Plates and Shells
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Module No # 09
Lecture No # 25
Membrane Analysis of Shells of Surface of Revolution

Hello everybody today I am starting module number 9 in that module this is the first lecture. This lecture will be delivered in a very common type of shell that is a doubly curved shells synclastic nature. I will use membrane theory of shell to analyses the particular type of shell and these type of shell are generally known as shell of surface of revolution. Examples are your domes maybe your spherical domes may be ellipsoid or may be your paraboloid of revolution and the hyperbolic of revolution is also a shell of surface of revolution.

But it is having different characters being synclastic in nature. The topic of today's lecture is membrane theory of shells of surface of revolution. Now in the last class I have introduced the membrane theory and you have seen that membrane theory of shell is nothing but moment less state of the stresses. So that is only the membrane stress in the shell structure will be considered and no bending moment and shearing forces is considered in the formulation of the problem.

So using this theory in the shell structure we obtain the membrane stresses which are in Cartesian coordinates system are N_x, N_y, N_{xy}, $N_{xy} = N_{yx}$ when the curvature is small compared to its thickness. So the membrane forces per unit length N_x, N_y, N_{xy} is our target and we have to obtain this. But in this kind of shell that is the surface is formed by rotating as plane curve about an axis of revolution it will be convenient to use the spherical coordinate system that means position of a point can be specified by 3 parameters. One is θ and another is angle φ and radial distance R so we will formulate the problem using the spherical coordinate system for shells of surface of revolution.

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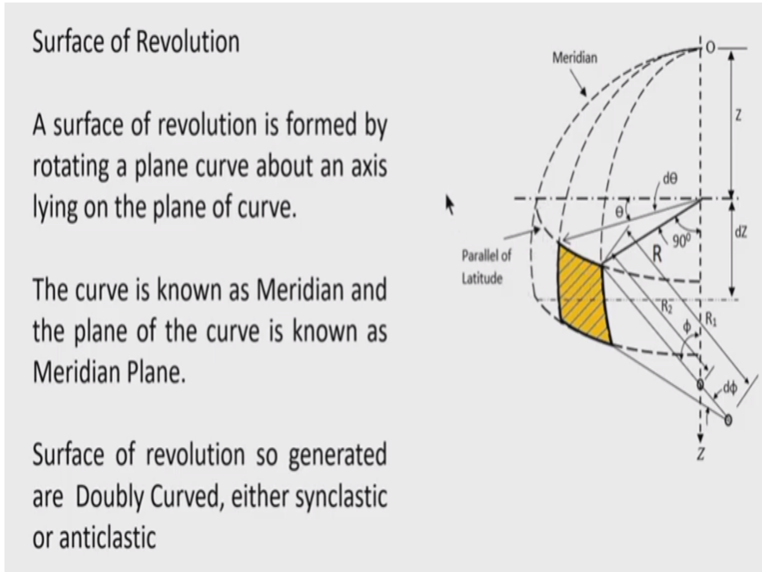

Now as I told the surface of revolution is formed by rotating a plane curve about an axis lying in a plane of curve. So say for example this is your one plane and this place curve is rotated about the axis of rotation. That is, you are seeing a vertical that is the *z* axis then a surface is formed and this surface is known as shells of surface of revolution. Depending on the nature of the curve the name of the surface of the revolution is also given.

For example, if a part of a circle or a circular arc which is a now in instant case it is a meridian and if it is rotated around the vertical axis of revolution then the spherical dome is formed. That is a spherical surface of revolution but when we rotate an ellipse, a meridian of elliptical curve then ellipsoid of revolution. Like that paraboloid of revolution etc., can be found. So any arbitrary curve can be chosen not only a part of circular or a parabolic curve or elliptical curve.

But any curve you can form and you can revolve it around the axis of rotation to form the surface of revolution. The curve which is rotated is known as Meridian, so its technical term is Meridian, this curve and the axis of rotation is this and this type of shell is generally doubly curved shell. And this is non-developable, so this type of shell is stronger compared to single curve shell which has developable surface.

So singly curved shell developable surface, examples are in cylindrical shell as well as in conical shell. So if you give a curve and then apply pressure then this surface can be flattened completely. So that is weaker than the doubly curved shell that is non-developable and here we will consider the non-developable shell that is the surface of revolution. This type of shell is characterized by 2 principle curvatures and because these are doubly curved, so the curvatures exist in 2 orthogonal directions and therefore we need to take into account the principle curvature into formulation. Now here you have seen that an element of a shell is taken which is shown by a yellow curve. And then with reference to some axis say here which is a horizontal x axis you can call it and then angle is measured which is known circumferential angle to locate the position of any point P.

Because this distance is taken as small distance and therefore the position of a point on this arc is defined by this angle θ that is circumferential angle. And also this point is located on the meridian say this in this meridian therefore the meridian angle for this point is ϕ. And radial distance is given by the R from the axis of reference, this again the R is related to the principle curvature of this surface.

Now you can see that when the meridian is curved is rotated around the vertical axis a surface is formed and the horizontal plane that is if you cut a section along the horizontal direction you will get if you rotate it 360° full then you will get circle. So this type of shell you get the plane that is known as parallel of latitude or parallels of circle sometimes it is called as parallels of circle.

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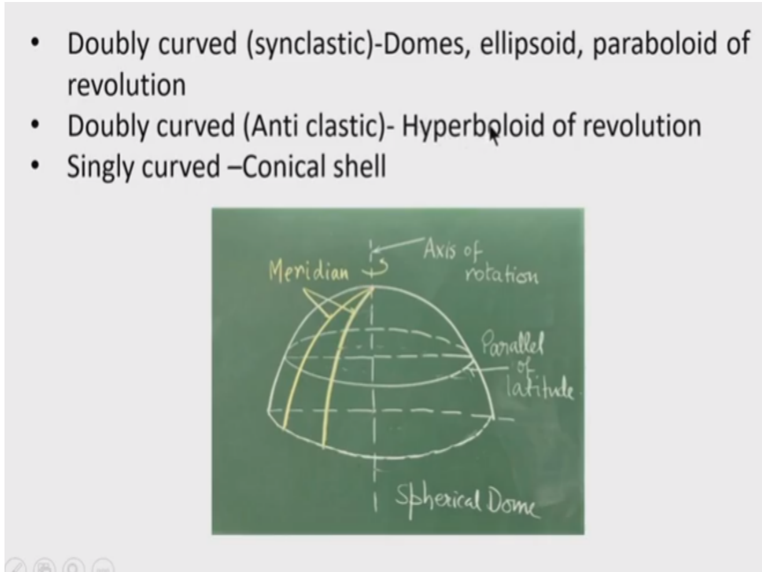

Now doubly curved shells are synclastic and very common examples are domes which are seen in case of temple, mosque, auditorium and other types of large buildings. I mean large building I mean that area is large to provide a column free space then we construct the thin shell like dome. The ellipsoid that is in place of this arc of circle as meridian if we take the elliptical curve then ellipsoid of revolution is formed.

Similarly, paraboloid of revolution is also there when a parabolic meridian is rotated. So doubly curved shells these are the examples of synclastic shell. And this is an example of spherical dome this arc is taken as a part of a circle. Then there is anticlastic shell in the doubly curved group that is known as hyperboloid of revolution and that is mainly used in the cooling tower.

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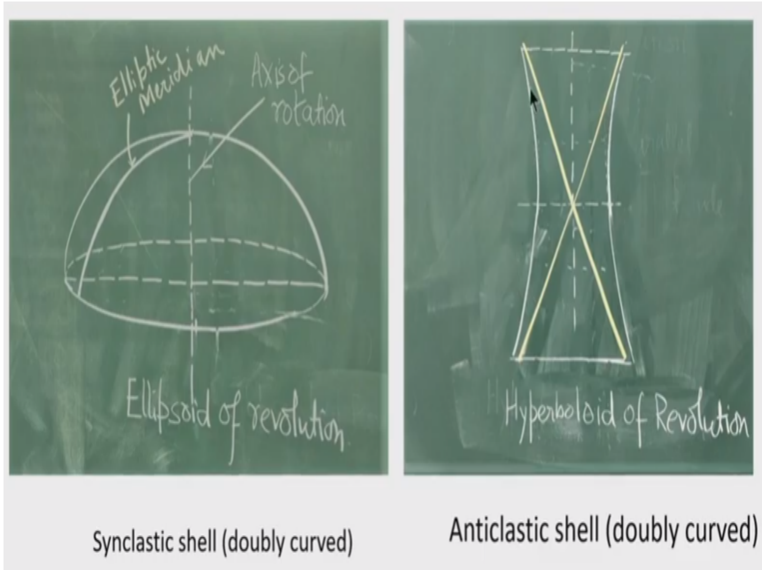

Examples are this. So hyperboloid of revolution that is this type of structure is used in cooling tower. Then this is also a synclastic shell that is instead of this circular meridian we have taken elliptic meridian to form the surface of the shell. Conical shells are the examples of synclastic shell and it is developable surface. But this also falling under the group of shells of surface revolution, conical shell is also formed in the groups of shells of surface revolution.

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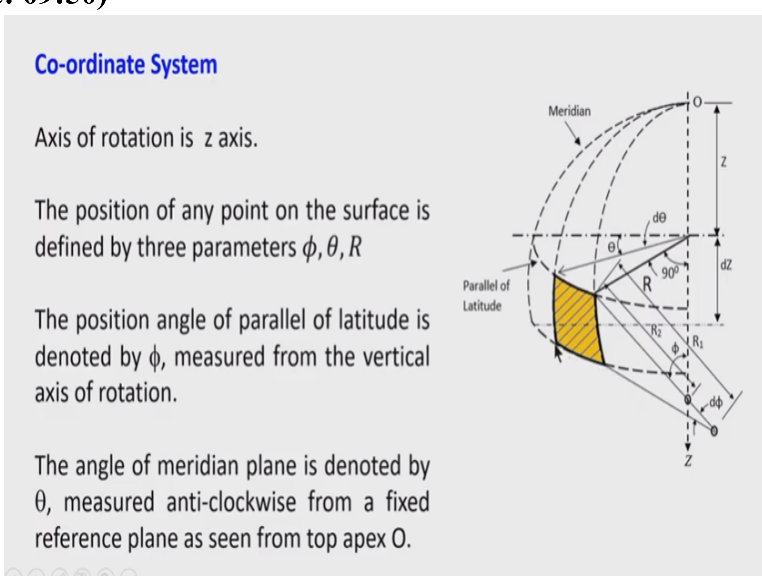

Now let us first introduce the coordinate system in which we will work out our problems. So axis of position here is z axis the position of any point on the surface is denoted by ϕ, θ, R. You can here note ϕ is the meridian angle by which the position of the point on a particular meridian is located. Then on this meridian the position of the point circumferential is also located by an angle θ which is known as circumferential angle. And the radial distance of the point from the center of the horizontal circle or parallel of latitude is R. This R_1 is the major principle radius is there and here you can see the center of curvature and R_2 is the second principle curvature but here this R_2 is shown as an intercept or distance between the point under consideration with the point of intersection with the axis of rotation. So that is here R_2. So these are that is the radius of the parallel circle is related to R_2 by meridian angle ϕ.

So one can readily see that $R = R_2 \sin \phi$. The angle of meridian plane is denoted by θ and measured anti-clock wise from a fixed reference plane and these reference plane is taken. Here this plane is say z-x plane if this is the x axis, so now the meridian and parallels of latitudes are actually lines of curvature. And this line of curvature this is parallels of latitude one lines of curvature in meridian is one line of curvature.

So these lines of curvature have the principle radius R_1 and R_2 so this second principle radius R_2 that I have discussed I have introduced to you it is nothing but a distance between the point under consideration and the point of intersection with the vertical axis of rotation.

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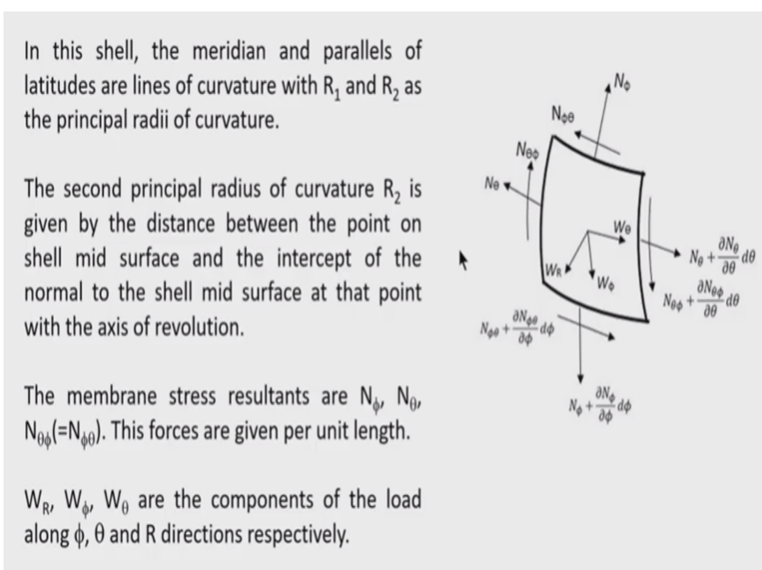

The membrane stress resultant on the element, we take small elements whose length here this will be your $Rd\theta$, width will be $R_1 d\varphi$. That can be readily verified from here that if this angle is $d\theta$ then this small arch will be $R \times d\theta$ if this radial line is R. Similarly, this arc if this angle is $d\varphi$ then this arc length will be this distance is R_1 so $R_1 \times d\varphi$. Membrane stress resultant are shown here in this element that is N_φ per unit length and in the meridional direction and this is N_θ in the circumferential direction.

So N_φ is the meridional stress and N_θ is the circumferential stress. $N_{\theta\varphi}$ and $N_{\varphi\theta}$ are the membrane shear forces that are shown on the edge of the element. On the opposite edges we consider the increment of this quantity. So suppose if this is N_θ, on the opposite edges this stress will be N_θ + increment. This increment is $\frac{\partial N_\theta}{\partial \theta} d\theta$ taking only first order term of the Taylor series expansion.

Similarly, the increment of N_φ on the opposite edges we will be $N_\varphi + \frac{\partial N_\phi}{\partial \phi} d\phi$ and increment of $N_{\varphi\theta}$ will be $N_{\phi\theta} + \frac{\partial N_{\phi\theta}}{\partial \phi} d\phi$. So you have seen that here on the element the forces acting along the edges in tangential direction that is the shear force and in the normal direction that is the direct membrane force, N_θ and N_φ. So we have to form the equilibrium conditions of this element to analyze this shell. And then we get a differential equation.

Now if the load acting on the shell are in general w and w may have any direction of action that means line of action. So line of action of load may be vertical or may be inclined in any direction. But ultimately for taking the equilibrium condition we have to resolve the component into 3 principle directions that is θ, φ and R. So, 3 principle coordinate direction will resolve the load and form the equilibrium equations.

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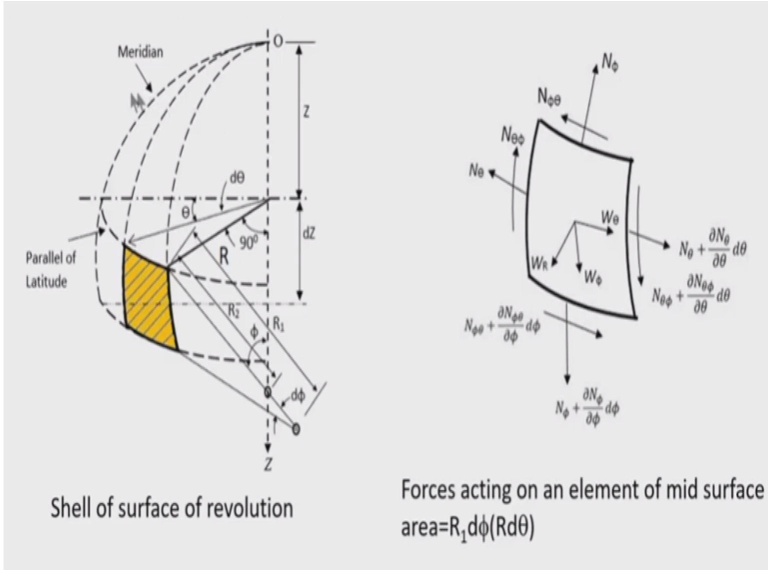

So in this figure it is illustrated that in this element total force is suppose in this edge total force will be N_φ into say this distance is R so it will be $Rd\theta$. So that means this distance will be $N_\varphi\ Rd\theta$ similarly here this force in this edge this force N_θ will be $N_\theta \times R_1 d\phi$. So in this way, we can find the total force on the element considering the distribution of the membrane forces on the small element is uniform. Because this elemental length is very small so therefore we do not consider any variation of this force along this small elemental length and therefore the normal stress or the meridional stress in this direction will be $N_\phi R d\theta$. Similarly, the membrane direct force in this edge will be $N_\theta R_1 d\phi$. So all the forces can be calculated and this total membrane shear force in this edge will be $N_{\phi\theta} R d\theta$. Here on the other edge, adjacent edge the membrane shear force is $N_{\theta\phi} R_1 d\phi$.

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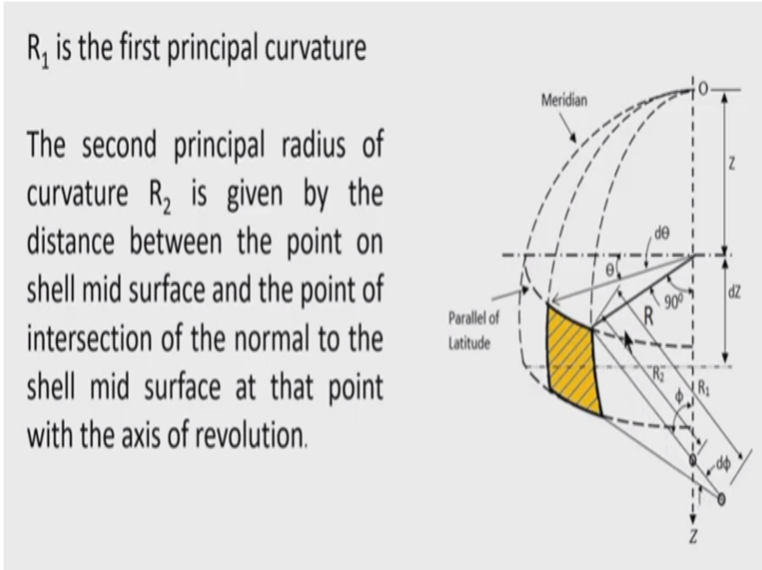

So principle curvatures R_1 and R_2 are introduced and this will be used to form the equation for the stress resultants.

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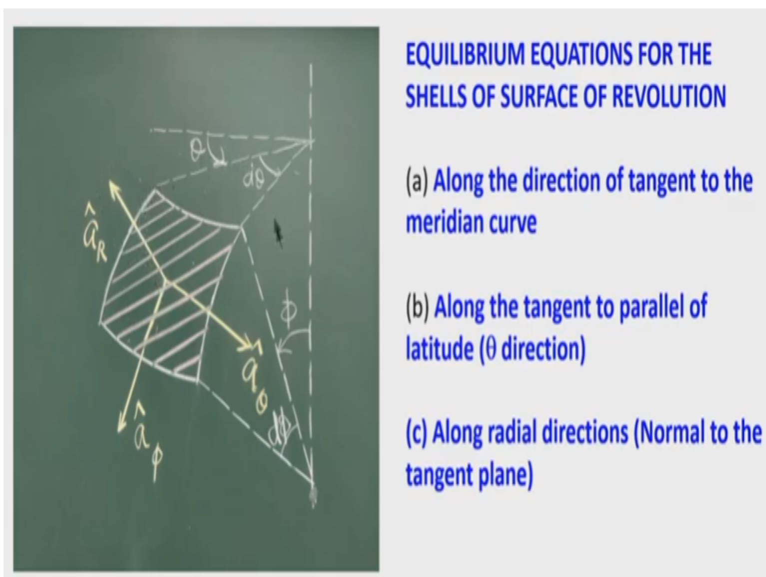

Now for this element, at any point on the shell surface we have 3 principle directions that is the θ, φ and R, R is the radial direction. So if I say a_θ is a unit vector along the circumferential direction, along with θ direction. Circumferential direction we generally we call it θ direction. So a_θ is the unit vector along the circumferential direction and a_ϕ is the unit vector along the meridional direction.

In the similar way the vector a_R is the unit vector along the radial direction so unit vector I mean the directions here are shown in the respective axis but the magnitude is 1. So here the shell element is formed bounded by the angle $d\phi$ and $d\theta$ so that this area is if this radial distance is R this area will be $Rd\theta$ × this arc length will be $R_1 d\phi$. So area of this element is $Rd\theta R_1 d\varphi$ so to form the equilibrium equation we actually collect all the component of the forces in the respective direction and then sum up with proper sign.

So along the direction of the tangent to the meridian curve what is meridian curve this is the meridian curve and this tangential direction is known as φ direction. So that is one direction will take to form the equilibrium equation that is one equilibrium equation will be formed after summing up all the components along the φ direction. φ direction is the meridian direction now next is along the tangent to the parallel of latitude. So the latitude here is seen that is the horizontal circle or this parallel of circle that you have seen.

So that in that direction we will again collect all the components and then we will sum up and that direction is known as θ direction. Similarly, in the radial direction if this is the center and radial direction is denoted in this positive direction of the axis are denoted with this arrow that is shown here. And in the radial direction again we collect all the component and sum up, equate to 0. So in this way we will form the equilibrium equations.

So if the resultant force is $\vec{R}$ then we can write this vector equation say $\sum F_\phi$ which denotes the sum of the components of all forces along the meridional direction. So along the meridian direction therefore we assign the unit vector $\hat{a}_\phi$ then + $\sum F_\theta$ which denote the forces, sum of the forces along the circumferential direction. And therefore we assign unit vector $\hat{a}_\theta$. Similarly, F_R

this summation of this all the forces in the radial direction will collect and we take it here and then assign a unit vector $\hat{a}_R$.

Now for equilibrium, resultant must vanish so therefore $\vec{R} = 0$ so that indicates that individually summation of $F_\phi = 0$, summation of $F_\theta = 0$ and summation of $F_R = 0$. So the principle of forming the equilibrium equation is that we have shown the differential element and the forces acting on the differential element and then we will resolve all the forces in the direction of meridian in the direction of meridian I mean tangent to the meridian and then another direction will take for this sum of all the forces that is in the direction of tangent to the parallel circle. And then another direction is along the normal to the parallel circle. So all these components will be taken and grouped in the respective direction and equated to 0 for forming the equilibrium equation. So let us now proceed to find the equilibrium equation one by one. First we will take the summation of forces along the direction of tangent to the meridional curve.

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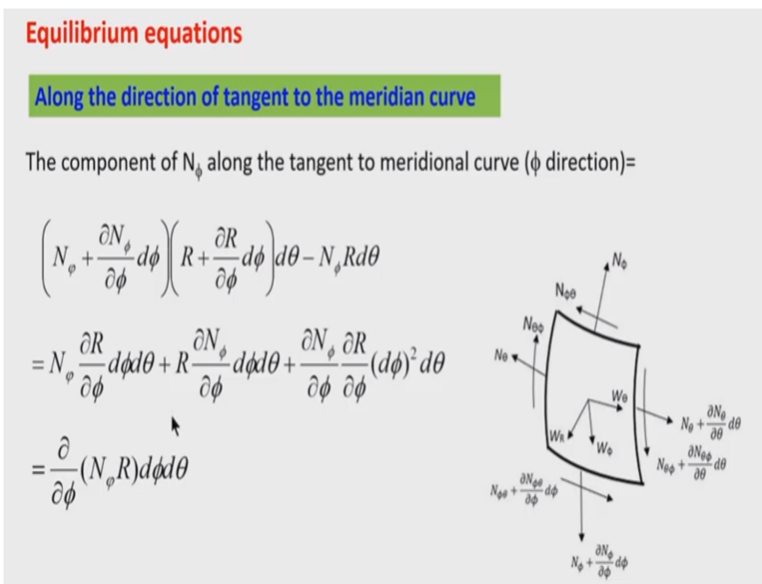

So along the direction of the tangent to the meridian curve or meridional curve meridional direction the membrane force is N_ϕ. So therefore the force quantity that I will write here $N_\phi R d\theta$ is the length of this arc. Now here we assign the negative sign because the positive direction of this unit vector $\hat{a}_{\phi}$ we have assigned in this downward direction. So that is we taken as negative sign now here you can see from the basic figure that radius of curvature of the parallel circle is variable.

That is the parallel circle here have radius say R but at this position the radius is not equal to the radius of this point. So radius is variable here and radius varies with the angle ϕ. So therefore we take the variation of the radius here and therefore the total meridional force in this arc will be $N_{\varphi} + \frac{\partial N_{\phi}}{\partial \phi} d\phi$ this is the force here. And these multiplied by the length of this element is $R + \frac{\partial R}{\partial \phi} d\phi$ why this length is changed? Because; the radius of curvature at this position and at this position is different as seen from this figure, radius of this parallel circle here and here will be different so therefore we take this variation of radius with ϕ. After simplification that is you multiply each term of this quantity with this quantity and then you neglect and cancel the common terms and neglect also this square of the small quantities $\partial\phi^2$, then ultimately you will get this term because these can be neglected as $d\phi^2$ is 1 and that is $N_{\phi} R d\theta$ is your first term and that is being cancelled with this term $N_{\phi} R d\theta$. So ultimately we are left with this term $N_{\varphi} \frac{\partial R}{\partial \phi} d\phi d\theta + R \frac{\partial N_{\phi}}{\partial \phi} d\phi d\theta$ and this is taken as 0 because this is a small quantity. So these 2 terms $N_{\varphi} \frac{\partial R}{\partial \phi} + R \frac{\partial N_{\phi}}{\partial \phi}$ can we written in this form.

Because R and N_{ϕ} both are dependent on ϕ so therefore it is written in the differential form $\frac{\partial}{\partial \phi}(N_{\varphi} R) d\phi d\theta$. So component of N_{ϕ} along the tangent to the meridional curve is obtained, that is ϕ direction. We obtain the component of N_ϕ in the ϕ direction. So we will keep it in store and then we will take it for summation.

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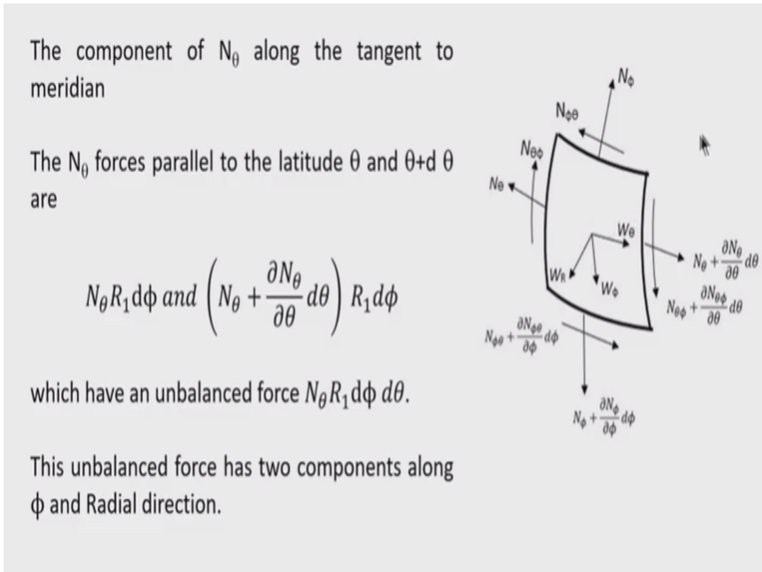

The component of N_θ along the tangent to the meridian now N_θ force is here in this edge N_θ total N_θ will be the $N_\theta \times R_1 d\phi$. In that edge the circumferential force will be $\frac{\partial N_\theta}{\partial \theta} d\theta \times R_1 d\phi$. Because $R_1 d\phi$ is the arc length so we have written these 2 forces on the opposite edges and if you see these 2 forces will have an unbalanced component towards the center.

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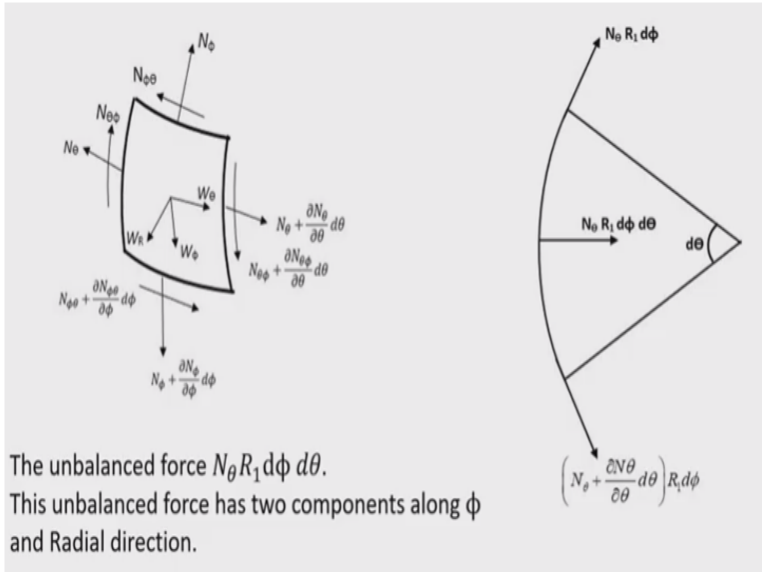

And that unbalanced component can be seen here because this is the component that we have found on the one edge total component $N_\theta R_1 d\phi$. On the other edge it is $\left(N_\theta + \frac{\partial N_\theta}{\partial \theta} d\theta\right) R_1 d\phi$. So resolving this towards the center of the parallel circle and if this angle is $d\theta$ then this component of this will be $N_\theta R_1 d\phi \sin d\theta/2$. Similarly, component of this force along this radius will be $N_\theta R_1 d\phi \times \sin d\theta/2$. If we neglect the contribution of $\frac{\partial N_\theta}{\partial \theta} d\theta\ R_1 d\phi \times \sin d\phi/2$. So neglecting the contribution of this quantity because, ultimately this product of 2 small quantities will be there. So therefore we can neglect safely and ultimately for small angle $d\theta$ we can take $\sin d\theta/2 = d\theta/2$. So therefore we get unbalanced component as $N_\theta R_1 d\phi\, d\theta$.

And this unbalanced component now can be resolved in the direction of tangent to the meridian as well as along the normal to the tangent. So this component will be resolved and unbalanced force we have calculated as $N_\theta R_1 d\phi\, d\theta$. This unbalanced force as 2 components along ϕ direction is the direction along the tangent to the meridian. And radial direction is the direction along the normal to the tangent to the meridian.

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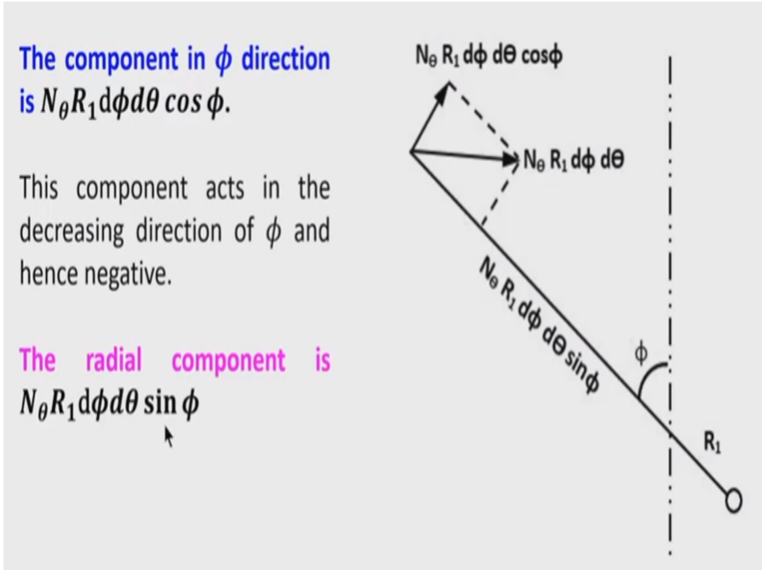

So these 2 components you are seeing here that is if this angle is ϕ then after resolving this force along the tangent direction to the meridian it becomes $N_{\theta}R_{1}d\phi d\theta \cos cos\, \phi$. Because if this angle is ϕ, this angle is 90 - ϕ and this will be again here ϕ so this component of this force along the tangent direction will be $N_{\theta}R_{1}d\phi d\theta \cos cos\, \phi$. Similarly, its component along the radial direction will be along the direction of principle radius that is the normal direction it will be $N_{\theta}R_{1}d\phi d\theta \sin sin\, \phi$.

So 2 components we have got and we have written it and we will take it when forming the summation of the forces along the respective direction. So radial component is this and this along the ϕ direction this is the component $N_{\theta}R_{1}d\phi d\theta \cos cos\, \phi$. In radial direction it will be $N_{\theta}R_{1}d\phi d\theta \sin sin\, \phi$.

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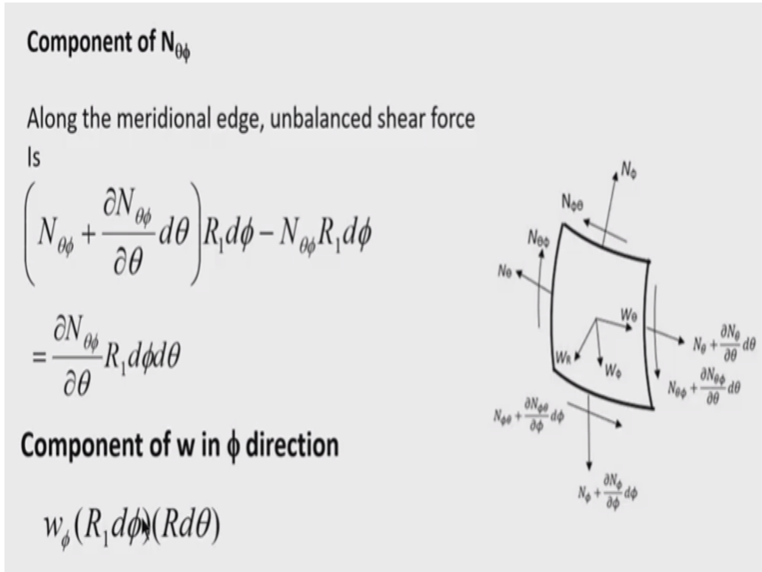

Now we come to the component of $N_{\theta\varphi}$ is in this direction along the tangent to the meridian. So therefore the component of $N_{\theta\varphi}$ along the tangent to the meridian is simply the algebraic sum of these forces acting on the 2 opposite edges. So you are seeing here $N_{\theta\varphi}$ × $R_1 d\phi$, $R_1 d\phi$ is the length of this arc. And similarly in this edge this force is $\left(N_{\theta\phi}+\frac{\partial N_{\theta\phi}}{\partial\theta}d\theta\right)R_1 d\phi$.

So after summing up and cancelling some term we get this the component of the $N_{\theta\varphi}$ tangent to the meridian that is the ϕ direction is $\frac{\partial N_{\theta\phi}}{\partial\theta}R_1 d\phi d\theta$. Now we have taken all the components whatever are there in the direction of tangent to the meridian. The external load component that have to be now taken. So component of *w*, if *w* is the load acting on the shell surface and its unit is force per unit area, surface area.

So in the ϕ direction we take the component as w_ϕ and therefore total force on this element along the ϕ direction will be w_ϕ into area of the element is $R_1 d\phi \times Rd\theta$. So we get all the components along the ϕ direction.

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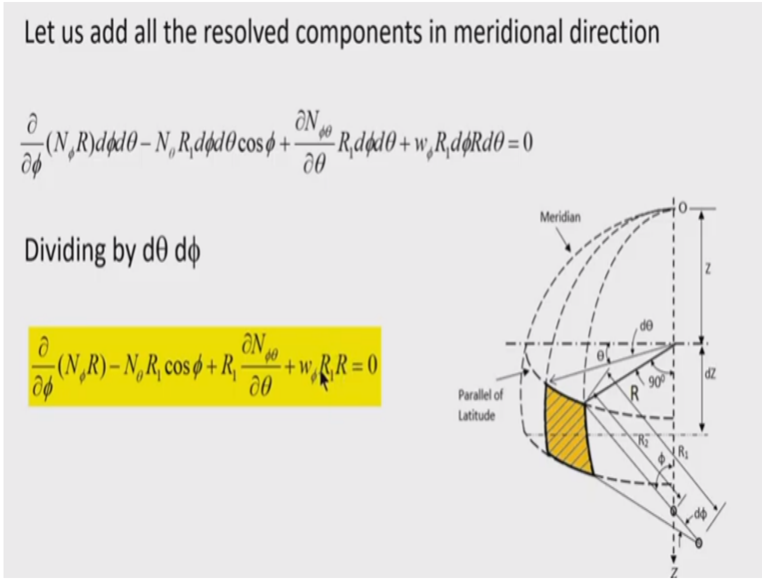

Now we sum up all the components along the ϕ direction and we get this equation $\frac{\partial}{\partial\phi}(N_\phi R)d\phi d\theta$. That is one component that we have found out earlier we have taken here. Similarly, other component we have taken and other component and then these components are actually one is along the ϕ direction another is components of N_θ along the ϕ direction, and the component of $N_{\phi\theta}$ along the ϕ direction and component of the external load.

So after cancelling the common term, common term is $d\theta d\phi$ so we can now write the equation as $\frac{\partial}{\partial\phi}(N_\phi R) - N_\phi R_1 \cos\phi + R_1 \frac{\partial N_{\phi\theta}}{\partial\theta} + w_\phi R_1 R$ where *R* is the radius of parallel of circle. And R_1 is the first principle radius of curvature. This *R* can be related to this second principle of curvature by the meridional angle ϕ.

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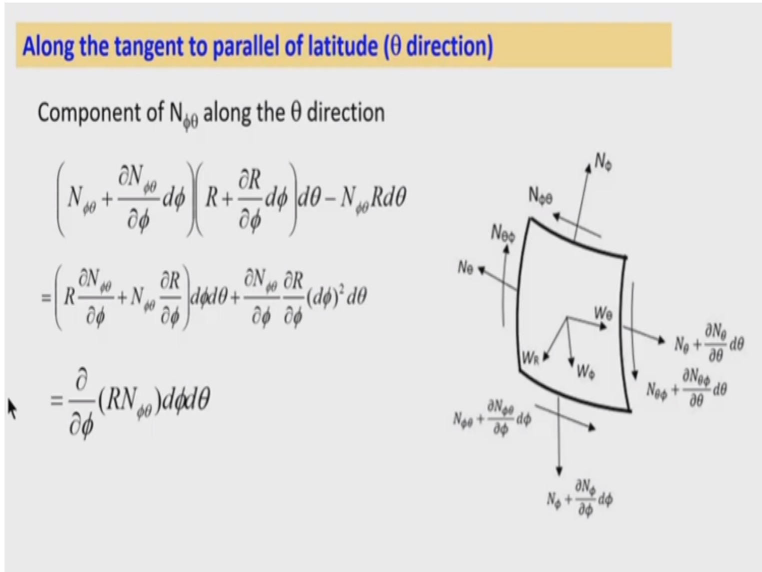

Now we get the first equation so let us obtain other 2 equations along the tangent to the parallel of latitude. Now the parallel of latitude is the θ direction so this direction the tangent to this direction is θ direction that is we can call it circumferential direction. So first let us take the component of $N_{\varphi\theta}$. So $N_{\varphi\theta}$ you are seeing that is actually in the direction of θ.

So therefore summing up the component that is acting on the 2 opposite edges we can get the quantity. So here the first quantity that is here it is in the positive direction so we take $N_{\phi\theta}+\dfrac{\partial N_{\phi\theta}}{\partial\phi}d\phi$. So that is one part then length of this arc is $\left(R+\dfrac{\partial R}{\partial\phi}d\phi\right)d\theta$. Because the radius here is R but here the radius is not R, radius is changed. Radius is variable with ϕ that is clear from the diagram earlier shown so therefore radius here will be $R+\dfrac{\partial R}{\partial\phi}d\phi$.

So arc length of this edge will be $\left(R+\dfrac{\partial R}{\partial\phi}d\phi\right)d\theta$. So this arc length is taken here and then $N_{\phi\theta}Rd\theta$ is the total force along these edges, and it is along the negative direction of the θ that is along the decreasing direction of θ therefore it is taken as negative. So; after multiplying term by term and cancelling some common term and also ignoring the term with power of small quantities, we ultimately get this as $\left(R\frac{\partial N_{\phi\theta}}{\partial\phi}+N_{\phi\theta}\frac{\partial R}{\partial\phi}\right)d\phi d\theta+\frac{\partial N_{\phi\theta}}{\partial\phi}\frac{\partial R}{\partial\phi}(d\phi)^2 d\theta$. So cancelling this term that is this is very small quantity we ultimately get this because this $\partial N_{\phi\theta}$ and this R these 2 can be combined and we get $R \times$ $N_{\phi\theta}$ and it is derivative with respect to ϕ, $\frac{\partial}{\partial\phi}$ and we get this $\frac{\partial}{\partial\phi}(RN_{\phi\theta})d\phi d\theta$. So, one component is obtained similarly other component of forces in the θ direction is to be obtained.

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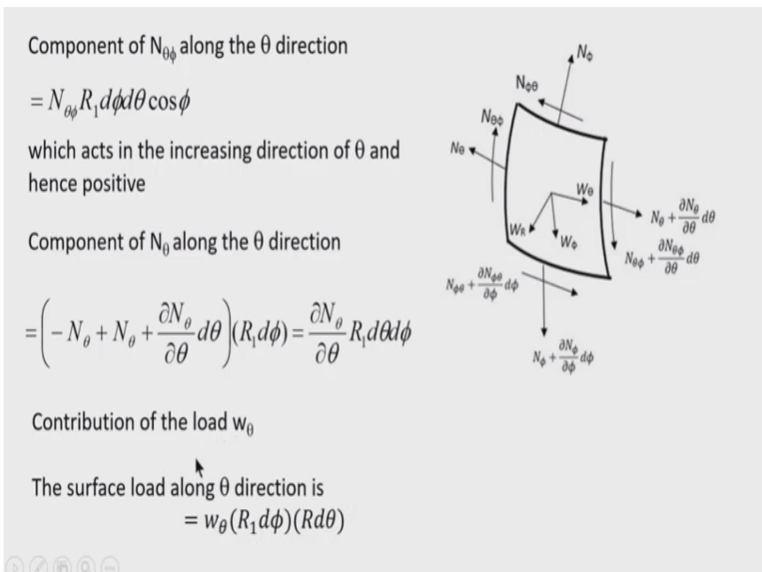

Component of $N_{\theta\phi}$ along the θ direction, $N_{\theta\phi}$. So $N_{\theta\phi}$ is acting here and on the edges you are seeing that the increment is given so $N_{\theta\phi}+\frac{\partial N_{\theta\phi}}{\partial\theta}d\theta$. So the component of these 2 forces along the θ direction that is in the circumferential direction is found in the similar way that we have found in earlier case, the component of N_θ along the radial direction and normal direction.

So in this way we get this component of $N_{\theta\phi}$ as $N_{\theta\phi}R_1 d\phi d\theta \cos\phi$ and this is along the increasing direction of θ and hence taken as positive quantity in the summation process to be taken in the later on. So this total component of $N_{\theta\phi}$ along the θ direction is now given $N_{\theta\phi}R_1 d\phi d\theta \cos\phi$. Component of N_θ along θ direction, N_θ is here N_θ is in this direction and on the opposite as this $N_\theta + \frac{\partial N_\theta}{\partial \theta} d\theta$.

So this component is found just after summation because these forces are acting along the tangent. So there is no need to resolve in to get the component so we take here because if this direction if positive direction then this quantity $N_\theta + \frac{\partial N_\theta}{\partial \theta} d\theta$ is one term. And then $-N_\theta$ because this is in the opposite direction but with 2 forces that is N_θ as $N_\theta + \frac{\partial N_\theta}{\partial \theta} d\theta$. The arc length is same so arc length $R_1 d\phi$ so after simplifying we get this force in the component of N_θ along the tangent to the parallel circle that is $\frac{\partial N_\theta}{\partial \theta} R_1 d\theta d\phi$. So we have got all the components along the θ direction except the component of w. So component of w along the θ direction let it be denoted by w_θ and therefore surface load along θ direction would be w_θ into area of the element which is $(R_1 d\phi)(R d\theta)$. So we get these components.

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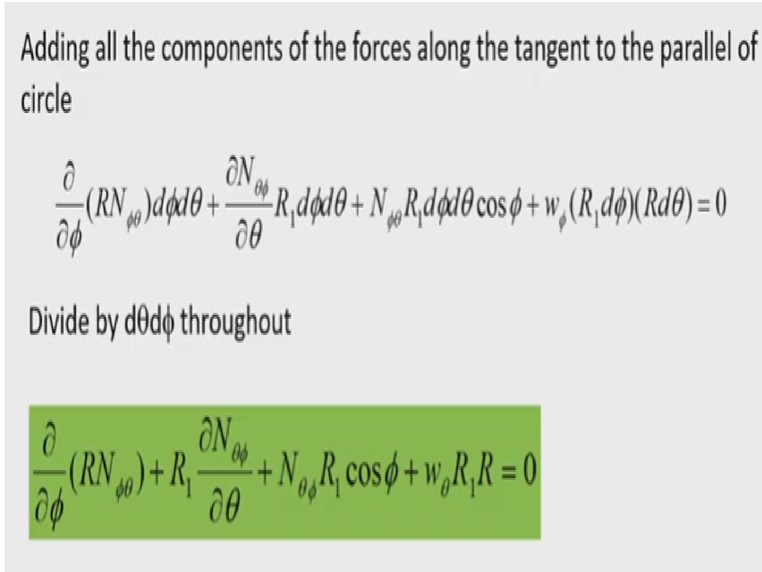

Then adding all the component of the forces along the tangent to the parallel circle we now get $\frac{\partial}{\partial \phi}(RN_{\phi\theta})d\phi d\theta + \frac{\partial N_{\theta\phi}}{\partial \theta}R_1 d\phi d\theta + N_{\phi\theta}R_1 d\phi d\theta \cos\phi + w_{\phi}(R_1 d\phi)(Rd\theta)$. And dividing throughout by $d\theta d\varphi$ we ultimately arrive at this equation $\frac{\partial}{\partial \phi}(RN_{\phi\theta})$. So $RN_{\phi\theta}$ is a term which is dependent on ϕ.

So therefore when we differentiate this quantity it will be one term will be $\frac{\partial R}{\partial \phi}N_{\phi\theta}$ and another term will be $R\frac{\partial N_{\phi\theta}}{\partial \phi}$. And this R_1 which is one of the principle radius of curvature is appearing here. And $\frac{\partial N_{\theta\phi}}{\partial \theta} + N_{\theta\phi}R_1 \cos\phi + w_{\theta}R_1 R$. So this is the second equation. This equation of equilibrium is in the direction of the tangent to the parallel circle.

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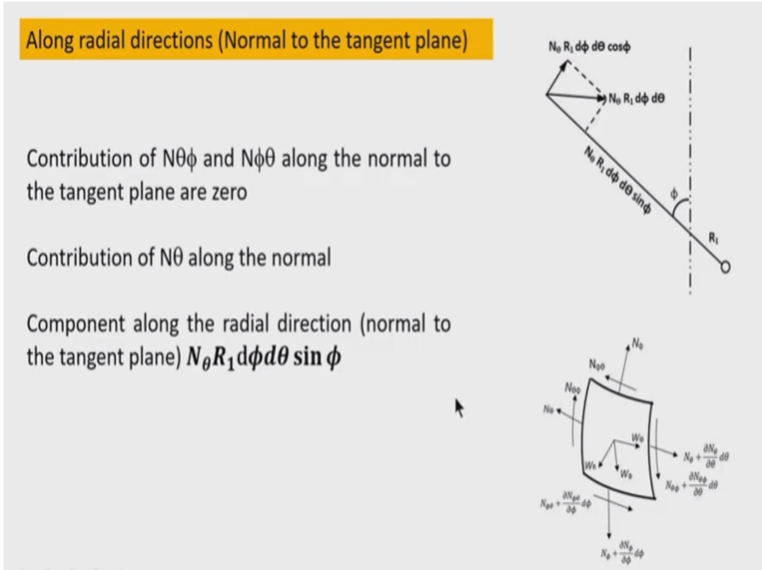

Having obtained this second equation now we proceed to obtain the third equation which is equation of equilibrium in the radial direction. Radial direction is your normal to the tangent plane and therefore after resolving the quantities or forces we get the components and then we can add it, sum it and can equate to 0 to get the equation of equilibrium. Now you can see if I take the normal direction that is radial direction you can tell it normal to the tangent. So in that case you can see that N_ϕ has no contribution and this $N_{\theta\phi}$ $N_{\phi\theta}$ along the normal when you resolve it, along the normal will have 0 value. So therefore these 2 forces have no contribution towards the equilibrium equation in the normal direction. Contribution of N_θ along the normal, so N_θ along the normal direction we have earlier obtained during forming the equilibrium equations along the ϕ direction.

And one component we have kept in store and now we are utilizing so component along the radial direction is $N_\theta R_1 d\phi d\theta \sin sin\, \phi$. This component we have obtained earlier but as seen here we did not use it because at that time we formed the equation in the ϕ direction. Now we are forming the equation in the radial direction so we take this quantity.

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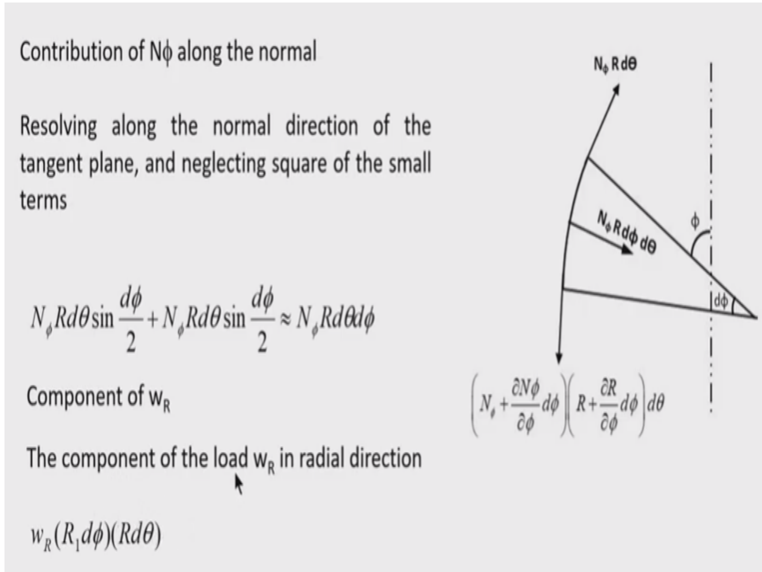

Contribution of N_ϕ along the normal. If you see the N_ϕ that is the meridional force at the 2 ends of the curve say here N_ϕ total force is $N_\phi R d\theta$ and here it will be $N_\varphi + \frac{\partial N_\phi}{\partial \phi} d\phi$ $\times$ $R d\theta$.

So now resolving these 2 forces $N_\phi R d\theta$ and $N_\varphi + \frac{\partial N_\phi}{\partial \phi} d\phi$ and the radius is here $R + \frac{\partial R}{\partial \phi} d\phi$ this because R was changing with ϕ.

So therefore this variation is taken and the small angle that is here with respect to your center at the parallel circle is $d\theta$. So therefore it is multiplied with $d\theta$ so arc length is here is $\left(R + \frac{\partial R}{\partial \phi} d\phi\right) d\theta$. But here arc length is $R d\theta$ because R is changing from here to here with respect to ϕ and this change is reflected here. So after resolving this along the normal direction then we get this $N_\phi R d\phi d\theta$.

So this direction is along the direction of the normal now if I resolve it then we get this $N_\phi Rd\theta \sin\frac{d\phi}{2} + N_\phi Rd\theta \sin\frac{d\phi}{2}$ and which is nothing but $N_\phi Rd\theta d\phi$. Because $\frac{d\phi}{2}$ or ϕ small quantity so $\sin\frac{d\phi}{2}$ is nothing but $\frac{d\phi}{2}$, so $\frac{d\phi}{2}$ is common factor here. And therefore it becomes $N_\phi Rd\theta d\phi$ so we got this component along the radial direction.

Now component of w_R along the radial direction will be say the load is w_R then load multiplied by surface area is $(R_1 d\phi)(Rd\theta)$. So this component we have got now to form the equation of equilibrium.

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Adding all the components along the normal to the tangent plane, i,e Radial Direction

$$N_\theta R_1 d\phi d\theta \sin\phi + N_\phi Rd\theta d\phi - w_R(R_1 d\phi)(Rd\theta) = 0$$

Taking $R=R_2 \sin\phi$

$$N_\theta R_1 d\phi d\theta \sin\phi + N_\phi R_2 \sin\phi d\theta d\phi - w_R(R_1 d\phi)(R_2 \sin\phi d\theta) = 0$$

Divide by $R_1 R_2 d\phi d\theta \sin\phi$

$$\boxed{\frac{N_\phi}{R_1} + \frac{N_\theta}{R_2} = w_R}$$

Equation of equilibrium is formed here say $N_\theta R_1 d\phi d\theta \sin\phi$ that is one component. Then another component of N_ϕ towards the normal is $N_\phi Rd\theta d\phi$, then this is the radial component of the load. Now in the definition of load when we define the load, we have taken the load component w_R away from the center. So therefore outwards, so outward direction we take the load as negative.

So therefore it is written as this $-w_R(R_1 d\phi)(Rd\theta)$ so total all the quantities are added up and then we can form the equilibrium equation. Now here interestingly you can see there is no differential term appearing in the equation of equilibrium. So no differential term is coming here because we will ultimately divide the quantities with the area of the element $d\phi d\theta$ $\times$ R_1 R.

So R_1 R $d\phi d\theta$ is the area of the element and taking $R = R_2 \sin \varphi$ that is clear here from the first diagram that this R is nothing but if this distance is R_2 and this angle is φ, $R = R_2 \sin \varphi$ so that relationship will use here $R = R_2 \sin \varphi$. And then we get this equation, $N_\theta R_1 d\phi d\theta \sin\phi + N_\phi$ instead of R we now substitute $R_2 \sin \varphi$, so $R_2 \sin\phi d\theta d\phi - w_R R d\theta (R_1 d\phi)$ and here we write $R_2 \sin \varphi$ and $d\theta$ is already there now it is equal to 0.

So divide throughout by $R_1 R_2 d\varphi d\theta \sin\varphi$ so after dividing all the terms by this factor $R_1 R_2 d\varphi d\theta \sin\varphi$, we ultimately arrive at this equation $\frac{N_\phi}{R_1} + \frac{N_\theta}{R_2} = w_R$. So you can see the equilibrium equation is now here in the third case that is when we summed up all the forces along the radial direction it is an algebraic equation not a differential equation.

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Equilibrium Equations for the Shell of surface of revolution for general condition of loading

$$\frac{\partial}{\partial \phi}(N_\phi R) - N_\theta R_1 \cos\phi + R_1 \frac{\partial N_{\phi\theta}}{\partial \theta} + w_\phi R_1 R = 0 \qquad (1)$$

$$\frac{\partial}{\partial \phi}(R N_{\phi\theta}) + R_1 \frac{\partial N_{\theta\phi}}{\partial \theta} + N_{\theta\phi} R_1 \cos\phi + w_\theta R_1 R = 0 \qquad (2)$$

$$\frac{N_\phi}{R_1} + \frac{N_\theta}{R_2} = w_R \qquad (3)$$

So now 3 equations of equilibrium we can state, 1 equilibrium equation that is along the direction of tangent to meridian that is the ϕ direction is the first equation that is $\frac{\partial}{\partial \phi}(N_\phi R) - N_\theta R_1 \cos\phi + R_1 \frac{\partial N_{\phi\theta}}{\partial \theta} + w_\phi R_1 R = 0$. Second equation is the equation along the θ direction that is direction of tangent to the parallel circle. So this equation is $\frac{\partial}{\partial \phi}(N_\phi R) - N_\theta R_1 \cos\phi + R_1 \frac{\partial N_{\phi\theta}}{\partial \theta} + w_\phi R_1 R = 0$. And third equation is algebraic equation $\frac{N_\phi}{R_1} + \frac{N_\theta}{R_2} = w_R$, so from this 3 equation we need to calculate these 3 quantities actually that is N_ϕ N_θ and $N_{\phi\theta}$. Of course for the shell whose thickness is small compared to the radius of curvature, then in that case $N_{\theta\phi}$ = $N_{\phi\theta}$. Now having found these 3 equations of equilibrium we now come to the case where the axi symmetrical loading conditions exist.

Now mainly the shell structures which are used for construction of roof say dome in an assembly hall or a religious spaces or any large hall is generally a thin structure and the loading there is only your, self-load. And very nominal quantity of live load. And occasionally wind load occurs and wind load of course it becomes asymmetrical distribution. And in addition to that there may be some snow load also but that snow load is also taken as the symmetrical load distribution.

So therefore axi symmetrical condition of this surface of revolution is most common case and the design we should take into account of the axi symmetrical condition.

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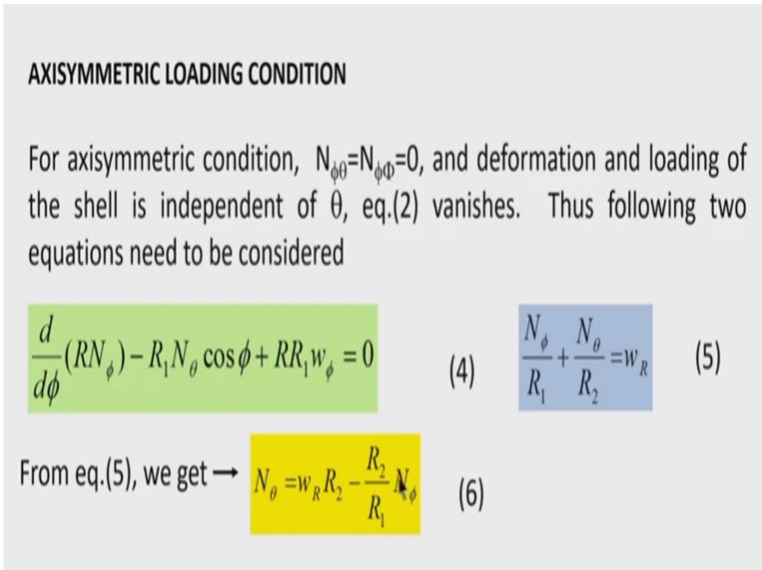

So for axi symmetrical condition $N_{\varphi\theta}$=$N_{\theta\phi}$ is 0 so therefore this equation this term will be 0 because $N_{\varphi\theta}$ is 0 and therefore first equation will be $\frac{d}{d\phi}(RN_\phi)-R_1N_\theta\cos\phi+RR_1w_\phi$. This term will have not contribution because for axi symmetrical condition $N_{\varphi\theta}$ is 0 if we look at this second equation because $N_{\varphi\theta}$ is 0. So this term is also not necessary, this is also not necessary and this is also not there.

So that means second equation is actually not necessary in case of axi symmetrical condition, first equation is necessary and the third equation is necessary. From the first equation, we can get this N_ϕ and once the N_ϕ is obtained we can get the N_θ from that equation. So we have only 2 equations of equilibrium in case of axi symmetrical condition and from this equation that is this equation number 5 here we can get $N_\theta = w_R R_2 - \frac{R_2}{R_1} N_\phi$.

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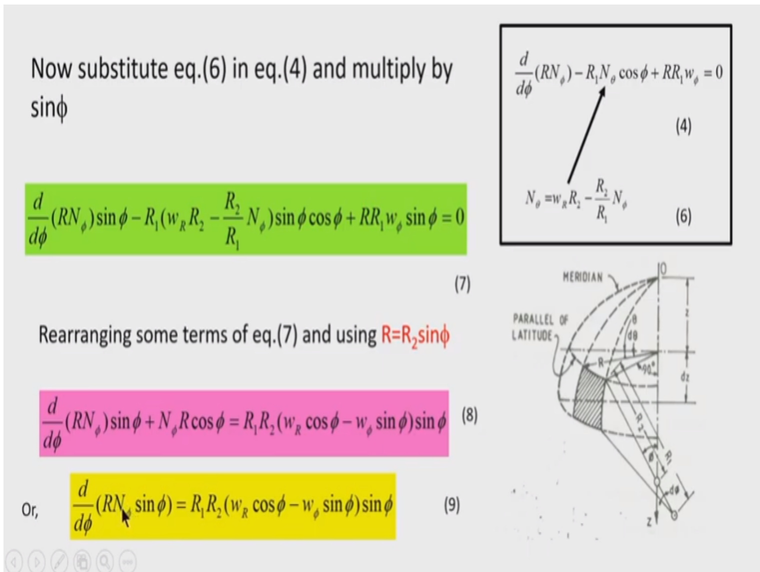

Then we substitute this equation that is obtained $N_\theta = w_R R_2 - \frac{R_2}{R_1} N_\phi$ into this differential equation. And then we get this equation as $d/d\phi$ we remove the partial derivative sign because it now becomes an axis symmetrical condition. So it is not dependent on these 2 quantities that is ϕ and θ. So ultimately it will be only dependent on ϕ, first equation so therefore we write $\frac{d}{d\phi}(RN_\phi)\sin\phi - R_1$ and this quantity is nothing but N_θ.

So N_θ we are substituting here as $w_R R_2 - \frac{R_2}{R_1} N_\phi$ and whatever quantity was there earlier we write it $\sin\phi\cos\phi$. Then $RR_1 w_\phi \sin\phi = 0$ now rearranging terms and using this expression so $R=R_2 sin\varphi$ you can see this R is here the radius of the parallels of latitude and it is related to second principle curvature R_2 as with the relation $R=R_2 sin\varphi$.

So substituting this here and rearranging some terms we get $\frac{d}{d\phi}(RN_\phi)\sin\phi + N_\phi R\cos\phi = R_1R_2(w_R\cos\phi - w_\phi\sin\phi)\sin\phi$. And then after arranging this term, these 2 terms can be retained here $\frac{d}{d\phi}(RN_\phi)\sin\phi$. So that is possible and after writing this we get a simple expression as this which can be integrated. So after integrating of this expression we get $RN_\phi\sin\phi$. That is now getting the value of 1 membrane stress that is N_ϕ.

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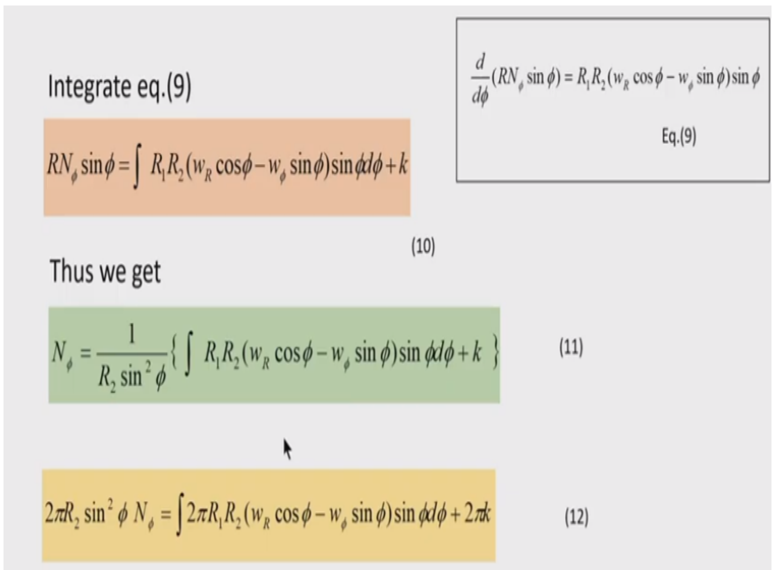

So integrating this expression we get $RN_\phi\sin\phi$, then $R_1R_2(w_R\cos\phi - w_\phi\sin\phi)\sin\phi d\phi + k$, k is a constant of integration. So from this equation we now obtain $N_\phi = \frac{1}{R_2\sin^2\phi}\{\int R_1R_2(w_R\cos\phi - w_\phi\sin\phi)\sin\phi d\phi + k\}$, k is a constant of integration which is to be found from edge condition. Now in; case of this surface of this revolution having a cut edges or having the known N_ϕ at 1 edge or at the other edge, we can find the value of k so that value of k when we find, then N_ϕ cannot be satisfied for the condition that exist at the other edge. So therefore a corrective term as to be applied. This is the drawback of the membrane theory because the membrane condition, the membrane state of condition may not be satisfied at the edges.

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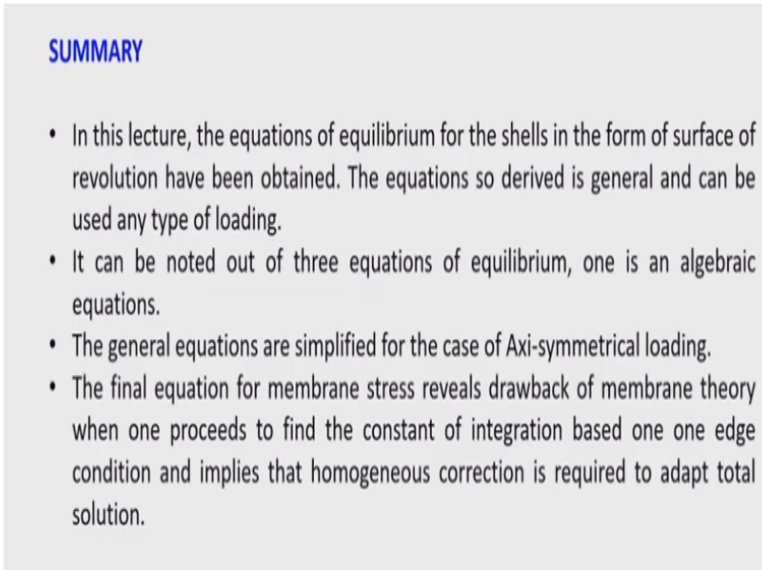

So today what we have done, we have discussed the equations of equilibrium for the shells in the form of surface of revolution and the equation so derived actually general and can be used for any type of loading. The 3 equations that I have derived first are general equations of equilibrium. And whether the loading is symmetrical or unsymmetrical you can use these 3 equations.

But we simplified these 3 equations in case of axi-symmetrical loading taking the condition of $N_{\phi\theta} = N_{\theta\phi}$ to be 0. And 3 equations are reduced to 2 equations and out of that we find that 1 is algebraic equations and in one case it is differential equation but it is an ordinary differential equation. So when the differential equation is solved then we can find this value of N_{ϕ} and substituting this value of N_{ϕ} in this second equation $\frac{N_{\phi}}{R_1} + \frac{N_{\theta}}{R_2} = w_R$ then we can get the second circumferential stress N_{θ}.

Because in case of axi-symmetrical loading this $N_{\theta\phi}$ is 0 so only 2 forces are of important one is N_{ϕ} and another is N_{θ}. So final equation from membrane stress that we have found reveal some drawback of membrane theory that I have told you earlier. And these are drawback you can note that when we want to calculate the constant of integration based on 1 edge condition and implies that constant may not be satisfied or that constant with the equation cannot be satisfied with the other edge, so that means a homogenous correction has to be applied to adopt total solution. So that drawback has to be removed to find out or to use the membrane solution near the edges. But away from the edges the membrane solution gives reasonably accurate results thank you very much.