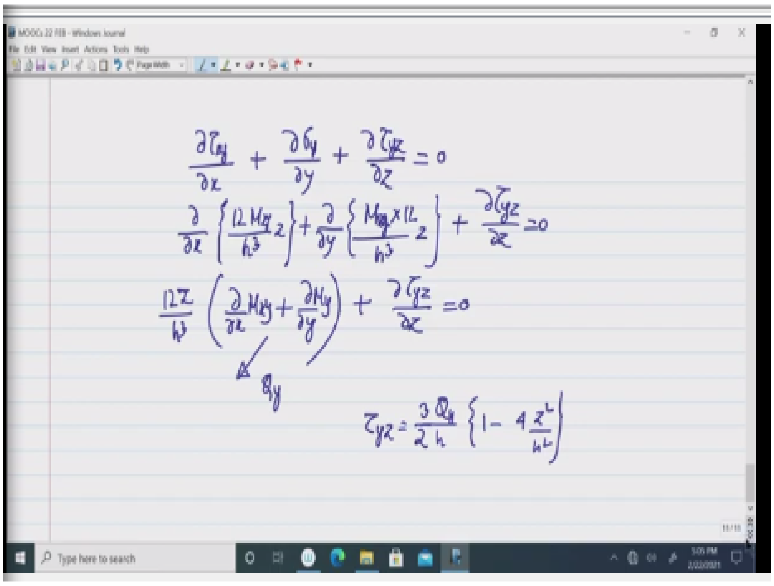

Now if I want to find for τ_{yz}, similarly I will carry out the integration of the second equation after substituting the value of M_y and this. So, second equilibrium equation is $\partial\tau_{xy}/\partial x + \partial\sigma_y/\partial y + \partial\tau_{yz}/\partial z = 0$. So here again, let us substitute this, this τ_{xy} we will substitute τ_{xy} is $12M_x/h^3 \times z$. And this σ_y we will substitute $M_y \times 12/h^3 \times z$ and this remain as it is.

So, similarly we can understand that, ultimately we get an equation that $\left\{\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}\right\}$ and 12 z/h^3 can be taken outside. And then we have $\partial\tau_{yz}/\partial z=0$, this quantity is known as Q_y. So, similarly after integrating we obtain this expression for τ_{yz} I am skipping the steps, because it is known to you from the previous integration.

So, therefore τ_{yz} can be written as finally is $\frac{3Q_y}{2h}\left(1-\frac{4z^2}{h^2}\right)$, so this expression we have written. So, all the expression we have written and then next expression is remained that is σ_z.

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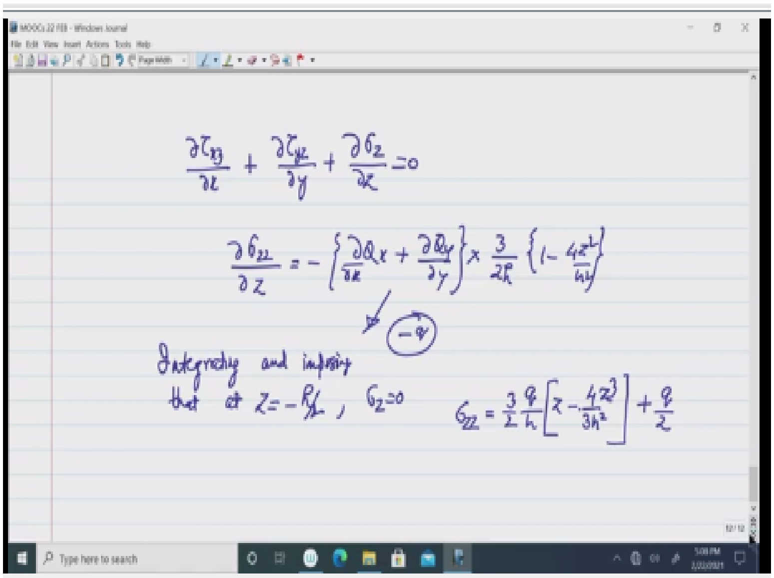

So, third equilibrium equation, let us take this third equilibrium equation $\partial\tau_{xz}/\partial x+\partial\tau_{yz}/\partial y+\partial\sigma_z/\partial z=0$. Substituting these value of τ_{xz} and τ_{yz} that we have obtained earlier. And we can integrate this quantity that is if I want to integrate then final expression that you will get. You will ultimately get an expression of σ after substituting this known value that we have obtained earlier; you will get a quantity like that.

And with that constant will be $\frac{3}{2h}\left(1-\frac{4z^2}{h^2}\right)$. So, all the value of τ_{xz} and τ_{yz} whatever I have obtained I have substituted here and I have taken isolated Q_x and Q_y. So, this is nothing but the expression that we have obtained is -q. So this expression is -q. So, then after integrating and imposing the condition and imposing at z = - h/2 when the bottom is free from any load -h/2, this σ_z is 0.

So based on that, we have obtained this final expression for σ_z as σ_{zz} or σ_z whatever you call 3/2 q/h, q is that transverse load on the plate, $z-4z^3/3h^2$. Ultimately after integration and substituting the constant, then you will get this is $4z^3/3h^2$, 3 will be there in the denominator. Then a constant term q/2 is coming.

So, you can see now at z = 0, that is at the mid plane, the σ_z the normal stress that is the crushing stress you can call it q/2. But at z = -h/2, it is 0 at z = +h/2, it will be q. So now I can show you the distribution fully for all stresses.

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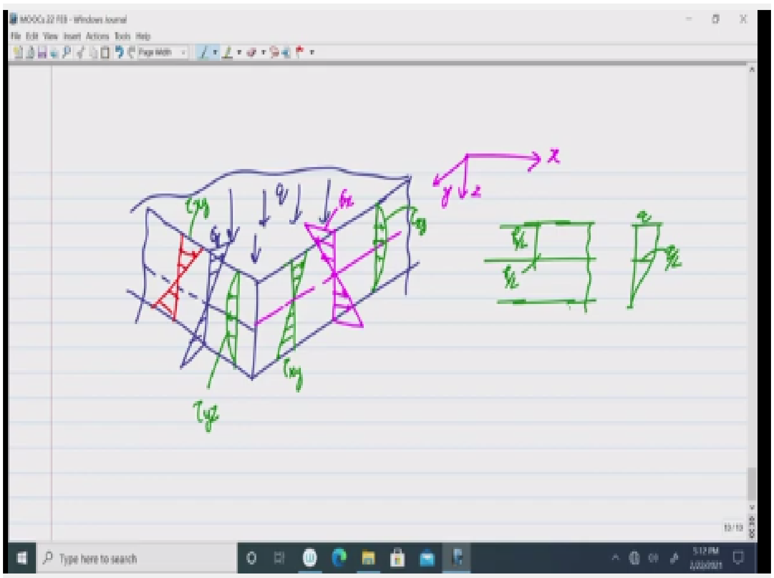

So, let me draw the plate. So, this is a part of a plate subjected to distributed load q and let me draw the stress distribution in a different colour. So, first this is the x axis, this is the y axis and this is z axis. So, we have this σ_x variation of normal stress along the x direction, if this is the neutral plane it will be like that linearly varying zero at this center and this is. So, this is the variation of σ_x, compressive at the top and this.

Then if I want to know the variation of τ_{xy}. Let me show the variation of τ_{xy} in a different colour, so that you are not confused, so this is the variation of τ_{xy}. Again this is your what is called it is linearly varying. So, on the other side also you can plot like that, there is no difficulty to plot this. On the other side, you can show this the variation of this along the middle plane, this is σ_y compressive.

And this if you want to see this variation of τ_{xy} it will be similar, but let me again show it, there is no harm in showing 2 times. So, you will get this variation as like that. But interestingly you will get the variation of these other 2 stresses that is similar to your beam shear stress that we get. So here, if I want to draw this τ_{xy}, it will be parabolic and maximum here. Similarly, if I want to draw the τ_{yz}, it will also be parabolic and maximum here.

So, y variation of τ_y and τ_{xy} or τ_{yx}, this is the variation of τ_{xz} and this is the variation of τ_{yz}. And for the σ_z, if you want to see the variation of σ_z because it will be clumsy, let us show it separately. This is the plate portion, this is the middle plane, this is h/2, this is h/2. So, here you will get the variation is like that. Here you will get this value q/2 and here it is q and here is 0. So, this shows the variation of this axis. Last problem let me see and I will give the hints and it can be solved. In the assignment problem, I will give the solution and you try the last problem, thank you.