# Plates and Shells

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## Lecture - 29

## Membrane Theory in Pressure Vessel in the Form a Torus and in a Tank of Arbitrary Meridian

Hello everybody. Today I am discussing again the membrane theory on pressure vessel, and this is my lecture to have the Model number 10. So, in the last class, I discussed how the pressure vessels can be analysed using the membrane theory. And pressure vessels are usually a surface of revolution. So, membrane theory is very well accepted here under the uniform internal pressure. And we have discussed four kinds of pressure vessel in the last class.

One is your circular cylindrical pressure vessel, then conical circular conical pressure vessel, then spherical pressure vessel and pressure vessel in the form of ellipsoidal of revolution. So, that we have discussed, using the membrane theory and necessary expressions for $N_\phi$, $N_\theta$, $N_\phi$ represents the meridional force and $N_\theta$ represents the circumferential force. So, these are the force per unit length and when you are required to convert into stress.

Then you divide $N_\phi$ and $N_\theta$ by thickness of the shell. You will get the quantity in terms of stress. So, today I will discuss another form of pressure vessel. That is called a torus and this torus is formed by revolving a section. Section may be circular or elliptical or it may be a rectangle also, around an axis of rotation and then it forms a surface of revolution. So, that kind of shell will take today for analysis.

And after that, we will again apply the membrane theory in a liquid retaining tank which is formed by any arbitrary meridian. Previously we have seen that when we discussed the spherical dome or you are this conical shell. We consider this meridian as part of a circle or it is a straight line. But, if it is other than this cup, say it may be any arbitrary cup, maybe a cubic parabola, maybe a quadratic parabola or it may be a cos hyperbolic type function.

That is a catenary function. So, all these types of arbitrary meridian can be handled by the general theory that I want to discuss today.

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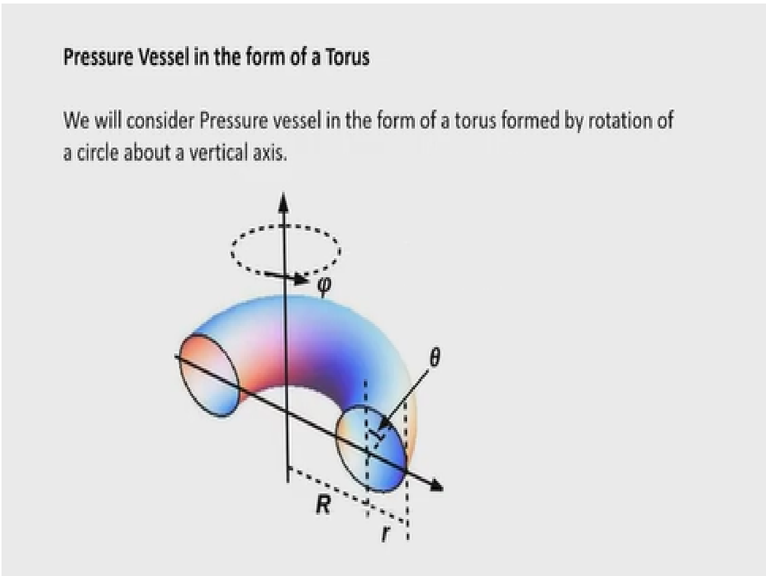


Now, pressure vessel in the form of torus is very common in industrial application and it is formed here by rotating a section. Here of course, in this figure as circular section is shown and it is rotating and vessel is formed which is known as the toroid. Sometimes, it is a pressure vessel in the form of a torus. So, it is a circular torus. If ellipse is rotated, elliptical section is rotated, then it will be elliptical torus.

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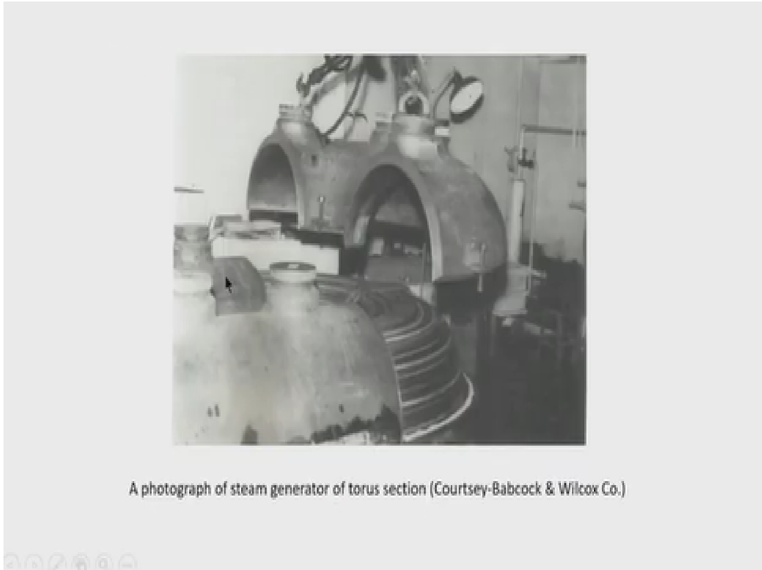


Now, the application of torus type of shell is common in industrial application. A photograph here you can see, it is a steam generator and we had the section of this is a circular, and it is a circular torus section.

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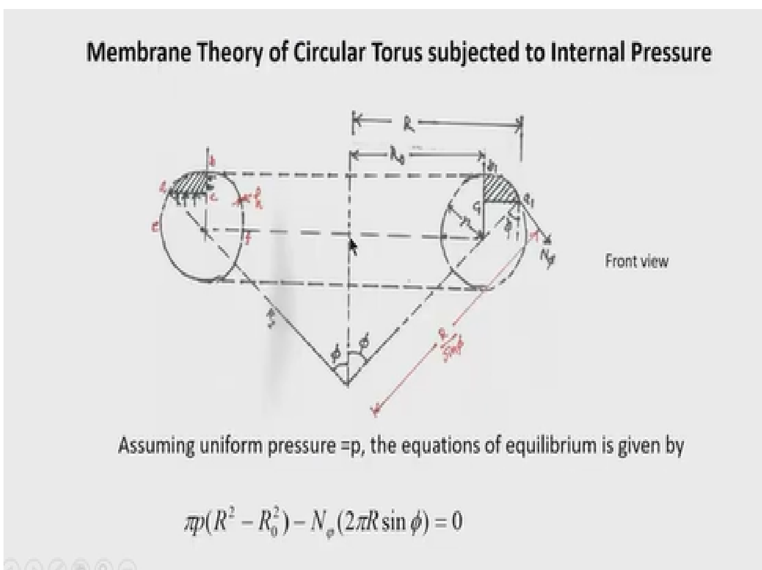


So, now, let us see the geometry of the torus. First, you have seen this torus in this form. So, if I see the front view, then it looks like that. Where it is the section of the torus, it is a circular section. And, you can see this, if it is rotated around a circle of radius $R$. So, this is radius $R$ and the centre of the section of the centre of the circular section is at a distance of $R_0$ from the vertical axis of rotation.

The meridional angle at any point on the torus is ϕ and this $R$ is the radius of the circular section. And we can find the principal radius of curvature from our general expression but it is a circular section. So, first principal radius of curvature is $R_1$ equal to radius of this circle. The second principal curvature is this $R_2$, $R_2$ is nothing but our $\frac{R}{\sin sin\ \phi}$. If $R$ is this distance, then this $\frac{R}{\sin sin\ \phi}$ will be your second principal radius of curvature.

So, this is your second principal radius of curvature is nothing but $\frac{R}{\sin sin\ \phi}$. Now, if I apply the equilibrium of the forces in the vertical direction then I get after resolving $N_\phi$ in the vertical direction, you can see here. The vertical component of $N_\phi$ per unit length is $N_\phi \cos cos\ (90 - \phi)$ which is nothing but $N_\phi \sin sin\ \phi$. So, $N_\phi \sin sin\ \phi$ is the meridional force per unit length.

Now, if I multiply it by the circumference of the torus, circumference of the torus can be easily found out say this is the circumference. So, if this is the radius, full radius is $R$, then circumference will be $2\pi R$. So, multiplying this vertical component $N_\phi \times 2\pi R \sin sin\ \phi$. We get this total vertical component of this meridional force. $N_\phi \sin sin\ \phi$ is the vertical component and $2\pi R$ is the circumference length of the torus.

So, this is to be equated or it has to be balanced by the pressure, internal pressure acting on this section. So, internal pressure is uniform pressure it is $p$, and if we see this it is acting on the section, that is given by this radius $R - R_0$. So, therefore, internal pressure will be

$\pi p \times (R^2 - R_0^2)$. So, balancing this total force due to internal pressure in the; vertical direction with the vertical component of meridional folds around the circumference.

Total vertical force of the meridional folds around the circumference of the torus we get this equation. From that equation, we can evaluate $N_\phi$.

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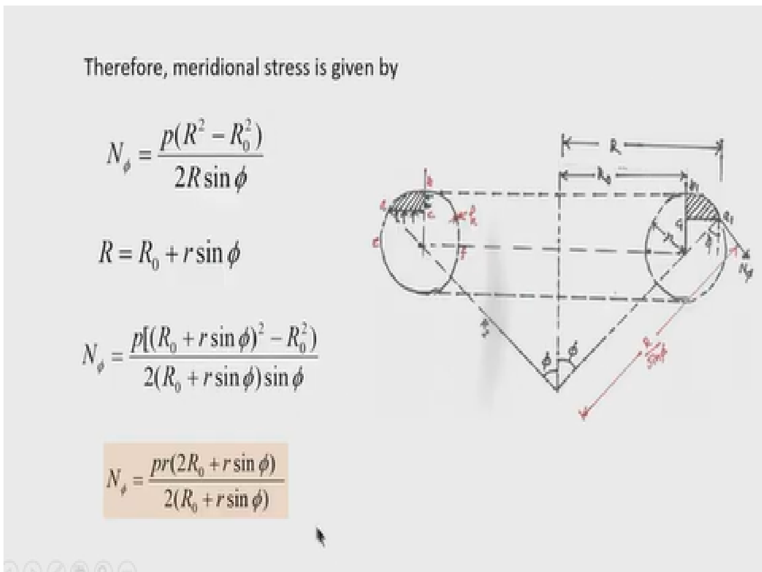


So, $N_\phi$ is now coming as $\frac{p\left(R^2 - R_0^2\right)}{2R\sin sin\,\phi}$. You can note here, that $R_0$ is the distance of the centre of this circular section and $R$ is the radius in which the circular section is rotated. So, again we can see that, the radius $R$ in which is the radius of the circular path in which the circular section is rotated. Then $R$ can be expressed as $R_0 + r \sin sin\,\phi$ . So, this is the $r$, this is also $r$. So, this quantity is $r \sin sin\,\phi$ .

So, therefore, $R_0 + r \sin sin\,\phi$ is the distance periphery of the torus, the extreme point of the torus from the vertical axis of rotation. Now, substituting this value here because from the geometry of the torus will be knowing say $R_0$ the distance of the circular section from the vertical axis of rotation as well as will know the radius of the circular section. So, we can substitute this in this expression that, $R = R_0 + r \sin sin \phi$ . Then we can simplify it.

So, using the algebraic formula that $a^2 - b^2 = (a + b)(a - b)$, we can now obtain this quantity. So, $R_0 + r \sin sin \phi$ square instead of this $R^2$ and $R_0^2$ is already there. So, applying this algebraic formula for factorization we now get the $N_\phi$, the meridional stress is $N_\phi = \frac{pr(2R_0 + r sinsin \phi)}{2(R_0 + r sinsin \phi)}$. So, here you can see that $p$ is the uniform internal pressure and this is constant.

So, there is no variation of this $p$ and $r$ is the radius of the circular section. Now, let us examine how these forces vary with the meridional angle $\phi$.

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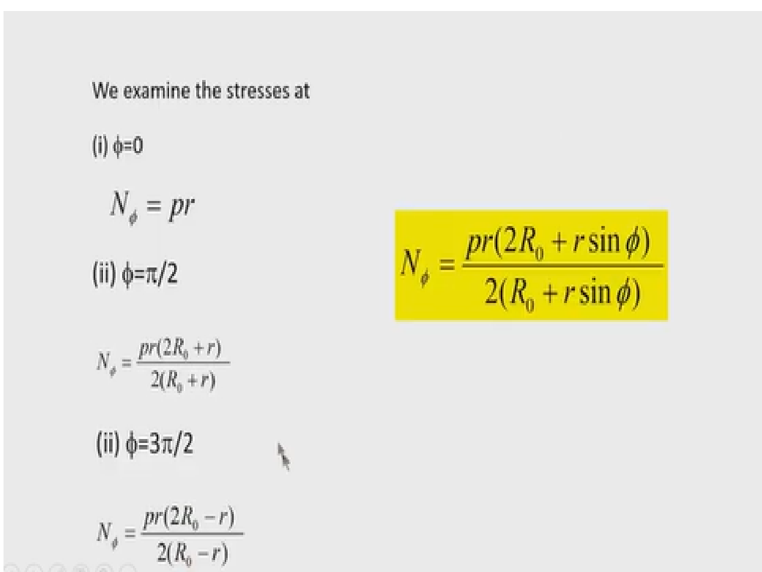

So, if I examine the meridional stress $N_{\phi}$ which is given by $= \frac{pr(2R_0 + r \sin sin \phi)}{2(R_0 + r \sin sin \phi)}$. Then, we get at $\phi = 0$, $\phi = 0$ put $\phi = 0$ in this expression, you will get $N_{\phi} = pr$. This is easily verified because this sin $sin$ $\phi$ will be 0 here and these $R_0$, $R_0$ will be cancelled. So, we will get simply $N_{\phi} = pr$. Now, the point is located here, $\phi = \frac{\pi}{2}$ at this point.

So, section is here now you can see this section is here this front view you are seeing the circle. But here you can see the circle from the side. So, that means, if we put $\phi = \frac{\pi}{2}$. Then this becomes $\frac{pr(2R_0 + r)}{2(R_0 + r)}$. So, this is expression of n phi at different position $\phi = \frac{\pi}{2}$. In all the cases, we are finding that the $N_{\phi}$ that is the meridional stress under the action of uniform compression, internal pressure is now a tensile force.

But it can reverse it sign at this angle when $\phi = \frac{3\pi}{2}$, if I locate a point on the toroid, with an angle measured from this vertical axis of rotation at $\frac{3\pi}{2}$. Then we can get here, this will be $-\ r$ because sin $sin$ $\phi$ will be $\frac{3\pi}{2}$, sin $sin$ $\frac{3\pi}{2}$ $=- 1$. So, we are getting $\frac{pr(2R_0 - r)}{2(R_0 - r)}$. So, this is the quantity of this meridional stress at different points on the toroid. Now, once the meridional stress is found, how the meridional stress is found?

Meridional stress is found by considering the equilibrium of the forces in the vertical direction. The equilibrium of the forces in the vertical direction; that the original equation that I have shown here is actually derived from the differential equation. Then differential equation was formed by resolving all the forces in the meridional reaction. But here when the differential equation was solved, I have shown in my previous lecture.

Then integrated results that, when it is integrated, differential equation is integrated, physical interpretation of this integral expression can be obtained simply by understanding that vertical component of $N_\phi$ is equal to the total vertical load at this level. So, whatever level we are considering for finding the $N_\phi$ and if we resolve the $N_\phi$ in the vertical direction total $N_\phi$. Then, it should be balanced by the total vertical load above that level.

So, this integral expression can be physically interpreted in a very simple way and this is very easy to apply in design of fish, instead of interrogation. So, we have applied here the physical meaning of the differential equation solution of the differential equation and then we get the meridional stress $N_\phi$. Now, second membrane stress that is $N_\theta$ that is in the circumferential direction has to be obtained.

From the equilibrium equation which is obtained in the radial direction by resolving the forces all forces in the radial direction. Radial direction, I mean that it is a normal direction along the tangent. Normal direction along the normal at the point located at the tangent. So, that means, if I resolve all the forces along the radial direction, we found that in case of these shells of surface of revolution. This equilibrium equation turns out to be an algebraic equation.

So, that is very interesting thing. The other two equations are the differential equation. But third equation is an algebraic equation. But second equation, of course, is not meaningful for the case, where the loading is axisymmetric and once the first equation is solved for the meridional stress and phi then, the third equation which is simply given by this.

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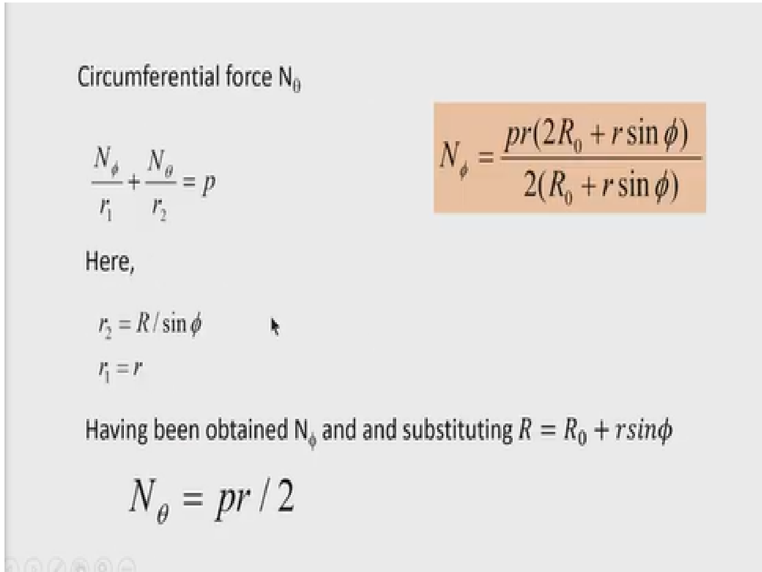


$\frac{N_\phi}{r_1} + \frac{N_\theta}{r_2} = p$, $p$ is the radial load. Here the internal pressure on the toroid is the radial load. So, from this algebraic equation, it can be easily found that $N_\phi$, $N_\theta$ can be calculated. Because $N_\phi$ is already evaluated as $\frac{pr(2R_0+r\sin sin\,\phi)}{2(R_0+r\sin sin\,\phi)}$. So, substituting this $N_\phi$ here, and $r_2$ we have seen that, second principal radius of curvature $r_2 = R/\sin sin\,\phi$ .

So, here substituting these $r_2 = R/\sin sin\,\phi$ and first principal radius of curvature, that is the radius of this circle. Then $r_1 = r$. So, if we substitute here and after the simplification, we get . Of course, we have to substitute this $R = R_0 + r\sin sin\,\phi$ . So, therefore, this very simple expression for $N_\theta$ is obtained as $\frac{pr}{2}$. Where $p$ is the internal pressure on the pressure vessel, $r$ is the radius of the circular section of the toroid.

And it is multiplied by 0.5 that is the $\frac{pr}{2}$. So, in this way, we can analyse the pressure vessels of different types that I have demonstrated. Four types of pressure vessels I have taken for this course. One is circular cylindrical pressure vessels and then second is circular conical pressure vessels then third is spherical pressure vessels, and fourth is the pressure vessels in the form of ellipsoid.

And now, we have discussed another form of pressure vessel that is in the form of toroid. So, the analysis is based on the membrane theory of this surface of evolution.

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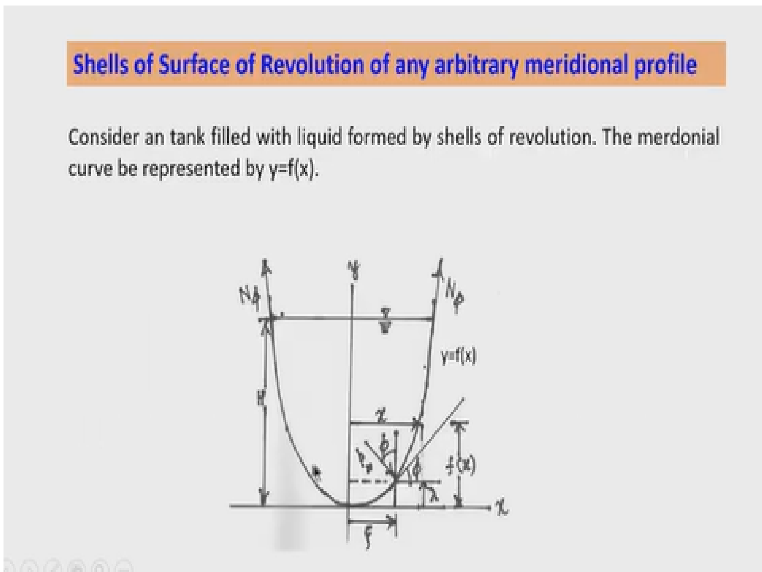


The second item that I will discuss today is a surface of revolution again when we are dealing with the surface of revolution of the shell. And then this I will be discussing with reference to a liquid retaining a structure. So, this surface of revolution is formed by rotating a meridional curve. This is a meridional curve whose equation in general, it is given as $y = f(x)$. It may be quadratic parabola say y may be equal to $kx^2$.

y may be equal to $kx^3$ , y may be equal to any polynomial expression of higher degree or it may be a hyperbolic function cause hyperbolic function different types of this meridional profile may be adopted. And, efficiency may also be compared. So, this problem is very interesting because it is a general formulation for this surface of revolution formed by any meridional profile.

And we can compare the efficiency of different meridional profile by evaluating the stresses. In some cases, you will find that meridional stress and hoopsters is lowest among all the types of meridian. So, I have demonstrated here a general expression. But then, I will use it for solving a numerical problem in which I will take a particular type of meridian. So here, you can see this water depth is up to H, H is the water depth $N_{\phi}$ is the component of the meridional stress.

And it is acting around the circumference. Then at any coordinate ξ, the x coordinates ξ, the y coordinate is λ. So, the slope of the meridional curve is defined by the angle ϕ, that is slope is $\tan tan\, \phi$ . So, this angle is ϕ. So, here again, this angle is ϕ and the pressure, that is acting on the wall of the tank, it is normal to the wall of the tank and has equal magnitude in all direction, according to Pascal law.

So, at any distance say x, we have the depth of water calculated as effects that is there depth of water is y. So, if I know the equation of y at any distance, at any coordinate x, the depth of water is calculated. But the diameter of the tank is $d$ that means the radius is $d\ by\ 2$, the limit of the axis 0 to $\frac{d}{2}$. Now, let us calculate the geometrical parameters and then we will proceed towards the analysis of the forces. The y axis here, the axis of rotation of the shell.

The shell is a surface of revolution and the membrane theory is well applicable except you will find that at this pole, that is at this point. There is some difficulty to apply the theory. But away from the shell, you will hear from the pole, you will find that membrane theory gives excellent results.

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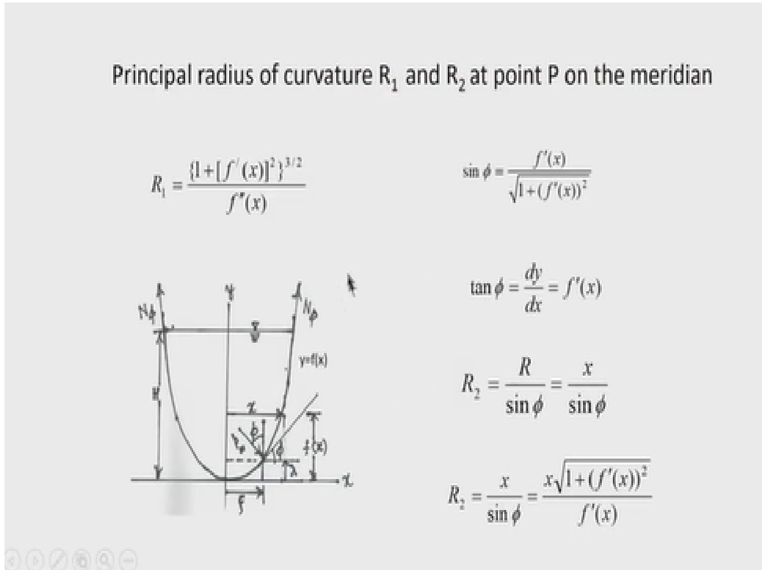


In principal radius of curvature is $R_1$ and $R_2$. And at any point $P$, suppose if I consider a point $P$ here. Then, this $R_1$ can be obtained as $\frac{\{1+[f'(x)]^2\}^{3/2}}{f''(x)}$, $f'(x)$ is the derivative of this $f(x)$ with respect to x. That means $f(x) = y$. Sometimes we can denote it as $y'$ also. This is nothing but $\frac{dy}{dx}$.

So, this is your principal radius of curvature $R_1$. So, once you know the tan $tan\,\phi$ then other quantities sin $sin\,\phi$ can be calculated, tan $tan\,\phi$ is $\frac{dy}{dx}$ that is $f'(x)$. So, sin $sin\,\phi$ and cos $cos\,\phi$ etcetera wherever necessary can be calculated. The second principal curvature $R_2$ is nothing but suppose this distance is your x, these distances your $R$ for example, at a distance ξ, then this $R$ can be calculated as $\frac{R}{\sin sin\,\phi}$.

So, this is ξ is the x coordinate, so in general, it is written $\frac{x}{\sin sin\,\phi}$. At appropriate location x, you put the numerical value and calculate this quantity $R_2$. Now, $R_2$ is given by $\frac{x}{\sin sin\,\phi}$, that is $\sin sin\,\varphi\ = \frac{f'(x)}{\sqrt{1+(f'(x))^2}}$ and this quantity is substitute area, we will get $R_2$.

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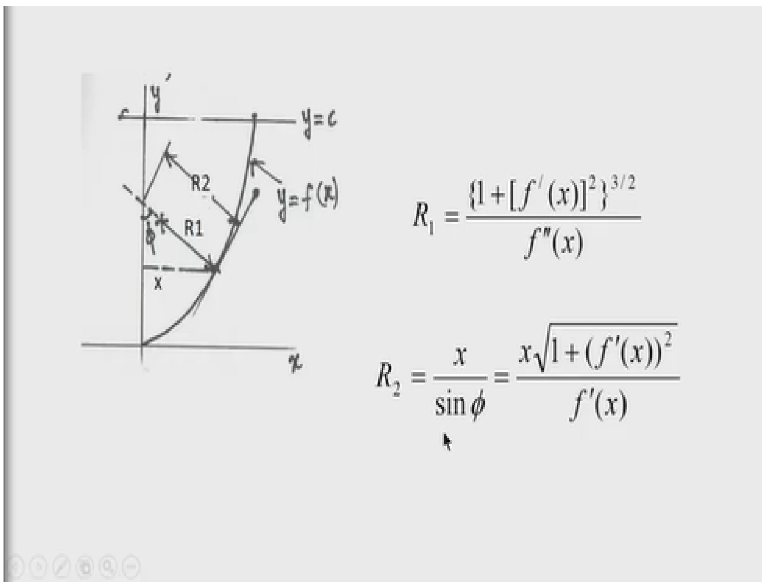


Now, the $R_2$ and $R_1$ is explained here with reference to your point again. So, $R_1$ is the principal radius of curvature of the meridional curve. You see this meridional curve is this, it defined by $y = f(x)$, and here y is your this depth of the tank which is a constant quantity. So now, this $R_1$ is up to here. So, this is the tangent. So, radius of curvature will be aligned along the direction normal to the tangent.

So, say for example radius of curvature, that is terminating here. So, this is the centre of curvature considering the curve at this point. Now, this second principal curvature $R_2$ is nothing but the intersection of these normal with the vertical axis of rotation. So, it is denoted by these distance $R_2$. So, $R_1$ is calculated as $\frac{\{1+[f'(x)]^2\}^{3/2}}{f''(x)}$, $f''(x)$ denotes the second derivative of y.

And $R_2 = \frac{x}{\sin sin\, \phi}$, that is clear from this diagram. So, therefore, this equal to $\frac{x\sqrt{1+(f^{'}(x))^2}}{f^{'}(x)}$.

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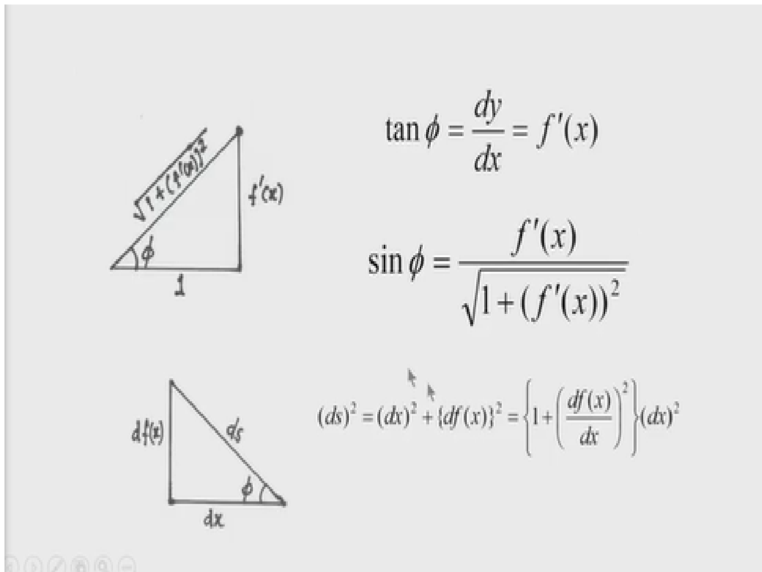


This is further clarified with the diagram a triangle, if I say calculate $dy\, dx$ and $dy\, dx$ I represents in a right in that triangle the basis one and you are this multitude is $dy/dx$. That is $f'(x)$. Then hypotenuse of this triangle is $\sqrt{1 + (f^{'}(x))^2}$. So, if this triangle is given that is all the sides of the triangle are specified relatively. Then, we can get the ratio of the sites to get the trigonometrical quantity.

That is $\sin sin\, \phi$ will be $\frac{f^{'}(x)}{\sqrt{1+(f^{'}(x))^2}}$. Similarly, the $\cos cos\, \phi$ can also be written, $\cos cos\, \phi$ will be equal to $\frac{1}{\sqrt{1+(f^{'}(x))^2}}$. Now, in this shell, if I take a small element at any point the curve element of the shell which is say we denote it by $ds$. Then the $ds$ on the other side if you take, it is a symmetrical tank. So, therefore, on the other side, it will be mirror image of that this curve.

So, if I take the length of the arc on the other side of the tank, which is denoted by $ds$ and then this angle this is the $dx$ and this is your $y$, $dy$. So, $dy$ is nothing but $df(x)$. So, $ds^2$ again using the Pythagoras rule, we can write $(ds)^2 = (dx)^2 + (dy)^2$. So, $(dy)^2$ is nothing but $\{df(x)\}^2$, here I am writing here with the functional form $\{df(x)\}^2$. So, taking $(dx)^2$ common, $\left\{1 + \left(\frac{df(x)}{dx}\right)^2\right\}(dx)^2$.

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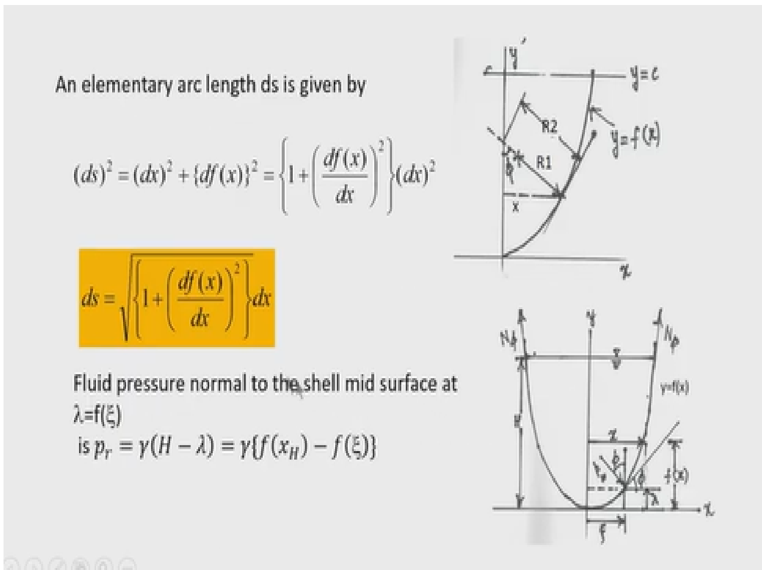


So, $ds$ will be now given by this root over of this quantity, root over this

$$ds = \sqrt{\left\{1 + \left(\frac{df(x)}{dx}\right)^2\right\}}dx$$

. So, why this $ds$ is necessary? Because $ds$ is necessary to calculate the width of the strip. Because, when you calculate the pressure on the shell total pressure acting on the shell, normal pressure or is component. We have to calculate the surface area. So, to know the surface area of annular ring, we should know the strip width.

So, strip width is nothing but $ds$. So, $ds$ is now obtained as this quantity. Now, fluid pressure acting at this level, this level say it is your the height from the basis λ and total depth of the water is H. So, according to this static pressure law, we can get this a γ is the density of the fluid then $\gamma(H - \lambda)$. So, this is the fluid pressure at this point and it is acting normal to the surface. So, this $\gamma H$, I can replace it as function $f\left(x_H\right)$.

Because if $x_H$ is the limit of the $x$, that is the tank is bounded from $-x_H$ to $+x_H$. So, this $x = x_H$ will get the $H$ or that will be equal to $y$ at $x = x_H$. Here similarly at λ, this ordinate corresponds to the ordinate, this coordinate ξ and therefore, if I evaluate the function at this below $x = $ ξ, we will get this λ. So, therefore, $H - \lambda$ is written as $\gamma\{f\left(x_H\right) - f(\ )\}$. So, therefore, fluid pressure, normal fluid pressure at this point is now $\gamma\{f\left(x_H\right) - f(\ )\}$.

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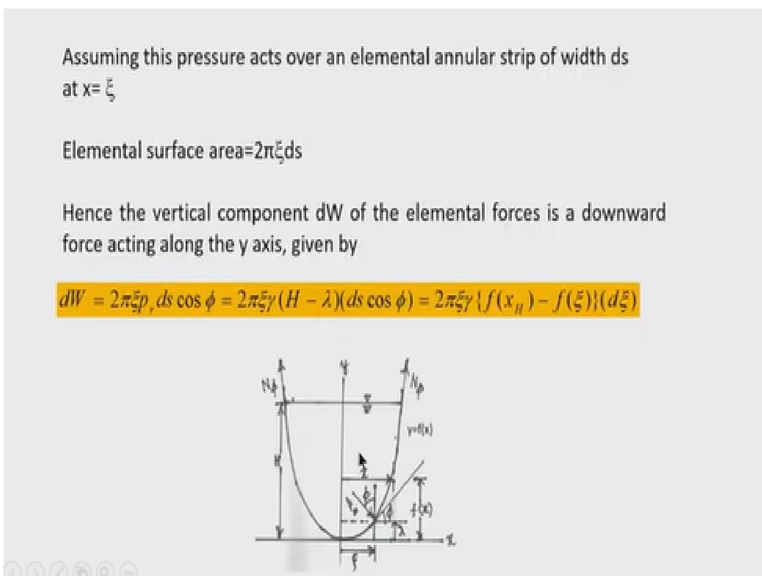


Assuming this pressure acts over an elementary annular strip of width ds. So, this elemental strip, that I have shown earlier. Here also it is ds, here also in the same this is ds. So, it is acting on the ds. But now we have to calculate the area of the annular ring, surface area of the annular ring. Now, surface area of the annular ring will be the diameter of the parallel circle at this point is 2πξ multiplied by the width of the strip will give you the surface area.

So, surface area is $2\pi\xi \times ds$. So, the total load that is, if I want to calculate total vertical component dW. That is vertical component dW is nothing but the vertical component of pr, radial pressure on this strip. So, if this angle is ϕ, the component of $pr$ along the vertical direction is $pr \cos cos\, \phi$ . So, $pr \cos cos\, \phi$ is the vertical force acting on this small element ds. So, now the surface area of this strip is $2\pi\xi$ is the circumference.

And ds is the width of the strip and $pr \cos cos\, \phi$ is the normal this vertical load on the strip. So, multiplying all these things, we get the component of the weight on the elementary strip. That is component of the $pr$ in the vertical direction on the elementary strip is now given by $2\pi\xi pr ds \cos cos\, \phi$ , $pr \cos cos\, \phi$ is the vertical component of this fluid pressure on the strip. Now, we substitute this value of other quantities that, we have earlier shown.

That $pr$ is now given as $\gamma(H - \lambda)$, at this point $\gamma(H - \lambda)$. So, $pr$ is substituted and $ds \cos cos\, \phi$ . And now, this $H$ is written as $f(x_H)$ and λ is written as $f(\xi)$. And, it is integrated with respect to ξ. So, ds is now converted in terms of $d\xi$. Because this is small distance so, we can write in terms of $d\xi$.

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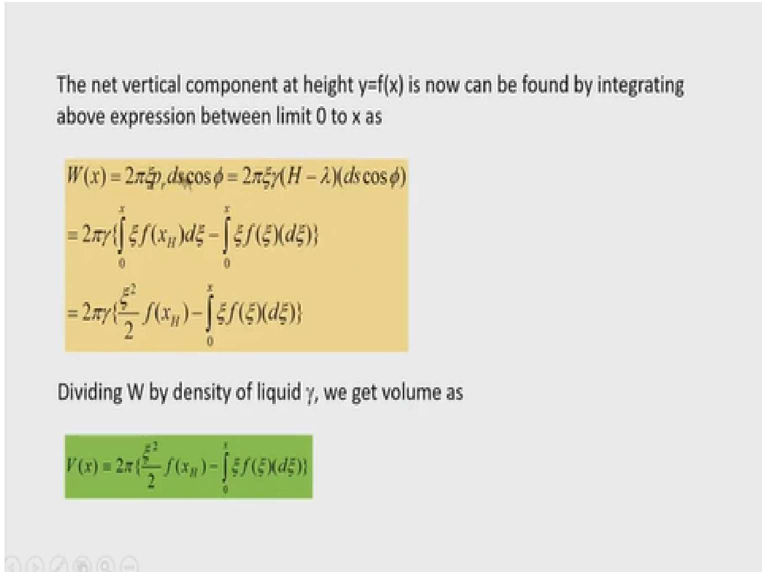


So, net vertical component at height $y = f(x)$ is now found out by integrating the above expression. So, we have found the vertical weight on the elementary strip, if we sum up all sides vertical width along the length of the arc, around the periphery integrate. Then we will get the total vertical load, total vertical load acting on the shell. So, that means, if I now want to calculate $W(x)$. So, $dW(x)$ is given by this.

So, integrating this expression now, we can get this quantity. So, this is the expression. So, this quantity is now integrated and after integration you can find that the final result is $2\pi\gamma\xi^2$ because it is integrated with respect to ξ. So, $\frac{\xi^2}{2} f(x_H)$ is constant quantities. So, it will remain as it is similarly here the $\xi f(\xi)$, because depending on the type of the meridional curve, the function is defined and then when it is multiplied by ξ.

It will convert it to another function. So, that function the product of the $\xi f(\xi)$, has to be integrated between the limits 0 to x. So, we get the total weight at this level, foot pressure is acting normal to the wall and the vertical component of the foot pressure at this level on the surface is nothing but, $2\pi\{\frac{\xi^2}{2} f(x_H)$, $f(x_H)$ is nothing but height of the tank, with the depth of the water 0 to $x$ integration, $\xi f(\xi)(d\xi)\}$.

Now, if I divide this $W/\gamma$, then we get this volume. So, dividing $W/\gamma$ we get the volume. So, $\gamma$ will be cancelled here. So, we get $2\pi\{\frac{\xi^2}{2} f(x_H) - \int_0^x \xi f(\xi)(d\xi)\}$.

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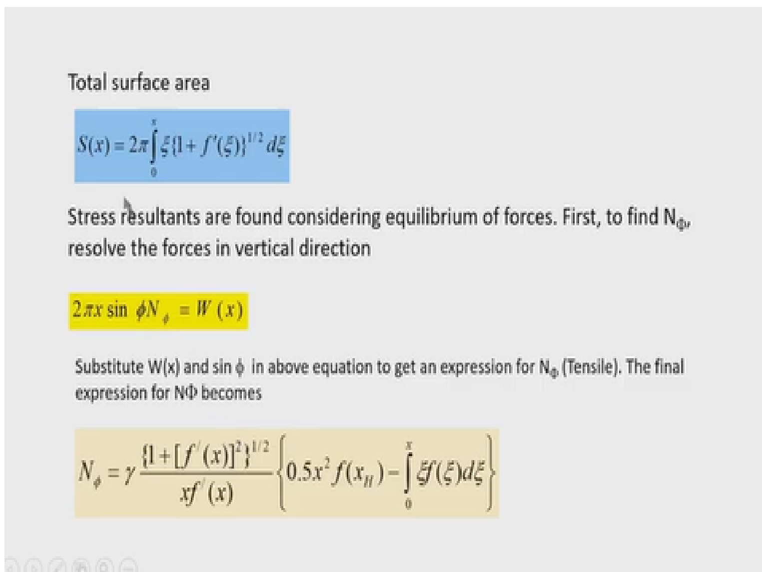


Total surface area is now $2\pi$ that, we actually earlier we got it, total surface area ds, and ds now will substitute in terms of this quantity. So, ds is this $(1 + y'^2)$ or $f'(x)^2$. So, substituting this, we get this total surface area, now total surface area of the shell. Now, if I take the vertical component of $N_\phi$ at this level, we can see this, if I take this $N_\phi$ at this level. At this level $N_\phi$ is acting along the tangent, so vertical component is $N_\phi$ sin $sin\ \phi$ .

So, now, the $N_\phi$ sin $sin\ \phi$ is the vertical component and $2\pi x$ is the circumference of this parallel circle at this point. So, if this is x at this point, set for this point or any point was coordinate is x coordinate is defined. So, $2\pi x$ is the circumference of the parallel circle. So, $2\pi x$ into particle component of $N_\phi$, will give you the total vertical force, meridional force and it has to balance by the total particle load acting on the shell due to internal pressure.

So, total vertical load is nothing but component of the pressure, fluid pressure in the vertical direction. So, we have obtained $W(x)$ and$\phi$ is also known because the once the curve be specified, we can find out tan $tan\ \phi$ and from tan $tan\ \phi$ we can evaluate$\phi$ . So, now, this $N_\phi$ can be easily calculated, so substituting $W(x)$ and$\phi$ in the above equation to get an expression of $N_\phi$. And then $N_\phi$ will be tensile here the final expression is calculated like that.

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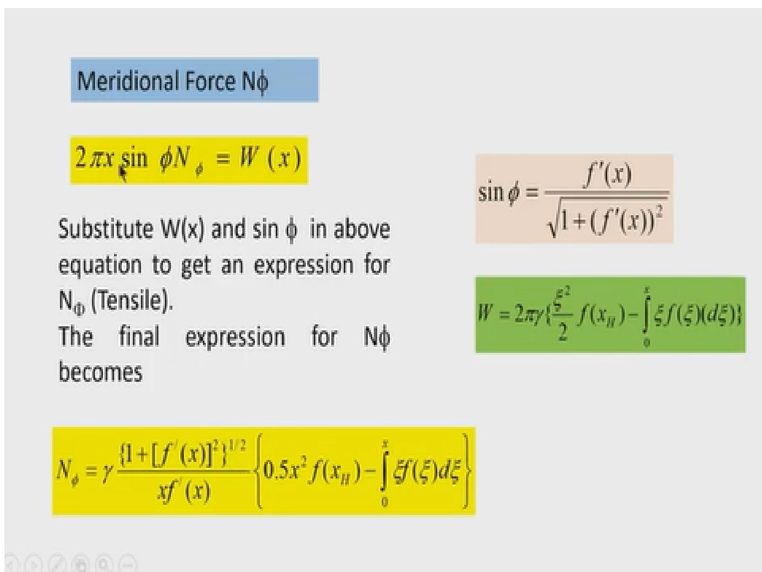


And you can see that, meridional force now that, I have discussed here. Now, here I am in detail I am giving and steps to be followed. Put sin $sin\ \varphi\ = \frac{f'(x)}{\sqrt{1+(f'(x))^2}}$ in this expression and W we have obtained earlier $2\pi\gamma\{\frac{\xi^2}{2} f(x_H)\ -\ \int_0^x \xi f(\xi)(d\xi)\}$.

So, that quantity is substituted here and $\phi$ is substituted here. So, when I substitute this quantity and that quantity and after simplification $N_{\phi}$ is obtained as

$$N_{\phi} = \gamma \frac{\{1+[f'(x)]^2\}^{1/2}}{xf'(x)} \left\{ 0.5x^2 f(x_H) - \int_0^x \xi f(\xi) d\xi \right\}$$

So, $\xi^2$ is nothing but x, $\xi$ is a specification of this coordinate x and y. So, at any distance x will put here x. Here $\xi$ is taken for the integration purpose and limit of integration is taken 0 to x. So, that the membrane stresses are given in terms of the coordinate x. Of course, the coordinate x is also related to the ordinate y and x and y are related because the meridional curve is known.

So, once this is given, then after simplification we can get this 0.5 is coming because it is 1 by 2 xi squared. So, 0.5x squared and this is f x H, you can write here also the height of the tank and then xi f xi, f xi is the meridional curve, that is a function, a continuous function is taken here, and it is multiplied with $\xi$ and then integrated. So, if it is a say for example, it is a cubic parabola, then the total the function will be here, $\xi f(\xi)$ will be a fourth order polynomial.

So, in that case the integration can be done, whether you take a trigonometrical function or algebraic polynomial in that case, the integration will be carried out and limit is 0 to x. So, that the $N_{\phi}$ is specified at any distance x. So, this is the expression for $N_{\phi}$. This integration is here left as it is because we do not know the type of the meridional curve. We have derived the general expression.

So, therefore, it is written in the integral form, once the curve is known, we can just substitute it and integrated. So, after knowing the $N_{\phi}$ will proceed to find $N_{\theta}$. Now, $N_{\theta}$ is the hoop stress, that is the stress along the circumferential direction. And, it is given by a very simple equation, that is the third equilibrium equation in the shells of surface of revolution and third equation is obtained by resolving all the forces along the radial direction.

And equating resolving the; forces in the radial direction and summing up them and equating to 0. That means, to establish the equilibrium condition along the radial direction. So, this situation turned out to be $\frac{N_\phi}{R_1} + \frac{N_\theta}{R_2} = p_R$, $p_R$ is the load or force along the radial direction. Now, $p_R = p$, because the foot pressure acts normal to the tank wall. So, it is directed along the centre of curvature so, it is taken as $p$.

$R_1$ we have obtained, that is the first principal curvature of the meridional curve curvature. So, from the formula known in the calculus, we can calculate $R_1$. So, $R_1$ is given by $\frac{\{1+[f^{'}(x)]^2\}^{3/2}}{f^{''}(x)}$, $f^{''}(x)$, $f^{''}(x)$ indicates that it is the second derivative of the function. Similarly, $R_2$ we have seen, $R_2$ is nothing but we have seen that $R_2$ is nothing but $\frac{R}{\sin sin\ \phi}$.

And you can see the $R$ is nothing but the radius of the parallel circle and it is defined by the x coordinate. So, we have written here $x$ and sin $sin\ \phi$ was earlier we have found out sin $sin\ \phi$ . It is $\frac{f^{'}(x)}{\sqrt{1+(f^{'}(x))^2}}$. So, this is sin $sin\ \phi$ so when the $R$ that is $\frac{x}{\sin sin\ \phi}$, there $R_2$ is turned out to be $\frac{x\sqrt{1+(f^{'}(x))^2}}{f^{'}(x)}$. Now substitute $R_1$, $R_2$, $p_R$ in this expression, and $N_\theta$, in the $N_\theta$ we want to find.

So, we earlier we have found $N_\phi$, so substitute $N_\phi$. So, $N_\phi$ whatever we have found out substitute $N_\phi$ here, $N_\theta$ we are targeting. So, all these quantities are substituted and after simplification, the final expression for $N_\theta$ becomes,

$$\gamma \frac{\{1+[f'(x)]^2\}^{1/2}}{f'(x)}\left[x\{f(x_H) - f(x)\} - \frac{f''(x)}{f'(x)\{1+[f'(x)]^2\}}\left\{0.5x^2 f(x_H) - \int_0^x \xi f(\xi) d\xi\right\}\right].$$

So, this is a long expression of this $N_\theta$ compared to $N_\phi$. But you see that expressions that we have obtained for $N_\phi$ and $N_\theta$ are general expressions. So, in that expression, there is no restriction that meridional curve should be of this form, for that form.

So, any type of meridional curve you can incorporate here provided their continuous function. So, say cubic parabola or quadratic parabola or even elliptical curve or a hyperbolic curve or any form of curve can be used here for the meridional profile. Theoretical calculation based on that different profiles have been done by some authors and the Alfred, Jigani one author, they have published a paper on that.

The membrane forces on the tank of revolution, surface of revolution with any arbitrary meridians. So, they compare the efficiency of different meridional profile, they have taken quadratic parabola, they have taken cubic parabola, they have taken this elliptical curve and they have taken hyperbolic function also. So, out of that all the types of meridian, they have found that cubic parabola is the most efficient type of meridian for the tank used as surface of evolution.

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Exercise: Show the variation of membrane stresses for the meridian in the form of cubic parabola

$f(x) = kx^3 \quad (x \geq 0)$

in a tank of surface of revolution in which depth of water (H)=4m, Diameter of parallel of latitude at water surface =6 m, Density of liquid=10 kN/m$^3$

Now, let us discuss a numerical problem. So, I want to show the variation of membrane stress for the meridian in the form of cubic parabola. So, as I noted in a research paper that, this cubic parabola is the most efficient type of meridian. So, I have taken the cubic parabola for demonstrating this result. So, I will calculate using our formulation and then I will plot this variation.

So, in a tank of surface of revolution, in which the depth of the water H is 4 meter and diameter of the parallel of latitude at water surfaces is 6 meter. That means, the water surface the diameter or radius of the parallel circuit is 3 meter. Density of the liquid is taken as $10\ kN/m^3$.

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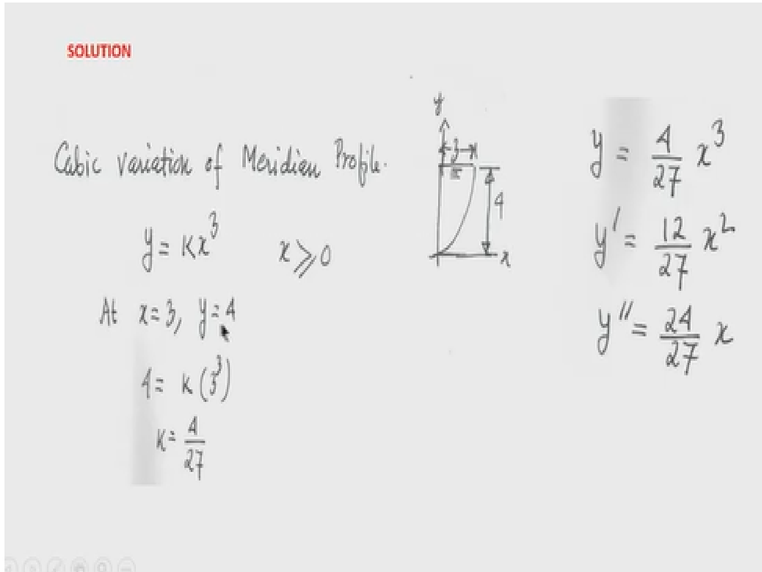


So, let us solve this example. So, you can see this is a half portion of the tank. Because it is a symmetrical structure and it is acted upon by axisymmetric loading. So, therefore, I have taken only the half portion to illustrate the problem. So, height of the water is full that is $f\left(x_H\right) = 4$ or H is equal to or capital H = 4. So, you can use 2 notation x function of x substitute x substitute as $x_H$ is also 4 and the total height depth of the water given is 4.

And, the equation of the parabola is given as $y = Kx^3$. The $x \geq 0$ here we are taking. Now, K is unknown here, K can be found out from the given below at x = 3 y = 4. So, that is known to us. So, substituting this x = 3, y = 4 we get $4 = K \times 3^3$. So, that means K is nothing but $\frac{4}{27}$. So, y is known completely, $y = \frac{4}{27}x^3$. So, $y'$ that is the first derivative of this meridional curve is now that, if I take the first derivative it will be $\frac{12}{27}x^2$.

Second derivative if I take, it will be $\frac{24}{27}x$. So, these two derivatives are needed in our formulation.

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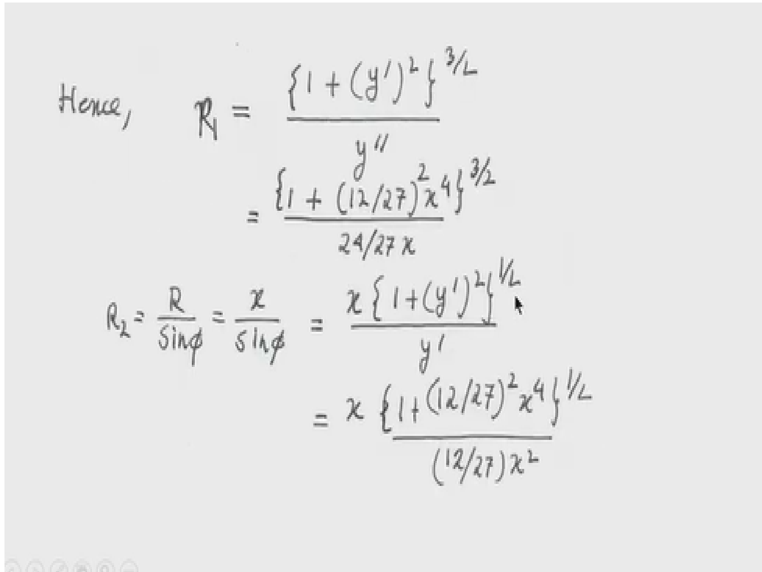


Now, $R_1$ that is the radius of curvature of the meridional curve is given by $\frac{\left\{1+(y')^2\right\}^{\frac{3}{2}}}{y''}$, $y'$ is nothing but $f'(x)$ which is nothing but $\frac{dy}{dx}$. So, we have obtained earlier $y'$ and $y''$, substitute this quantity here, we get $R_1 = \frac{\left\{1+\left(\frac{12}{27}\right)^2 x^4\right\}^{\frac{3}{2}}}{\frac{24}{27}x}$ .$R_2 = \frac{R}{\sin sin\,\phi}$.

Now, x we know, x is a distance that we have to take into account for finding the membrane stresses and $\sin sin\,\phi$ because this card is known. So, this quantity represents this $\tan tan\,\phi$ . So, once you know the $\tan tan\,\phi$ you can calculate the $\sin sin\,\phi$ , that I have shown with a triangle. So, $\sin sin\,\phi$ is known so therefore, this $R_2 = \frac{x\left\{1+(y')^2\right\}^{\frac{1}{2}}}{y'}$.

So, after substituting the value of $y'$, we now get the $R_2 = \frac{x\left\{1+\left(\frac{12}{27}\right)^2 x^4\right\}^{\frac{1}{2}}}{\frac{12}{27}x^2}$. So, knowing these two principal radii of curvature now we will proceed.

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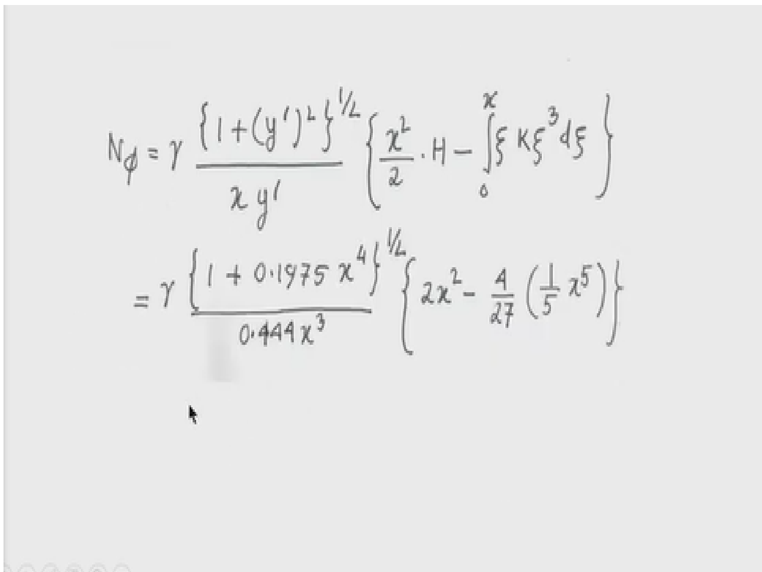


So, $N_\phi$ expression that we have obtained earlier you can see this expression that we have obtained earlier; we will make use of this expression. So, using this expression now, after simplification of this expression, substituting the value of y prime and integrating this expression because, $k = \frac{4}{27}$, so $\frac{4}{27}$ is there, and it is $\xi^4$. So, when you integrate $\xi^4$ with respect to ξ.

Then, it will become $\frac{\xi^5}{5}$, and after putting the limit, it will become $\frac{1}{5}x^5$. And $f\left(x_H\right)$ is nothing but H, and H is given as 4 meter. So, this quantity becomes $2x^2$. So, after substituting this and simplifying this $N_\phi$ is this quantity.

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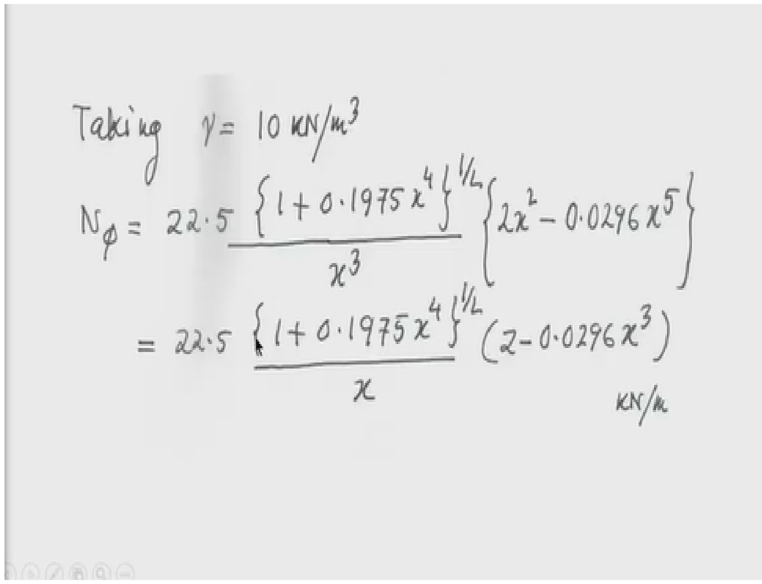


Then further simplification is done taking gamma as $10kN/m^3$. So, $N_\phi$ now becomes finally $\frac{22.5\{1+0.1975\,x^4\}^{\frac{1}{2}}}{x} \times (2 - 0.0296\,x^3)$. Now, you can see that near the pole, pole is the bottom of the tank, pole is the bottom of the tank. At this point the membrane stress becomes unbounded. Membrane stress becomes unbounded.

So, will not take this value for the design purpose. So, that is the disadvantage of membrane theory that requires collection. So, this is at $x = 0$, that is at the pole origin this $N_\phi$ is not bounded. So, away from this pole membrane values will be reasonably accurate. The forces will be reasonably accurate.

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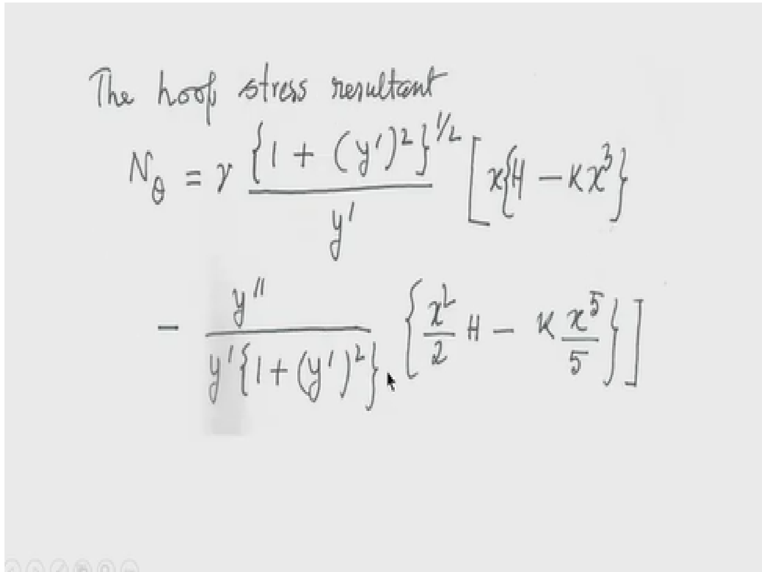


Hoop stress resultant is found as $\gamma\frac{\left\{1+\left(y^{'}\right)^{2}\right\}^{\frac{1}{2}}}{y^{'}}\times[...]$. So, the $\frac{1}{2}$, this power is written as $\frac{1}{2}$. And this quantity whole quantity is multiplied by this bracket is closed here, the bracket is started from here and it is closed here. So, under this bracket these quantities are $x\{H - Kx^{3}\}$, K we know now, because it was the equation of the curve.

So, as the K is known as $\frac{4}{27}$. So, we write this and then, $-\frac{y^{''}}{y^{'}\left\{1+\left(y^{'}\right)^{2}\right\}}\times\{\frac{x^{2}}{2}H - K\frac{x^{5}}{5}\}$.

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$$N_\theta = \gamma \frac{\left\{1 + (12/27)^2 x^4\right\}^{\frac{1}{2}}}{(12/27)x^2} \left[ x\left\{4 - (12/27)x^3\right\} - \frac{(24/27)x}{(12/27)x^2\left\{1 + (12/27)^2 x^4\right\}} \times \left\{2x^2 - \frac{4}{27 \times 5} x^5\right\} \right]$$

So, after simplification that, after substituting the value of γ and doing all these calculations, we now finally arrived the expression for $N_\theta$ is this. And further simplifying that these ratios are calculated $\frac{12}{27}$ is calculated $\frac{12}{27}$ squared is substituted here. Then $\frac{\frac{24}{27}}{\frac{12}{27}}$, so, it becomes 2. So, and after simplification, it turns out to be this quantity.

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After simplification and taking density of fluid $\gamma$=10 kN/m$^3$, we get

$$N_\theta = 22.5 \frac{\left\{1 + 0.1975 x^4\right\}^{\frac{1}{2}}}{x^2} \left[ x(4 - 0.44x^3) - \frac{2x}{(1 + 0.1975x^4)} (2 - 0.0296x^3) \right]$$

kN/m

So, final expression for entities $22.5\times\frac{\{1+0.1975\,x^4\}^{\frac{1}{2}}}{x^2}\times\left[x\left(4-0.44\,x^3\right)-\frac{2x}{(1+0.1975\,x^4)}\times(2-0.0296\,x^3\right]$. So, this is the expression for $N_\theta$ and now, if I plot this $N_\theta$ with respect $N_\theta$ $N_\phi$. First let us plot $N_\phi$ with respect to, $N_\phi$ with respect to this final expression of $N_\phi$ with respect to depth of the tank.

So, depth of the tank is again with the depth of the water. So, depth of the water at any level can be calculated from the given function. So, $y = Kx^3$. So, at any x we can calculate the depth of the water. Here depth is measured from the pool that is the origin upwards. So, here we will take that y is measured from the pool. So, this $Kx^3$ is the meridional curve equation. For example, the limit of x is 3 meter.

So, if I want to calculate this depth of the water at 1 meter, then I will calculate this $\frac{4}{27}x^3$. So, in place of x, I will substitute 1. So, $\frac{4}{27}$ is the depth of water at $x = 1$. Here depth of water is measured from the origin which is the bottom point of the tank. So, if I calculate and plot this.

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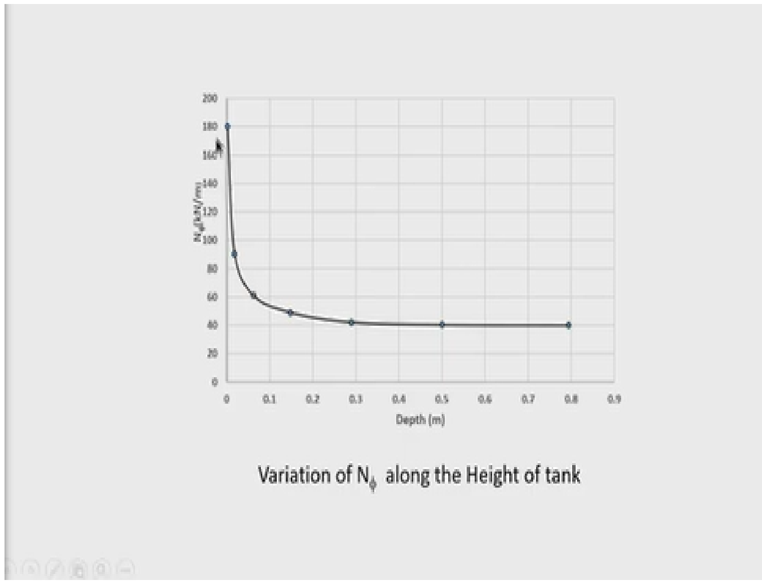


Then you can see the variation of $N_\phi$ is like that and it is found to be tensile for all values up this depth. Now, initially when near the pole, pole is 0. Along the x-axis depth is given and along the y axis, the values are $N_\theta$ in $kN/m$. So, near the pole, the value is very high. So, at 0 of course, it becomes unbounded and it gradually decreases and you can find that after certain depth, it becomes almost uniform.

The $N_\phi$ becomes almost uniform in that cubic parabola, and is value is also very low. So, it is 40, around 40 to it varies 40 to 60, in most of the region. So, this is the variation of $N_\phi$ the meridional stands in the tank, along the height of the tank.

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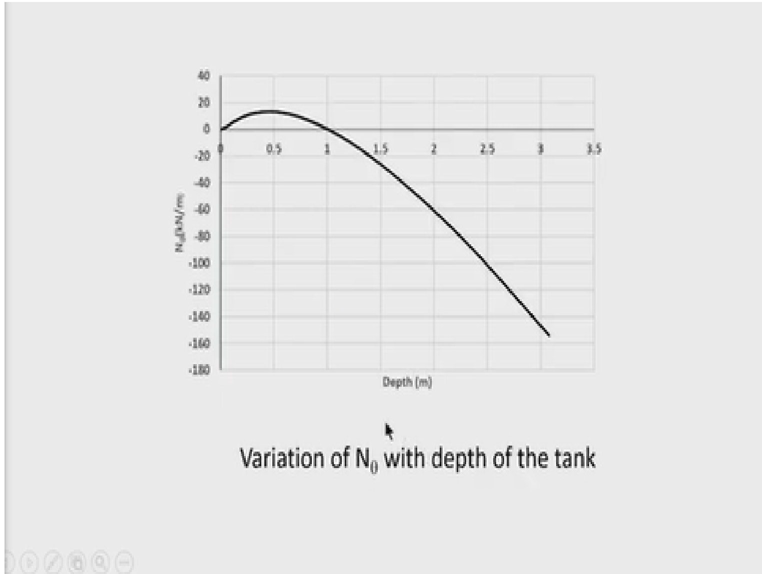


Then, if I plot $N_\theta$, we can see it changes sign. So, except 0, we can plot this and we will find that at certain depth the tensile force reverses its sin, that is tensile force now becomes compressive. But $N_\phi$ is always found to be tensile and $N_\theta$ is tensile for small region and then it becomes compressive.

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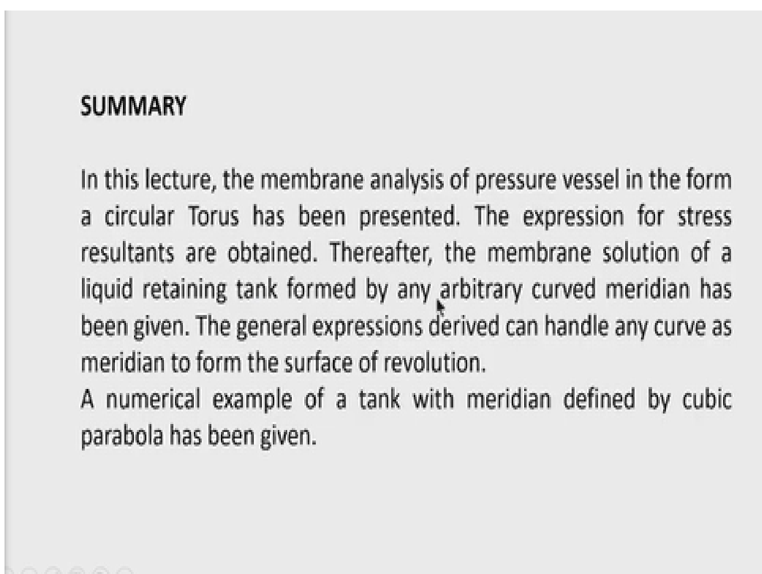


So, let us summarize the today's lecture. In this lecture, the membrane analysis of pressure vessel in the form of a circular torus has been presented. The theory that I presented is illustrated with a circular torus, but it can be used for other form of torus. That is, if I take a elliptical torus also, I can use this formulation. That is the general formulation that theory, that the vertical component of this meridional stress to be balanced by the vertical component of the internal pressure at this level.

So, the expressions for stress resultant are obtained. And thereafter, we have presented a membrane solution of a liquid retaining tank, formed by any arbitrary curved meridian, that general expressions was derived and that general expression can handle any curve as meridian, to form the surface of revolution. A numerical example has been illustrated in today's class with meridian defined by cubic parabola. Thank you very much.