Plates and Shells

Prof. Sudip Talukdar
Department of Civil Engineering
Indian Institute of Technology, Guwahati

Lecture - 36

Some Applications of Symmetrical Bending of Circular Cylindrical Shell

Hello everybody, today it is my third lecture of the module 12. In the earlier lecture, I introduced the simplified bending theory of the shell, specially, cylindrical shell. So, in this theory, we have seen that when the shell is subjected to radial load, then the bending moment is produced at the support. So, bending moment actually propagates as an edge disturbance and membrane solution becomes the particular integral.

So, that is the specialty of the simplified bending theory. Membrane theory has to be here used as a particular solution. So, we will show how the membrane theory and bending theory that is specially the bending theory of this cylindrical shell can be used in some practical examples to show that the bending moment produced by the radial deformation of the shell is like an edge disturbance.

So, it will gradually die out at a larger distance from the edge, but the membrane solution will exist and predominates the behaviour of the shell as a particular solution. So, that will be illustrated today by two examples that I have selected for this symmetrical bending of circular cylindrical shell.

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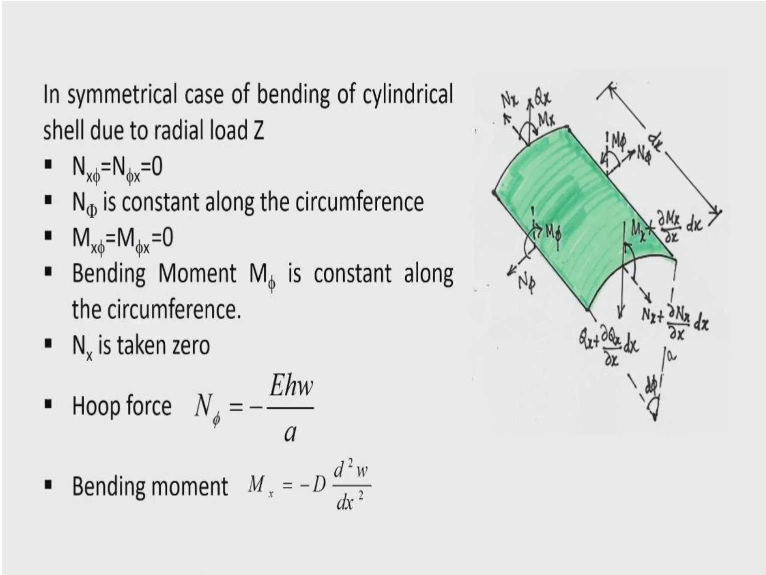

Now, let us see a shell element subjected to load that is radial to the surface normal to the surface. So, I mean that the component of the load along the x-axis and along the y-axis that is the tangential direction to the surface is neglected. So, only loads that are important to cause the stresses in the shell are the load normal to the surface, that is in the radial direction and it is an axisymmetrical type of loading.

So, by virtue of that, we assume that the membrane shearing forces $N_{x\phi} = N_{\phi x} = 0$ and N_{ϕ} is constant along the circumference. So, N_{ϕ} at any location you will find that N_{ϕ} is constant along the circumference. At any circumferential angle if N_{ϕ} is evaluated, you will get the same value. It is dependent only on the x, there is the longitudinal distance from the point of reference.

The bending moment M_{ϕ} that is in the tangential direction is constant along the circumference. So, that is another assumption in this theory that is the shell subjected to axisymmetrical loading and only the loading is in the radial direction the longest general membrane force that is N_x is taken 0. Hoop force $N_{\phi} =-\frac{Ehw}{a}$ where w is the deformation of the shell.

That is the radial deformation of the shell, which causes the change of diameter, a is the radius of the shell. Bending moment along any longer general distance measured from the reference point that is the origin is given by $M_x = - D\frac{d^2w}{dx^2}$. Now, here D is the flexural rigidity of the shell. And you can find the similarity with the beam bending equation. But in beam bending equation that EI was the flexural rigidity.

Here the flexural rigidity will contain these Poisson ratio term also. So, D is nothing but equal to $\frac{Eh^3}{12(1-\upsilon^2)}$. So, with that background we will try to apply this theory to solve 2 problems of interest, these 2 problems I will take up one by one. One is your cylindrical pipe subjected to uniform pressure, internal pressure and it is supported at the end that means the boundaries are specified.

So, we will see if the length of the pipe is very long, then the how the solution becomes and when the shell is of finite length, how the solution is obtained. So, after that, I will apply this theory to a cylindrical water tank that is very common in our application cylindrical container containing fluid or retaining fluid or any gaseous substance also. So, because of that, the internal pressures are developed and cylindrical wall, the wall is fixed at the base plate.

So, there will be some bending moment generated at the junction of the base plate and the cylindrical work wall. And that bending moment we will see how it varies along the length of the or along the height of the tank.

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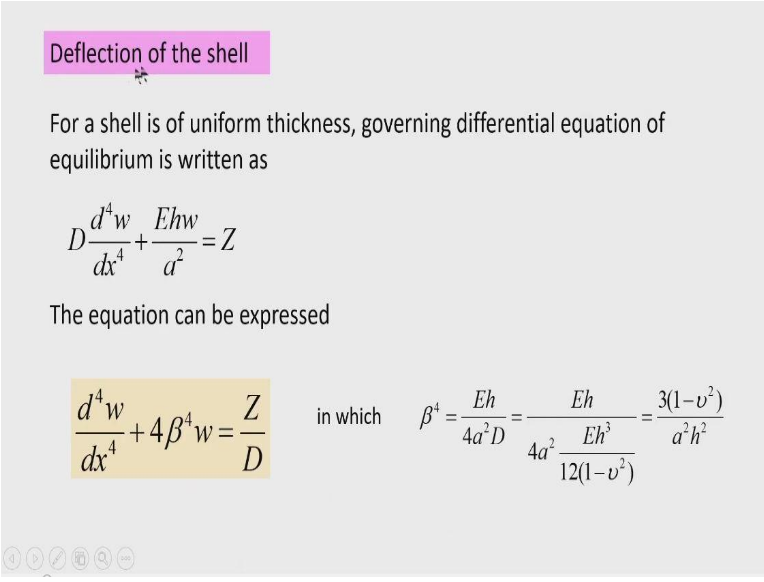

So, that two problems we will discuss today we take the deflection of the shell equation the governing equation for the deflection of the shell. If, the thickness of the shell is uniform, then governing differential equation for the bending of shell is given by $D\frac{d^4w}{dx^4} + \frac{Ehw}{a^2} = Z$, where Z is the radial load that is load acting in the radial direction normal to the surface.

Now, this equation can be expressed in a form that we can easily find out the roots of the characteristic equation. So, dividing both sides by d we now obtain $\frac{d^4w}{dx^4} + 4\beta^4 w = \frac{Z}{D}$. So, you can see here Z is the load that is acting along the normal to the surface and it is we call it a radial load.

And D is the flexural rigidity of the shell whereas β is a characteristic parameter that contains the material properties as well as the shell dimension. The β^4 that expression is given by $\frac{Eh}{4a^2D}$, a is the radius of the shell and D is the flexural rigidity. Now, substituting $D = \frac{Eh^3}{12(1-\upsilon^2)}$ and simplifying we get, $\beta^4 = \frac{3(1-\upsilon^2)}{a^2h^2}$.

So, once we find the β^4, β can be found out and the unit of β is the inverse of length. That is if the length is expressed in meter, then unit of β will be m^{-1}. So, with this differential equation, we shall now proceed to apply the solution in some problems that I mentioned shell have finite length supported at the ends and a cylindrical tank.

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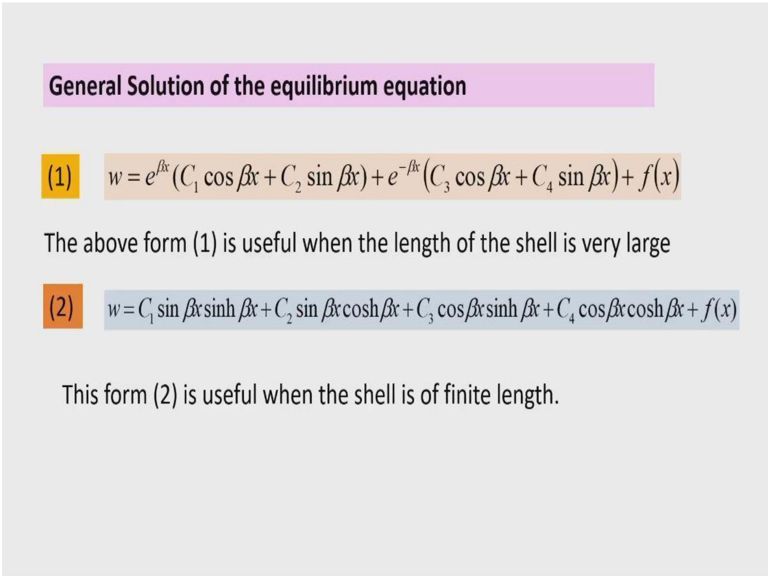

So, general solution of this equilibrium equation differential equation, have been found and it can be expressed in two forms. The one form contains the exponential term and the sinusoidal or cosine functions. So, if I see this form number one that I can see that $w = e^{\beta x}(C_1 \cos cos\, \beta\, x + C_2 \sin sin\, \beta\, x) + e^{-\beta x}\left(C_3 \cos cos\, \beta\, x + C_4 \sin sin\, \beta\, x\right) + f(x)$. Why it is another function $f(x)$ is added?

Because the effects is added due to the particular solution. The $f(x)$ is the particular solution of the differential equation and you can note that this differential equation is non-homogeneous equation and therefore, according to theory of linear differential equation, we have to find the solution of homogeneous part and then it has to be superimposed on the particular solution. If Z is 0 then no particular solution exists.

So, due to the forcing term present in the differential equation of equilibrium and forcing term is due to this pressure that is acting normal to the shell surface. And it is the solution of that particular integral is $f(x)$. Later on, we will prove that $f(x)$ is nothing but the membrane solution of the shell. Now, here one interesting thing you can note that $e^{\beta x}$, this term can amplify the deflection or can decrease the deflection if it is associated with minus sign.

Because β is always positive because it contains the material properties and the shell dimensions which are positive quantity. So, therefore, this term there is first expression in case the deflection with the increase of x, but here you can see that $e^{-\beta x}$, here βx as a damping factor. So, in that case, this expression actually tried to diminish or reduce the deflection of the shell with the increase of distance.

So, in a long shell, we have seen that if a load is applied at a particular section only then at a very large distance the effect of the load is negligible. So, therefore, we neglected the constant C_1 and C_2 for a very long shell, but shells of finite length has to be solved containing these four constants of integration. The four constants of integration can be found by applying the boundary condition at the ends of the shell.

So, one number one equation is useful when the length of the shell is very large if I drop the constant C_1 and C_2. Number 2 form that is $w = C_1 \sin sin\, \beta\, x \sinh sinh\, \beta\, x + C_2 \sin sin\, \beta\, x \cosh cosh\, \beta\, x + C_3 \cos cos\, \beta\, x \sinh sinh\, \beta\, x + C_4 \cos c$. These terms that you are noting here return in terms of trigonometric and hyperbolic function can be easily obtained.

When we substitute $e^{\beta x} = \frac{\cosh cosh\, \beta x + \sinh sinh\, \beta x}{2}$. And also, this term $e^{-\beta x} = \frac{\cosh cosh\, \beta x - \sinh sinh\, \beta x}{2}$. So, with the; use of this exponential term in the form of hyperbolic function and carrying out term by term multiplication and rearranging the constants.

We can write w in this form this is the second form which contains the combination of trigonometrical and hyperbolic functions. Here again, effects is the particular integral due to the load acting on the shell. This form two is useful when the shell is of finite length.

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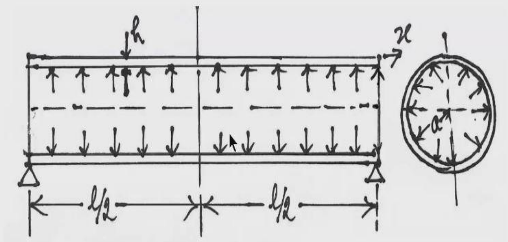
For shells of finite length and subject to uniform pressure with supported ends

$$w = C_1 \sin \beta x \sinh \beta x + C_2 \sin \beta x \cosh \beta x + C_3 \cos \beta x \sinh \beta x + C_4 \cos \beta x \cosh \beta x + w_p$$



Now, let us apply this theory or solutions that we have written here in the earlier slide for a shell of finite length and subjected to uniform pressure with supported ends. So, here I consider a shell of length l is a closed-cell and it is subjected to inform internal pressure the thickness of the shell is h and the length of the shell is l. So, for convenience, we take the centre of the shell as the origin.

There is a point of reference from which we measured the distance x, the cross-section of the shell is shown here and it is subjected to internal pressure that is acting radially. Now, if the origin is taken at the centre of the pipe or shell here that you are noting in this figure, then we can take advantage of the symmetrical terms in the expression of deflection. So, since the shell is symmetrical with respect to the centre of the shell in respect of loading geometrical parameters and support.

Then we can only assume that the deflection of the shell contain only the event functions. So, let us identify what are the even functions in this expression. Even function in this expression, you can see that all the terms are product of two functions, one is trigonometrical function and another is hyperbolic function. Now, let us examine the first function $\sin sin \beta x$ $\sinh sinh \beta x$ that is the product.

$\sin sin \beta x$ is an anti-symmetrical function $\sinh sinh \beta x$ is also anti symmetrical function. So, product of two anti-symmetrical functions is a symmetric function. Similarly, here you can see $\cos cos \beta x$ is a symmetrical function and $\cosh cosh \beta x$ is also symmetrical function. So, product of two symmetrical function is again symmetrical function, but if you examine these two terms $\sin sin \beta x$ is a anti-symmetrical function whereas, $\cosh cosh \beta x$ is symmetrical function.

So, product of these two is anti-symmetrical function. Similarly, this $\cos cos \beta x$ is symmetrical function and $\sinh sinh \beta x$ is anti-symmetrical function. So, we shall drop this constant C_2 and C_3 that means, we need not consider this term containing the coefficient C_2 C_3 because our problem is symmetrical and we take only the event function in the expression of deflection.

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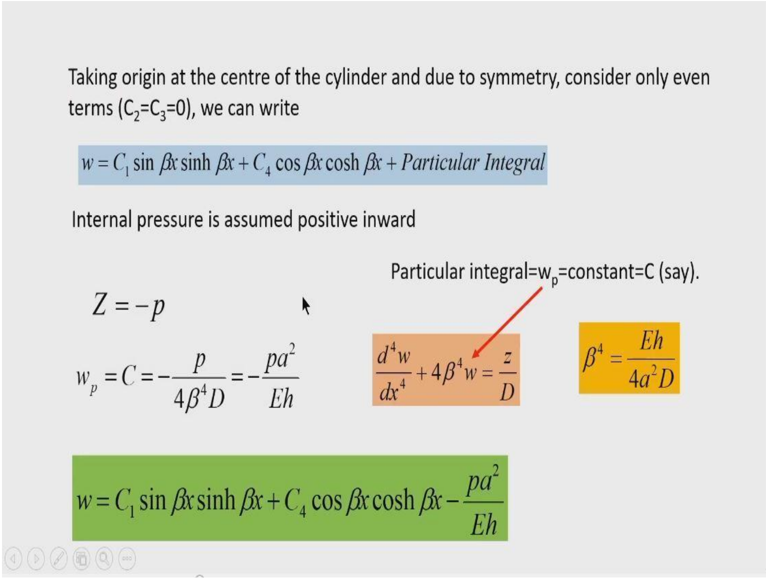

So, with that advantage, we can now reduce the deflection equation with two constants of integration. So, what are the two constants of integration? C_1 and C_4 that were associated with the sin function and sin hyperbolic function and cos function and cos hyperbolic function. So, both are now symmetrical function associated with a particular integral. Internal pressurization positive inward.

So, it is acting outward because the pressure that is internal pressure fluid pressure or whatever maybe it is acting outward so, therefore, it is taken with a negative sign. So, Z is taken as $-p$ now, since the forcing term is a constant term there is a constant here. So, we expect that particular solution is also constant. So, according to theory of undetermined coefficients in the linear differential equation, we now assume that particular integral is w_p is a constant C.

So, substituting C in this exhibition because C is a constant. So, when it is differentiated, you will get 0 here, but here you will get $4\beta^4 C$ and β^4 is nothing but $\frac{Eh}{4a^2 D}$. So, w_p that is the particular solution is nothing but C, now comes out as $C = -\frac{p}{4\beta^4 D}$, where D is the structural rigidity. Substituting the value of β^4 and D we now simplify $w_p = -\frac{pa^2}{Eh}$.

So, general solution of this differential equation of the bending of the cylindrical pipe now reduces to $w = C_1 \sin sin\, \beta x\ \sinh sinh\, \beta x\ + C_4 \cos cos\, \beta x\ \cosh cosh\, \beta x\ - \frac{pa^2}{Eh}$. This term the last term is due to the load in the radial direction that is the internal pressure p. Now, here you are seeing the two constants of integrations are appearing in this solution.

So, that have to be found out considering the condition at the ends, since the shell is a finite length and the boundaries are defined in the problem. We can now apply the boundary condition.

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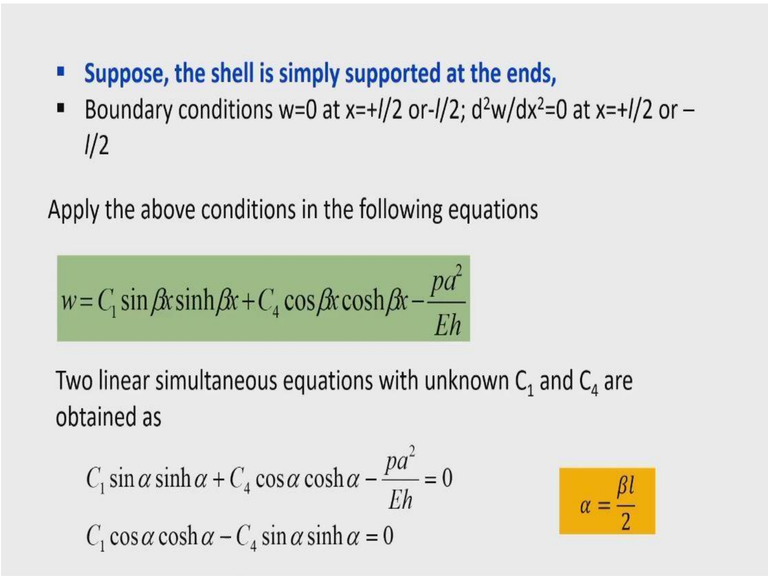

In this problem, we assume that shell is simply supported at the ends. So, boundary conditions now becomes at $x =+ l/2$ or $- l/2$ say, if x is measured positive towards the right from the origin, then right end support of the shell is denoted by located at $x =+ l/2$, whereas, left-hand support of the shell is located by the distance or coordinate that is $x =- l/2$.

And w is 0, $- l/2$ locate the support at the left end and $+ l/2$ locate the support at the right end, w is 0 as well as bending moment is 0 because it is simply supported condition. So, applying the above conditions in the following equations, following equation is this containing two terms of two constants of integration that is C_1 and C_4. Now, the solution will contain the combination of trigonometrical function as well as hyperbolic function.

So, sin sin βx sinh $sinh$ βx and cos cos βx cosh $cosh$ βx . So, first, substitute the condition that deflection condition at $x = l/2$. So, substituting this condition an $x = l/2$ now, we get C_1 sin sin α sinh $sinh$ α $+ C_4$ cos cos α cosh $cosh$ α $- \frac{pa^2}{Eh}$. So, this is from this equation that directly we substituted $x = l/2$ and we assume this parameter $\alpha = \frac{\beta l}{2}$.

So, you can see here when I substitute $x = l/2$ this term becomes C_1 sin $sin \frac{\beta l}{2}$. And $\times$ sinh $sinh \frac{\beta l}{2}$ $+ C_4$ cos $cos \frac{\beta l}{2}$ cosh $cosh \frac{\beta l}{2}$ $- \frac{pa^2}{Eh}$. So, this is the result of the substitution of $x = l/2$ and assuming $\alpha = \frac{\beta l}{2}$. So, first equation we got from the first boundary condition that $w = 0$. Second boundary condition is obtained when we consider the bending moment at the support $+ l/2$ or $- l/2$ is 0.

So, considering this condition that is we have to now obtain the second derivative of this equation and after substituting $x = l/2$, again we got C_1 cos cos α cosh $cosh$ α $- C_4$ sin sin α sinh $sinh$ α $= 0$, because of differentiation this particular integral term that is a constant here will disappear. So, therefore, you are getting the term containing C_1 and C_4.

This equation can be solved because it contains two unknown quantities C_1 and C_4 other quantities are known. So, we can solve two simultaneous equations with two unknowns.

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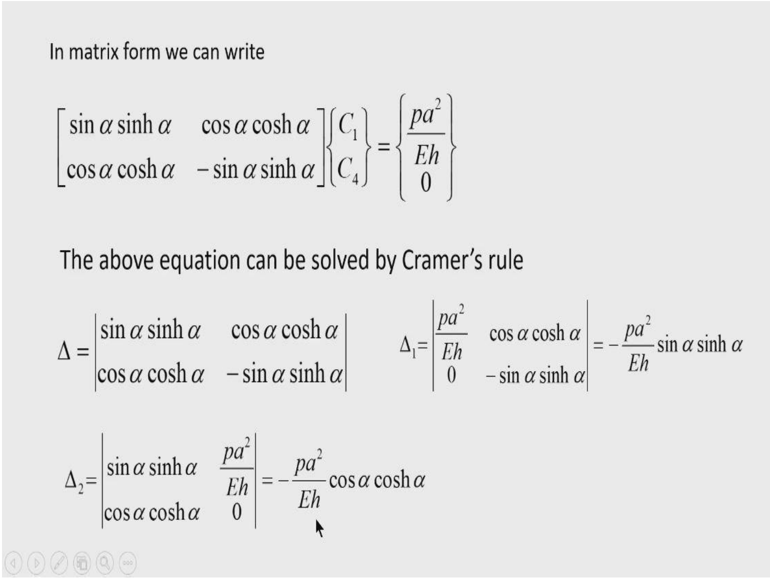

So, here I used the solution with Cramer's rule. So, we write the equation in matrix form this equation is written in matrix form. So, if I write this equation in matrix from then coefficient matrix becomes $[\sin sin\ \alpha\ \ \sinh sinh\ \alpha\ \ \ \cos cos\ \alpha\ \ \cosh cosh\ \alpha\ \ \ \cos cos\ \alpha\ \ \cosh cosh\ \alpha\ \ -\ \sin sin\ \alpha\ \ \sinh sinh\ \alpha\ ]\{C_1\ C_4\}$. C_1 and C_4 are the unknown constants of integration that have to be evaluated by solving this matrix equation.

So, using Cramer's rule, we now define this parameter delta that is the determinant form using the coefficient of this matrix that is I call the coefficient matrix. So, $|\sin sin\ \alpha\ \ \sinh sinh\ \alpha\ \ \ \cos cos\ \alpha\ \ \cosh cosh\ \alpha\ \ \ \cos cos\ \alpha\ \ \cosh cosh\ \alpha\ \ -\ \sin sin\ \alpha\ \ \sinh sinh\ \alpha\ |$. So, that is the determining using the coefficients of the matrix 2×2 matrix.

Then, Δ_1, I define as $\left|\frac{pa^2}{Eh}\ \ \cos cos\ \alpha\ \ \cosh cosh\ \alpha\ \ 0\ \ -\ \sin sin\ \alpha\ \ \sinh sinh\ \alpha\ \right|$. And if I expand this determinant, you can see this expansion and expansion of determinant will lead to the equation $-\frac{pa^2}{Eh}\sin sin\ \alpha\ \ \sinh sinh\ \alpha$. Δ_2, we now write it in this way $\left|\sin sin\ \alpha\ \ \sinh sinh\ \alpha\ \ \frac{pa^2}{Eh}\ \ \cos cos\ \alpha\ \ \cosh cosh\ \alpha\ \ 0\right|$.

And we expand this determinant we now get it $-\frac{pa^2}{Eh}\cos cos\, \alpha\ \cosh cosh\, \alpha$. So, Δ_1 and Δ_2 are obtained, we can now have to find this determinant Δ.

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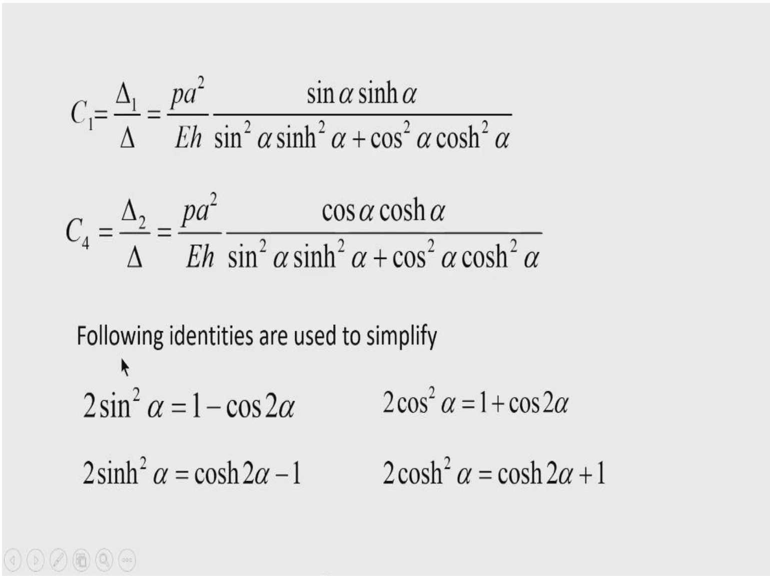

So, determinant Δ is found and after substituting these, $C_1 = \frac{\Delta_1}{\Delta}$ we now obtain $C_1 = \frac{pa^2}{Eh}\frac{\sin sin\, \alpha \sinh sinh\, \alpha}{\alpha\, \alpha + \alpha\, \alpha}$. Similarly, we get, $C_4 = \frac{\Delta_2}{\Delta} = \frac{pa^2}{Eh}\frac{\cos cos\, \alpha \cosh cosh\, \alpha}{\alpha\, \alpha + \alpha\, \alpha}$.

So, the numerator term is that you are seeing is directly coming from this expansion of determinant with Δ_1 and Δ_2 whereas, denominator will contain the expanded form of the determining Δ. So, the denominator that you are seeing here $\alpha\, \alpha\ +\ \alpha\, \alpha$ is nothing but expansion of the determinant formed by the coefficient of the matrix equation, 2×2 matrix.

Following identities are used to simplify further the constant of integration. So, what are the identities? One identity is trigonometrical identity that is $2\alpha\ =\ 1\ -\ \cos cos\ 2\alpha$ and another is $2\alpha\ =\ 1\ +\ \cos cos\ 2\alpha$. Then other 2 are this hyperbolic function identity that is $2\alpha\ =\ \cosh cosh\ 2\alpha\ -\ 1$ and second one is $2\alpha\ =\ \cosh cosh\ 2\alpha\ +\ 1$.

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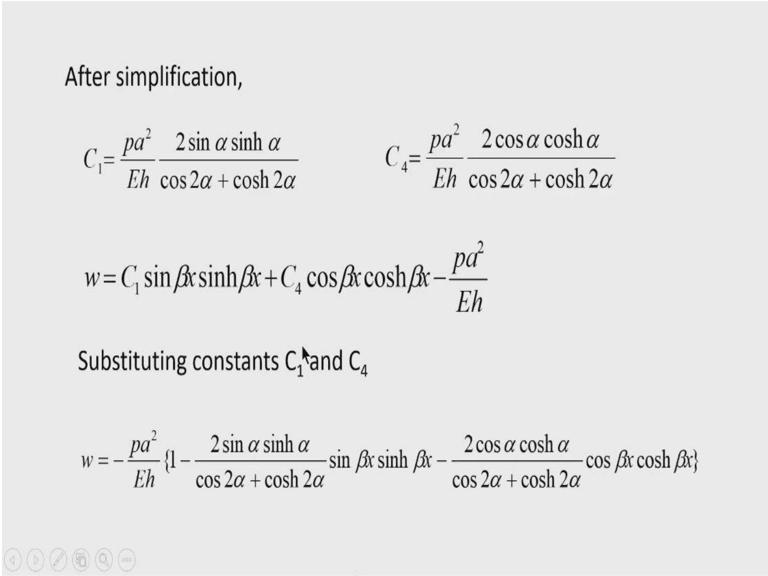

So, using these identities here, we can now express the C_1 and C_4 in simplified form. So, C_1 contains $C_1 = \frac{pa^2}{Eh}\frac{2\sin sin\,\alpha\ \sinh sinh\,\alpha}{\cos cos\,2\,\alpha + \cosh cosh\,2\,\alpha}$. Similarly, C_4 contains $\frac{pa^2}{Eh}\frac{2\cos cos\,\alpha\ \cosh cosh\,\alpha}{\cos cos\,2\,\alpha + \cosh cosh\,2\,\alpha}$. So, we can now write the complete solution.

So, complete solution is $C_1 \sin sin\,\beta\,x \sinh sinh\,\beta\,x + C_4 \cos cos\,\beta\,x \cosh cosh\,\beta\,x - \frac{pa^2}{Eh}$. C_1 and C_4 are now completely known. So, therefore, substituting C_1 as this quantity and C_4 as this quantity in this expression, we can now get the equation in this form $w = -\frac{pa^2}{Eh}\{1 - \frac{2\sin sin\,\alpha\ \sinh sinh\,\alpha}{\cos cos\,2\,\alpha + \cosh cosh\,2\,\alpha}\sin sin\,\beta x\ \sinh sinh\,\beta x\ -$, this term.

Then the other term is $\frac{2\cos cos\,\alpha\ \cosh cosh\,\alpha}{\cos cos\,2\,\alpha + \cosh cosh\,2\,\alpha}\cos cos\,\beta x\ \cosh cosh\,\beta x$. So, you will now get the complete equation of the deflection.

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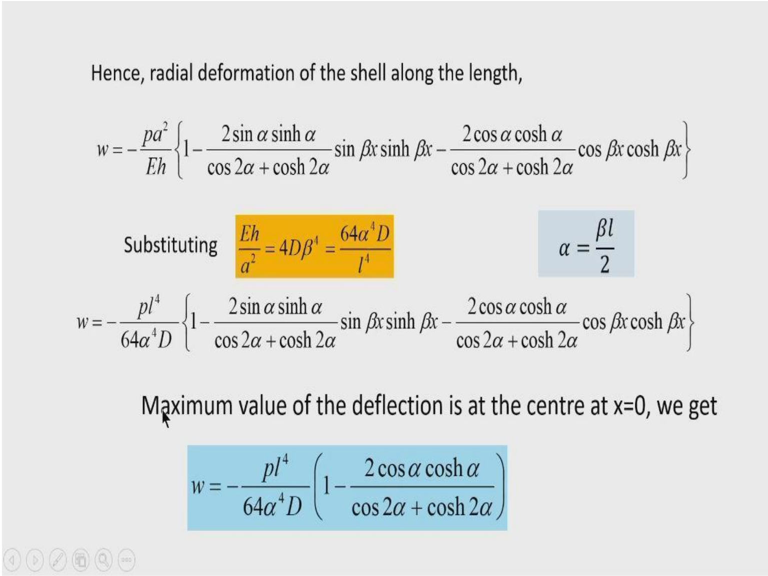

So, it is written again here and substituting this $\frac{Eh}{a^2} = 4D\beta^4$ and using these, $\beta = \frac{\alpha 2}{l}$, we can find $\beta = \frac{2\alpha}{l}$, we can now express $\frac{Eh}{a^2}$ that is appearing in this equation is $\frac{64\alpha^4 D}{l^4}$. So, using this parameter we now express the deflection as $w = -\frac{pl^4}{64\alpha^4 D}\left\{1 - \frac{2 \sin sin\ \alpha\ \sinh sinh\ \alpha}{cos cos\ 2\ \alpha + cosh cosh\ 2\ \alpha} \sin sin\ \beta\ x\ \sinh sinh\ \beta\ x - \frac{2 \cos cos\ \alpha\ \cosh cosh\ \alpha}{cos cos\ 2\ \alpha + cosh cosh\ 2\ \alpha} \cos cos\ \beta\ x\ \cosh cos\right.$

. So, since this shell is supported symmetrically and loading is also symmetrical and the shell thickness is also symmetrical, that means, it has a uniform flexural rigidity.

So, the maximum deflection we obtain at the centre of the shell. So, at the centre of the shell substituting $x = 0$. In the final expression of the deflection, you can see this term will vanish will be vanished, because when you put $x = 0$ here this will be 0 and as well as this will be 0 and cos $cos\ \beta x$ when it is evaluated with the $x = 0$ it will be 1, cosh $cosh\ \beta x$ also will be 1.

So, the maximum deflection is obtained as $w = -\frac{pl^4}{64\alpha^4 D}\left(1 - \frac{2 \cos cos\ \alpha\ \cosh cosh\ \alpha}{cos cos\ 2\alpha + cosh cosh\ 2\alpha}\right)$. That means, second term is a hyperbolic function, so cosh $cosh\ 2\alpha$.

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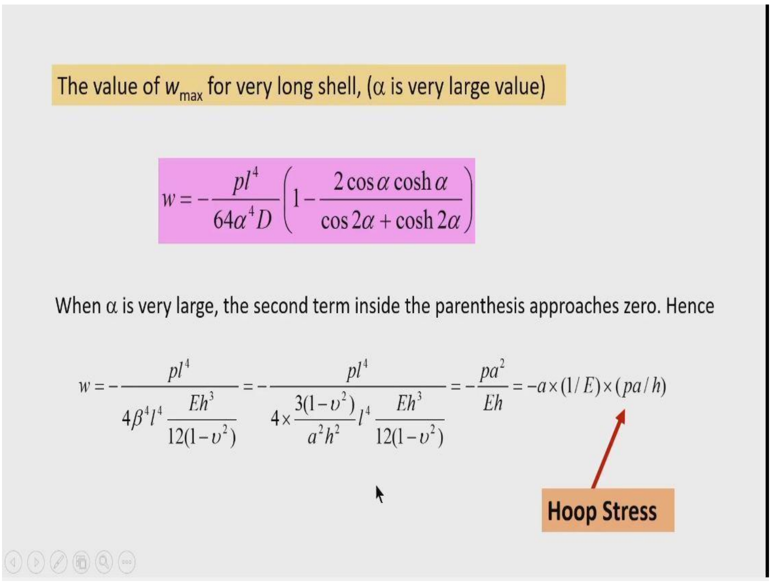

Now, some important conclusion will draw from this expression. Let us consider the shell is very long. So, in the long shell that we use this $\alpha = \frac{\beta l}{2}$, we use the $\alpha = \frac{\beta l}{2}$. So, naturally, α parameter will be very large for very long shell. So, now if you see, if you substitute here cos $cos\ \alpha$, which is very high value then this term becomes very large value denominator will be very large. So, naturally this fraction will be negligible.

So, hence maximum deflection for a very long shell is obtained as $\frac{pl^4}{64\alpha^4 D}$ because this term can be ignored for the reason that the cosh $cosh\ 2\ \alpha$ will be very large because of this large length $\alpha = \frac{\beta l}{2}$. So, second term inside the parenthesis approaching 0 and therefore, we write the expression of deflection for the long shell at the centre with this expression $w = -\frac{pl^4}{4\beta^4 l^4 \frac{Eh^3}{12(1-\upsilon^2)}}$.

And in place of D, we now substituted $\frac{Eh^3}{12(1-\upsilon^2)}$ and also in place of alpha we brought this parameter α is assumed as $\frac{\beta l}{2}$. So, β is appearing here. Now, substituting β^4 as we have obtained earlier as $\frac{3(1-\upsilon^2)}{a^2h^2}$ and then l^4 is already there and we substituted this D here also is substituted written here, then D is also written here $\frac{Eh^3}{12(1-\upsilon^2)}$.

So, after simplifying this expression you will find that $w =- \frac{pa^2}{Eh}$. Now, this expression can be a examine say pa/h is nothing but hoop stress in a cylindrical shell subjected to internal pressure. So, stress divided by strain that is divided by modulus of elasticity that is E will give you the strain. So, stressed by E that is σ/E will give you strain and strain multiplied by radius will give you the changing radius.

So, that is the radial deformation. So, you can see that w that we get here for very long shell is nothing but your membrane solution $\frac{pa^2}{Eh}$.

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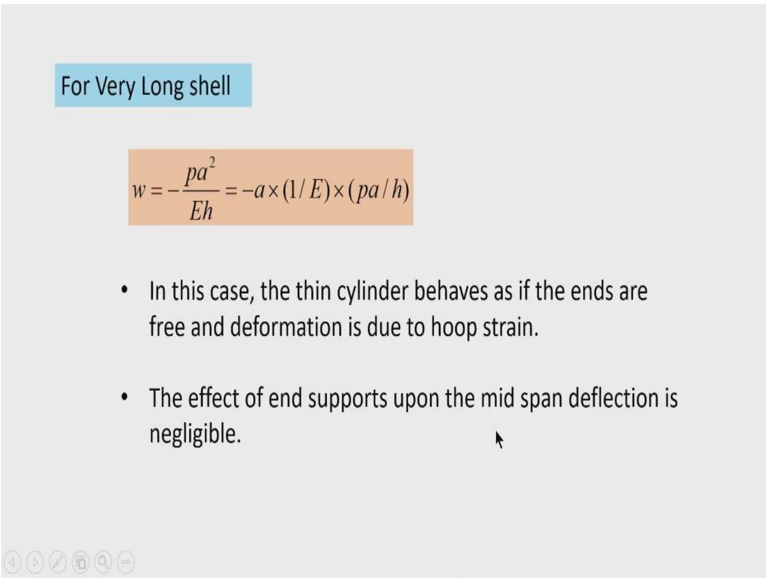

So, for what very long shall we now conclude that $w =- \frac{pa^2}{Eh}$ and two interesting things we can note here. If the shell is very long and it is a thin shell cylindrical shell, of course, we are dealing with the cylindrical shell behaves as if ends are free and deformation is due to hoop stain. So, that is a bias for a very long shell the effect of bending moment can be neglected. The effect of end supports upon the mid-span deflection is also negligible.

So, if the particular integral does not appear in the shell deflection equation, then the effect of this support that is there is no question of this applying the boundary condition because the shelves are very long and the boundary condition will not affect the deflection at the mid-span. So, that to conclusion; we can arrive from this expression.

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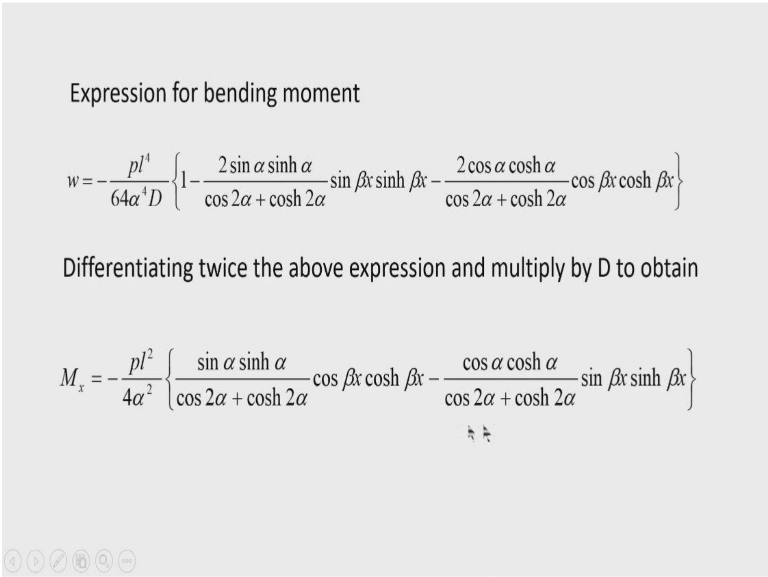

Expression for any moment can be obtained by differentiating the deflection equation two times with respect to x. So, differentiating twice the expression with respect to x, of course, you have to multiply the differentiated quantity with the flexural rigidity of the shell. It is also multiplied with D and finally, we get the bending moment expression.

As

$M_x = -\frac{pl^2}{4\alpha^2}\left\{\frac{\sin sin\ \alpha\ \sinh sinh\ \alpha}{\cos cos\ 2\ \alpha + \cosh cosh\ 2\ \alpha}\cos cos\ \beta\ x\ \cosh cosh\ \beta\ x - \frac{\cos cos\ \alpha\ \cosh cosh\ \alpha}{\cos cos\ 2\ \alpha + \cosh cosh\ 2\ \alpha}\sin sin\ \beta\ x\ \sinh sinh\ \beta\ x\right\}$

. So, this is the expression for bending moment of the shell.

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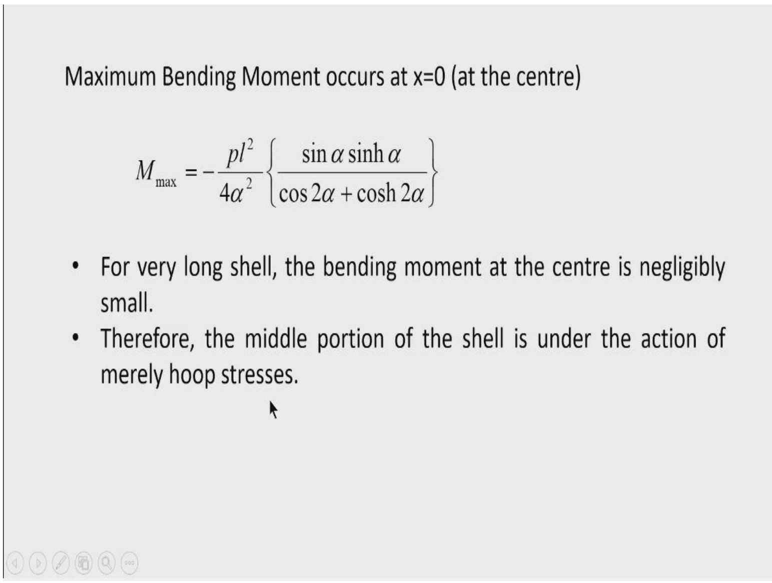

Maximum bending moment occurs at the centre again because of symmetrical problem and maximum bending moment we get from the earlier expression from the earlier expression when is substitute x = 0. So, if I substitute x = 0, this term will not be coming into picture and this term $\cos cos\, \beta x$ $\cosh cosh\, \beta x$ will be 1. So, therefore, maximum bending moment we will get $M_{max} = \frac{pl^2}{4\alpha^2}\left\{\frac{\sin sin\, \alpha\ \sinh sinh\, \alpha}{\cos cos\, 2\alpha + \cosh cosh\, 2\alpha}\right\}$.

So, one interesting thing you can see there again if alpha is large that is for very long shell this quantity approximate 0 or becomes very small quantity. So, therefore, very long shell the bending moment at the centre is negligibly small. So, that it is observed very clearly from this expression if $\cosh cosh\, 2\alpha$ is large. Therefore, the medial portion of the shell is under the action of merely hoop stresses.

And in the middle portion of the shell and his vicinity the membrane state of stress exists. So, that is an important conclusion for very long shell.

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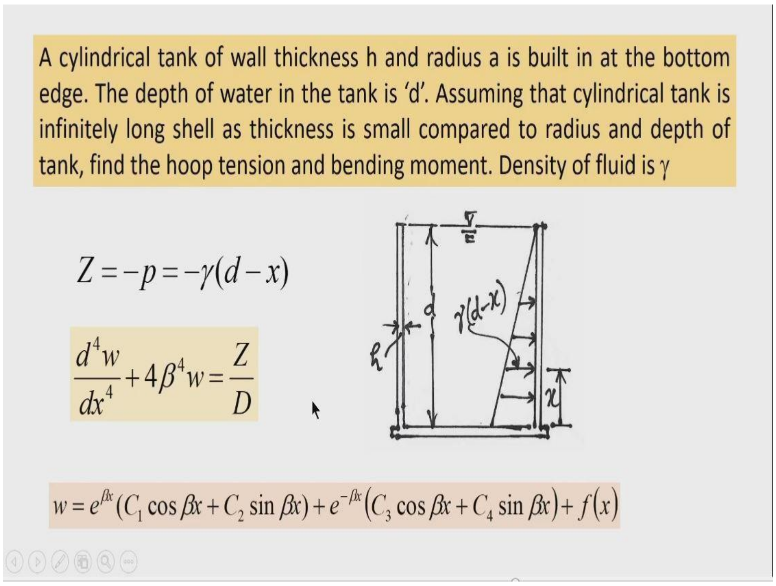

Now, let us apply this solution to another problem that I have told in the beginning of this lecture that a cylindrical shell has to be analysed. And that type of shell is frequently used as water-retaining structure or storage tanks in many applications and the for the solution, we take the wall thickness is uniform, but it is not necessary that wall thickness should be uniform, it may also vary.

So, in that case, differential equation for the deflection of the shell will contain the coefficient which is variable in x radius of the shell is a and it is built in at the bottom edge. So, the shell is fixed at the bottom edge. The depth of the water in the tank is d. Water or any fluid whose density is taken as gamma, now assuming that cylindrical tank is infinity log shell will use this condition that shell is infinitely long, but it is not necessary also.

If the conditions at the two ends are defined and the shell is treated as a tank of finite length, then also the equation or solution can be obtained. So, our intention is to find the hoop tension and bending moment in the shell. So, we proceed with that differential equation $\frac{d^4w}{dx^4} + 4\beta^4 w = \frac{Z}{D}$,where Z is the internal pressure in the tank.

And at a distance x, x is measured from this base. Base is taken as the reference plane. So, if I take this as the origin N_x is measured upward positive. So, at a distance x, the depth of the water causing the pressure is $d - x$. So, $(d - x)\gamma$ will be the pressure at this level and the variation of the pressure you can see it is linear hydrostatic pressure. The governing differential equation of this shell now can be written as $\frac{d^4w}{dx^4} + 4\beta^4 w =- \frac{(d-x)\gamma}{D}$.

Instead of Z, we can substitute this quantity. Solution of that equation is given earlier and it is written here again $w = e^{\beta x}(C_1 \cos cos\, \beta\, x + C_2 \sin sin\, \beta\, x) + e^{-\beta x}\left(C_3 \cos cos\, \beta\, x + C_4 \sin sin\, \beta\, x\right) + f(x)$, $f(x)$ is the particular solution due to internal pressure.

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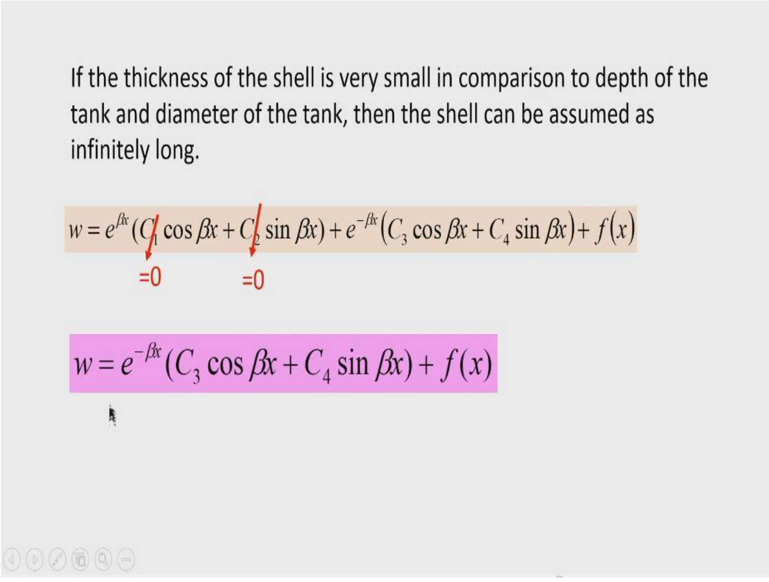

Because the shell is very long. So, as we have treated earlier these two terms have to be dropped otherwise, the deflection and bending moment at the other end will be unbounded. Because this quantity $e^{\beta x}$ will increase with x because β is a positive quantity. So, therefore, we have to retain only these two constants of integration C_3 and C_4 and the solution now written as $w = e^{-\beta x}(C_3 \cos cos\, \beta\, x + C_4 \sin sin\, \beta\, x) + f(x)$.

So, this is the final solution and two constants of integration now have to be determined from the end condition.

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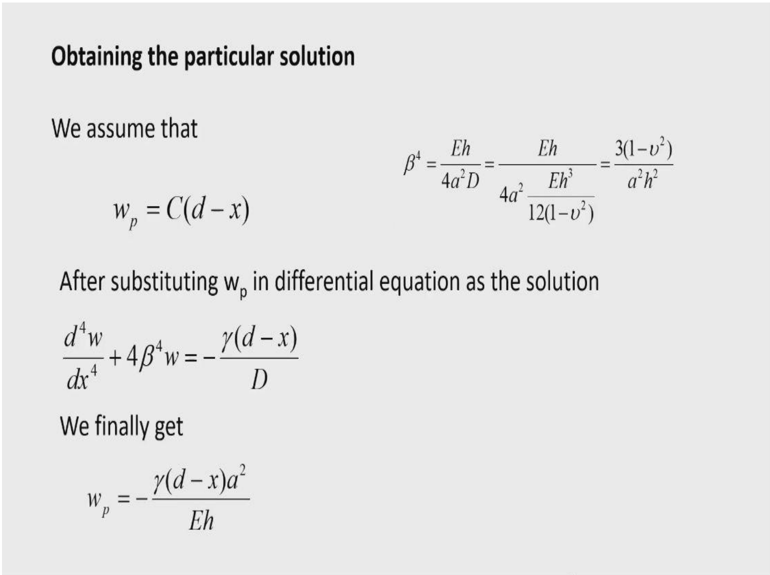

So, homogeneous solution we have obtained and particular solution now, we can obtain using this the theory that earlier I have explained, because the particular integral is due to the forcing function which is a linear function. And therefore, we assumed $w_p = C(d - x)$. So, we assume $w_p = C(d - x)$ and substituting these w_p in the differential equation, we finally get the particular solution as $w_p = -\frac{\gamma(d-x)a^2}{Eh}$.

So, this is the particular solution. Simplification, of course, is made by using this quantity $\beta^4 = \frac{Eh}{4a^2D}$.

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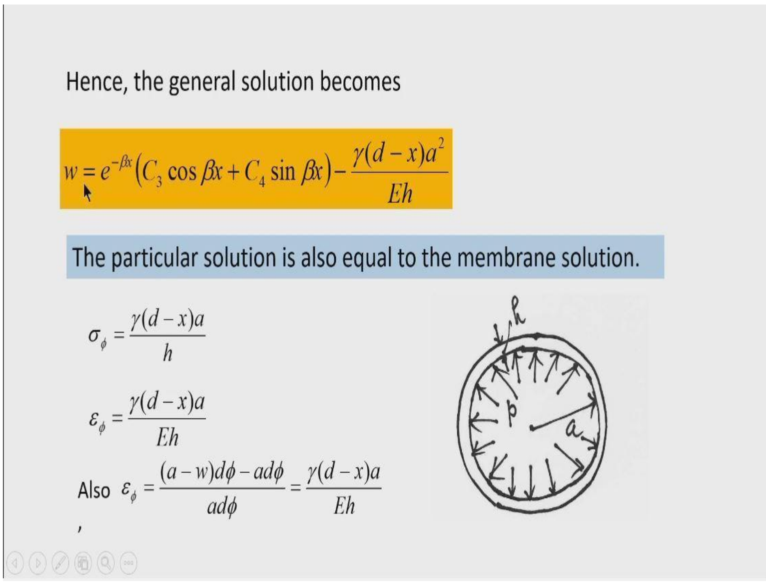

So, now complete solution is written as $w = e^{-\beta x}\left(C_3 \cos cos\, \beta\, x + C_4 \sin sin\, \beta\, x\right) - \frac{\gamma(d-x)a^2}{Eh}$. So, this is the particular solution. And again, we will prove that particular solution is also the membrane solution. So, let us prove this, so at any level say x that is defined, x is measured from the bottom of the tank the hoop stress is given by $\frac{\gamma(d-x)a}{h}$.

So, hoop strain is $\frac{\gamma(d-x)a}{Eh}$. Also, we can write hoop strain as the change in circumference divided by the original circumference for a small element of the cell, which subtends an angle $d\phi$ at the centre. So, length of the part of the circumference of the shell which subtends an angle $d\phi$ at the centre is $ad\phi$ and this length is change due to radial deformation.

So, changed length of this element is $(a - w)d\phi$ along the circumference, an original length was $ad\phi$. So, if I take the ratio of this to $\frac{(a-w)d\varphi - ad\varphi}{ad\varphi}$ this will in the hoop strain. So, hoop strain is now obtained as $\frac{\gamma(d-x)a}{Eh}$. Now, from this quantity, if I want to solve w. So, this is an expression which connects w with the hoop strain. So, if I solve this equation for w then what do we get?

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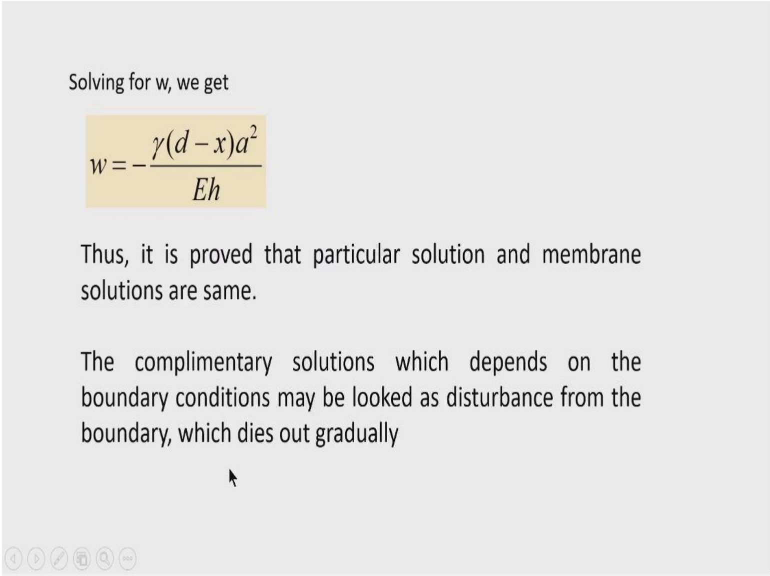

We get $w = -\frac{\gamma(d-x)a^2}{Eh}$, you can see from this equation this equation $d\phi$ will get cancel. So, therefore, this circumferential strain will be nothing but $-w/a$. So, $-w/a$ is nothing but $\frac{\gamma(d-x)a}{Eh}$. So, w can be readily found as $-\frac{\gamma(d-x)a^2}{Eh}$. So, it is again proved that particular solution and membrane solution are same.

The complementary solutions actually depend on the boundary conditions and it may be looked at as a disturbance emanating from the edges. So, but these disturbance dies out gradually because we have taken a damping like factor $e^{-\beta x}$.

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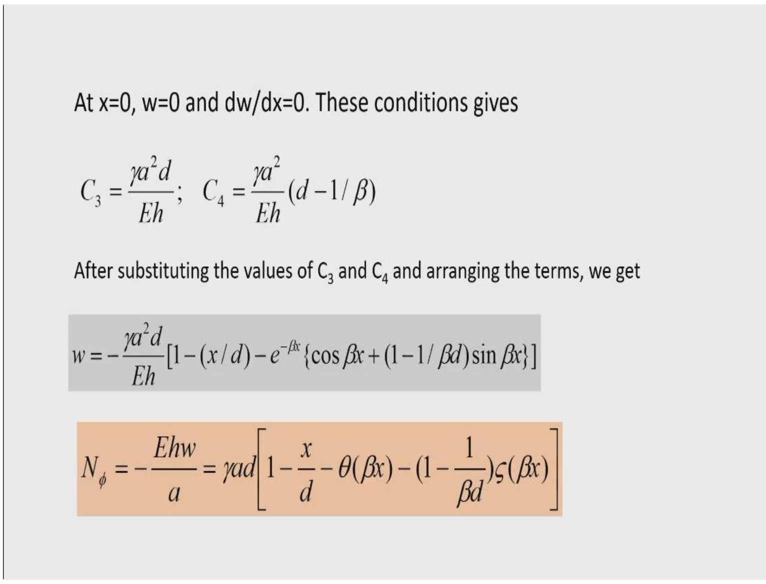

At x = 0, $w = 0$, $\frac{dw}{dx} = 0$. So, this is our given boundary condition. And applying these boundary condition in the deflection equation that we obtained here. Here, if I apply the boundary condition $w = 0$, $\frac{dw}{dx} = 0$ at x = 0, we obtain now, two constants $C_3 = \frac{\gamma a^2 d}{Eh}$ and $C_4 = \frac{\gamma a^2}{Eh}(d - 1/\beta)$. So, after substituting C_3 and C_4, we can now write w as,

$$w = -\frac{\gamma a^2 d}{Eh}[1 - (x/d) - e^{-\beta x}\{\cos cos\,\beta x + (1 - 1/\beta d)\sin sin\,\beta x\}]$$

.

So, this is the complete expression of the deflection where all the parameters are known, because d is the depth of the fluid or depth of the tank. If the fluid is up to the full height of the tank, then d will be height of the 1tank, h is the thickness of the shell a is the radius of the cylindrical shell and other quantities, you will know that beta is the characteristic parameters that we have found earlier.

So, once you will know w then the membrane stresses membrane stresses only one membrane stress is of importance that is N_ϕ, others are insignificant in the problem and these are neglected. So, $N_{\phi} = -\frac{Ehw}{a}$ and substituting w here we get N_{ϕ}, $\gamma ad\left[1 - \frac{x}{d} - \theta(\beta x) - \left(1 - \frac{1}{\beta d}\right)\zeta(\beta x)\right]$.

Now θ, ζ are the functions which contains this exponential term with cosine terms or the sin term.

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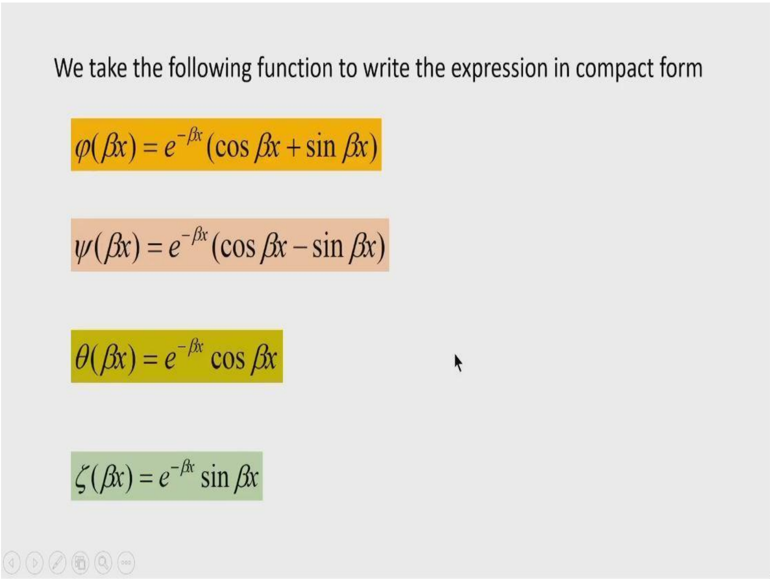

So, these functions are defined as there are several functions which contain this combination of exponential term and cosine and sin term. So, here you can see I have written four functions $\phi(\beta x) = e^{-\beta x}(\cos cos\, \beta\, x + \sin sin\, \beta\, x)$, $\psi(\beta x) = e^{-\beta x}(\cos cos\, \beta\, x - \sin sin\, \beta\, x)$, etcetera. So, here we abused only these two functions $\theta(\beta x)$ and $\zeta(\beta x)$.

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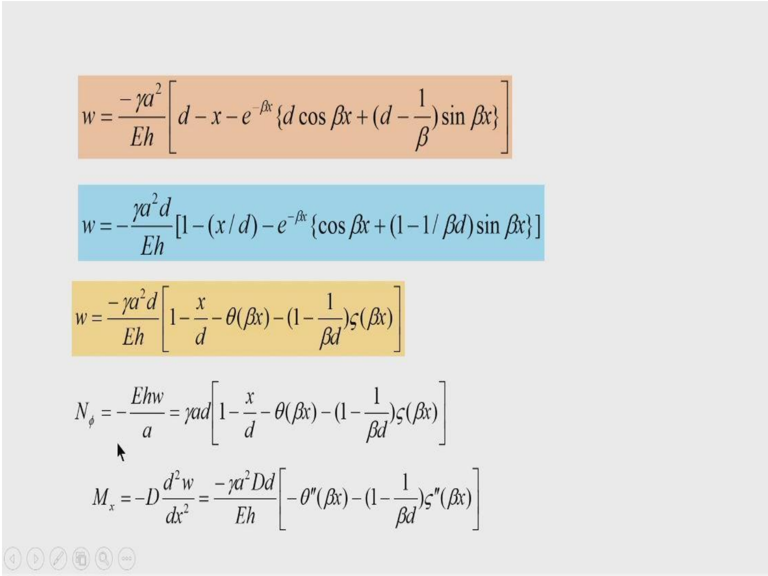

So, it is written in the functional form and the maximum deflection will be obtained somewhere but, this deflection you have to calculate by plotting the graph or you can impose the condition what the slope is 0 to get the maximum deflection. The membrane force N_ϕ can be obtained as $-\frac{Ehw}{a}$ which is nothing but $\gamma a d\left[1 - \frac{x}{d} - \theta(\beta x) - (1 - \frac{1}{\beta d})\varsigma(\beta x)\right]$.

Bending moment is obtained by calculating the second derivative of these deflection functions and then multiplying the second derivative with the flexural rigidity. So, bending moment is obtained as $\frac{-\gamma a^2 Dd}{Eh}\left[-\theta''(\beta x) - (1 - \frac{1}{\beta d})\varsigma''(\beta x)\right]$. You should distinguish between these two symbol capital D is the flexural rigidity of the shell whereas, small d is the depth of the tank or depth of the water in the tank.

θ'' indicates that second derivative of this quantity, because second derivative of these two terms will be 0.

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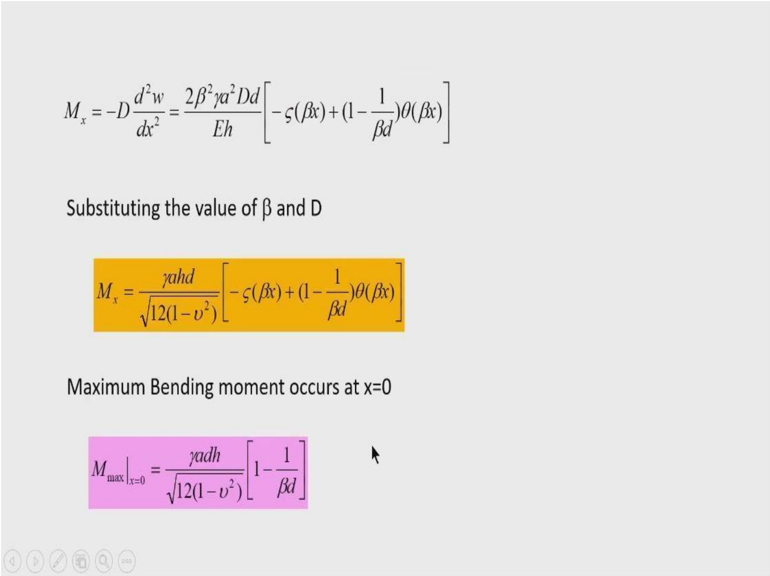

So, M_x is written after simplification that is putting the value of these various parameters that are appearing here d flexural rigidity that $\frac{Eh^3}{12(1-\nu^2)}$ and bringing them in terms of β that is the characteristic parameter of the shell without right the bending moment in the shell as $\frac{2\beta^2\gamma a^2 Dd}{Eh}\left[-\zeta(\beta x) + (1 - \frac{1}{\beta d})\theta(\beta x)\right]$.

Substituting the value of β and D again it can be expressed in another form. So, $M_x = \frac{\gamma ahd}{\sqrt{12(1-\upsilon^2)}}\left[-\varsigma(\beta x) + (1 - \frac{1}{\beta d})\theta(\beta x)\right]$. So, M_x can be calculated now, along the height of the tank and maximum bending moment you can see is only at the fixed end.

So, fixed end provide restraints. So, maximum bending moment is calculated at x = 0. So, maximum value is obtained by putting x = 0 here and we can obtain that maximum value is nothing but $M_{max}|_{x=0} = \frac{\gamma adh}{\sqrt{12(1-\upsilon^2)}}\left[1 - \frac{1}{\beta d}\right]$.

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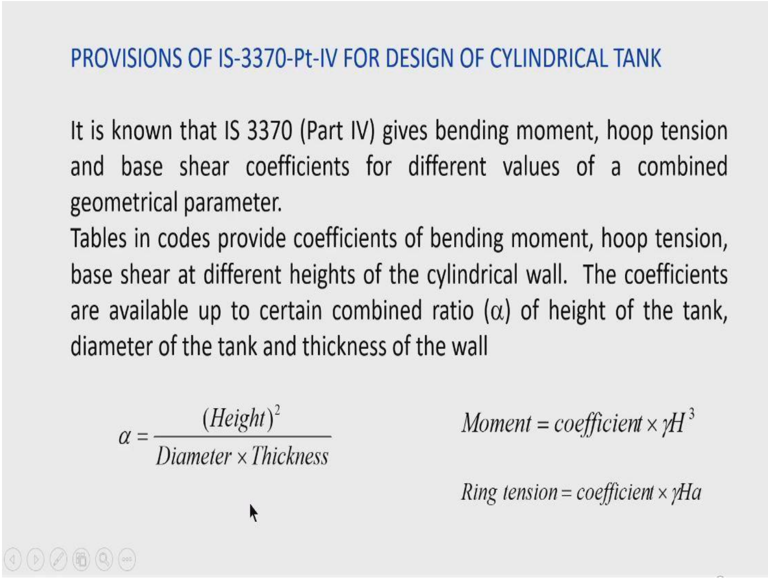

Now, such types of tanks are very common in our construction industry, whether you use it for steel tank or these reinforced concrete tanks. The provisions of various quotes for design have been available. So, here is such one provision that is given by the code IS Indian standard 3370 part 4. So, in this code the bending moment hoop tension and base shear coefficients for different values of a combined geometrical parameter are given.

Now tables in the codes provide coefficients for bending moment hoop tension base shear at different heights of the cylindrical shell. Base shear of course, at the base and other quantity you can obtain at different height. The coefficients are available up to certain combined ratio of height of the tank, diameter of the tank and thickness of the wall. So, this is the actually limitation of this code.

So, you will get these coefficients up to a parameter that I have defined as I call it as α, $\alpha = \frac{(Height)^2}{Diameter \times Thickness}$. So, if this parameter is greater than 16, no coefficients are available in the code. However, using the general theory, the coefficients are generated for different values of alpha also. And then once the coefficients are known from the code the moment is given by coefficient into γH^3.

H is the depth of water and it is with a cubic quantity so, H cube ring tension that is the hoop tension developed is nothing but $= coefficient \times \gamma Ha$.

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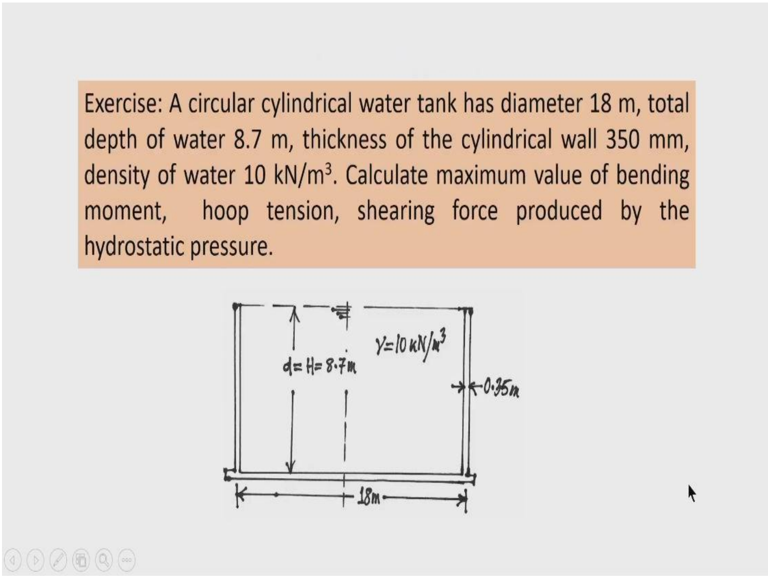

Let us discuss a numerical problem a circular cylindrical water tank has a diameter 18 meter, total depth of the water 8.7 meter, thickness of the cylindrical wall 350 millimetre density of the water 10 kilo Newton per meter cube. We have to calculate the maximum value of the bending moment hoop tension and sharing force, of course, you can calculate produced by hydrostatic pressure.

Now, let us see these tank. The diameter of the tank is 18 meter, the height of the water here is 8.7 meter, density of the water is 10 kilo Newton per meter and thickness of the wall is 0.35 meter.

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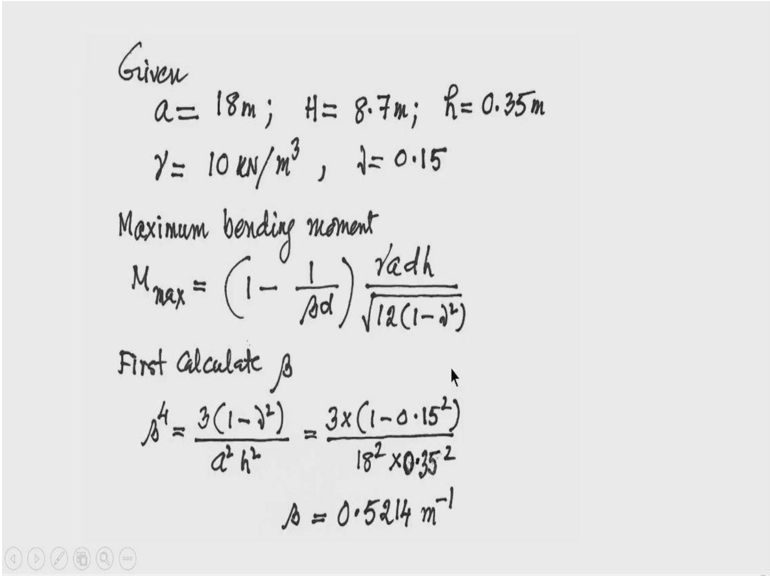

So, with this given data, I use the expression for maximum bending moment $M_{max} = \frac{\gamma adh}{\sqrt{12(1-\upsilon^2)}}\left[1 - \frac{1}{\beta d}\right]$. After substituting the numerical value, we can obtain, first let us calculate the beta. So, $\beta^4 = \frac{3(1-\nu^2)}{a^2h^2}$ And after substituting the numerical value Poisson ratio for concrete it is RCC tank.

So, we have taken as 0.15, substituting the value of this parameter beta comes out as $0.5214m^{-1}$.

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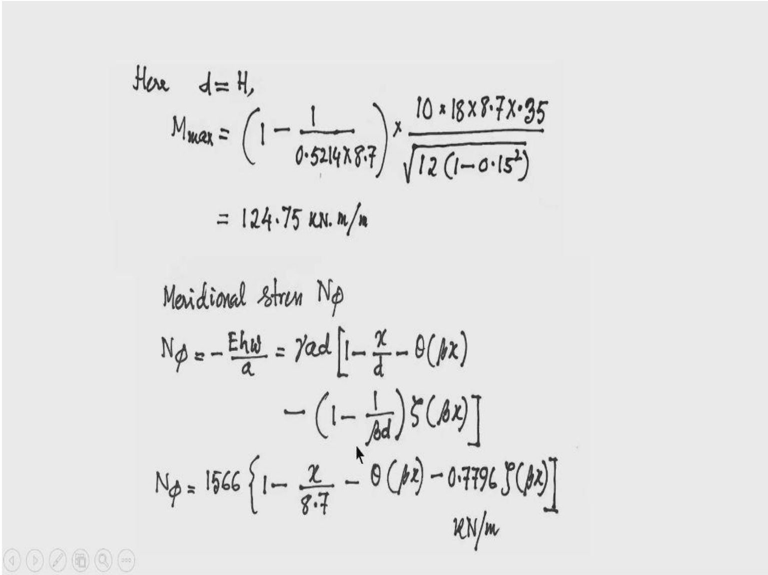

So, now maximum bending moment will be at the base and here we take d = H in the formulation. So, substituting all the parameters in this maximum bending moment expression here, we now obtain the maximum bending moment as $124.75\ kN.m/m$. So, this is the result of maximum bending moment. Meridional stress hoop stress that can be calculated $N_{\phi} =- \frac{Ehw}{a}$ and the expression after substituting w.

And return using the special function θ and ζ, we can now express $N_{\phi} = \gamma ad\left[1 - \frac{x}{d} - \theta(\beta x) - (1 - \frac{1}{\beta d})\varsigma(\beta x)\right]$. And after substituting the numerical value here we now an expressing all the quantities that is the force unit in kilo Newton and length unit in meter. We now get $N_{\phi} = 1566\times\left\{1 - \frac{x}{8.7} - \theta(\beta x) - 0.7796\times\zeta(\beta x)\right\}$ and it is unit is kN/m. So, the variation of N_{ϕ} is obtained for different values of x along the height of the tank.

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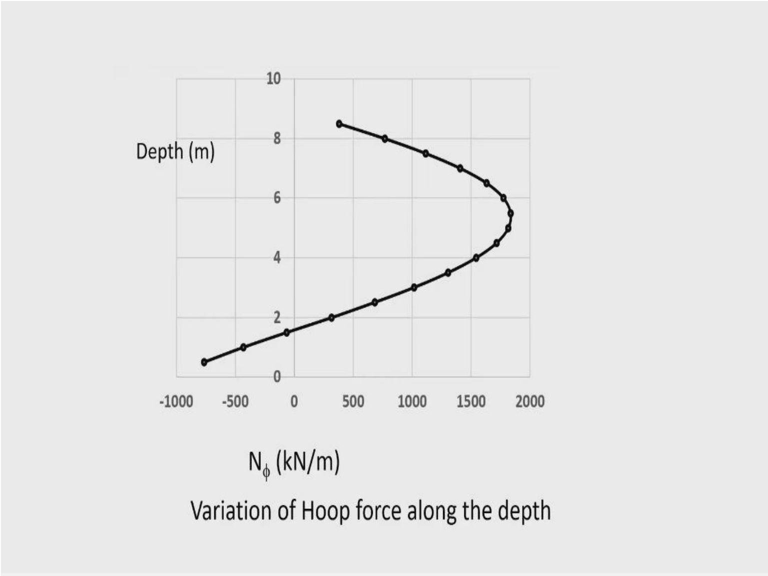

Variation of Hoop force along the depth

And that variation is plotted here and you can see the maximum Hoop force is obtained here. Say at a distance of these around 5.7 and variation Hoop of force is maximum at this point.

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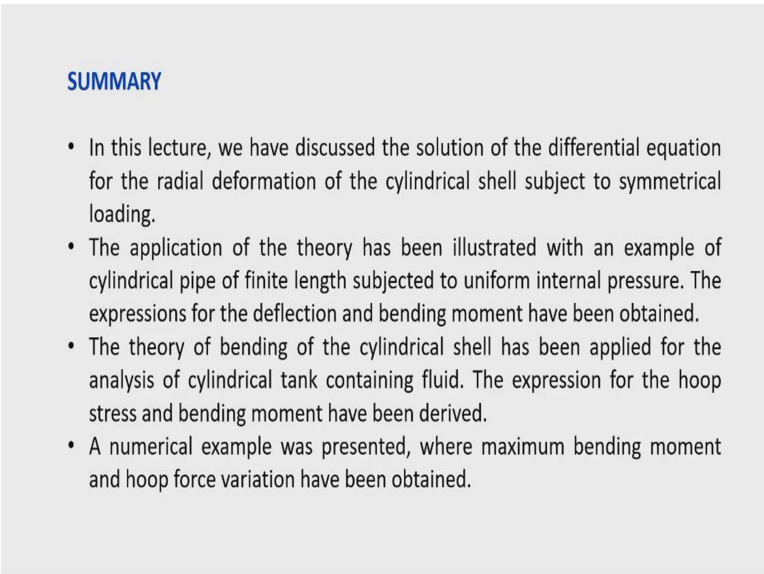

So, let us summarize today's lecture. In this lecture, we have discussed the solution of the differential equation for the radial deformation of the cylindrical shell subjected to symmetrical loading. The application of the theory has been illustrated with an example of cylindrical pipe of finite length subjected to internal pressure. The theory of bending of cylindrical shell has been further applied for the solution of a analysis of a cylindrical tank containing fluid. Lastly, I have given a numerical example to illustrate the theory. Thank you very much.