Plates and Shells

Prof. Sudip Talukdar
Department of Civil Engineering
Indian Institute of Technology-Guwahati

Lecture-24
Stress resultants and couples in shells

Hello everybody, today I am delivering the 3rd lecture of the module 8, and the topic is the plates and shell; we have covered the plate, and then we have entered the chapter shell. So, so far, I have introduced this shell to you as a theme structural elements edges stress skin. So, I have shown you how the membrane analysis of a stress skin can be done, and the differential equation of a membrane is obtained, which was under the uniform tension.

And we obtain the deformation under the uniform pressure on the membrane. The membrane is given as an example of a stress skin, and then we discussed the classification of various shells. So, various shells, you have seen that classification was done mainly based on the curvature because curvature is the characteristic of the shell. That means shell is actually known by it is curvature; if there is no curvature, shell is a plate.

So, depending on the nature of curvature, we have group the shell into different categories and the surface that is formed by translation of a straight line or of a curve; everything was discussed and given you the full description of the shell of different types. Now today, I want to discuss the stresses resultants and couples in thin shell. Now the shell that is mainly used in civil construction as a roof structure is subjected to your self-weight, that is, the dead load, live load.

Live load is nominal because in the roof structure, except for maintenance, no live load is significant. So, maintenance live load is very low, so therefore the dead load predominates, and in some region where snowfall is more, then snow load has to be considered in the analysis and design of shell structure. Now in case of other applications, say pressure vessels or these aircraft structures.

Aircraft, the skin that is the fuselage skin which is acting as a membrane, is subjected to wind pressure, and therefore the stresses are developed on the shell structure. Similar is the case of pressure vessel, which is subjected to internal pressure, and as a result, the stresses are developed. So, I have here mentioned the stress resultants and couples, which includes the force as well as moment in the thin shell in response to the external loading.

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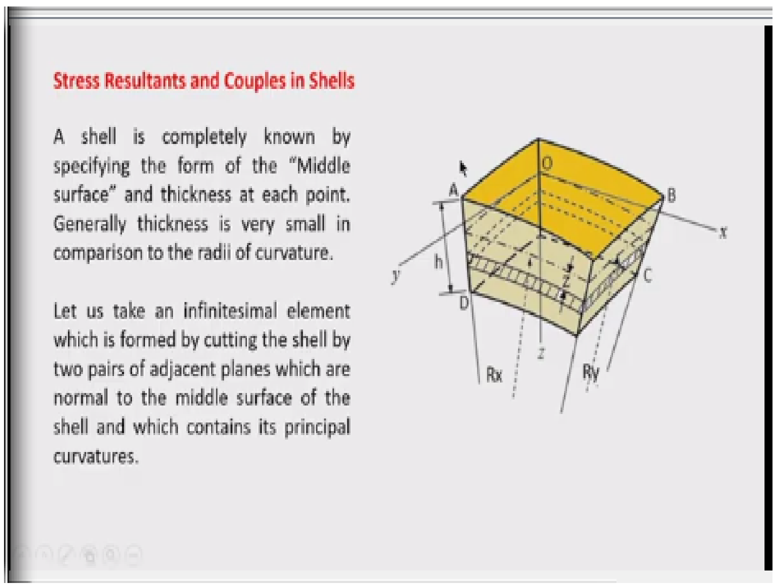

Now let us take a shell element which is of certain curvature, and curvature exist in both direction; it is an element of doubly curved shell. And the middle surface of the shell is the surface with bisects the thickness of the shell. So, thickness of the shell is denoted by h, and thickness is very small compared to radius of curvature in general. So, radius of curvature in 2 orthogonal directions are denoted by R_x and R_y.

And middle surface here it is shown as dotted line, as you are noting here. And we take an element of the shell across the thickness at a distance z of say this thickness of the element is D_z at a distance z. So, we will consider this for deriving this stress resultant. Let us consider 2 adjacent planes which are normal to the middle surface. So, these adjacent 2 planes are normal to the middle surface at which the principal curvature exists.

So, principal radius of curvature in these 2 edges in normal planes you are seeing, that in this plane the principal radius of curvature R_x will lie and here also in this plane the another principal curvature the R_y will lie.

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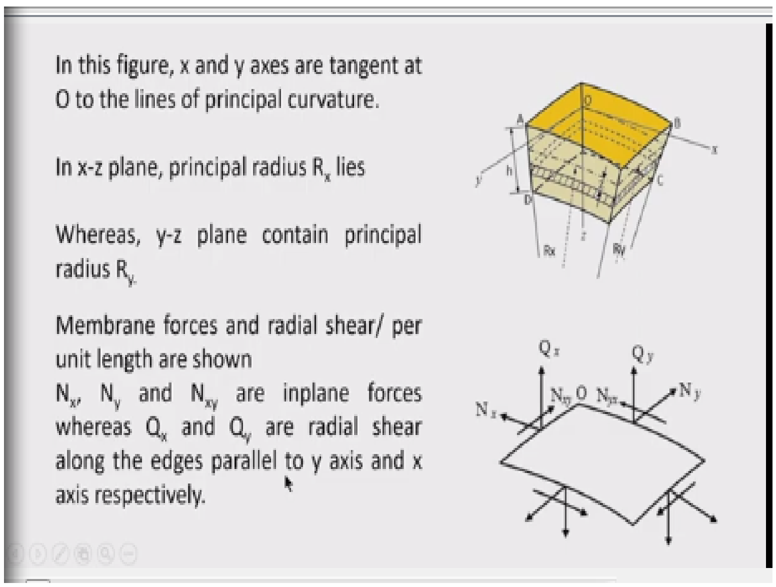

Now in this figure, the x-axis and y-axis are tangential at point O to the lines of principal curvature. So, principal curvature, as you are seeing here, it is along the middle surface, so this x-axis is tangential to the principal curvature along the x-direction; similarly, y-axis is tangential along the principal curvature in the y-direction. Now in the x-z plane, the x-z plane, the principal radius R_x will lie.

Similarly, in the y-z plane, the principal radius, y-z plane that you are seeing here y-z, this is y, and this is z. And y-z plane, the principal radius R_y will be there, and in another plane, x-z, the principal radius R_x will be there. Now in the shell, due to external loading, the in-plane shear as well as radial shear will be developed and also bending moment. So, let us see the in-plane shear in the x-direction is N_x, and in-plane, shear in y-direction is N_y.

Membrane shear force is N_{xy}, and N_{yx} are also there in an element. And we are also noting that vertical shear along the edge, which is parallel to y-axis, is Q_x and on the other edge, which is parallel to y-axis, is Q_y. So, N_x, N_y and N_{xy} are in-plane forces, whereas Q_x and Q_y are radial shear along the edges parallel to y-axis and x-axis, respectively.

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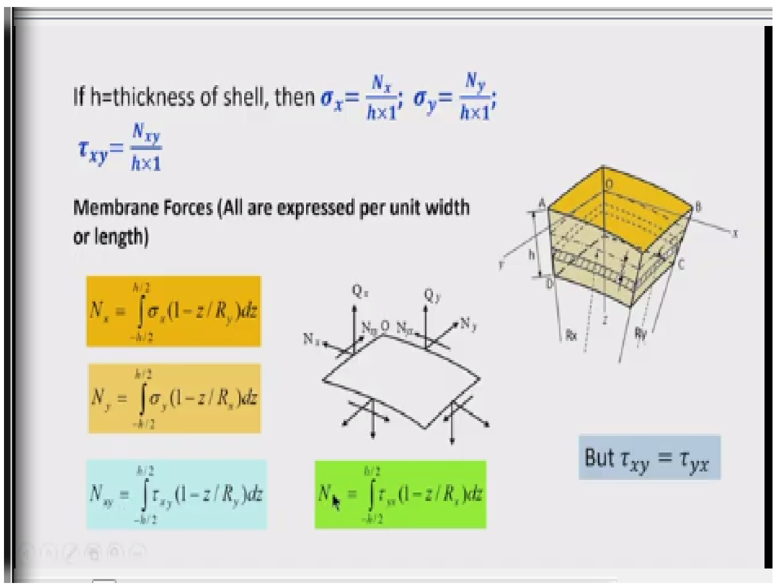

Now let us take the thickness of the shell is h. Now, as I have mentioned, this like plate all the quantities the in-plane forces, then your bending moment, twisting moment etcetera, everything will be denoted per unit length. So, therefore N_x is also expressed in terms of per unit length. For example, the force unit is Newton, then N_x will be Newton per meter if the length units taken is meter on millimetre, whatever you call.

Then if I want to calculate stress, distribution of normal stress across the thickness, $\frac{N_x}{h\times1}$,$h\times1$ is the area of the strip. So, because the unit would these taken for the shell in the analysis and design of shells. So, we multiply thickness by 1 to get the area of the element. Similarly,

$\sigma_y = \frac{N_y}{h\times1}$ and $\tau_{xy} = \frac{N_{xy}}{h\times1}$. Now we call in-plane forces as the main forces; let us see what are the membrane forces and how these are evaluated?

So, first, let us see the N_x is the membrane force in the x-direction, and you are seeing here in this figure, this N_x is along the x-direction, x-direction is this, x-axis shown here. So, N_x is calculated as $\sigma_x(1 - z/R_y)dz$. So, if there is no curvature, that means the adjacent edges that you are seeing is actually in trapezoidal shape. So, this is due to curvature and therefore, to account for curvature, this term is there $- z/R$.

And in the appropriate direction, say we are calculating this normal stresses along the x-direction. So, along the x-direction, that is actually your y-z plane, and in this y-z plane, you can see that R_y radius of curvature is lying. So, therefore $\int_{-h/2}^{h/2} \sigma_x(1 - z/R_y)dz$ and limit will be $- h/2$ to $h/2$ because h is the thickness of the shell and if we take an element at a distance z from middle surface.

Then we have to integrate it from $- h/2$ to $+ h/2$ if middle surface is taken as the reference. Similarly, N_y, the membrane force in the y-direction is calculated as $\int_{-h/2}^{h/2} \sigma_y(1 - z/R_x)dz$, integration limit is $- h/2$ to $+ h/2$. N_{xy} will be developed because of shear stress. So, shear stress τ_{xy} is taken here, and N_{xy}, you are seeing that N_{xy} is acting along this edge which is parallel to the y-axis.

So, therefore these R_x factor is coming here, that $- z/R_x$, this term is there and $(1 - z/R_x)$, this is actually due to curvature. So, when this is curvature is very small, that means the radius of curvature is very large. In that case, z/R_y or z/R_x can be neglected because this will be small in comparison to 1, and this factor can be omitted. So, this N_{xy} the another component of membrane force that is a membrane shear force is given by $\tau_{xy}\left(1 - \frac{z}{R_y}\right)dz, - h/2$ to $+ h/2$ is the limit of integration.

Similarly, N_{yx}, N_{xy} is the membrane shear force along the direction that is parallel to x-axis So, N_{yx} that you are getting here $\tau_{yx}\left(1 - \frac{z}{R_x}\right)dz$. Now $- h/2$ to $+ h/2$ are the limits of the integration. Now here, τ_{yx} should be equal to τ_{xy}, but you can see that N_{xy} in general is not equal to N_{yx} because the 2 curvature may be different. Because here $- z/R_y$ and here you are seeing $- z/R_x$, so when these factor, these z/R_x or z/R_y is to be taken in shell analysis, then your N_{xy} and N_{yx} is not required.

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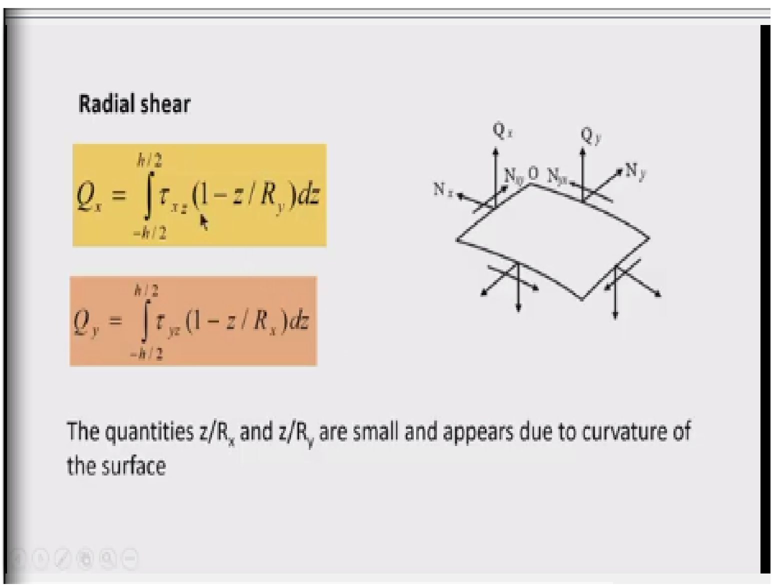

Radial shear, we express in terms of again integration because the vertical shear stress that is τ_{xy} is taken here. And if the distribution is known along the thickness, then the Q_x is developed due to τ_{xy}. So, τ_{xy} is this force that is the vertical shear stress, so $\tau_{xy}(1 - z/R_y)dz$ and it is integrated in the limit $- h/2$ to $+ h/2$. Similarly, Q_y, Q_y is the radial shear along the edge, which is parallel to x-axis.

Then we are getting $\tau_{yz}\left(1-\frac{z}{R_x}\right)dz$. So, these are the radial shear, so what are the stress resultant that we have got? We have got N_x, N_y, N_{xy}, N_{yx}, Q_x and Q_y. Now quantity z/R_x, z/R_y influence the magnitude of this stress resultant, that is, your membrane forces as well as shear forces. But when the thickness of the shell is small compared to the radius of curvature, then z/R_x, z/R_y are negligible in compared to unity.

So, in that case, this quantity $\frac{z}{R_x}$ and $\frac{z}{R_y}$ can be neglected. For example, for a small thin shell this z/R_y for a certain data, it becomes 1×10^{-6}. So, in that case, we can ignore this term, and we can simply write tau xz into dz, $-h/2$ to $+h/2$ is the integration limit and then find the result. So, similar is the case with other stress resultant.

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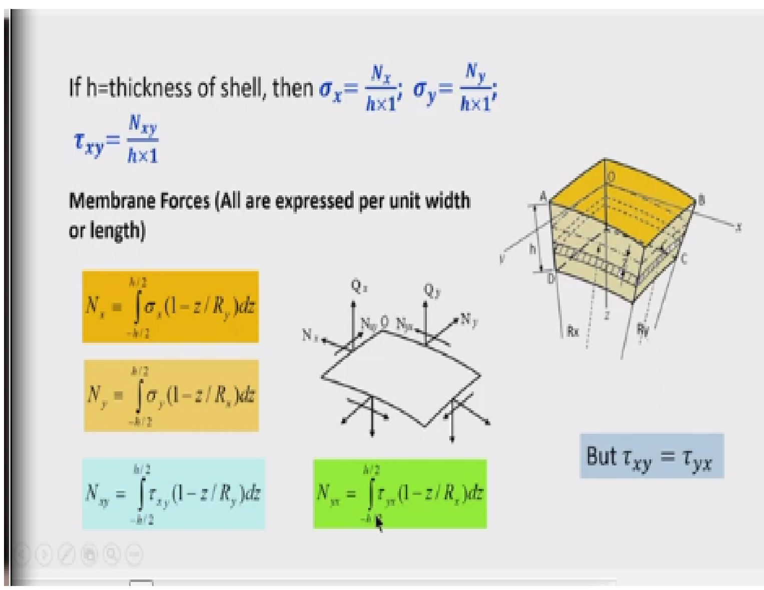

That means if z, the thickness of the shell is very, very small compared to the radius of curvature. Then this term can be neglected because it is very low value compared to 1. So, in that case, the magnitude of in-plane forces that can be found simply by integration of $\sigma_x\times dz$ and N_y can be found as simply by integration of $\sigma_y\times dz$, N_{xy} can be found by integration of $\tau_{xy}\times dz$, N_{yx} is simply found $\tau_{yx}\times dz$.

Now when we neglect these quantities, then we can easily verify that $N_{xy} = N_{yx}$, which quantity? That $\frac{z}{R_y}$ $and\ z/R_x$. If these quantities are approximately are tending to 0, then N_{xy} will be equal to N_{yx} because of the complimentary nature of the shear stress τ_{xy} and τ_{yx}. So, we got all stress resultants in the shell except the bending moment, but bending moment effect in the shell in many theories are neglected.

And it is found that the bending moment influence is mostly predominant near the support. And away from the support, the membrane state of stress in-plane forces are mostly significant and can be taken for the design. And in simplifies the design procedure and calculation. However, at this support, the bending moment effect is seen that is if the support when the shell joint with the edge beam, then there will be a moment. And this moment will alter this test condition near the boundary. So, that condition have to be taken in the shell theories by knowing the expression for the bending moment.

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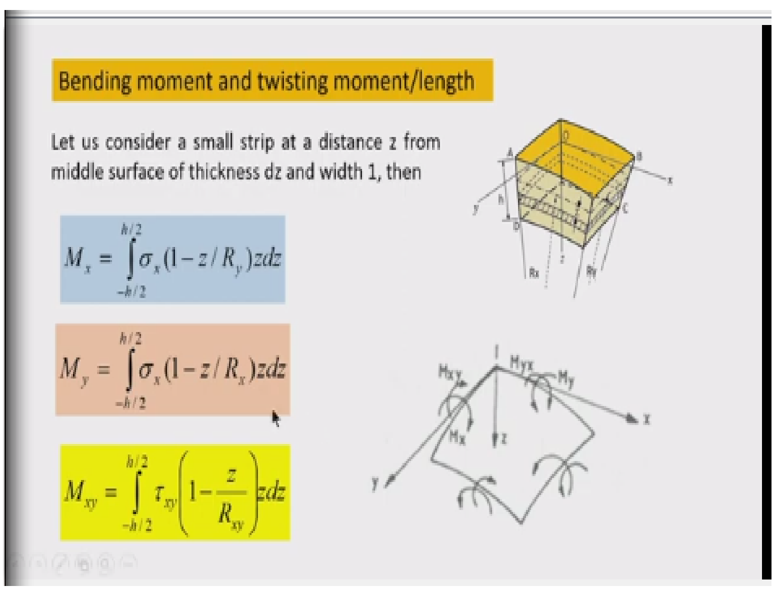

So, now we will discuss the bending moment expression. Let us take a small element of the shell at a distant z from the middle surface, and this element has a unit width, for example. So, therefore area is $1 \times dz$, so $(1 \times dz \times \sigma_x)$ is the force along the x-direction. So, now this force multiplied by the lever arm z that is, we are taking the moment of the forces about the neutral axis as about the middle surface, which is analogous to the neutral axis.

So, the neutral axis in case of beam, so here we are taking it as a middle surface. But the properties of middle surface is that they are the stresses are 0, and all the quantities are actually referred to in the middle surface. So, here if we take z is the distance of the element from the middle surface, then moment of the force is $\sigma_x \times 1 \times dz$, and this factor is taken because of curvature.

So, in the x-direction, if I see this curve, the radius of curvature is R_y, so R_y factor is there. Now we can write the expression for M_y if we know the normal stress along the y-direction, so in that case, $\sigma_y(1 - z/R_x)zdz$. So, here are the principal radius is R_x, will be your M_y, and M_{xy} is developed due to shear stress. So, twisting moment is given by $\tau_{xy}\left(1 - \frac{z}{R_{xy}}\right)zdz$.

Now all the quantities that you are seeing R_y, R_x and R_{xy} is due to curvature. So, when the thickness of the shell is small compared to the radius of the curvature, then these quantities are negligible, and we get a very simplified expression. Now here, the quantity R_{xy} is nothing but your twist curvature. So, that twist curvature we have seen in case of small deflection that is the if z is the shell surface then $\frac{\partial^2 z}{\partial x \partial y}$ is the twisting curvature.

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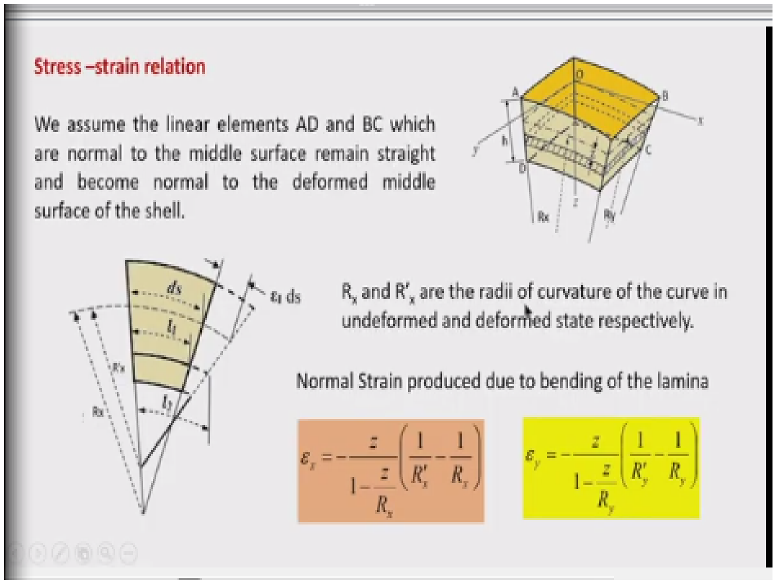

Now let us discuss what will be the stress-strain relation in case of thin shell. We assumed the linear elements AD and BC, which are normal to the middle surface, remain straight and become normal to the deform middle surface of the shell. So, this assumption is familiar to you as we have seen in case of plate also. So, linear element AD and BC, AD is an element you are seeing here, so along the edges or 2 adjacent edges are AD is a common element.

So, similarly element BC, ER, you are seeing the element BC and know which are originally normal to the deformed a middle surface before deformation. After deformation also remains normal and also straight and it is also further assumed. So, there will be no extension of this normal, that means normal are inextensible; in that case, the strain in this direction normal direction is to be taken as 0.

Now this R_x and $R_x^{'}$ are the radii of curvature or curve in the undeformed and deformed states, respectively. So, now we distinguish the curvature by 2 notation; one is before deformation because shell has original curvature. So, this is a element which has initial curvature. So, R_x is the original radius of curvature of the shell and $R_x^{'}$ is now the curvature which is formed after deflection of the shell.

So, if this is so, then normal strain produced due to bending of the lamina; if we take a lamina in this middle surface, then we can see that $\epsilon_x =$. That we can see that normal strain due to bending is $-z \times curvature$. Now here, curvature is modified, this $1/R_x^{'}$ as modified as $\frac{1}{R_x^{'}} - \frac{1}{R_x}$. So, 2 radius of curvatures are here, one indicates the radius of curvature after deformation, and another represents the radius of curvature before deformation.

And this is the factor that we have taken originally due to effect of curvature, so $1 - z - R_x$. Now suppose the thickness is very small compared to your radius of curvature, then these quantities will be neglected, this $z - R_x$ will be neglected. So, in that case, the normal strain produced due to bending will be $\epsilon_x =- z\times$(curvature along the x-direction). So, this is you can tell the effective curvature, you can take it as effective curvature as the difference of the curvature after deformation and the curvature before deformation.

Similarly, we can say that ϵ_y the normal strain produced due to bending along the y-direction = $-\frac{z}{1-\frac{z}{R_y}}\left(\frac{1}{R_y^{'}} - \frac{1}{R_y}\right)$. So, again, this factor is due to curvature effect and one is one term that is $\frac{1}{R_y^{'}}$ is the curvature after deformation and $\frac{1}{R_y}$ is the curvature before deformation. So, when you neglect the contribution of this thickness that is $z - R_y$, that term is small because the thickness is small in comparison to radius of curvature, then it becomes a simple expression $\epsilon_y =- z\times($ curvature along the y-direction).

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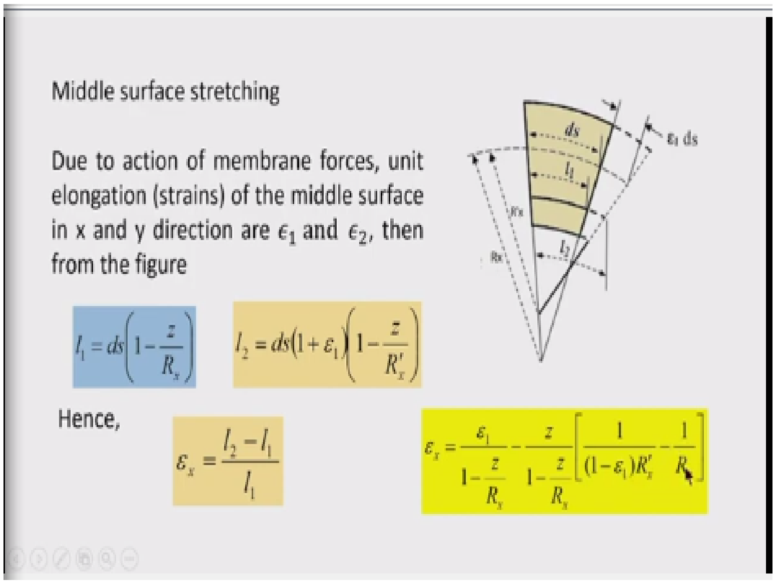

So, now let us consider the middle surface elongation due to action of membrane forces unit elongation of the middle surface in x and y-direction are ϵ_1 and ϵ_2. Previously we have seen that ϵ_x and ϵ_y are the strain or unit elongation produce due to bending of the lamina, middle surface. Now here we will tell this ϵ_1 and ϵ_2 are 2 strain quantities which refers to this strain due to elongation of the middle surface in x and y-direction, respectively.

So, in this figure, a section is taken that is along the x-direction. So, you can see here that l_1 is the original length of the fibre, and after deformation, you can see that the length becomes l_2, the elongation in normal the strain produced in the normal direction now will be $\frac{l_2 - l_1}{l_1}$. But l_1 that we can see here, l_1 is nothing but the arc length $ds \times (1 - z/R_x)$, so this factor is taken because of the existence of the curvature in the shell.

If there is no curvature, then $l_1 = ds$ or z/R_x is very small, then also $l_1 = ds$. So, here the length of the element that elongates after the middle surface undergoes in-plane strain, then this element ds will be increased. So, the increment of ds will be found out as $ds(1 + \varepsilon_1)$. So, this is the new length of the element ds, and therefore l_2 is found as $ds\left(1 + \varepsilon_1\right)\left(1 - \frac{z}{R'_x}\right)$, R'_x is taken because this refers to the radius of curvature after deformation of the shell.

So, then the unit elongation along the x-direction due to middle surface stretching is nothing but $\frac{l_2 - l_1}{l_1}$. Now substitute the value of l_2 here from this equation to here and also the value of l_1 to this equation and here. Then after simplification, you can write ϵ_x as $\varepsilon_x = \frac{\varepsilon_1}{1 - \frac{z}{R_x}} - \frac{z}{1 - \frac{z}{R_x}}\left[\frac{1}{(1 - \varepsilon_1)R'_x} - \frac{1}{R_x}\right]$.

Now you can this expression is obtained just by substituting this l_1, l_2 in this expression and simplification. Now while simplifying this result, you will find that here, in this term, inside the bracket first term, you will get $\frac{(1 + \epsilon_1)}{R'x}$. This factor was originally not there; it will be the factor $1 + \epsilon_1$ will be appearing in the numerator and denominator no factor was there only R'_x was there.

But if I multiply $\left(1 + \epsilon_1\right)$ by $(1 - \epsilon_1)$, that is, if I multiply the numerator and denominator of this expression by $\left(1 - \epsilon_1\right)$. Then in numerator, this factor will be $\left(1 - \epsilon_1^2\right)$ and denominator $\left(1 - \epsilon_1\right)$. Now seen this strain is small, ϵ_1 is a small quantity, so ϵ_1^2 is neglected. So, ultimately final expression becomes this $\frac{\varepsilon_1}{1 - \frac{z}{R_x}} - \frac{z}{1 - \frac{z}{R_x}}\left[\frac{1}{(1 - \varepsilon_1)R'_x} - \frac{1}{R_x}\right]$.

So, this term that you are seeing $1 - \frac{z}{R_x}$ or here $1 - \frac{z}{R_x}$ is due to your the curvature of the element. Now when $\frac{z}{R_x}$ is small, that means when it becomes small, when the shell is thin? If it is a thin shell, then the thickness of the shell is very, very less compared to the radius of curvature. So, in that case, $\frac{z}{R_x}$ is a negligible quantity, that means it is very small compared to 1. So, in that case, the expression becomes $\epsilon_1 - z\left[\frac{1}{(1-\varepsilon_1)R'_x} - \frac{1}{R_x}\right]$.

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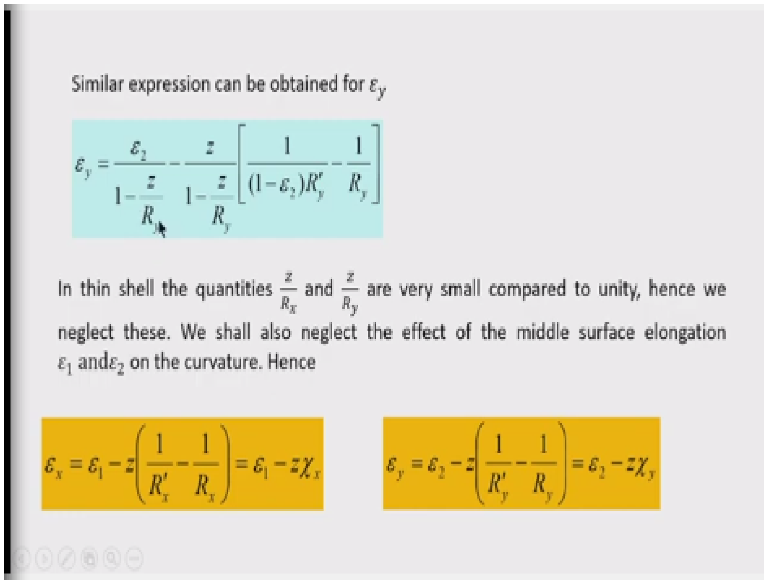

So, in a similar fashion, we can calculate the strain ϵ_y. So, ϵ_y is calculated as $\frac{\varepsilon_2}{1-\frac{z}{R_y}}$, R_y is the radius of curvature in this x-z plane. Then $\frac{z}{1-\frac{z}{R_y}}\left[\frac{1}{(1-\varepsilon_2)R'_y} - \frac{1}{R_y}\right]$. So, R'_y and R_y, these are the principal radius of curvature and 1 denotes the radius of curvature after deformation, that is **we** denoted by prime and another is before deformation.

Now in thin shell structures, the quantity z/R_x and z/R_y are small compared to unity. Therefore, we can neglect this, and hence in simplified form, the ϵ_x and ϵ_y can be written as $\varepsilon_1 - z\left(\frac{1}{R'_x} - \frac{1}{R_x}\right) = \varepsilon_1 - z\chi_x$, what is χ_x? χ_x is the effective curvature. So, this curvature term is taken χ_x as the curvature, which is taken as the difference of 2 curvature; what are the 2 curvature? One curvature is curvature after deformation, and another is curvature before deformation.

So, we get now the expression of ϵ_x, including the middle surface, stretching, that is $\varepsilon_1 - z\chi_x$. Similarly, ε_y can be calculated as $\varepsilon_2 - z\left(\frac{1}{R'_y} - \frac{1}{R_y}\right)$, R'_y is the again your the curvature of the middle surface in the x-z plane. And R' denotes the curvature, R' denotes the radius of curvature after deformation and R_y denotes the radius of curvature before deformation. So, in a similar way, we can write $\epsilon_y = \varepsilon_2 - z\chi_y$. So, χ_y is the effective curvature in the y-direction, which is nothing but $\frac{1}{R'_y} - \frac{1}{R_y}$.

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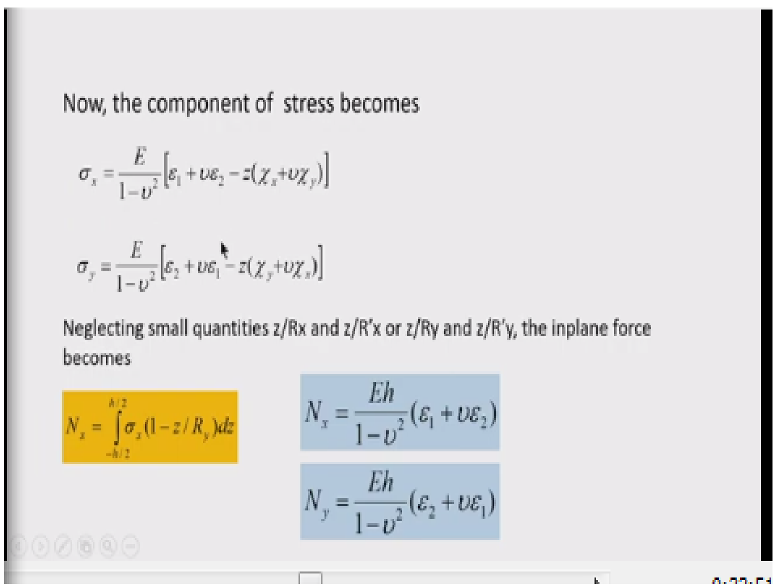

Now the component of stress we can write because knowing the strain, we can now form a stress-strain relationship using the Hooke's law because it is a linear elastic problem. So, we can use the Hooke's law, and we can write $\sigma_x = \frac{E}{1-\upsilon^2}\left[\varepsilon_1 + \upsilon\varepsilon_2 - z(\chi_x + \upsilon\chi_y)\right]$. Similarly, σ_y, we can write $\frac{E}{1-\upsilon^2}\left[\varepsilon_2 + \upsilon\varepsilon_1 - z(\chi_y + \upsilon\chi_x)\right]$.

Now here you can see that the ν is a Poisson ratio and E is a Young's modulus of elasticity, ϵ_1, ϵ_2 are the normal strains due to in-plane forces, whereas this χ_x and χ_y are the curvatures and $- z\times\chi_x$ and also $- z\times\nu\times\chi_y$, ν is due to the Poisson effect, so that quantity it denotes the normal strain due to bending. So, the total normal stress is written as this.

Similarly, $\sigma_y = \frac{E}{1-\upsilon^2}\left[\varepsilon_2 + \upsilon\varepsilon_1 - z(\chi_y + \upsilon\chi_x)\right]$ and bracket close. Now, if we neglect these small quantities z/R_x and z/R_x' or z/R_y and z/R_y', why we neglect this? Because the shell structure is very thin, thickness of the shell is very small. So, therefore in relation to radius of curvature, this quantity z/R_x, this ratio will be very small compared to 1, so therefore we neglect this.

And therefore, the in-plane forces N_x now becomes $\sigma_x(1 - z/R_y)$, and N_x will be this quantity that we have found out, σ_x if we neglect this R_x, and R_x', these quantities. Then we can simply write this here, and after integration, you will find that $N_x = \frac{Eh}{1-\upsilon^2}(\varepsilon_1 + \upsilon\varepsilon_2)$. Similarly, $N_y = \frac{Eh}{1-\upsilon^2}(\varepsilon_2 + \upsilon\varepsilon_1)$.

Here you can see that this in-plane force as N_x, N_y are expressed as the force per unit length. So, when we divide it by h, then we get σ_x and σ_y, but this relationship is obtained on the assumption that z/R_x and z/R_y and z/R_x' or z/R_y' are small compared to 1.

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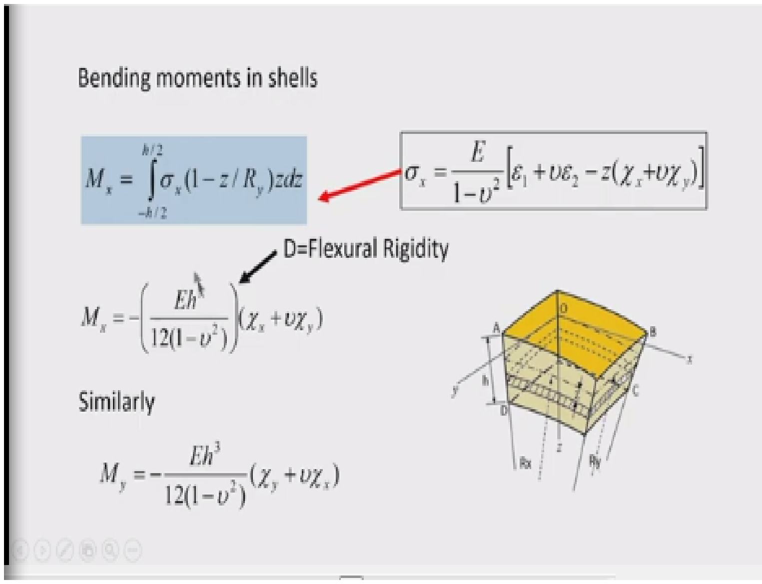

So, bending moment expression in this shell. So, we have earlier shown that for a small element dx, the bending moment or moment of resistance can be written as $\sigma_x\left(1 - \frac{z}{R_y}\right)z\ (dz \times 1)$, that is the area and lever arm is z. So, after integration in the limit $-\frac{h}{2}\ to + \frac{h}{2}$, we get the expression of M_x. Now substituting the value of σ_x here, we have got the σ_x earlier $\frac{E}{1-\upsilon^2}\left[\varepsilon_1 + \upsilon\varepsilon_2 - z(\chi_x + \upsilon\chi_y)\right]$.

So, that expression, if substituted in the expression for M_x the integral expression, and after integration, we get this term $M_x = -\left(\frac{Eh^3}{12(1-\upsilon^2)}\right)(\chi_x + \upsilon\chi_y)$. So, this term is obtained, and you know that this term resembles the same quantity that we obtain in case of this plate and $\frac{Eh^3}{12(1-\upsilon^2)}$ is nothing but flexural rigidity of the shell.

Flexural rigidity of the shell and flexural rigidity of the plate has similar kind of expression. So, in that case, only material properties are of importance, and there is no curvature parameter. Though we neglect in some analysis this bending moment in the shell, but the flexural rigidity is also important for shell. Because shell is very rigid structure and the highest strength weight ratio for the shell is a salient feature.

So, even the bending moment is neglected in case of shell in thin shell the flexural rigidity is important parameter in the shell and at the edges when the shell joins the edge beam or a support. Then for calculating the bending moment, we need to use the flexural rigidity term. Similarly, in the direction of y, this M_y is written as $-\frac{Eh^3}{12(1-\upsilon^2)}(\chi_y + \upsilon\chi_x)$.

These are the curvatures term, χ_y is the curvature along the y-direction, and χ_x is the curvature along the x-direction. Now here curvature term can be elaborately written in case of shell with 2 parameters, one is the curvature before deformation, and another is curvature after deformation, so χ_y will be $\frac{1}{R'_y} - \frac{1}{R_y}$. Similarly, χ_x can be written as $\frac{1}{R'_x} - \frac{1}{R_x}$.

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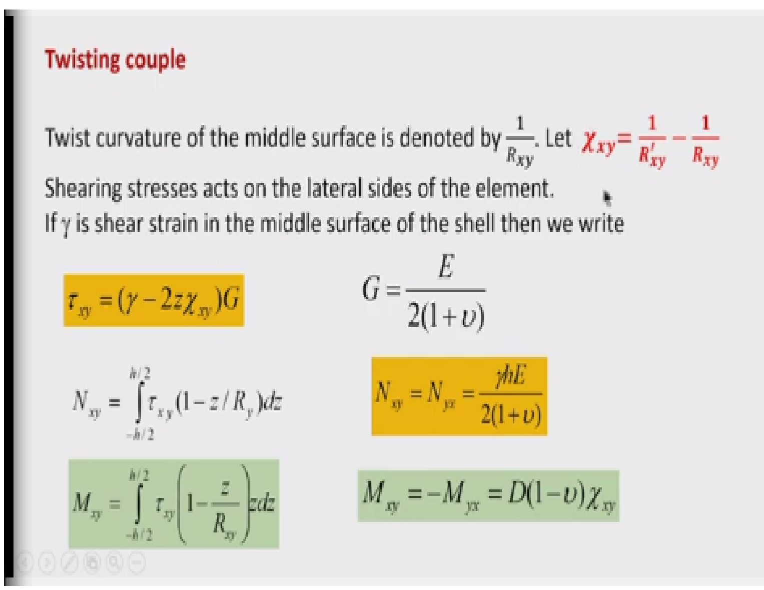

Now twisting couple, twisting curvature of the middle surface denoted by this factor, and in some cases, the twisting couple is of importance in the analysis of the shell. So, χ_{xy}, that is, the effective twisting curvature is also found as $\frac{1}{R'_{xy}} - \frac{1}{R_{xy}}$, where R'_{xy} refers to the deformed surface, and R_{xy} is the twisting curvature of the surface before deformation.

So, shearing stress act on the lateral sides of the element that we have shown you, and if gamma is the shear strain in the middle surface of the shell, then we can write this τ_{xy}, now $\gamma - 2z\chi_{xy}$. So, this factor is taken due to shearing stress produced during bending. So, gamma minus this factor will represent the shearing strain in the shell subjected to in-plane forces and twisting moment.

So, therefore the $\tau_{xy} = (\gamma - 2z\chi_{xy})G$, G is the shear modulus of the shell and which is denoted by $\frac{E}{2(1+\upsilon)}$. Now, if I want to calculate this membrane shear force N_{xy}, then τ_{xy} term is taken here, and it is integrated within the limit $-\frac{h}{2}$ to $+\frac{h}{2}$. Of course, this factor is there $(1 - z/R_y)$, which represents the effect of curvature. So, after integration and neglecting this term z/R_y in comparison to 1, then we can get $N_{xy} = N_{yx}$ will be $\frac{\gamma hE}{2(1+\upsilon)}$.

Because N_{xy} will be N_{yx} when the z/R_x or z/R_y is neglected in comparison to 1. So, in that case, N_{xy} will be N_{yx}; otherwise, this 2 factors are here for N_{xy}, it is R_y, for N_{yx}, it will be R_x, so z/R_x and z/R_y have to be equal otherwise N_{xy} will not be equal to N_{yx}. So, twisting moment we calculate now as $\tau_{xy} \times zdz$, $\tau_{xy}\ dz$ is the shear force into multiplied by z and then integrated within the limit $-\frac{h}{2}$ to $+\frac{h}{2}$, we get the expression of M_{xy}.

So, M_{xy} is nothing, but $-M_{yx}$ is found as $D(1 - \upsilon)\chi_{xy}$, where χ_{xy} denotes the effective twisting curvature. That means taking account of the curvature after deformation and before deformation. Because shell is already curved surface, so we have some curvature before the shell undergoes any deformation. So, that curvature is simply $1/R_{xy}$, and after deformation, the curvature is taken as $1/R'_{xy}$ for the deflected surface produced due to loading. So, difference of these 2 will give you the effective twisting curvature, effective twisting curvature is χ_{xy}.

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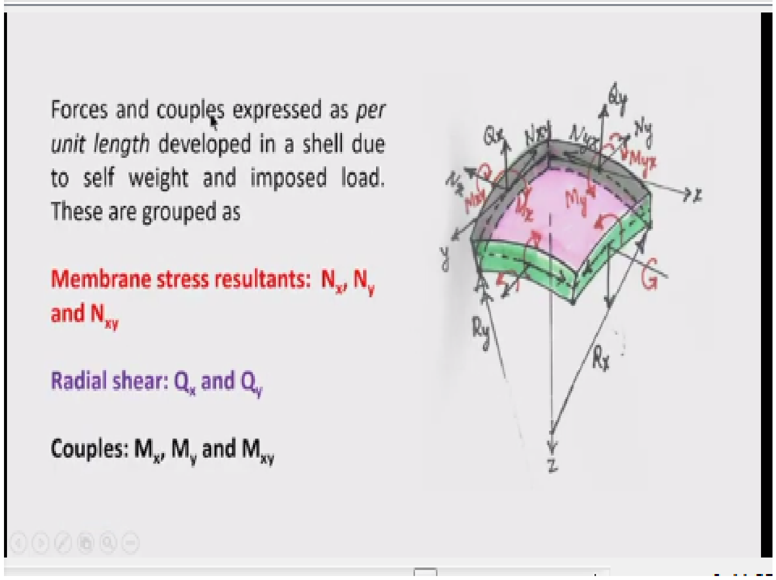

Now we have obtain the forces and couple the expression for forces, and couple are now to be used in other problems. And we have obtained what are the stress resultant one group is membrane stress resultant, that is, N_x, N_y and N_{xy}. So, this state of stress in shell is mostly accepted in design. Because these are found to be predominant in case of thin shell, then radial shear is Q_x, Q_y and bending moment or twisting moment is M_x, M_y and M_{xy}.

So, you can see altogether there are 8 stress resultants in a shell element so that we have to find out by analysis. But after certain assumption, this number of stress resultants will be reduced based on the different theories. Now here, you can see in a element of shell the N_x is the membrane shear force along the x-direction, then N_y is the membrane shear force along the y-direction, and N_{xy} and N_{yx} are the membrane shear force.

N_x and N_y are the actually it is normal force, so it is the membrane force, and N_{xy} and N_{yx} are called the membrane shear force, in-plane membrane force N_x, N_y these are in the normal direction, and these are direct stresses, whereas N_{xy} and N_{yx} are the shear stresses. Then we have the radial shear in the element, which is Q_x and Q_y, Q_x the radial shear per unit length x along the edge, which is parallel to y-axis, and Q_y is the shear force that acts along the edge, which is parallel to x-axis.

Then we have coupled M_x, M_y and M_{xy}, M_x is the bending moment along the x-direction, M_y is the bending moment along the y-direction, and then M_{xy} is that twisting moment. So, M_{xy} you are seeing here it is shown, in the opposite side this M_{xy} will appear with increment. R_x is the radius of curvature, and R_y is the radius of curvature of another curve, the tangent to this curve is parallel to x-axis, and R_x is the radius of the curvature or radius of the curve in which the tangent is parallel to the y-axis.

So, if the shell element is acted upon the vertical forces or any other forces not necessary vertical. So, then knowing the component of the σ_x and then N_x is related to σ_x, N_y will be related to your σ_y, and Q_x will give you τ_{xz}, Q_y will give you τ_{yz}, and this N_{xy} will be related to τ_{xy}, and N_{yx} will be related to τ_{yx}.

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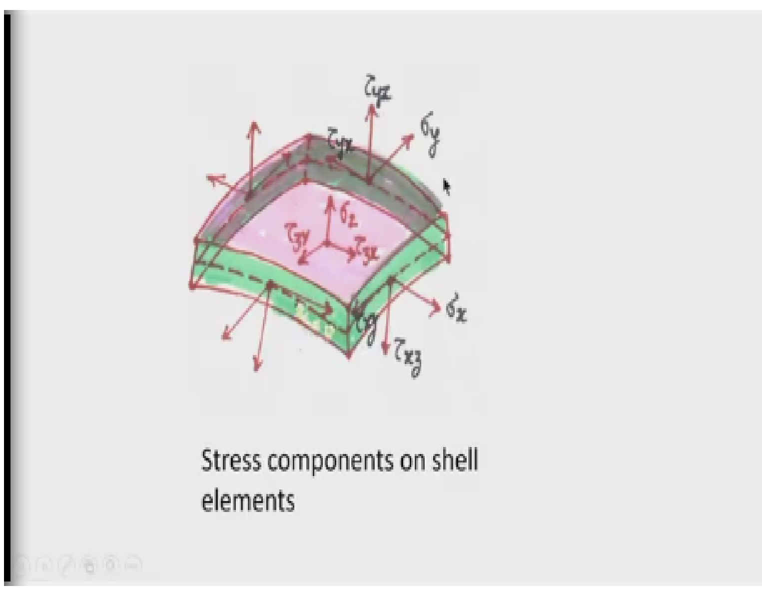

So, we get a state of stress like that in a shell element. So, σ_x, τ_{xy} and σ_y these 2 are the main stress component. And in some theories, and we neglect this, τ_{yz}, τ_{xz} and σ_z is neglected always in thin shell theories, and τ_{zy}, τ_{zx} is also neglected. So, main stress component that are taken into analysis of the shell structure and design of that is σ_x, σ_y, τ_{xy} or τ_{yx}.

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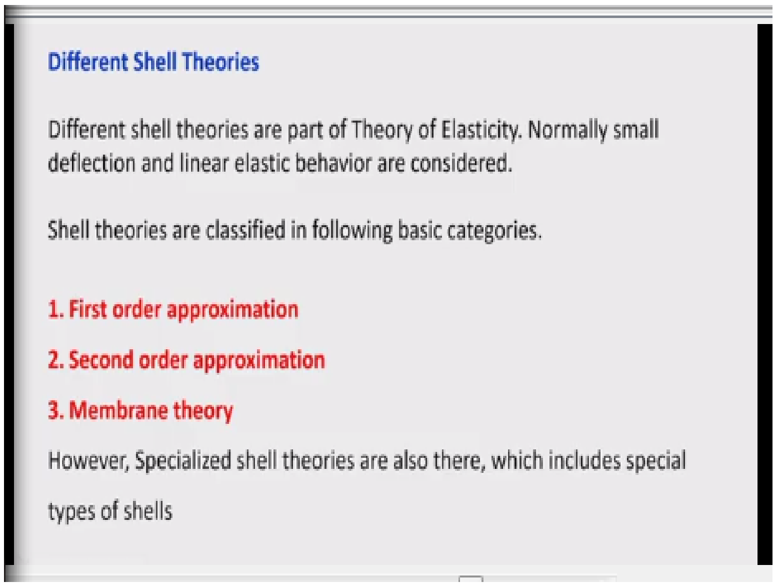

Now there are different shell theories based on which the analysis of the shell is carried out, and design is also based on that theories. So, differential theories are part of actually the theory of elasticity. Normally small deflection is assumed, and linear elastic behaviour are consider; that I have shown you that using the Hooke's law, we obtain the stress-strain relationship. Shell theories are classified in the following basic categories.

But there are other theories also, I have mentioned only 3 theories because 3 theories have significant difference in between them. And we can group them into separate class, and each theory will find that there is different aspects, and sometimes, the theory gives improved results and some theories are simplified but gives acceptable result. So, let us first consider the first-order approximation theory.

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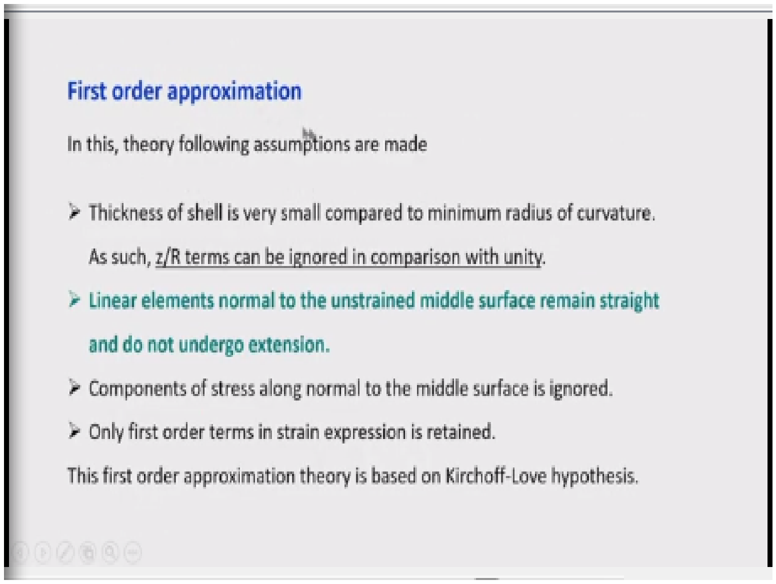

These theories based on following assumption. You will find some assumptions are common to these Kirchhoff's law plate theory that we have discussed earlier. So, in the first-order approximation theory of shell, the thickness of the shell is assumed to be small compared to minimum radius of curvature because, in a doubly curved shell, we have 2 curvature in 2 orthogonal directions.

So, therefore the minimum radius of curvature is taken and compared with the thickness. If the ratio of thickness to minimum radius of curvature is small, then we can use the first-order approximation theory. So, in that case, $\frac{z}{R}$ terms that I have shown you in different expression of the strain that we have obtain in my earlier slides $\frac{z}{R}$ terms becomes small compared to 1, and therefore these terms can be neglected.

So, this is the salient features of this first-order approximation. Linear elements normal to the unstrained middle surface that is before deformation, remain straight and do not undergo extension. So, this assumption is similar to Kirchhoff's law of hypothesis that we have earlier found. And also, the component of stress along normal to the middle surface is ignored. That means if I see the stress component normal to the middle surface, that is, σ_z is neglected, and then τ_{yz} is neglected, and τ_{xz} is neglected, so this 3 quantities are neglected.

So, these are the salient features of the first-order approximation theory, and these are in similar line as we have found in the Kirchhoff's plate theory Kirchhoff's in case of thin plate only first product terms in the strain expressions is written. That means strain expressions say, for example, the normal strain expression $\frac{\partial u}{\partial x}$, that is the linear term that we only consider if higher-order terms of these derivative are there in the strain expression we neglected. So, these are some points of the first-order approximation theory.

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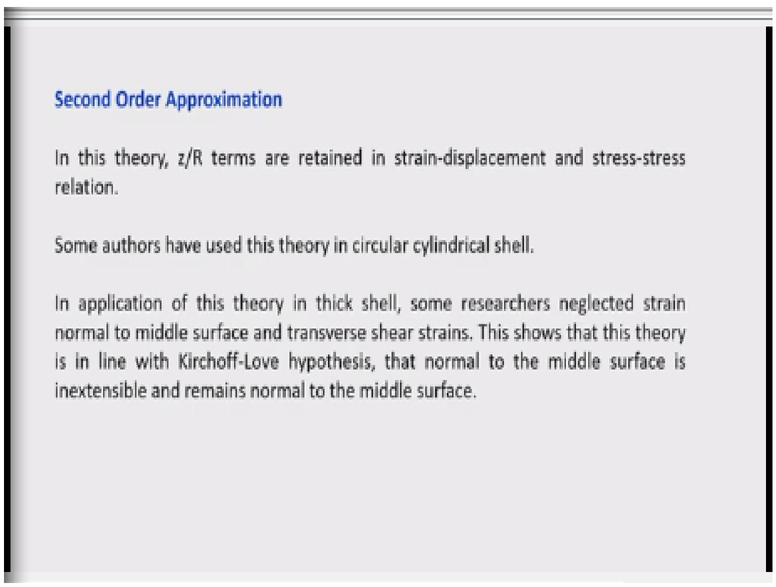

In this second-order approximation theory, $\frac{z}{R}$ terms are to be retained in the stain displacement and stress-strain relationship. We have earlier simplified this stress-strain relationship neglecting the $\frac{z}{R}$ terms in first-order theory. But in the second-order theory full expression, that we have found the stain has that quantity that $1 - z/R_y$ or $1 - z/R_x$; this term appears in the denominator, so these terms should not be neglected. So, we take this z/R terms and carry out the analysis. So, some authors have used this theory in circular cylindrical shell.

But very simplified analysis or in simple cases, these theories are applied. In application of this theory in thick shell, some researcher neglected strain normal to the middle surface and transverse shear strain. So, if one neglect this strain normal to the middle surface and transverse shear strain, this shows that this theory is in line with the Kirchhoff's law hypothesis. And that shows the normal to the middle surface remains normal after deformation, and it is inextensible and remains straight to the normal and straight to the middle surface.

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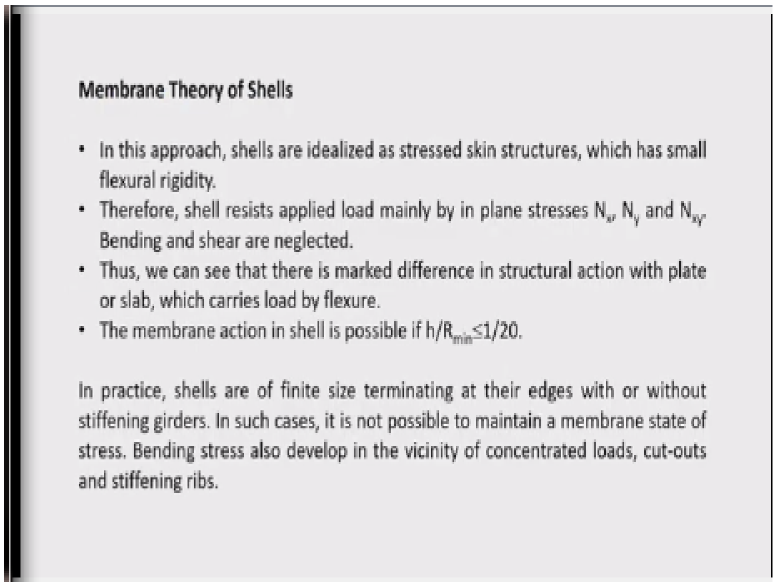

The 3rd theory that is the membrane theory of shell, is very popular in the analysis and design of shell. In most of the design of shells, the membrane theory of shells are adopted because this gives simplification in the analysis and also produces the result which are reasonably accurate and acceptable for practical design. So, in this approach, shells are idealized as stress skin which has small flexural rigidity.

That means the bending resistance of the shell is neglected. So, the load is resisted mainly by the in-plane forces. So, in-plane forces that are denoted by N_x, N_y and N_{xy}, N_x, N_y are the direct stresses and N_{xy} are these shear stresses. So, N_x, N_y and N_{xy} will be responsible for resisting the load that acts on the shell. Bending and shear that is Q_x, Q_y, M_x, M_y and M_{xy} are neglected in this theory.

So, this theory gives a very simplified approach for the designer and the results are obtained without difficulty. These calculations becomes less, at the edges, this membrane stress condition may not be fully satisfied. But away from the edges, the results are fairly accurate and give the reasonable result and can be accepted to determine the thickness and in case of reinforced concrete shell, the reinforcement steel area based on this direct stresses N_x, N_y.

Thus we can see that there is marked difference in structural action with plate or slab, which carries the load by flexure. That in case of plate or slab that we have seen earlier, that is mainly a flexural element. And the bending moments are developed, the moment of resistance are developed to basis the load, but this is not the case of thin shell. In thin shell membrane forces, the in-plane forces are main resisting forces for the external load.

The membrane action of the shell is possible if $\frac{h}{R_{min}} < \frac{1}{20}$, so that is prescribed by this various code of practice and we can take this tender that membrane analysis can be done is $\frac{h}{R_{min}} < 0.05$. But in practical situation, shells are actually finite size terminating at their edges, and sometimes the stiffening girders may be there or maybe without stiffening girders.

In such cases to maintain the membrane state of stress is not possible, so bending stress have to be considered in the vicinity of the support so that we have to take. And membrane theory of shells are very well adopted in case of, say spherical dome or a surface of revolution of thin elements and accept near the edge and also in case of pressure vessel. Say a spherical vessel subjected to internal pressure, the uniform state of stress is produced, and as a hook tension, the load will be resistant.

But if the ends of the shell are fixed, then the disturbance at the edges will be there, and this will be actually producing the bending effects in the shell. And therefore, membrane state of stress will not be valid near the supports. However, the membrane theory of shells are well accepted and used for design of many important shell structure also. Sometimes designer without rigorous calculation, in case the thickness and reinforcement near the edges to take care of the bending effects, thank you very much.