Plates and Shells
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Lecture - 35
General Bending Theory of Cylindrical Shell

Hello everyone. Today I am delivering the second lecture of the module 12. You have seen in my earlier classes, I first introduced the membrane theory of cylindrical shell and then in the last class that is first lecture of module 12, I introduced the bending moment in the shell. But a simplified approach was adopted to carry out the analysis. In this lecture, I will try to show you that bending moment in the shell due to deformation in the shell wall subjected to the action of radial load can be found out.

And this can be taken into design or many important structures specially, for the circular cylindrical water tank then cylindrical pressure vessels, cylindrical pipes, circular pipe which are very common in applications in many engineering field will be analysed or solved using the theory that I will discuss today. So, today's topic will be theory of cylindrical shell including the bending moment due to radial deformation.

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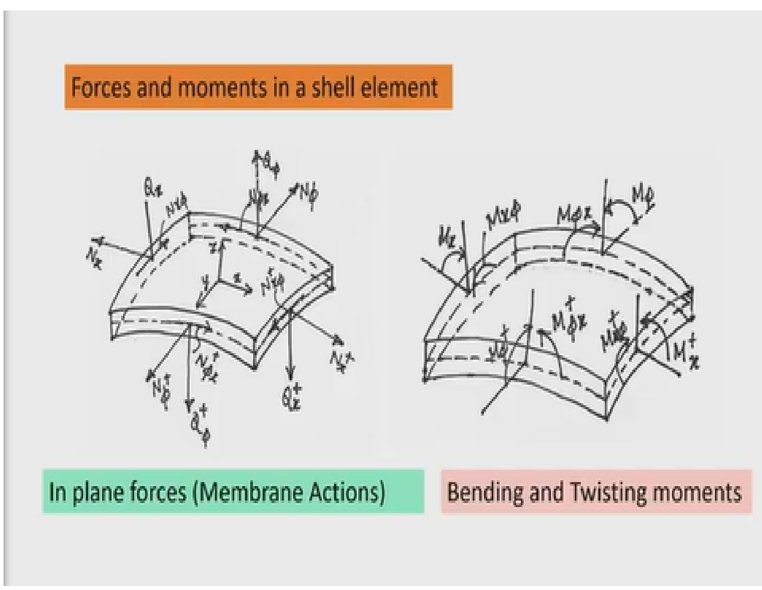

As we have seen that in general a shell element has force resultants and also the moment. So, among the forces resultants the membrane forces or in plane forces are these along the longtitudal direction it is N_x along the circumferential or tangential direction that we call is N_φ and then your this membrane sharing force $N_{x\varphi}$ $N_{\varphi x}$ and vertical shear force. Vertical shear force it is not in plain force out of plane force.

So, vertical shear force Q_x along the edges perpendicular to the x axis and adjacent edges it will be Q_φ. On the opposite edges, the increments are noted here. So, N_φ^{+} means, N_φ is change here due to some increment of this membrane force at this edge. So, one thing may be noted that these forces are things are all in per unit width or length of the shell. If I go to the moment actions in this shell, then I can find that M_x is one moment that is along the x axis.

And on the opposite forces of obviously there will be incremental quantity $M_x + \frac{\partial M_x}{\partial x} dx$ this length is dx. Then on the circumferential direction that is M_φ and M_φ^{+} and the twisting moment $M_{x\varphi}$, $M_{\varphi x}$ are also present. So, in general the shell element contains so, many stress resultants including the imprint forces, vertical shear forces and the bending moment as well as twisting moment.

Now, different simplifications have been done one of the cases that you have learned in my earlier lecture is the simplified beam theory which takes into account of your beam action and arch action in the shell. And acceptable results are obtained also using the beam theory and it is also still popular in many designs offices.

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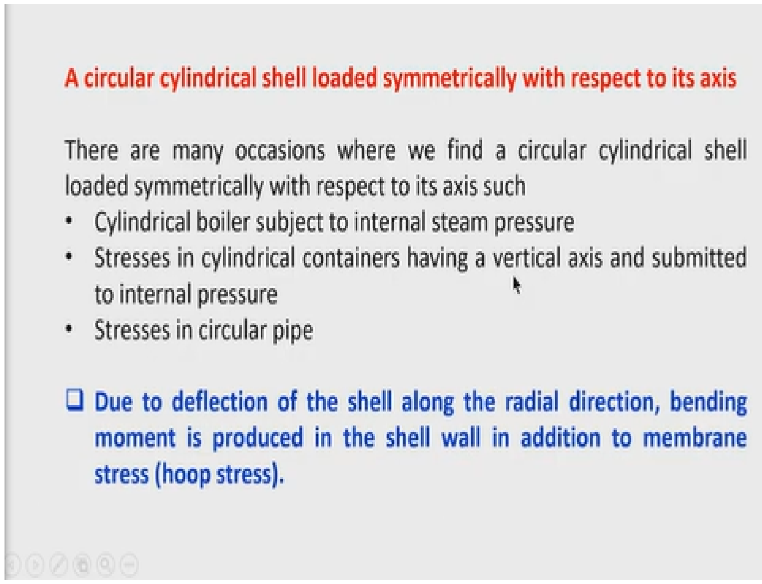

Now, we shall analyse the cylindrical shell subjected to a loading which is in the radial direction. Other loads will be neglected and shells will be symmetrically loaded with respect to its axis. So, that type of situation may arise in cylindrical boiler shell subjected to internal steam pressure. Then cylindrical containers having vertical axis and subjected to internal pressure that is very common in our design of liquid retaining structures and then stresses in circular pipe carrying fluid.

These are also various applications of the theory of cylindrical shell subjected to Radial route. Now, due to deflection of the shell along the radial direction bending moment is produced in the shell wall in addition to membrane stress. So, as a result of radial deformation due to application of the internal pressure or external pressure whatever may be the case the bending moment is produced and memory stresses are also present.

Now, we will analyse the action of the shell with respect to bending moment and this membrane stresses. But simplification will also be done here using certain assumptions which are also realistic. And then these some of the force resultant will have insignificant effect. So, we will drop this in our differential equation. Actually, in earlier classes when we use the strength of material approach, there was no necessity of solving a differential equation.

Now, here we will solve the differential equations of equilibrium of the shell subjected to radial load.

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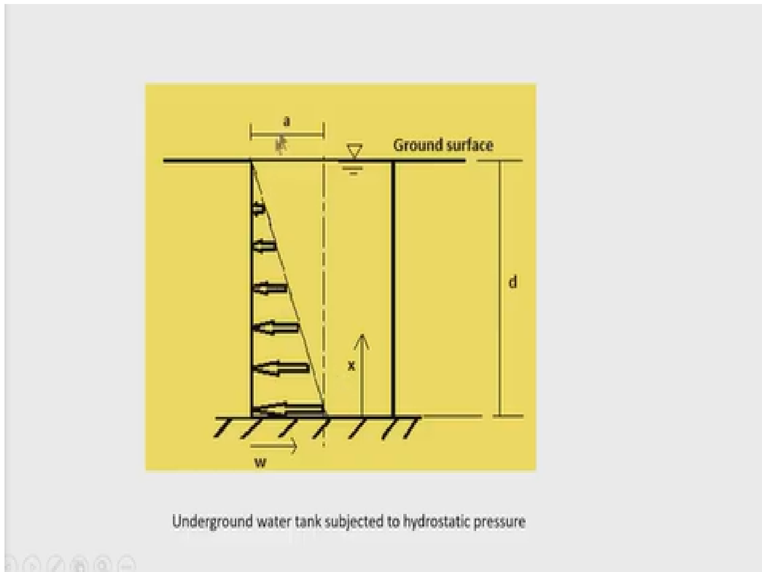

So, a common application you can see here a tank whose cross section is circular. So, it falls under this category and subjected to radial load and this load may be uniform or in case of say hydrostatic pressure that is it is a linearly varying load maximum at the bottom and also since it is underground trend when the art pressure is also have to be taken into account in such cases. And if such underground trends are analysed including the seismic forces.

Then hydrodynamic force that is induced in case of this tank filled with liquid or water that can also be taken into account.

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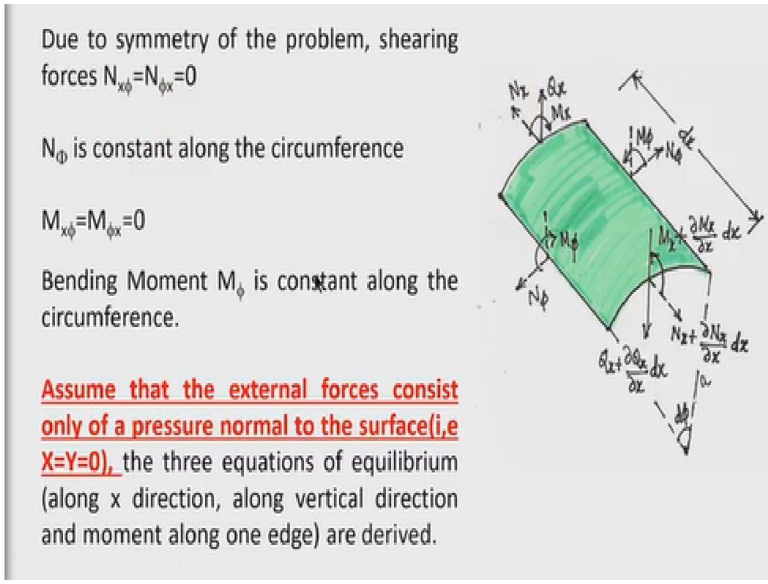

So, let us go to the theory now, we take a shell element, length is dx an angle subtended by the element at the centre is $d\varphi$. So, that this arc length is $Rd\varphi$, if the radius is a here, I will symbolize the radius with a. So, $a \times d\varphi$ is the arc length into dx will be area of the shell surface. Now, due to symmetry of the problem because the; loading is symmetrical with respect to excess of the shell.

Now, in this case the element is taken here, which is a part of cylinder and the axis is horizontal. For example, here x is horizontal. So, loading is symmetrical about the axis. So, therefore, shearing forces will not come into picture. So, therefore, will omit N x phi = N phi x and also the deformations that are produced in this shell are small and the distance of the shell element from the middle surface is very, very small compared to the radius of curvature.

So, therefore, $N_{x\varphi}$ = $N_{\varphi x}$ =0 as well as $M_{\varphi x}$ the twisting moment $M_{x\varphi}$ = $M_{\varphi x}$. So, that two things are possible only because, this z that is the distance measured from the middle surface of the shell is very, very small compared to the radius of the curvature. So, z by art tends to 0. So, in that case only we get this identity that is $N_{x\varphi}$ = $N_{\varphi x}$ =0 and $M_{x\varphi}$ = $M_{\varphi x}$. But due to symmetry we will take $N_{x\varphi} =$ $M_{\varphi x} = 0$ as well as $M_{x\varphi}$ and $M_{\varphi x}$ will be twisting moment will also be vanished.

Now, bending moment M_{φ} is assumed to be constant along this arc length so, the M_{φ} is constant here and M_x will vary along the axis. Now assume that external forces consist only of pressure normal to the surface. Now, here the load whatever load may be will decompose or will resolve into three components, one is along the x direction that is capital X, we call this load as it capital X component of the load as capital X.

Then along the tangential direction that component will call that call as capital Y and in the radial direction radial direction is your z direction. So, component of the load along z direction is termed as capital Z. Now, the pressure is normal to the surface. So, we have no component has capital X and capital Y. So, capital X and capital Y are 0 and only the load component that exists along the radial direction is capital Z.

So, that will take into account in formulating the problem. So, this figure shows the stress resultants at the edges and load is capital Z on the element.

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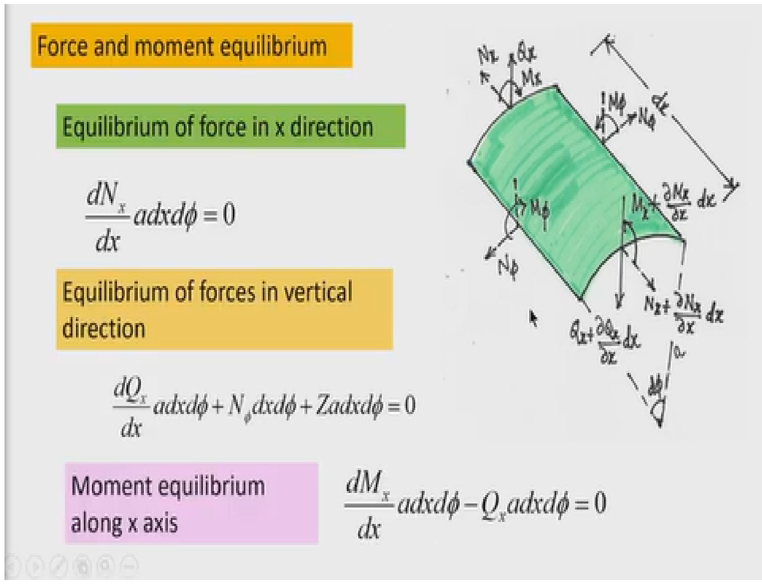

Now, let us derive the equilibrium equation. So, forces and moment equilibrium are considered here. So, equilibrium of the force in the x direction. See this picture figure that is one end this longitude forces N_x at the other N with increment and N_x will appear $N_x + \frac{\partial N_x}{\partial x}dx$. And there is no shearing force member and sharing force as I have told because of symmetry symmetrical cases.

So, therefore, if I take the equilibrium of the forces that is if I take the summation of forces in the x direction and then equal to 0 to maintain the equilibrium of the forces in the x direction, then I will get this $\frac{dN_x}{dx}adxd\varphi = 0$, where $adxd\varphi$ is the area of the shell. This N_x in this direction you are finding it N_x in the opposite direction $N_x + \frac{\partial N_x}{\partial x}dx$. But remember these forces all the forces quantity or moment quantities that are shown here are expressed as per unit length.

So, therefore, if I establish the equilibrium, I have to find the total forces acting along the edges. So, $N_x \times ad\varphi$ is the force here along this because this length is $ad\varphi$. Similarly, in this edge the force along the x direction is $\left(N_x + \frac{\partial N_x}{\partial x} dx\right) \times ad\varphi$. So, summing these force components along the x direction, we ultimately arrived at this equation $\frac{dN_x}{dx} adxd\varphi = 0$.

So, this is one equilibrium equation. Second equilibrium equation will obtain considering the summation of forces in the vertical direction and equating the sum to 0. So, let us see what are the forces in the vertical directions? So, vertical directions you are finding the forces that Q_x is here and $Q_x + \frac{\partial Q_x}{\partial x} dx$ is here and component of N_φ along the vertical direction will also be there. Suppose N_φ is acting tangential to this direction.

Then N_φ can be resolved into radial direction actually it is not vertical this word should be actually radial. Because, we are considering the equilibrium in the radial reaction and component of the load capital Z is in the radial direction. So, there is actually the vertical force equilibrium means, it should imply the equilibrium of the forces in radial direction. Now, if I see the equilibrium of the forces in the radial direction, then $Q_x ad\varphi$ is the force here apart.

And in that edge curved edge, whose length is again $ad\varphi$ total force in the radial direction will be $\left(Q_x + \frac{\partial Q_x}{\partial x} dx\right) \times ad\varphi$. Then component of N_φ have to be taken here and the total load, radial load on this element will be Z, Z is the pressure intensity of pressure in the area of the surface. Area of the surface, I told you that the art length is $ad\varphi$ and dx is the length. So, $adxd\varphi$ is the total area of the surface.

$$\frac{dQ_x}{dx} adxd\varphi + N_\varphi dxd\varphi + Zadxd\varphi = 0$$

So, $Zadxd\varphi$ is the total load on the surface and it is added to the other component of the forces in the radial direction. So, this is one equilibrium equation where we relate the shearing force and this membrane force one membrane force and in N_φ another membrane forces appearing here and the membrane shearing forces 0 in that case and the radial looked. So, simplification can be done after dividing both sides by $adxd\varphi$ because $adxd\varphi$ you are seeing the common everyone.

Here of course, it is not there, but you can divide it. So, N_φ / a will come anyway. So, let us see the moment equilibrium equation along the x axis. So, M_φ will be balanced by M_φ so, there is it is identically satisfied the moment equilibrium in the tangential direction. In the longer general direction if I take the moment equilibrium then I can find that M_x is the bending moment at the said $M_x \times ad\varphi$ is the bending moment along this edge.

On the opposite edges the bending moment will be $M_x + \frac{\partial M_x}{\partial x} dx$ again $ad\varphi$. So, this is the moment. Now moment of the vertical forces that are vertical forces to equal and opposite forces are shown here and this increment can be neglected. So, they will form a couple of obviously they will form a couple and this couple is given us $Q_x adxd\varphi$. So, considering the equilibrium of the moment along the x direction, we now get this equation $\frac{dM_x}{dx} adxd\varphi - Q_x adxd\varphi = 0$.

So, three equations of equilibrium we have obtained. Other equations of equilibrium are not coming into picture because some of the forces are vanished and some are identically satisfied. For example, here $N_\varphi = N_\varphi$ so, it is identically satisfied.

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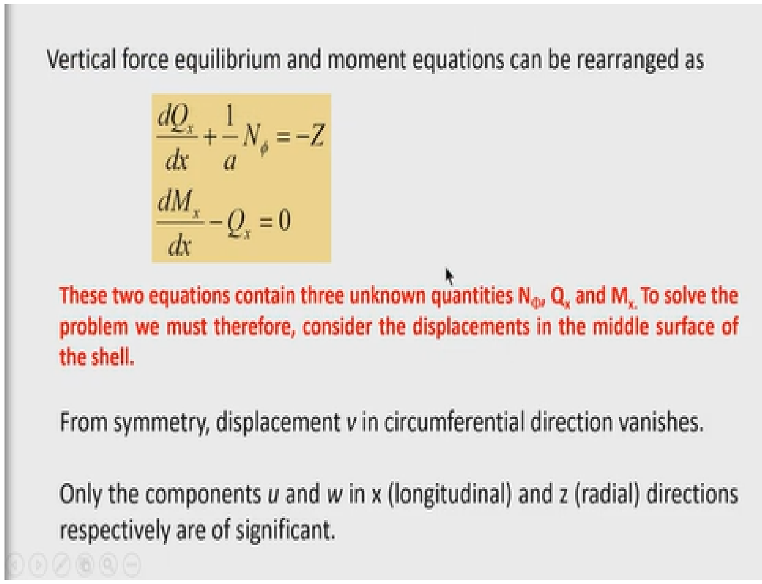

So, particle force equilibrium and momentum equation equilibrium now can be rearranged. Another interesting thing here you can see that first equation. If I simplify this equation

$$\frac{dN_x}{dx} = 0$$

because, if I divide both sides by $adxd\varphi$, then only one quantity will be present. So, $\frac{dN_x}{dx} = 0$ is the only quantity. So, it indicates that derivative of N_x with respect to x is 0 that means, N_x is constant along the edges.

So, that thing will interpret later. First let us see, interpret the vertical force equilibrium actually it is radial force equilibrium and moment equilibrium. So, for that two equations if you see, this second equation if I divide both sides by $adxd\varphi$, then I will get $\frac{dQ_x}{dx} + \frac{1}{a} N_\varphi = -Z$. So, that equation relates the membrane force in the tangential direction, there is the hoop force very important force in the design of cylindrical shell and Q_x is the shearing force.

So, we have related shearing force with the membrane force via the load that is acting. Then second equation if you see, this third equation again, if you divide it divide both sides by $adxd\varphi$ you will get $\frac{dM_x}{dx} = Q_x$. So, that is arrived here $\frac{dM_x}{dx} - Q_x = 0$. So, two equations that is containing the vertical equilibrium moment equilibrium are written here. Now from symmetry of the problem, the displacement v in circumferential direction vanishes.

Now, one thing you can note here there are two equations of equilibrium, but unknown quantities are three Q_x N_φ and M_x. So, two equations three unknown so, it imposes difficulty. So, therefore, we have to bring the displacement relationship to solve the problem completely. Now, from symmetry displacement v in the circumferential direction can be ignored. Only the component you in the longitudinal direction x and w in the radial direction that is in the z direction.

Z direction I call it radial direction not a vertical direction. So, w in the radial direction are of significant. So, u and w are only the displacement that can be considered to establish the strange tense relationship.

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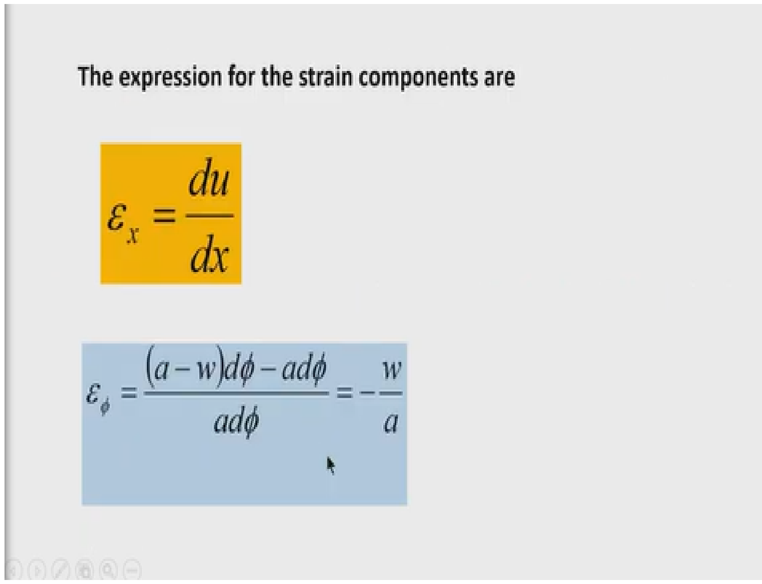

Now, let us see the expression strain components. In the longitudinal direction the strain ε_x is given by $\varepsilon_x = \frac{du}{dx}$. Then circumferential string that is the tangential string that is obtained in this way that I am giving the explanation suppose, due to action of radial load that is the shell is compressed and therefore, original circumference length circumferential length to us $ad\varphi$.

Now, due to compression that is the deformation w in the radial direction the change radius is $(a-w)$. So, $(a-w)d\varphi$ is the change circumference circumferential length. So, difference of this length divided by original circumferential length $ad\varphi$ will give you the circumferential strain. So, this is obtained as $\varepsilon_\varphi = \frac{(a-w)d\varphi - ad\varphi}{ad\varphi} = -\frac{w}{a}$

So, that two strains are now will be utilized to obtain the membrane stress relations in this problem.

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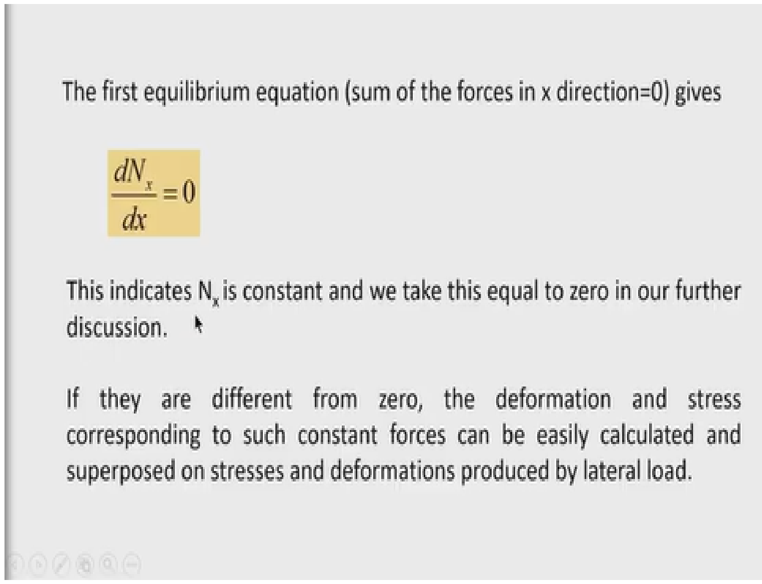

The first equation of equilibrium that I have told you that it is a single term equation and after dividing this term with $adxd\varphi$ we ultimately get $\frac{dN_x}{dx} = 0$. So, this indicates that N_x is constant and we take this constant equal to 0 in our further discussion. Now, if they are different from 0 the deformation and stress corresponding to such constant forces can be easily calculated and superimposed on the stresses and deformation that are produced by lateral load.

So, that is also possible. Now, we take for our calculation that N_x is constant and we take this constant equal to 0 in our further derivation. So, we are concerned only with one membrane forces N_φ. So, our mainly stress resultant are the N_φ then bending moment M_x, M_φ and shear force Q_x.

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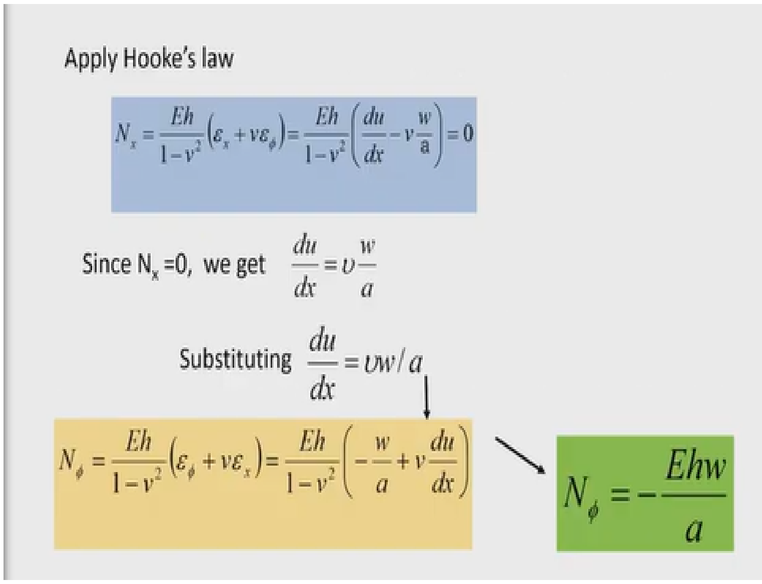

So, apply Hooke's law, if N_x is the membrane force in plane forces along the x direction, we can write this $N_x = \frac{Eh}{1-v^2}\left(\varepsilon_x + v\varepsilon_\varphi\right)$, Eh is the thickness of the shell divided and v is the Poisson ratio. Now substituting the value of ε_x and ε_φ. We can now see that this expression becomes

$$N_x = \frac{Eh}{1-v^2}\left(\frac{du}{dx} - v\frac{w}{u}\right) = 0$$

Now, since it is 0, we now get $\frac{du}{dx} = \upsilon\frac{w}{a}$. So, that is one relation we get it. Here $\left(\frac{du}{dx} - v\frac{w}{u}\right)$ is the circumferential strain that we have obtained here. We obtain here $\frac{w}{a}$ is that circumferential strain epsilon phi. Substituting $\frac{du}{dx} = \upsilon\frac{w}{a}$ in the expression of N_φ. Now N_φ can be written similarly, $N_\varphi = \frac{Eh}{1-v^2}\left(\varepsilon_\varphi + v\varepsilon_x\right)$

So, this is equal to $N_\varphi = \frac{Eh}{1-v^2}\left(-\frac{w}{a}+v\frac{du}{dx}\right)$. Now, you can easily verify it that $\frac{du}{dx}$ is nothing but $\upsilon\frac{w}{a}$ where υ is the Poisson ratio. So, if I substitute this here then and after simplification, we will get N_φ is nothing but $N_\varphi = -\frac{Ehw}{a}$. So, this is very important relationship and which can determine the hoop stresses in this cylindrical shell.

Hoop stress is very important element in the design means, you have to find out by whatever method you consider the hoop stress whether it is compressive or tensile due to nature of the forces and then based on that the thickness of the shell or if it is a reinforced concrete shell hoop reinforcement can be calculated. Now, one thing you can note here the w is the radial deformation that are not determined till now. So, we will now go for determining w.

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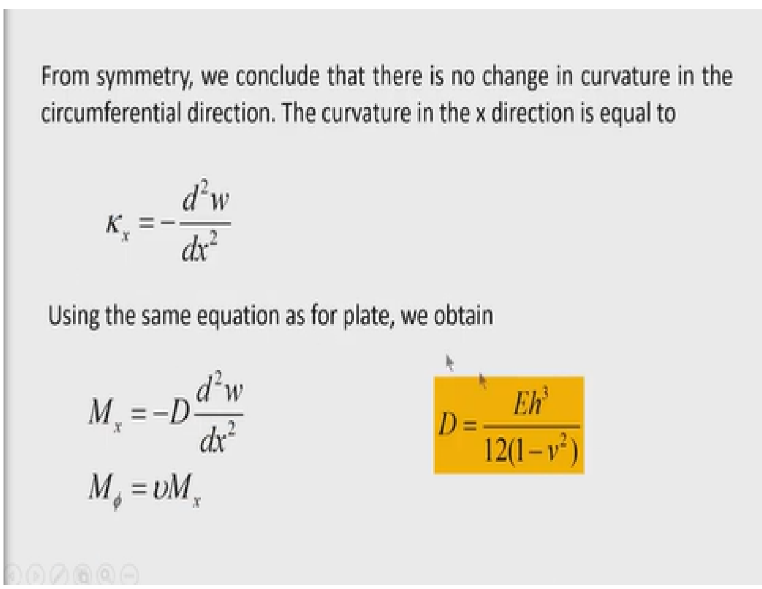

From symmetry we conclude that there is no change in curvature in the circumferential direction. So, curvature in the x direction is taken edge κ_x is the curvature in the x direction it is nothing but $\kappa_x = -\dfrac{d^2w}{dx^2}$. So, this is the curvature and it is constant along the circumferential direction. But it is varies along the x direction. So, using the same equation as for plate now, if you remember the plate equation the bending moment in the x direction was given by - D × (Curvature in the x direction) + υ ×(Curvature in y direction).

Now, since curvature in y direction is not taken into account. So, bending moment M_x is nothing but $M_x = -D\dfrac{d^2w}{dx^2}$, where D is the flexural rigidity of the shell $\dfrac{Eh^3}{12(1-\nu^2)}$. Now, we can note here the flexural rigidity of the shell and plate has the same expression. All those shells are carved element but there is no change in expression of the flexural rigidity.

And M_φ can be obtained similarly, because in that case this curvature in the y direction cannot be taken. So, ultimately M_φ will be $\upsilon \times M_x$. So, that two bending moment expression we obtain.

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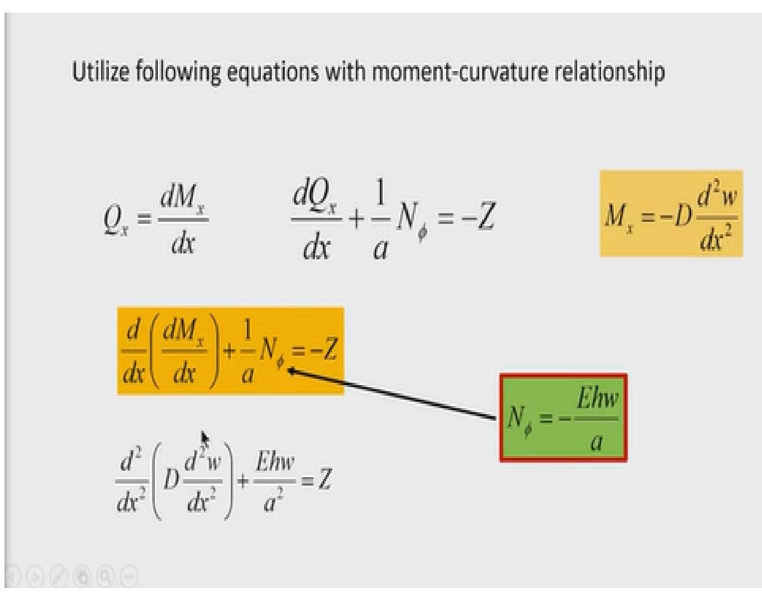

Now go to the shear force expressions. Shear force was given by $\left(\frac{dM_x}{dx}\right)$ and we found this equation from the equilibrium of the forces in the radial direction. So, that equation was $\frac{dQ_x}{dx}+\frac{1}{a}N_\varphi=-Z$. Where Z is the radial route. Now, substitute $Q_x=\frac{dM_x}{dx}$. So, this equation we rewrite here like that $\frac{d}{dx}\left(\frac{dM_x}{dx}\right)+\frac{1}{a}N_\varphi=-Z$. Substitute $N_\varphi=-\frac{Ehw}{a}$ that we have obtained earlier.

So, here it is substituted and after simplification that is M x is also substituted here you see M_x is brought here N_φ is brought here. So, ultimately the equation of equilibrium now is obtained relating the deflection w with the radial load. So, we are getting the $\frac{d^2}{dx^2}\left(D\frac{d^2w}{dx^2}\right)+\frac{Ehw}{a^2}=Z$. Now D we have written inside the bracket in a general sense because D may be also a function of x and y. So, therefore, but if D is constant, we can take it outside.

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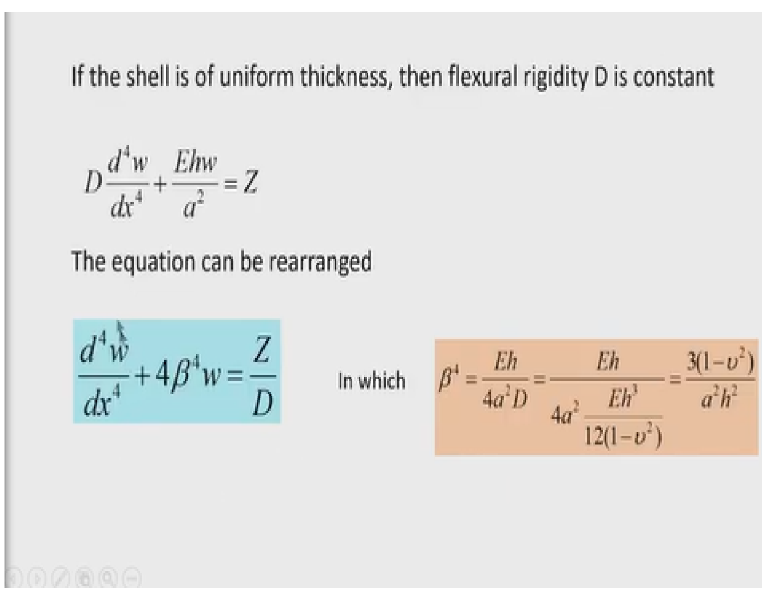

And we can write the differential equation as for constant D as

$$D\frac{d^4w}{dx^4}+\frac{Ehw}{a^2}=Z$$

So, this is the governing differential equations of equilibrium of the cylindrical shell subjected to radial load. This equation can be rearranged suppose if I divide both sides by D then I will get $\frac{d^4w}{dx^4}+4\beta^4w=\frac{Z}{D}$

And substituting D as $\frac{Eh^3}{12(1-\upsilon^2)}$. β^4 is given as $\frac{3(1-\upsilon^2)}{a^2h^2}$. The β is very important parameter for this shell deformation and it is responsible for the stresses that are generated in this shell. Because β is governed by you can see is the material properties also there and shell geometry is also involved here.

So, it is obvious now that we have to formulate a method or device or approach for solving the radial deformation w. Once the radial deformation w is found, then we can easily find the hoop stresses and after taking secondary derivative we can take bending moment and third derivative also will induce the shear force like that. So, first let us go to find this radial deformation w.

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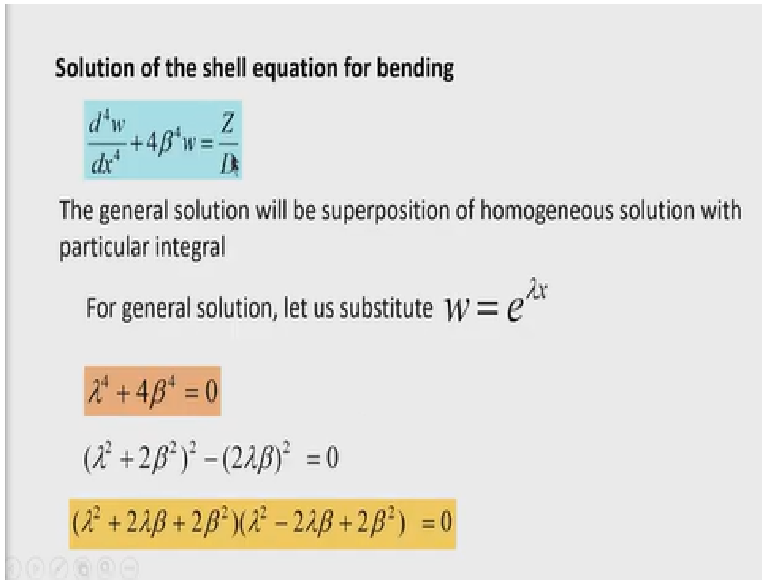

Solution of this differential equation has to be obtained. Now, this differential equation is a non-homogeneous ordinary differential equation of fourth order having constant coefficient. So, it will contain the general solution will contain the homogeneous solution and particular integral, because it is a linear equation. So, take $w=e^{\lambda x}$ for finding the homogeneous solution first.

So, after substituting $w=e^{\lambda x}$ in this equation, we get $\lambda^4+4\beta^4=0$, we take homogeneous case, this right hand side is treated as 0 to find the homogeneous solution. And we substituted this into $e^{\lambda x}$ this is the conventional method of finding the homogeneous solution of the linear differential equation. Then we arrive at this characteristic equation $\lambda^4+4\beta^4=0$.

We have to find the full roots of the λ then we can write down the homogeneous solution. Now, this equation can be simplified or rearranged in this $(\lambda^2+2\beta^2)^2-(2\lambda\beta)^2=0$. Again, this equation can be written as the product of two expressions that is

$$(\lambda^2+2\lambda\beta+2\beta^2)(\lambda^2-2\lambda\beta+2\beta^2)=0$$

So, it is actually written as product of two expressions inside the bracket that you are seeing. Now, to find out the roots lambda it is obvious that either $(\lambda^2+2\lambda\beta+2\beta^2)=0$ or $(\lambda^2-2\lambda\beta+2\beta^2)=0$.

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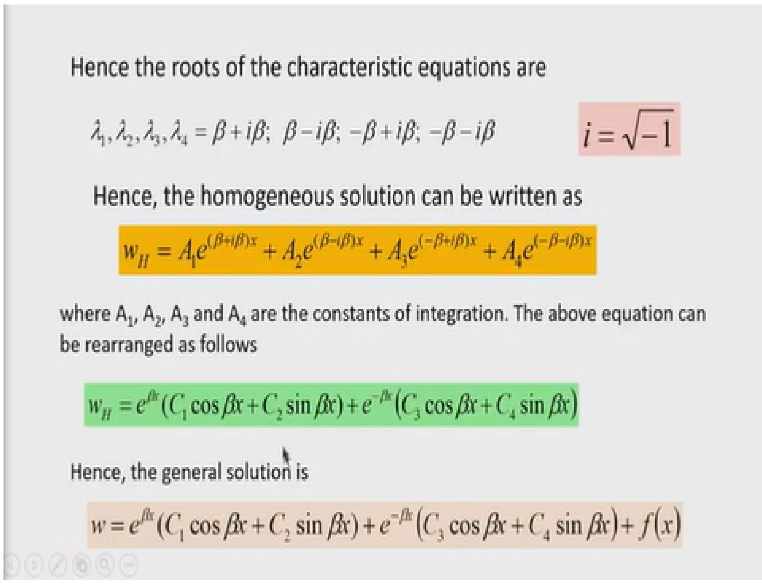

So, if we take these two equations, then we can get after solving the quadratic equation we get 4 roots of λ which is denoted here as $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ and it is values are $\beta+i\beta;\ \beta-i\beta;\ -\beta+i\beta;\ -\beta-i\beta$ where i is the imaginary unit and it is given $i=\sqrt{-1}$. You can see this equation the solution of the characteristic equation is yielding the complex roots, but these are appearing as complex conjugate.

Now, homogeneous solution now can be written as $A_1e^{(\beta+i\beta)x}+A_2e^{(\beta-i\beta)x}+A_3e^{(-\beta+i\beta)x}+A_4e^{(-\beta-i\beta)x}$

So, we have written the homogeneous solution A_1, A_2, A_3, A_4 are constants of integration. Now, it is obvious that $e^{(\beta+i\beta)x}$ any term you take it can be written as $e^{\beta x}\times e^{i\beta x}$ .Now, $e^{i\beta x}$ can be decomposed by the Mobius theorem in trigonometry as $\cos\beta x+i\sin\beta x$. So, like all the terms can be written where $e^{\beta x}$ or $e^{-\beta x}$ will come as a factor. So, therefore, this equation can be rearranged in this form $w_H = e^{\beta x}(C_1 \cos \beta x + C_2 \sin \beta x) + e^{-\beta x}\left(C_3 \cos \beta x + C_4 \sin \beta x\right)$. So, this is the homogeneous solution and depending on the nature of the load, we can now write in general the particular solution as $f\left(x\right)$. So, complete solution of the fourth order shell equation simplified one will be this.

$$w = e^{\beta x}(C_1 \cos \beta x + C_2 \sin \beta x) + e^{-\beta x}\left(C_3 \cos \beta x + C_4 \sin \beta x\right) + f\left(x\right)$$

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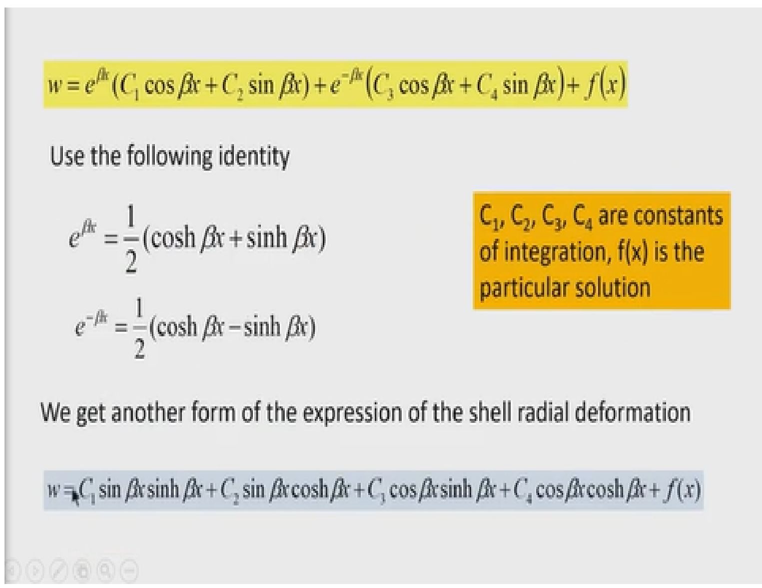

One thing you can observe that β that we have given it is a positive quantity, β is always a positive quantity. So, that you should not and it is unities also as is dimension will be length to the power - 1, so, that you should note. Then interestingly, you can see that since β is positive.

This w can be increased with in case of x or w deflection can decrease with the increase of x. So, these two things we can note here. Now, this equation can also be written in another form. Using this trigonometrical identity as $e^{\beta x} = \frac{1}{2}(\cosh \beta x + \sinh \beta x)$

$$e^{-\beta x} = \frac{1}{2}(\cosh \beta x - \sinh \beta x)$$

So, utilizing these two here instead of $e^{\beta x}$ if I substitute this and instead of $e^{-\beta x}$ if I substitute this quantity and renaming the constants as C_1 , C_2, C_3, C_4 etc. We can now express the equation in another form

$$w = C_1 \sin \beta x \sinh \beta x + C_2 \sin \beta x \cosh \beta x + C_3 \cos \beta x \sinh \beta x + C_4 \cos \beta x \cosh \beta x + f(x)$$

Where, $f(x)$ is the particular solution.

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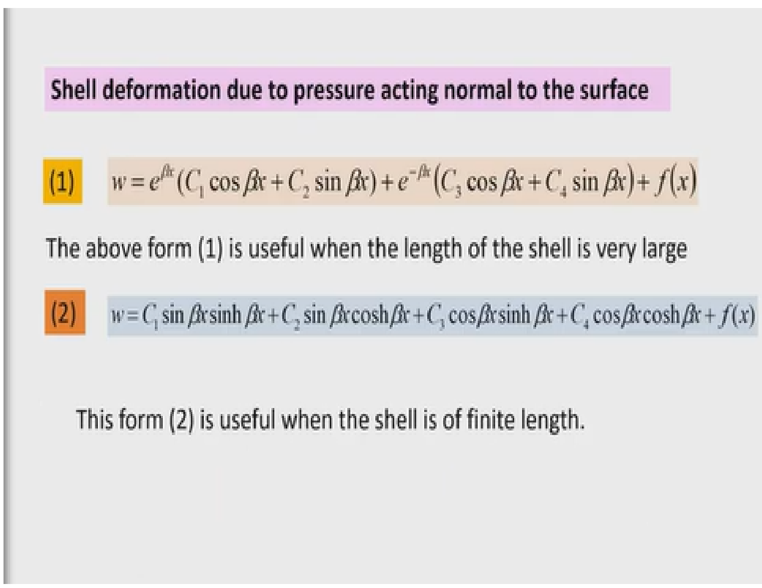

So, two forms of equation we have found but these are identical. So, sometimes you will find that for some problems the first form that is given in exponential with exponential term is useful. And some kind of problem you will find that second form is useful. From our experience and solution of different problems that are found in the textbook. It is seen that the number one form that is written here is useful when the length of the shell is large, very large.

And second form is useful when the shell is having a finite length and these boundaries are completely defined boundary conditions are defined. And the advantage of symmetrical terms and into symmetrical terms can also be invoked depending on the loading pattern. If the loading is symmetrical for example, throughout the cell and if we take the origin at the centre of the cylinder or shell, then we can see that the anti symmetrical term should not come into the deflection equation.

So, in this equation you can see so, many the terms are either symmetrical or anti symmetrical. You can see here $\sin\beta x \sinh\beta x$ is a symmetrical term because $\sin\beta x$ is anti-symmetrical and $\sinh\beta x$ is also anti symmetrical. So, product of two anti symmetrical term is symmetrical. Similarly, you can see this is another term that is symmetrical and symmetrical also symmetrical term these two terms are anti symmetrical.

So, depending on the nature of the loading and other support condition you can take the advantage of the symmetrical or anti symmetrical term to reduce the computational time that is number of constants of integration will be reduced.

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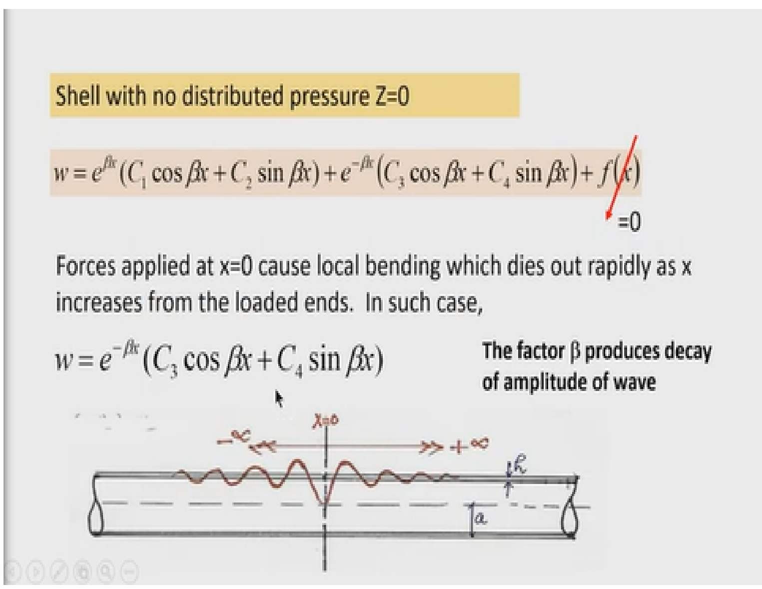

Now, let us see a shell which is long and which has no distributed pressure means, if this shell is subjected to internal pressure through this length that is another problem another kind of problem. But that problem where there is no distributed pressure and only the shell is acted upon by a load uniformly distributed along the circumference symmetrically only at certain point. Then, we can use the equation one the form one.

So, this form is exponential form

$$w = e^{\beta x}(C_1 \cos \beta x + C_2 \sin \beta x) + e^{-\beta x}\left(C_3 \cos \beta x + C_4 \sin \beta x\right) + f\left(x\right)$$

now seen Z is 0 $f\left(x\right)$ we particulars solution we take in 0. So, equation is this. So, deformation is this, due to pressure acting at the point only. So, there is no distributed pressure. So, if the force is applied at x = 0, so, say it is a long cylindrical shell at this if this is the origin.

And if a force is applied distributed along the circumference then no other force or pressure is existing along the length of the infinite shell. So, then you can see that due to application of the load at certain point deflection and slope or curvature will gradually vanish. And therefore, if we consider this term then this will be against the physical meaning of this problem. So, if we take this term then the deflection etcetera and bending moment deflexion slope curvature or other quantities will increase with x.

So, that is not possible realistic. So, therefore, we drop this constant the C_1 and C_2 and for these type of cases we get $w = e^{-\beta x}(C_3 \cos \beta x + C_4 \sin \beta x)$

Here you can see that it acts like a deformation is seemed like a damped wave extending in both directions from the point of application of the load. The effect of β acts like a damping factor.

The factor β produces decay of amplitude of the wave. Actually, the deflated curve will be in the form of wave because cos and sin term are appearing.

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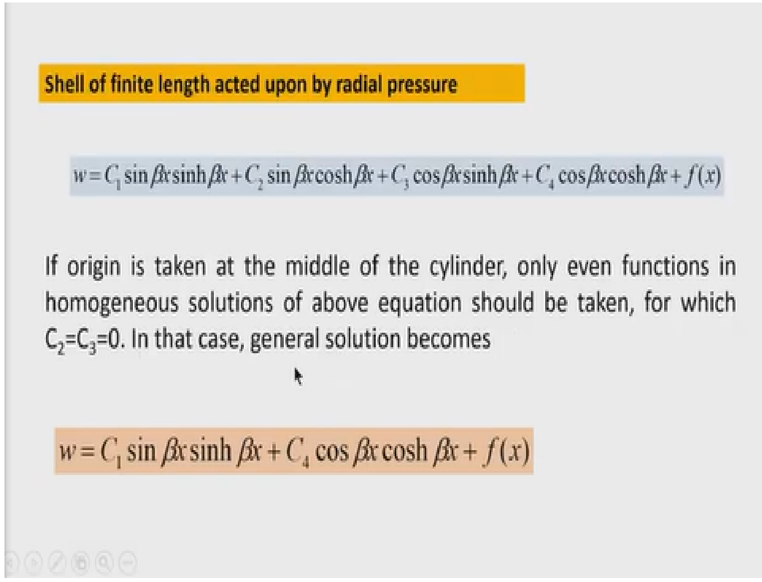

When this shell of finite length is acted upon by radial pressure, we take this expression and when the origin is taken at the centre of the cylinder then only even terms can be taken and we are obviously write $w = C_1 \sin\beta x \sinh\beta x + C_4 \cos\beta x \cosh\beta x + f(x)$

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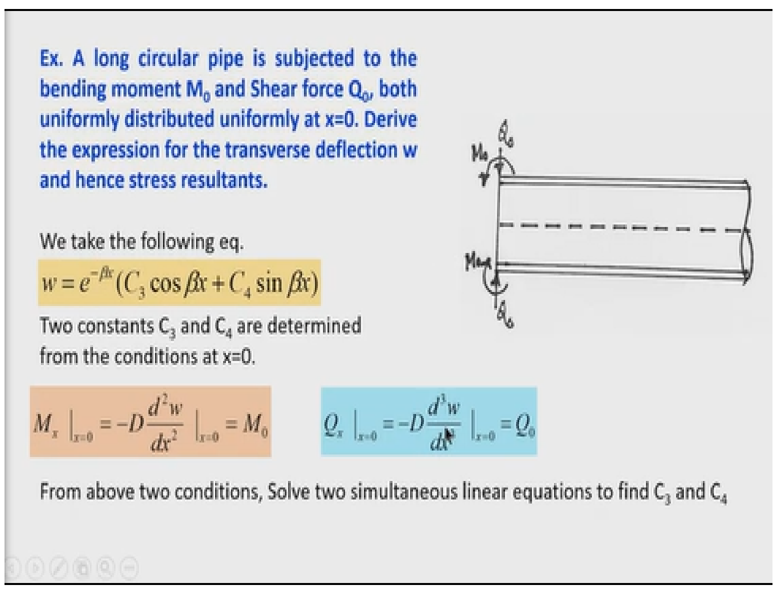

First let us consider a problem where the shell is acted upon by the shearing force and bending moment at certain point only and there is no distributed pressure. So, oviously, we will utilize this solution $w = e^{-\beta x}(C_3 \cos\beta x + C_4 \sin\beta x)$. As e to the power $e^{\beta x}$ terms will be not coming here. Because if it is present there, then deflection and curvature etcetera will increase with x that is not possible.

So, therefore, this expression will only be utilized. Now two constants C_3, C_4 have to be determined. Now, since this shell is long the boundary condition is not specified. So, we have to impose the condition that is given at the point of application of the load and moment. So, at x = 0 the bending moment $-D\frac{d^2w}{dx^2}$ is nothing but the given quantity M_0. Similarly, at x = 0 the shearing force $-D\frac{d^3w}{dx^3}$ is nothing but the Q_0 that is given here.

So, ultimately utilizing this equation we get two equations linear equation for solving these constants C_3 and C_4.

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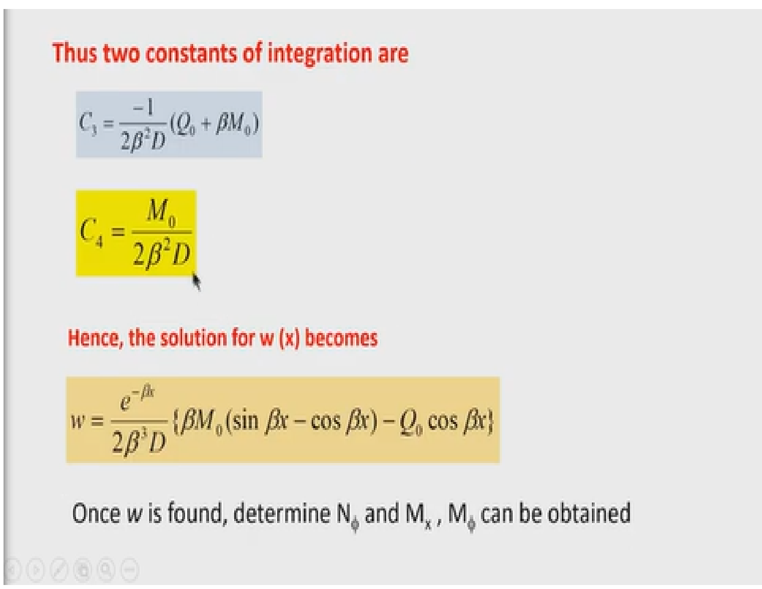

By solving C_3 and C_4 we now get $C_3 = \frac{-1}{2\beta^2 D}(Q_0 + \beta M_0)$ and $C_4 = \frac{M_0}{2\beta^2 D}$. So, how C_3 and C_4 are coming here you will have understood it by applying the condition at x = 0 bending moment is as M_0 and shear force at x = 0 as Q_0. And utilizing this expression of w, we arrived two equations with two unknown C_3 and C_4 and after solving that, we get C_3 and C_4.

So, hence, the solution for w now becomes

$$w = \frac{e^{-\beta x}}{2\beta^3 D}\{\beta M_0(\sin\beta x - \cos\beta x) - Q_0\cos\beta x\}$$

Once w is found determine N_φ because N_φ is nothing but $\frac{Ehw}{a}$. So, you can determine N_φ and then after taking second derivative you can determine M_x and M_φ is nothing but υM_x. So, all these quantities can be obtained.

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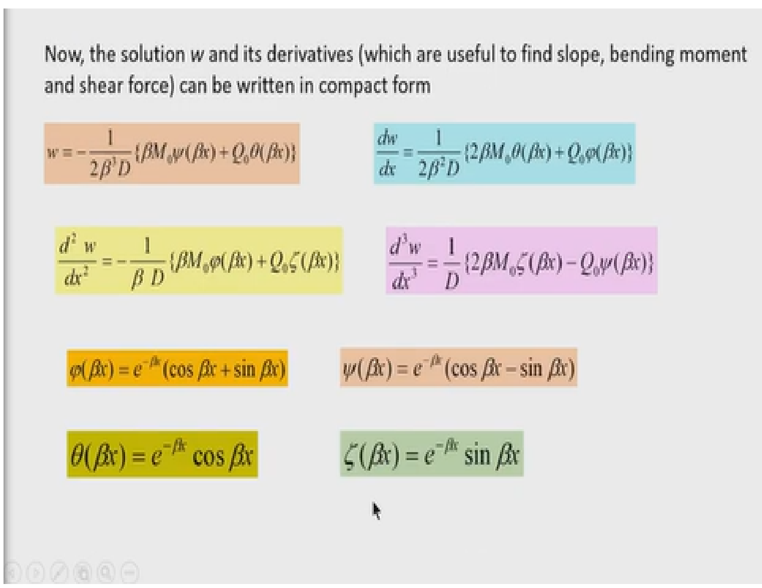

So, w is this and slope, if you want to find you can find after differentiating w. Bending moment of course sequence the second derivative. So, second derivative is calculated and you can see second derivative is written but in the expression that I have written here. You may carefully note that I have brought the functions ψ, β and then φ then ζ all function that I brought here. That function has very interesting characteristics.

All functions are with decaying factor $e^{-\beta x}$ you can see that $\phi(\beta x)$ is function $e^{-\beta x}(\cos\beta x+\sin\beta x)$. $\psi(\beta x)=e^{-\beta x}(\cos\beta x-\sin\beta x)$, $\theta(\beta x)=e^{-\beta x}\cos\beta x$, $\zeta(\beta x)=e^{-\beta x}\sin\beta x$ So, these are written using this function in the compact form.

But the quantities are obtained after taking the derivative. We need the derivative of 3rd order to determine the shear force and derivative of second order to determine the bending moment and this is of course, the slope.

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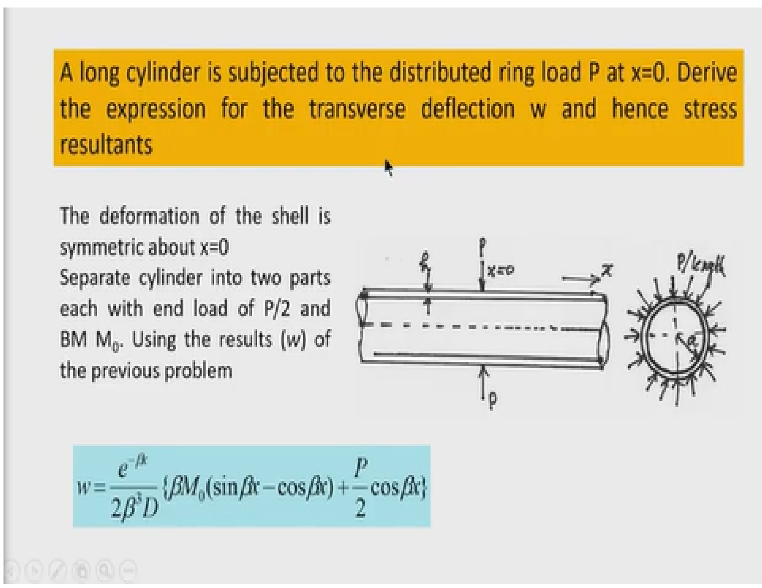

Let us apply this known expression that we obtained here to solve a problem of long cylindrical pipe subjected to a distributed ring load. The pipe is distributed load here you can see the ring load which is P per unit length. The question is to derive the exhibition for the transverse deflection w and the stress resultant stress. Stress resultants are nothing but bending moment and N_φ, M_φ. And M_φ of course is dependent on a M_x

Deformation of the shell is symmetrical about this x = 0 if I take the origin at x = 0 that is the point of application of the load. We now utilize this equation that we earlier obtain, but before that, we separate the cylinder into two parts each with equal load $\frac{P}{2}$ and $\frac{P}{2}$. And bending moment M_0 because of action of the load when we cut the cylinder the bending moment M_0 will also come but M_0 is undetermined here.

So, only the load that is P is given we take half of the load in the left side and half of the load in the right side. So this, Q_o we can take it take as $\frac{P}{2}$ so, utilizing the earlier expression this w is like that w is this. So, instead of Q_o we write $\frac{P}{2}$ and $-\frac{P}{2}$ of course. So, this expression becomes $w = \frac{e^{-\beta x}}{2\beta^3 D}\{\beta M_0(\sin\beta x - \cos\beta x) + \frac{P}{2}\cos\beta x\}$

Now, one thing is that M_0 is still not determined we know P. So, let us go to find M_0 by imposing the condition at x = 0.

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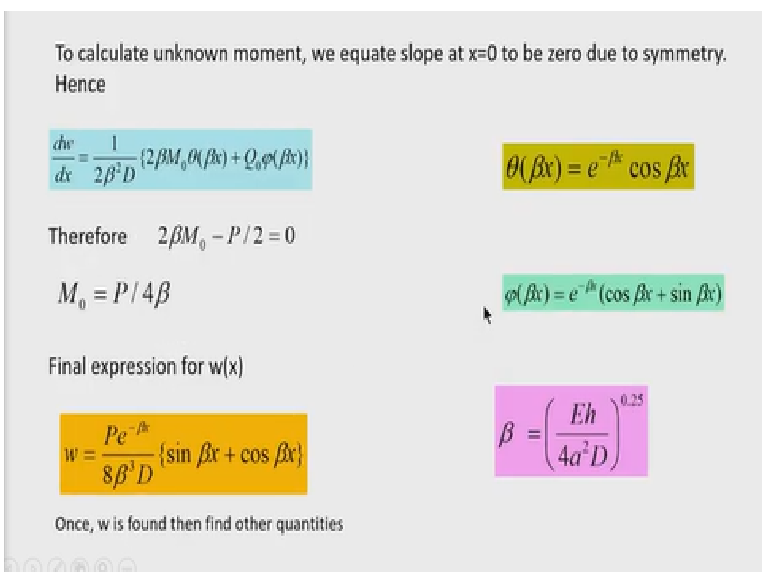

Because of symmetry you can see that slope at x will be 0 because the problem is symmetrical problem the slope at x = 0. So, if I take slope, slope we have written earlier this expression if I take this slope and equate to 0 of course, substituting Q_o as $-\frac{P}{2}$. So, we will get this expression for M_0. So, after writing these functions are written here these θ, φ etc. all the functions are written in this form this functional form are given earlier.

This θ, φ etc. so, utilize this utilize this function and then we have written in compact form using the function that we had defined earlier. And here it is given $\theta(\beta x) = e^{-\beta x}\cos\beta x$. So, we get ultimately at x = 0 by substituting x = 0 because we are imposing the condition on x = 0, we ultimately get this equation $2\beta M_0 - P/2 = 0$ and then we got $M_0 = P/4\beta$

So, the deflection equation now becomes $w = \frac{Pe^{-\beta x}}{8\beta^3 D}\{\sin\beta x + \cos\beta x\}$

Now, once of the deflections is found other quantities can be calculated and remember that beta is this factor, the characteristic factor. So, that is very important or you can call it decaying factor also.

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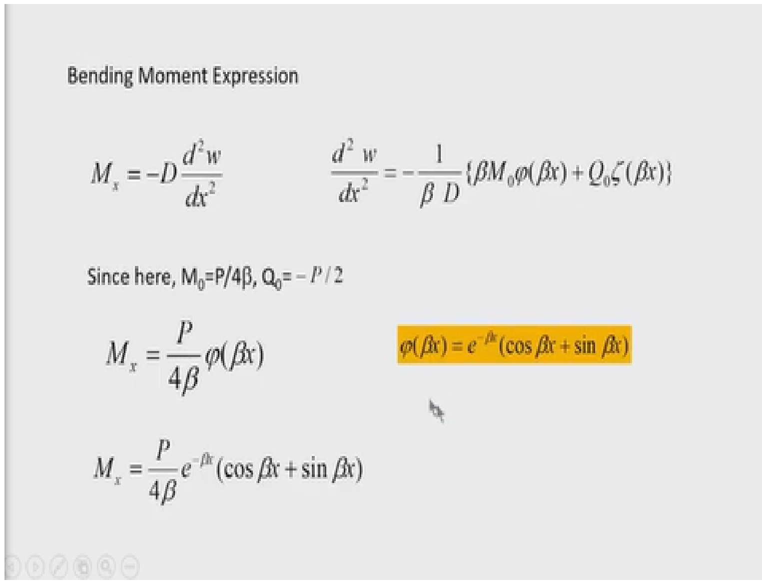

Binding moment expression is nothing but $M_x = -D\frac{d^2w}{dx^2}$ and we get $M_x = \frac{P}{4\beta}\phi(\beta x)$. So, we can now write $M_x = \frac{P}{4\beta}e^{-\beta x}(\cos\beta x + \sin\beta x)$. Two interesting thing is there the maximum value of deflection and bending moment are obtained at the point of application of the loads. So, if you put x = 0 the maximum deflection comes out as $\frac{P}{8\beta^3 D}$.

So, this is the maximum deflection of the pipe. Similarly, maximum bending moment you can see here if you put x = 0 maximum bending moment is $\frac{P}{4D}$.

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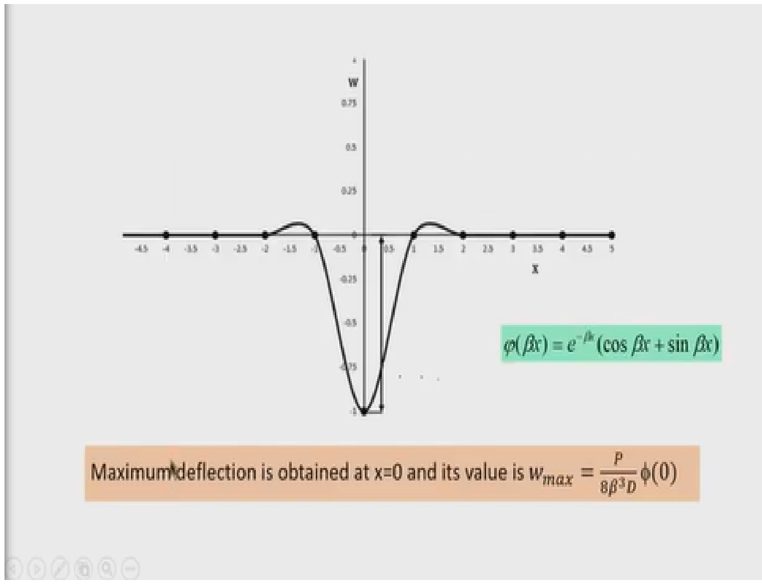

So, if I draw the graphs of the deflection of the shell along the length, then you will find that it is a dumped wave, but dumping is a dumping factor because of high rigidity of the shell, you will find that decaying is very fast. Of course, it depends on the thickness of the shell and the modulus of elasticity, the beta factor depends on that. So, if you improve the material the waves will be dumped earlier. And the weak material loss shell is thin then the deflection will continue for longer distance.

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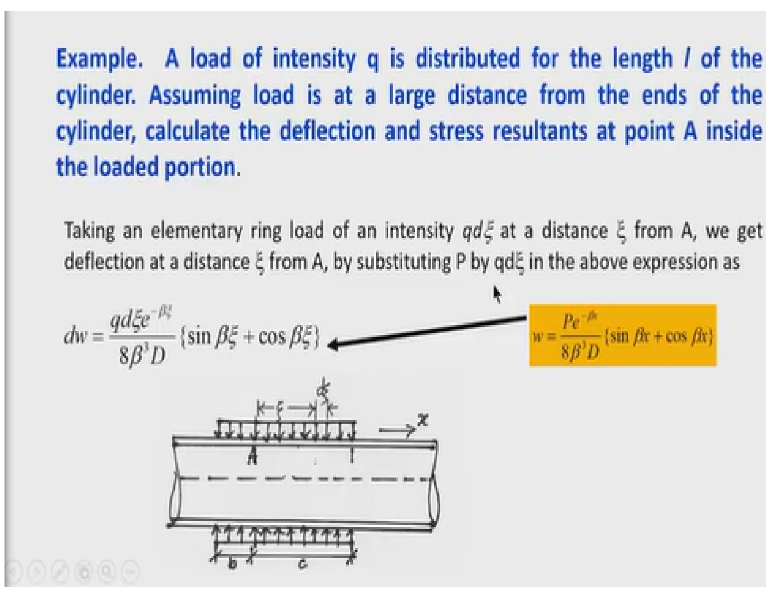

Let us see another application. A load of intensity q, it is not like a point load it is distributed over certain length. The length of the load is b + c and it is required to determine the deflection at A. So, we want to utilize the earlier this result that we know for deflection. So, that result will utilize for point load. How will you utilize here? You see these distance is given and it is required to determine the deflection at point A.

So, we will consider a small element at a distance ξ the length of the element in $d\xi$. So, at this element the concentrated load that is acting is $q \times d\xi$. So, in the formula in the earlier formula for deflection, we write here P as $q \times d\xi$ and other factors are the $e^{-\beta\xi}$ divided by $8\beta^3 D$ whole thing multiplied by $\{\sin\beta\xi + \cos\beta\xi\}$ and this is the deflection of this element.

$$dw = \frac{q d\xi e^{-\beta\xi}}{8\beta^3 D}\{\sin\beta\xi + \cos\beta\xi\}$$

Now, to find out the contribution of the full load that is acting on the shell element, we have to integrate it.

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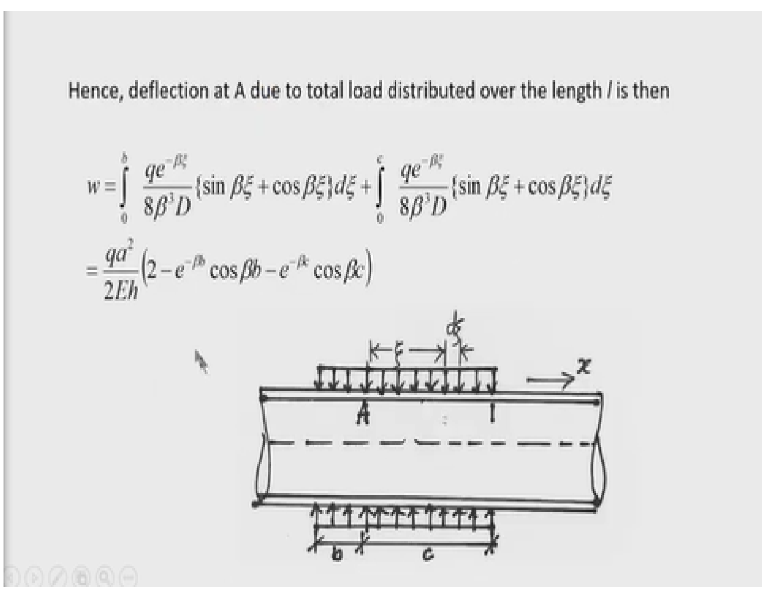

So, integrating in the limit, so, here is a 0 to c and towards the right it will be 0 to b. So, integrating these two expression we now get $w = \frac{qa^2}{2Eh}\left(2 - e^{-\beta b}\cos\beta b - e^{-\beta c}\cos\beta c\right)$

So, this is the deflection of the point A. Due to a strip load you can call it that load acting for a certain length it is not a load at a particular section only.

So, that problem we have solved utilizing the formulation of the deflection that we have already derived for the load acting at a certain point.

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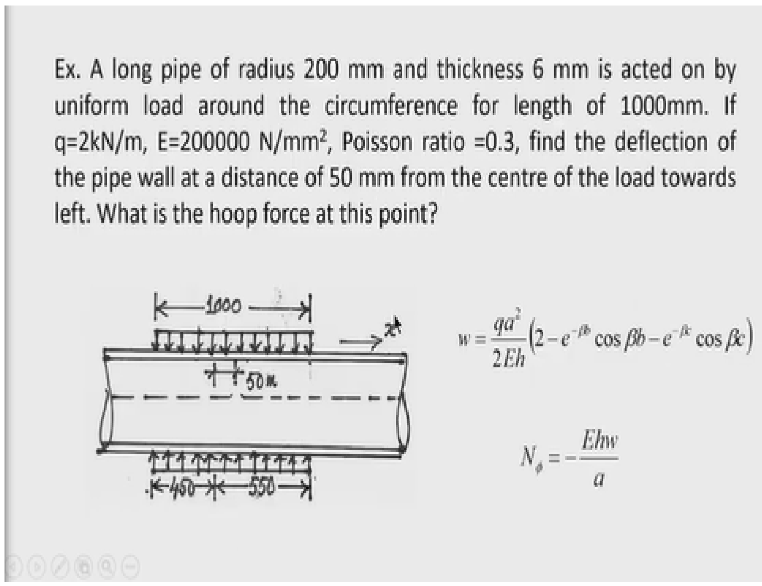

Let us see a numerical problem of the application of the theory. A long pipe of radius 200 mm and thickness 6 mm is acted on by uniform load around the circumference. So, this load actually is acting around the circumference uniformly and length of the load is 1000 millimetre, if q is that is 2 $\frac{kN}{m}$ modulus of elasticity is 2×10^5 Poisson ratio is 0.3, find the deflection of the pipe wall at a distance of 50 mm from the centre of the load towards left.

So, centre of the load is here. So, we have to find the deflection here. So, this point is now of this distance the b is 450 and c is 550. So, in our earlier formulation that you recognize b and c, so, these two distances are found out here. So, w we write as

$$w = \frac{qa^2}{2Eh}\left(2 - e^{-\beta b}\cos\beta b - e^{-\beta c}\cos\beta c\right).$$

And once the w is found we can find $N_\varphi = \frac{Ehw}{a}$.

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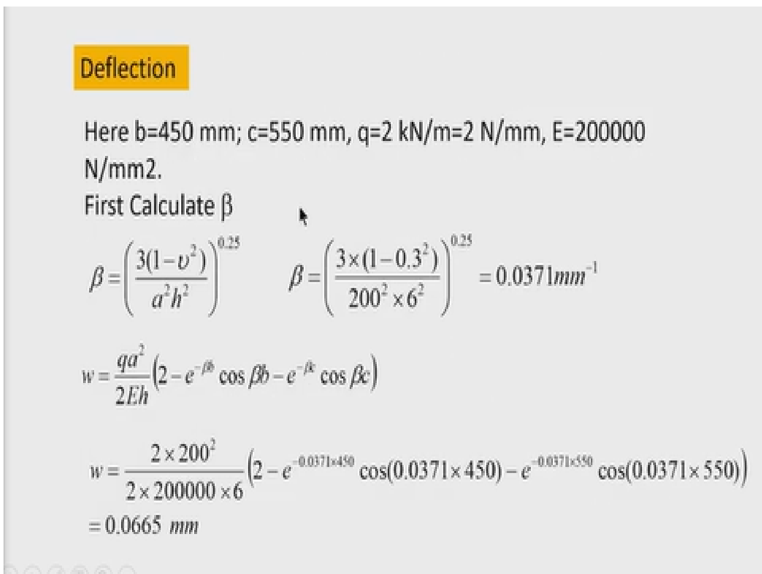

So, first step is to find the characteristic parameter beta. So, here b is 450 c is 550 q is 2 kilonewton per meter there is 2 $\frac{kN}{m}$ E is this 2×10^5 $\frac{N}{mm^2}$. So, β is calculated by using this expression that I have shown you earlier $\beta = \left(\frac{3(1-\upsilon^2)}{a^2h^2}\right)^{0.25}$. So, β is coming out as 0. 0371 millimetre to the power - 1. Substituting the value properly here w is

$$w = \frac{2\times 200^2}{2\times 200000 \div 6}\left(2 - e^{-0.0371\times 450}\cos(0.0371\times 450) - e^{-0.0371\times 550}\cos(0.0371\times 550)\right)$$
$$= 0.0665\ mm$$

One thing you should mind it, that when we evaluate the cos or sin function that quantity has to be taken into radian. Here it is product so, the denominator is 2 Eh so $2\times 2\times 10^5 \times 6$, 6 is the thickness i.e h.

After performing the numerical calculation very carefully the question is that you have to take this as the radian if you take in degrees then your result will be wrong. So, w is coming as 0.0665 millimetre.

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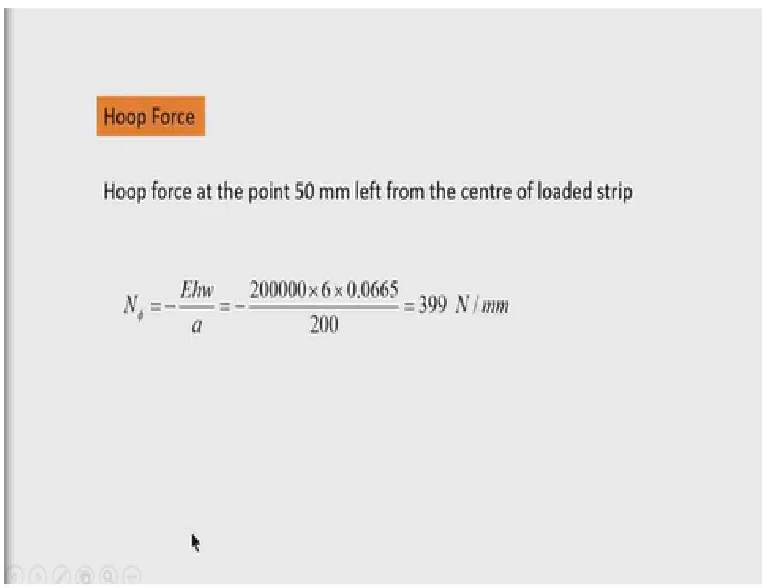

So, let us find out now hoop force, hoop forces depending on this deflection here it is compressive force. So, $N_\varphi = -\frac{Ehw}{a}$. So, here we have found $2\times 2\times 10^5 \times 6$ multiplied by 0.0665 divided by 200. So, after calculating this it is coming 399 $\frac{N}{mm}$.

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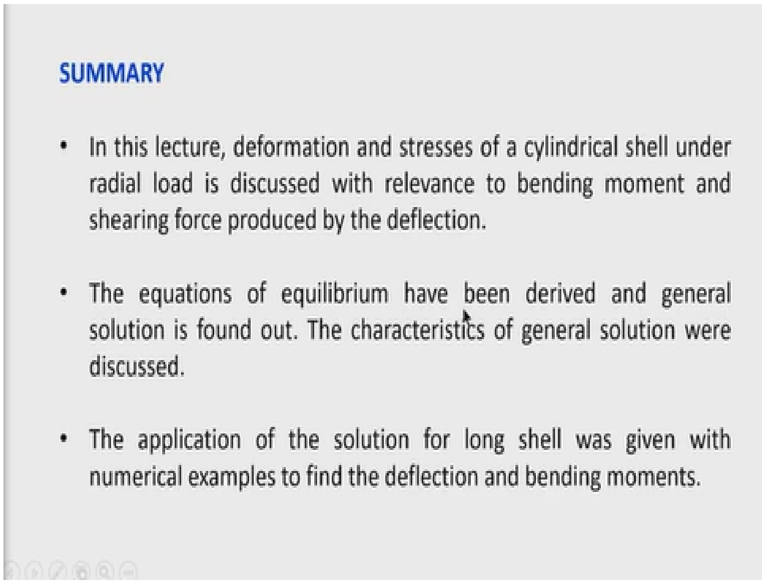

So, let us summarize what we have done today. In this lecture deformation and stresses of a cylindrical shell under radial load is derived with relevance to bending moment and shearing force produced by the deflection. The equations of equilibrium have been derived and general solution is found. So, procedure for finding the gentle solution is completely described and two cases are actually shown.

One is with long shell this is exponential form and another with shells of finite length where boundaries are specified. Then we will use this hyperbolic trigonometric and hyperbolic form. So, two forms of equations are derived and we have used one form that is for long shell and application of the solution for long shell was given with numerical examples. Thank you very much.